Generalized Riemann Functions, Their Weights and Their Evaluation on Complete Graphs

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Abstract

By a *Riemann function* we mean a function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ such that $f(\mathbf{d})$ is equals 0 for $d_1 + \cdots + d_n$ sufficiently small, and equals $d_1 + \cdots + d_n + C$ for a constant, C, for $d_1 + \cdots + d_n$ sufficiently large. By adding 1 to the Baker-Norine rank function of a graph, one gets an equivalent Riemann function, and similarly for related rank functions.

To each Riemann function we associate a related function $W\colon \mathbb{Z}^n \to \mathbb{Z}$ via Möbius inversion that we call the *weight* of the Riemann function. We give evidence that the weight seems to organize the structure of a Riemann function in a simpler way: first, a Riemann function f satisfies a Riemann-Roch formula iff its weight satisfies a simpler symmetry condition. Second, we will calculate the weight of the Baker-Norine rank for certain graphs and show that the weight function is quite simple to describe; we do this for graphs on two vertices and for complete graphs.

For complete graphs, we build on the work of Cori and Le Borgne who gave a linear time method to compute the Baker-Norine rank of the complete graph. The associated weight function has a simple formula and is extremely sparse (i.e., mostly zero). Our computation of the weight function leads to a new linear time algorithm to compute the Baker-Norine rank, via a new formula likely related to one of Cori and Le Borgne, but seemingly simpler for general $\mathbf{d} \in \mathbb{Z}^n$, namely

$$r_{\mathrm{BN},K_n}(\mathbf{d}) = -1 + \left| \left\{ i = 0, \ldots, \deg(\mathbf{d}) \;\middle|\; \sum_{j=1}^{n-2} \bigl((d_j - d_{n-1} + i) \bmod n \bigr) \leqslant \deg(\mathbf{d}) - i \right\} \right|.$$

However, the formula of Cori and Le Borgne, which requires $\mathbf{d} \in \mathbb{Z}^n$ to be a sorted parking function, is easier to evaluate for such $\mathbf{d}$.

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Our study of weight functions leads to a natural generalization of Riemann functions, with many of the same properties exhibited by Riemann functions.

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Contents

1 Introduction

The main goal of this article is to give a combinatorial study of what we call *Riemann functions* and their *weights*. Our main motivation is to gain insight into the special case that is the Graph Riemann-Roch formula of Baker and Norine [6]; the Baker-Norine formula has received a lot of recent attention [9, 4, 18, 7], as has its generalization to *tropical curves* and other settings in recent years [4, 12, 14, 15, 1, 17, 2, 8].

We were first interested in weights to address a question posed in [6] regarding whether or not their Graph Riemann-Roch formula could be understood as an Euler characteristic equation; this is partially answered in [11]. However, weights are interesting for a number of purely combinatorial reasons: first, a Riemann-Roch formula is simpler to express in terms of the weight of the Riemann function. Second, the weights of the Riemann-Roch functions of certain graphs are very simple to write down. For example, in this article we build on the methods of Cori and Le Borgne [9] to give a very simple formula for the weights of the Baker-Norine rank function of a complete graph; in addition, this leads to a new formula for the Baker-Norine rank of a complete graph, that can be written is a simple fashion for general elements of $\mathbb{Z}^n$. [By contrast, the Cori and Le Borgne formula is faster to evaluate, although their formula requires the element of $\mathbb{Z}^n$ to be a "sorted parking configuration," which requires some precomputation for a general element of $\mathbb{Z}^n$; the two formulas seem to reflect different aspects of the Baker-Norine rank for complete graphs.] Given the above, as well as the connection of weights to sheaves and Euler characteristics in [11], we suspect that weights may be a useful way to describe many Riemann functions.

This article has two types of results: foundational results on Riemann functions and Riemann-Roch type formulas, and calculations of the weights of Baker-Norine rank functions of two types of graphs. Let us briefly summarize the results, assuming some terminology that will be made precise in Section 2.

1.1 Riemann Functions and Weights

By a *Riemann function* we mean a function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ (where $\mathbb{Z}$ denotes the set of integers) such that $f(\mathbf{d}) = f(d_1,\ldots,d_n)$ is *initially zero*, meaning $f(\mathbf{d}) = 0$ for $\deg(\mathbf{d}) = d_1 + \cdots + d_n$ sufficiently small, and *eventually*—meaning for $\deg(\mathbf{d})$ sufficiently large—equals $\deg(\mathbf{d}) + C$ for a constant, $C \in \mathbb{Z}$, which we call the *offset of* f. By adding 1 to the Baker-Norine rank function of a graph [6], one gets an equivalent Riemann function, and similarly for related rank functions.

If $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is any function that is initially zero, then there is a unique, initially zero W such that

$$f(\mathbf{d}) = \sum_{\mathbf{d}' \leqslant \mathbf{d}} W(\mathbf{d}')$$

where $\leqslant$ the usual partial order on $\mathbb{Z}^n$ (i.e., $\mathbf{d}' \leqslant \mathbf{d}$ means $d'_i \leqslant d_i$ for all $i = 1,\ldots,n$); we call W the *weight* of f. If f is a Riemann function, then W is also eventually zero; much of what we prove about Riemann functions also holds for *generalized Riemann functions*, which we define as any initially zero function f whose weight is eventually zero.

Returning to a Riemann function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ with offset C, for any $\mathbf{K} \in \mathbb{Z}^n$ there exists a unique function $f^\wedge_{\mathbf{K}}$ such that for all $\mathbf{d} \in \mathbb{Z}^n$ we have

$$f(\mathbf{d}) - f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + C, \tag{1}$$

and we refer to as a *generalized Riemann-Roch formula*; $f^\wedge_{\mathbf{K}}$ is also a Riemann function. Furthermore, if $f^\wedge_{\mathbf{K}} = f$ for some f, K, then the formula reads

$$f(\mathbf{d}) - f(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + C,$$

which is the usual type of Riemann-Roch formula, both the classical formula of Riemann-Roch, and the Baker-Norine analog. Hence, our view of Riemann-Roch formulas is more "happy-go-lucky" than is common in the literature: for each $f, \mathbf{K}$ there is a generalized Riemann-Roch formula (1); we study any such formula, and view the case where $f^\wedge_{\mathbf{K}} = f$ as a special case which we call *self-duality*.

We are interested in weight functions, W, for a number of reasons:

1. the weights of the Baker-Norine rank (plus 1) of the graphs we study in this article turn out be be simple to describe and very sparse (i.e., mostly 0); by contrast, at least for the complete graph, the Baker-Norine function is more difficult to compute. Hence the weights may be a more efficient way to encode certain Riemann functions of interest.

2. For a Riemann function $f\colon \mathbb{Z}^n \to \mathbb{Z}$, the weight of $f^{\wedge}_{\mathbf{K}}$ turns out to equal $(-1)^n W^*_{\mathbf{L}}$, where $\mathbf{L} = \mathbf{K}+\mathbf{1}$ (where $\mathbf{1} = (1,\ldots,1)$), and $W^*_{\mathbf{L}}$ is the function $W^*_{\mathbf{L}}(\mathbf{d}) = W(\mathbf{L}-\mathbf{d})$; hence it seems easier to check self-duality using the weight, W, rather than directly on f. A referee for the paper has pointed out that the map $\mathbf{d} \mapsto \mathbf{L}-\mathbf{d}$ occurs in [6], as the map giving a bijection from v_0-reduced divisors to v_0-critical divisors ([6], see Lemma 5.6, where K^+ there equals $\mathbf{L} = \mathbf{K}+\mathbf{1}$ in this article).

3. In [11], we model Riemann functions by restricting $f\colon \mathbb{Z}^n \to \mathbb{Z}$ to two of its variables, while holding the other $n-2$ variables fixed; if f satisfies self-duality, a two-variable restriction, $\widetilde{f}\colon \mathbb{Z}^2 \to \mathbb{Z}$, of f will generally not be self-dual; however for any $\widetilde{\mathbf{K}} \in \mathbb{Z}^2$, $\widetilde{f}^{\wedge}_{\widetilde{\mathbf{K}}}$ can be described as a restriction of $f^{\wedge}_{\mathbf{K}}$ (for any $\mathbf{K} \in \mathbb{Z}^n$). Since self-duality isn't preserved under restrictions, but generalized Riemann-Roch formulas behave well under restrictions, it seems essential to work with generalized Riemann-Roch formulas (1) in [11] or, more generally, whenever we wish to work with restrictions of Riemann functions to a subset of their variables.

4. In certain Riemann functions of interest, such as those considered by Amini and Manjunath [2], self-duality does not generally hold, and yet one can always work with weights and generalized Riemann-Roch formulas.

5. The formalism of weights applies to generalized Riemann functions, which is a much wider class of functions, and we believe likely to be useful in future work to model other interesting functions. In this case (1) is replaced by

$$f(\mathbf{d}) - f^{\wedge}_{\mathbf{K}}(\mathbf{K}-\mathbf{d}) = h(\mathbf{d}),$$

where h is the unique *modular function* that eventually equals f (see Section 3, specifically Subsection 3.3). One might expect such formulas to hold when, for example $f = f(\mathbf{d})$ is the sum of even Betti numbers of a sheaf depending on a parameter $\mathbf{d} \in \mathbb{Z}^n$, whose Euler characteristic equals a modular function h.

1.2 The Weight of the Baker-Norine rank for Two Types of Graphs

The second type of result in this article concerns the explicit calculation of the weights of the Baker-Norine rank function (plus 1) for two types of graphs, namely graphs on two vertices and the complete graph, K_n, on n vertices. Both types of weight functions are quite simple and very sparse (i.e., mostly 0). For K_n we build on the ideas of Cori and Le Borgne [9] to compute the weight of the Baker-Norine rank. A side effect of this computation is a formula for the Baker-Norine rank:

$$r_{\mathrm{BN},K_n}(\mathbf{d}) = -1 + \left| \left\{ i = 0,\ldots,\deg(\mathbf{d}) \;\middle|\; \sum_{j=1}^{n-2} \bigl((d_j - d_{n-1} + i) \bmod n \bigr) \leqslant \deg(\mathbf{d}) - i \right\} \right|,$$

where the "mod" function above returns a value in $\{0,\ldots,n-1\}$; this looks different from a formula given by Cori and Le Borgne; we compare the two formulas in Subsubsection 6.7.3.

We also explain that—like the Cori and Le Borgne algorithm—there is an algorithm that computes this function in time $O(n)$. Our proof of this formula is self-contained, although uses some of the observations of Cori and Le Borge including one short and rather ingenious idea of theirs regarding the Baker-Norine function on a complete graph.

1.3 Organization of this Article

The rest of this article is organized as follows. In Section 2 we give some basic terminology, including the definition of a *Riemann function* and some examples, which (after subtracting 1) includes the Baker-Norine rank. In Section 3 we discuss what we mean by the *weight* of a Riemann function; this leads to a notation of *generalized Riemann functions*, which share many of the properties of Riemann functions. In Section 4 we define what we mean by a Riemann-Roch formula; we describe the equivalent condition on weights, which is simpler; these ideas generalize in a natural way to the setting of generalized Riemann functions. In Section 5 we compute the weight of the Baker-Norine rank for graphs on two vertices, joined by any number of edges. In Section 6 we compute the weight of the Baker-Norine rank for a complete graph on n vertices, and we give a formula for the Baker-Norine rank, which—like a related formula of Cori and Le Borgne—allows the rank to be computed in linear time in n. In Section 7 we prove our main theorems—stated earlier—that characterize *modular functions* used to define generalized Riemann functions.

1.4 Acknowledgements

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2 Basic Terminology and Riemann Functions

In this section we introduce some basic terminology and define the notion of a Riemann function. Then we give some examples of Riemann functions.

2.1 Basic Notation

We use $\mathbb{Z}, \mathbb{N}$ to denote the integers and positive integers; for $a \in \mathbb{Z}$, we use $\mathbb{Z}_{\leqslant a}$ to denote the integers less than or equal to a, and similarly for the subscript $\geqslant a$. For $n \in \mathbb{N}$ we use $[n]$ to denote $\{1, \ldots, n\}$. We use bold face $\mathbf{d} = (d_1, \ldots, d_n)$ to denote elements of $\mathbb{Z}^n$, using plain face for the components of $\mathbf{d}$; by the *degree* of $\mathbf{d}$, denoted $\deg(\mathbf{d})$, we mean $d_1 + \ldots + d_n$. We use the usual L^1-norm

$$\|\mathbf{d}\|_{L^1} \stackrel{\text{def}}{=} |d_1| + \cdots + |d_n|$$

(generally on $\mathbb{Z}^n$), and for $\mathbf{d}, \mathbf{d}'$ we also write $\rho_{L^1}(\mathbf{d}, \mathbf{d}')$ for $\|\mathbf{d} - \mathbf{d}'\|_{L^1}$.

We set

$$\mathbb{Z}^n_{\deg=0} = \{\mathbf{d} \in \mathbb{Z}^n \mid \deg(\mathbf{d}) = 0\},$$

and for $a \in \mathbb{Z}$ we similarly set

$$\mathbb{Z}^n_{\deg=a} = \{\mathbf{d} \in \mathbb{Z}^n \mid \deg(\mathbf{d}) = a\}, \quad \mathbb{Z}^n_{\deg\leqslant a} = \{\mathbf{d} \in \mathbb{Z}^n \mid \deg(\mathbf{d}) \leqslant a\}.$$

We use $\mathbf{e}_i \in \mathbb{Z}^n$ (with n understood) be the i-th standard basis vector (i.e., whose j-th component is 1 if $j = i$ and 0 otherwise), and for $I \subset [n]$ (with n understood) we set

$$\mathbf{e}_I = \sum_{i \in I} \mathbf{e}_i; \tag{2}$$

hence in case $I = \emptyset$ is the empty set, then $\mathbf{e}_\emptyset = \mathbf{0} = (0, \ldots, 0)$, and similarly $e_{[n]} = \mathbf{1} = (1, \ldots, 1)$.

For $n \in \mathbb{N}$, we endow $\mathbb{Z}^n$ with the usual partial order, that is

$$\mathbf{d}' \leqslant \mathbf{d} \quad \text{iff} \quad \forall i \in [n],\ d'_i \leqslant d_i,$$

where $[n] = \{1, 2, \ldots, n\}$.

If S is a set, we use $|S|$ to denote the cardinality of S.

2.2 Riemann Functions

In this section we define *Riemann functions* and give examples that have appeared in the literature.

Definition 1. We say that a function $f \colon \mathbb{Z}^n \to \mathbb{Z}$ is a Riemann function if for some $C, a, b \in \mathbb{Z}$ we have

1. $f(\mathbf{d}) = 0$ if $\deg(\mathbf{d}) \leqslant a$; and
2. $f(\mathbf{d}) = \deg(\mathbf{d}) + C$ if $\deg(\mathbf{d}) \geqslant b$;

we refer to C as the *offset* of f.

Remark 2. The referee has pointed out that although the offset C in the above definition is uniquely determined, and values of a, b are not. One could uniquely determine a, b by insisting that a, b be the "best" possible such values, i.e., that a be largest possible such value and b be the smallest (with some convention in the degenerate case where $f(\mathbf{d}) = \deg(\mathbf{d}) + C$ for all $\mathbf{d}$). Such "best possible" a, b would be interesting invariants, although in this article we do not study them.

In our study of Riemann functions, it will be useful to introduce the following terminology.

Definition 3. If f, g are functions $\mathbb{Z}^n \to \mathbb{Z}$, we say that f *equals* g *initially* (respectively, *eventually*) if $f(\mathbf{d}) = g(\mathbf{d})$ for $\deg(\mathbf{d})$ sufficiently small (respectively, sufficiently large); similarly, we say that that f is *initially zero* (respectively *eventually zero*) if $f(\mathbf{d}) = 0$ for $\deg(\mathbf{d})$ sufficiently small (respectively, sufficiently large).

Therefore $f \colon \mathbb{Z}^n \to \mathbb{Z}$ is a Riemann function iff it is initially zero and it eventually equals the function $\deg(\mathbf{d}) + C$, where C is the offset of f.

2.3 The Baker-Norine Rank and Riemann-Roch Formula

In this article we study examples of the Baker-Norine rank for various graphs. In this subsection we briefly review its definition and its properties; for more details, see [6].

We will consider graphs, $G = (V, E)$ that are assumed to be connected and may have multiple edges but no self-loops. At times we write $G = (V_G, E_G)$ to emphasize G, although we often drop the subscript G. Recall that if $G = (V, E)$ is any graph, then its *Laplacian*, Δ_G equals $D_G - A_G$ where D_G is the diagonal degree counting matrix of G, and A_G is the adjacency matrix of G.

Definition 4 (The Baker-Norine rank function of a graph)**.** Let $G = (V_G, E_G) =(V, E)$ be a connected graph without self-loops (but possibly multiple edges) on n vertices that are ordered as $v_1, \ldots, v_n$. Hence we view its Laplacian, Δ_G, as a map $\mathbb{Z}^n \to \mathbb{Z}^n$. Let $L = \mathrm{Image}(\Delta_G)$. Since $\Delta_G = D_G - A_G$ as described above, each element of the image of Δ_G has degree 0, i.e., $L \subset \mathbb{Z}_{\deg=0}$. We say that $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^n$ are *equivalent*, written $\mathbf{d} \sim \mathbf{d}'$, if $\mathbf{d} - \mathbf{d}' \in L$ (in which case $\deg(\mathbf{d}) = \deg(\mathbf{d}')$ since $L \subset \mathbb{Z}_{\deg=0}$); we say that $\mathbf{d}$ is *effective* if $\mathbf{d} \geqslant \mathbf{0}$. Let $\mathcal{N}$ be the elements of $\mathbb{Z}^n$ that are not equivalent to an effective element of $\mathbb{Z}^n$; in particular

$$\deg(\mathbf{d}) < 0 \Rightarrow \mathbf{d} \in \mathcal{N}.$$

Consider

$$f(\mathbf{d}) = \rho_{L^1}(\mathbf{d}, \mathcal{N}) = \min_{\mathbf{d}' \in \mathcal{N}} \|\mathbf{d} - \mathbf{d}'\|_{L^1}, \tag{3}$$

where $\| \cdot \|_{L^1}$ is the usual L^1-norm

$$\|(x_1, \ldots, x_n)\|_{L^1} = |x_1| + \cdots + |x_n|.$$

We also write $f = f_G$, to emphasize the graph G, although its definition as a function $\mathbb{Z}^n \to \mathbb{Z}$ also depends on the ordering $v_1, \ldots, v_n$ of its vertices. The *Baker-Norine rank* of $\mathbf{d}$, denoted $r_{\mathrm{BN}}(\mathbf{d})$, is $f(\mathbf{d}) - 1$.

Since $f(\mathbf{d}) = 0$ iff $\mathbf{d} \in \mathcal{N}$, which is the case if $\deg(\mathbf{d}) < 0$, it follows f is initially zero, and hence $r_{\mathrm{BN}}(\mathbf{d})$ initially equals -1.

The Baker-Norine *Graph Riemann-Roch* formula states that for all $\mathbf{d}$ we have

$$r_{\mathrm{BN}}(\mathbf{d}) - r_{\mathrm{BN}}(\mathbf{K}_G - \mathbf{d}) = \deg(\mathbf{d}) + 1 - g \tag{4}$$

where

1. $g = 1 + |E_G| - |V_G|$ (which is non-negative since G is connected), called the *genus of* G and

2. $\mathbf{K}_G = \big(\deg_G(v_1) - 2, \ldots, \deg_G(v_n) - 2\big)$, where $\deg_G(v)$ is the degree of v in G, i.e., the number of edges incident upon v in G.

It follows that for all $\mathbf{d} \in \mathbb{Z}^n$

$$f(\mathbf{d}) - f(\mathbf{K}_G - \mathbf{d}) = \deg(\mathbf{d}) + 1 - g. \tag{5}$$

It follows that for $\mathbf{d}$ such that

$$\deg(\mathbf{d}) > \deg(\mathbf{K}_G) = \sum_i \Big(\deg_G(v_i) - 2\Big) = 2|E| - 2|V|$$

we have $\deg(\mathbf{K}_G - \mathbf{d}) < 0$, and hence $f(\mathbf{K}_G - \mathbf{d}) = 0$; hence

$$\deg(\mathbf{d}) > 2|E| - 2|V| \quad \Rightarrow \quad f(\mathbf{d}) = \deg(\mathbf{d}) + 1 - g, \tag{6}$$

i.e., $f(\mathbf{d})$ eventually equals $\deg(\mathbf{d}) + 1 - g$. Hence f is a Riemann function with offset $C = 1 - g = |V_G| - |E_G|$.

The Baker-Norine formula is an analog of the classical Riemann-Roch formula for algebraic curves or Riemann surfaces; we briefly discuss this in Subsection 2.6.

2.4 Slowly Growing and Periodic Riemann Functions

In the examples we give of Riemann functions in this section, here are two properties found in many of our examples.

Definition 5. We say that a function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is

1. *slowly growing* if for all $\mathbf{d} \in \mathbb{Z}^n$ and $i \in [n]$ we have

$$f(\mathbf{d}) \leqslant f(\mathbf{d} + \mathbf{e}_i) \leqslant f(\mathbf{d}) + 1, \tag{7}$$

and

2. *p-periodic* for a $p \in \mathbb{N}$ if for all $i, j \in [n]$ and all $\mathbf{d} \in \mathbb{Z}^n$ we have

$$f(\mathbf{d} + p\,\mathbf{e}_i - p\,\mathbf{e}_j) = f(\mathbf{d}).$$

Let us show that the Baker-Norine rank is both periodic and slowly growing.

It is well known that $\mathbb{Z}_{\deg=0}/L$ for L above is a finite group (called $\mathrm{Pic}_0(G)$ in [5], see Section 1.3 there, and the references to [19, 3]), whose size is $\det'(\Delta_G)$ (the product of the nonzero eigenvalues of Δ_G). It follows that any element of $\mathbb{Z}_{\deg=0}$ has order divisible by $p = |\mathbb{Z}_{\deg=0}/L|$. Hence for any distinct $i, j \in [n]$, $\mathbf{e}_i - \mathbf{e}_j$ has order divisible by p, i.e., $p(\mathbf{e}_i - \mathbf{e}_j) \in L$. Since $r_{\mathrm{BN}}(\mathbf{d})$ depends only on the class of $\mathbf{d}$ modulo L, it follows that r_{BN} has period p.

To show that the Baker-Norine rank is slowly growing, let us prove some stronger properties for a more general class of functions.

Proposition 6. *Let $\mathcal{N} \subset \mathbb{Z}^n$ be nonempty and a* downset*, i.e., $\mathbf{d} \in \mathcal{N}$ and $\mathbf{d}' \leqslant \mathbf{d}$ implies that $\mathbf{d}' \in \mathcal{N}$. Let*

$$f(\mathbf{d}) \stackrel{\mathit{def}}{=} \rho_{L^1}(\mathbf{d}, \mathcal{N}). \tag{8}$$

Then for any $\mathbf{d}$ we have:

1. *$f(\mathbf{d}) = \rho_{L^1}(\mathbf{d},\mathcal{N})$ is finite;*

2. *if $\mathbf{d}' \in \mathcal{N}$ is a nearest point in L^1-distance to $\mathbf{d}$ in $\mathcal{N}$, i.e., $\rho_{L^1}(\mathbf{d},\mathcal{N}) = \|\mathbf{d}-\mathbf{d}'\|_{L^1}$ then $\mathbf{d}' \leqslant \mathbf{d}$;*

3. *if $f(\mathbf{d}) \geqslant 1$ (i.e., $\mathbf{d} \notin \mathcal{N}$), then $f(\mathbf{d}) = m+1$ where m is the largest integer such that for any $\mathbf{a} \geqslant \mathbf{0}$ and of degree m we have that $\mathbf{d}-\mathbf{a} \notin \mathcal{N}$;*

4. *if $f(\mathbf{d}) \geqslant 1$ (i.e., $\mathbf{d} \notin \mathcal{N}$), then*

$$f(\mathbf{d}) = 1 + \min_{i\in[n]} f(\mathbf{d}-\mathbf{e}_i). \tag{9}$$

Moreover, f is slowly growing.

Proof. To prove (1), $\mathcal{N}$ is nonempty, and hence contains a point $\mathbf{d}'$. Hence $\rho_{L^1}(\mathbf{d},\mathcal{N}) \leqslant \rho_{L^1}(\mathbf{d},\mathbf{d}') = \|\mathbf{d}-\mathbf{d}'\|_{L^1}$ is finite.

To prove (2), for the sake of contradiction assume that $\mathbf{d}' \leqslant \mathbf{d}$ does not hold; then for some $i \in [n]$ we have $d'_i > d_i$, and hence

$$\|\mathbf{d}-\mathbf{d}'+\mathbf{e}_i\|_{L^1} < \|\mathbf{d}-\mathbf{d}'\|_{L^1};$$

but then $\mathbf{d}'-\mathbf{e}_i \in \mathcal{N}$, since $\mathcal{N}$ is a downset, and $\mathbf{d}'-\mathbf{e}_i$ is closer to $\mathbf{d}$ (in the L^1 distance) than is $\mathbf{d}'$, which is impossible.

To prove (3), if $\mathbf{a} \geqslant \mathbf{0}$ is of degree at most $f(\mathbf{d})-1$, then $\deg(\mathbf{a}) = \rho_{L^1}(\mathbf{d},\mathbf{d}-\mathbf{a}) < f(\mathbf{d})$ and hence $\mathbf{d}-\mathbf{a} \notin \mathcal{N}$. Hence $f(\mathbf{d})-1 \geqslant m$. On the other hand, if $\mathbf{d}' \in \mathcal{N}$ is a closest point in $\mathcal{N}$ to $\mathbf{d}$, hence of distance $f(\mathbf{d})$ to $\mathbf{d}$, then (2) implies $\mathbf{d} \leqslant \mathbf{d}'$; since $f(\mathbf{d}) \geqslant 1$, $\mathbf{d} \neq \mathbf{d}'$, and hence for some $i \in [n]$ we have $d_i > d'_i$. Therefore $\mathbf{a} = \mathbf{d}-\mathbf{d}'+\mathbf{e}_i \geqslant \mathbf{0}$ satisfies $\mathbf{d}-\mathbf{a}$ is of distance $f(\mathbf{d})-1$ to $\mathbf{d}$, and hence $m \geqslant f(\mathbf{d})-1$. Combining this with $f(\mathbf{d})-1 \geqslant m$ obtained above implies that $f(\mathbf{d})-1 = m$.

To prove (4), let $\mathbf{d}'$ be a nearest point in $\mathcal{N}$ to $\mathbf{d}$. Then by (2), $\mathbf{d} \geqslant \mathbf{d}'$, and since $\mathbf{d} \notin \mathcal{N}$ we have $d_i > d'_i$ for some $i \in [n]$. Hence $\mathbf{d}-\mathbf{e}_i$ is of distance at most $f(\mathbf{d})-1$ to $\mathcal{N}$, and hence

$$\text{for some } i \in [n], \quad f(\mathbf{d}-\mathbf{e}_i) \leqslant f(\mathbf{d})-1. \tag{10}$$

However, the triangle inequality implies that for all $i \in [n]$,

$$\rho_{L^1}(\mathbf{d},\mathcal{N}) \leqslant 1 + \rho_{L^1}(\mathbf{d}-\mathbf{e}_i,\mathcal{N}),$$

and therefore

$$\forall i \in [n], \quad \rho_{L^1}(\mathbf{d}-\mathbf{e}_i,\mathcal{N}) \geqslant \rho_{L^1}(\mathbf{d},\mathcal{N})-1,$$

or equivalently

$$\forall i \in [n], \quad f(\mathbf{d}-\mathbf{e}_i) \geqslant f(\mathbf{d})-1, \tag{11}$$

Combining this with (10) yields (9).

Finally, let us prove that f is slowly growing: first, for any $i \in [n]$ we may replacing $\mathbf{d}$ with $\mathbf{d} + \mathbf{e}_i$ in (11) and infer that

$$\rho_{L^1}(\mathbf{d}, \mathcal{N}) \geqslant \rho_{L^1}(\mathbf{d} + \mathbf{e}_i, \mathcal{N}) - 1,$$

i.e.,

$$f(\mathbf{d} + \mathbf{e}_i) \leqslant f(\mathbf{d}) + 1.$$

Second, if $\mathbf{d}' \in \mathcal{N}$ is the nearest point in $\mathcal{N}$ to $\mathbf{d} + \mathbf{e}_i$, then

$$\rho_{L^1}(\mathbf{d}, \mathbf{d}' - \mathbf{e}_i) = \rho_{L^1}(\mathbf{d} + \mathbf{e}_i, \mathbf{d}') = f(\mathbf{d}),$$

and since $\mathcal{N}$ is a downset, $\mathbf{d}' - \mathbf{e}_i \in \mathcal{N}$. Hence

$$f(\mathbf{d} + \mathbf{e}_i) \geqslant f(\mathbf{d}).$$

Hence for all $i \in [n]$ we have

$$f(\mathbf{d}) \leqslant f(\mathbf{d} + \mathbf{e}_i) \leqslant f(\mathbf{d}) + 1,$$

i.e., f is slowly growing. □

Corollary 7. *The Baker-Norine rank r_{BN} of any graph is slowly growing.*

The above corollary is easy to show directely for the Baker-Norine rank; Proposition 6 shows that this property holds for a much wider class of functions, f.

Proof. We have $r_{\mathrm{BN}}(\mathbf{d}) = f(\mathbf{d}) - 1$, where $f(\mathbf{d}) = \rho_{L^1}(\mathbf{d}, \mathcal{N})$, and $\mathcal{N}$ is the set of $\mathbf{d}$ that are not equivalent to an effective divisor. If $\mathbf{d}' \leqslant \mathbf{d}$ and $\mathbf{d}'$ is equivalent to an effective divisor, then so is $\mathbf{d}$; hence $\mathcal{N}$ above is a downset. Hence f is slowly growing, and hence so is $r_{\mathrm{BN}} = f - 1$. □

2.5 Generalizations of the Baker-Norine Rank

Many variants of the Baker-Norine rank have been studied. We remark that in literature that generalizes that Baker-Norine rank, e.g., [2], one typically studies the function $r = f - 1$ where f is the distance to a wider class of sets $\mathcal{N}$ in (3) than in [6]; hence r is initially -1 instead of initially 0.

Example 8. Amini and Manjunath [2] generalized Definition 4 by taking $L \subset \mathbb{Z}^n_{\deg=0}$ be any lattice of full rank in $\mathbb{Z}^n_{\deg=0}$ (i.e., rank $n-1$); in this case the definitions of "equivalent," "effective," and of $\mathcal{N}$ in Definition 4 (i.e., the set of elements of $\mathbb{Z}^n$ not equivalent under L to an effective divisor) carry over. In Theorem 1.4 there they establish conditions on L under which a Riemann-Roch type formula (5) holds. Notice that here, too, $\mathcal{N}$ is a non-empty downset, and hence f is slowly growing; notice also that since L is of full rank, $p = |\mathbb{Z}_{\deg=0}/L|$ is finite, and hence f is p-periodic.

Next we prove that the functions in Example 8 are Riemann functions. To prove this, let us give some conditions on a subset $\mathcal{N} \subset \mathbb{Z}^n$ which ensure that f in (3) gives a Riemann function. Unlike [6, 2], we do not address the question of whether or not f satisfies a Riemann-Roch type formula (5).

Proposition 9. *Let $n \in \mathbb{N}$ and $\mathcal{N} \subset \mathbb{Z}^n$ such that*

1. *for some $m, m' \in \mathbb{Z}$ we have*

$$\mathbb{Z}^n_{\deg \leqslant m} \subset \mathcal{N} \subset \mathbb{Z}^n_{\deg \leqslant m'}, \tag{12}$$

   and

2. *if M is the largest degree of an element of $\mathcal{N}$ (which exists and is finite by (1)), then there exists a C such that if $\mathbf{d}' \in \mathbb{Z}^n_{\deg = M}$, then some $\mathbf{d} \in \mathcal{N} \cap \mathbb{Z}^n_{\deg=M}$ has $\|\mathbf{d}' - \mathbf{d}\|_1 \leqslant C$ (i.e., $\rho_{L^1}(\mathbf{d}', \mathcal{N} \cap \mathbb{Z}^n_{\deg=M}) \leqslant C$).*

Then f as in (3), i.e., $f(\mathbf{d}) = \rho_{L^1}(\mathbf{d}, \mathcal{N})$, is a Riemann function with offset $-M$.

Proof. Since $\mathbf{d} \in \mathcal{N}$ for $\deg(\mathbf{d}) \leqslant m$, we have that f is initially zero. By induction on $\deg(\mathbf{d})$, we easily show that for any $\mathbf{d}$ with $\deg(\mathbf{d}) > M$, the L^1 distance from $\mathbf{d}$ to $\mathbb{Z}_{\leqslant M}$ is at least $\deg(\mathbf{d}) - M$. Hence

$$f(\mathbf{d}) \geqslant \deg(\mathbf{d}) - M; \tag{13}$$

let us show that equality holds for $\deg(\mathbf{d}) \geqslant M + Cn$.

First say that $\deg(\mathbf{d}) = M + Cn$. Then $\widetilde{\mathbf{d}} = \mathbf{d} - C\mathbf{1}$ has degree M, and hence there is a $\mathbf{d}' \in \mathcal{N} \cap \mathbb{Z}^n_{\deg=M}$ with $\|\widetilde{\mathbf{d}} - \mathbf{d}'\|_1 \leqslant C$. Hence $C\mathbf{1} + (\widetilde{\mathbf{d}} - \mathbf{d}') \geqslant \mathbf{0}$, and therefore

$$\|C\mathbf{1} + \widetilde{\mathbf{d}} - \mathbf{d}'\|_{L^1} = \deg\bigl(C\mathbf{1} + \widetilde{\mathbf{d}} - \mathbf{d}'\bigr) = Cn + M - M = Cn. \tag{14}$$

Hence

$$\rho_{L^1}(\mathbf{d}, \mathcal{N}) \leqslant \|\mathbf{d} - \mathbf{d}'\| = \|C\mathbf{1} + \widetilde{\mathbf{d}} - \mathbf{d}\|_{L^1} = Cn$$

Therefore

$$f(\mathbf{d}) = \rho_{L^1}(\mathbf{d}, \mathcal{N}) \leqslant Cn = \deg(\mathbf{d}) - M,$$

and in view of the reverse inequality (13), we have

$$f(\mathbf{d}) = \deg(\mathbf{d}) - M$$

when $\deg(\mathbf{d}) = M + Cn$.

Now we show that (13) holds with equality when $\deg(\mathbf{d}) > M + Cd$, i.e., when $\deg(\mathbf{d}) = M + Cn + b$ with $b > 0$. Then $\mathbf{d} - b\mathbf{e}_1$ has degree $M + Cn$ and L^1 distance b to $\mathbf{d}$, and using the triangle inequality we have

$$f(\mathbf{d}) = \rho_{L^1}(\mathbf{d}, \mathcal{N}) \leqslant b + \rho_{L^1}(\mathbf{d} - b\mathbf{e}_1, \mathcal{N}) = b + Cn = \deg(\mathbf{d}) - M.$$

Hence, in view of the reverse inequality (13), we have

$$f(\mathbf{d}) = \deg(\mathbf{d}) - M$$

when $\deg(\mathbf{d}) > M + Cn$. ☐

Let us make some further remarks on examples provided by Proposition 9.

Proposition 10. *Consider any case of Example 8, hence: $n \in \mathbb{N}$; $L \subset \mathbb{Z}^n_{\deg=0}$ is a sublattice (i.e., a subgroup) of full rank $n-1$ and hence $p = |\mathbb{Z}^n_{\deg=0}/L|$ is finite. We say that $\mathbf{d} \in \mathbb{Z}^n$ is* effective *if $\mathbf{d} \geqslant \mathbf{0}$; for $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^n$ we say $\mathbf{d}, \mathbf{d}'$ are* equivalent, *written $\mathbf{d} \sim \mathbf{d}'$, if $\mathbf{d} - \mathbf{d}' \in L$; and we set $\mathcal{N} \subset \mathbb{Z}^n$ to be the set of elements of $\mathbb{Z}^n$ that are not equivalent to an effective element of $\mathbb{Z}^n$. Then the conditions of Proposition 9 are satisfied: namely (1) we have*

$$\mathbb{Z}^n_{\deg\leqslant -1} \subset \mathcal{N} \subset \mathbb{Z}^n_{\deg\leqslant m'}$$

where $m' = (n-1)(p-1)-1$, and (2) if $\mathbf{d}^\star \in \mathcal{N}$ has maximum degree M, then there is a constant such that any $\mathbf{d} \in \mathbb{Z}^n_{\deg=M}$ is within a distance C of some element of $\mathcal{N}$. Hence $f(\mathbf{d}) \stackrel{def}{=} \rho_{L^1}(\mathbf{d}, \mathcal{N})$ is a Riemann function.

Proof. Since any effective $\mathbf{d}$ has $\deg(\mathbf{d}) \geqslant 0$, and equivalence preserves the degree, we have $\mathbb{Z}^n_{\deg\leqslant -1} \subset \mathcal{N}$.

Since $p = |\mathbb{Z}_{\deg=0}/L|$ is finite, for any $i \in [n-1]$, the order of $\mathbf{e}_i - \mathbf{e}_n$ is divisible by p, and hence $p(\mathbf{e}_i - \mathbf{e}_n) \in L$. It follows that any $\mathbf{d} \in \mathbb{Z}^n$ is equivalent to some $\mathbf{d}'$ where $0 \leqslant d'_i \leqslant p-1$ for $i \in [n-1]$; in this case

$$d'_n = \deg(\mathbf{d}') - d'_1 - \cdots - d'_{n-1} \geqslant \deg(\mathbf{d}') - (n-1)(p-1) = \deg(\mathbf{d}) - (n-1)(p-1).$$

Hence if $\deg(\mathbf{d}) \geqslant (n-1)(p-1)$, then $\mathbf{d}$ is equivalent to a $\mathbf{d}' \geqslant \mathbf{0}$. Hence

$$\mathcal{N} \subset \mathbb{Z}^n_{\deg\leqslant (n-1)(p-1)-1}.$$

Finally, let $\mathbf{d}_1, \ldots, \mathbf{d}_p$ be a set of representatives of $\mathbb{Z}_{\deg=0}/L$, and let $C = \max_i \|\mathbf{d}_i\|_{L^1}$. Then if $\mathbf{d}^\star \in \mathcal{N}$ is of any degree (including of maximum degree), and $\mathbf{d}$ has the same degree as $\mathbf{d}^\star$, then $\mathbf{d} - \mathbf{d}^\star$ has degree 0 and is therefore lies in $\mathbf{d}_i + L$ for some $i \in [p]$; hence $\mathbf{d} - \mathbf{d}^\star + \mathbf{d}_i \in L$, and hence $\mathbf{d} + \mathbf{d}_i \in \mathbf{d}^\star + L \subset \mathcal{N}$, since $\mathcal{N}$ is invariant under translations by L, and $\mathbf{d}^\star \in \mathcal{N}$. Hence $\mathbf{d} + \mathbf{d}_i \in \mathcal{N}$, and so $\mathbf{d}$ is within a distance C of an element of $\mathcal{N}$. □

Remark 11. The proof of Proposition 10 shows that if $\mathcal{N}$ is any set that is invariant under translation by elements in a subgroup $L \subset \mathbb{Z}^n_{\deg=0}$ of full rank, then Condition (2) of Proposition 9 holds.

Remark 12. By contrast with the above remark, for any $n \in \mathbb{N}$ consider

$$\mathcal{N} = \mathbb{Z}^n_{\deg\leqslant 0} \cup \mathbb{Z}^n_{\deg=7}$$

and $f(\mathbf{d}) = \rho_{L^1}(\mathbf{d}, \mathcal{N})$; then f is a Riemann function but for $\mathbf{d}$ with $4 \leqslant \deg(\mathbf{d}) \leqslant 6$ and any $i \in [n]$ we have that $f(\mathbf{d} + \mathbf{e}_i) < f(\mathbf{d})$. Hence f is not slowly growing.

Remark 13. We remark that if $L \subset \mathbb{Z}^n_{\deg=0}$ is not of full rank in Example 8, then condition (1) of Proposition 9 fails to hold. For example, if $n = 3$ and

$$L = \mathbb{Z}(-1,1,0) = \{(m,-m,0) \mid m \in \mathbb{Z}\} \subset \mathbb{Z}^3_{\deg=0},$$

then for all $a \in \mathbb{Z}$ all elements of $(a,0,-1)+L$ have their third component equal to -1; hence $(a,0,-1)$ is not equivalent modulo L to an effective divisor; hence $(a,0,-1) \in \mathcal{N}$, where $\mathcal{N}$ is as defined in Example 8. Hence there are elements of $\mathcal{N}$ of arbitrarily high degree, and hence there is no $m' \in \mathbb{Z}$ such that $\mathcal{N} \subset \mathbb{Z}^3_{\leqslant m'}$; in other words, $f(\mathbf{d}) \stackrel{\text{def}}{=} \rho_{L^1}(\mathbf{d},\mathcal{N})$ is not a Riemann function. The general case of $L \subset \mathbb{Z}^n_{\deg=0}$ not being of full rank is similar: then some $\mathbf{u} \in \mathbb{Z}^n_{\deg=0}$ is orthogonal to all elements of L, and hence so is $\mathbf{u}' = \mathbf{u} - \mathbf{1}(\min_i(u_i))$, which has all non-negative components, one of which — say the j-th component — is 0. Hence $a\mathbf{e}_j - \mathbf{u}'$ has arbitrarily high degree, but any element of $a\mathbf{e}_j - \mathbf{u}' + L$ has negative dot product with $\mathbf{u}'$, and hence cannot be effective.

2.6 Examples Based on Riemann's Theorem

All the discussion below is based on the classical *Riemann's theorem* and *Riemann-Roch theorem*. However, we use these examples only for illustration, and they are not essential to our discussion of the Baker-Norine rank functions and of most of the rest of this article.

[The material below is discussed in the context of Riemann surfaces in Appendix A of [6]; the reader may prefer this treatment. The case of Riemann surfaces is well-known to be equivalent to algebraic curves over $\mathbb{C}$ that we discuss below (see [16], Section 3.1).]

Let X be an algebraic curve over an algebraically closed field k, and K be its function field; one understands either (1) K is a finite extension of $k(x)$ where x is an indeterminate (i.e., transcendental) and X is its set of discrete valuations (e.g., [16], Section 1.2), or (2) X is projective curve in the usual sense (e.g., [13], Section 4.1), and K is its function field. (For $k = \mathbb{C}$ one can also view X as a compact Riemann surface, and K as its field of meromorphic functions; see, e.g., Section 3.1 of [16].) To each $h \in K \setminus \{0\}$ one associates the divisor (i.e., Weil divisor) equal to $(h) = \sum_{v \in X} \mathrm{ord}_v(h) v$ [16][1]. For each divisor D one sets

$$L(D) = \{0\} \cup \{h \in K \mid (h) \geqslant -D\},$$

(we can omit $\{0\}$ above provided that we regard $0 \in K$ as having divisor $(0) \geqslant -D$ for all D, i.e., 0 having order $+\infty$ at each point of X); this makes $L(D) \subset K$ a k-linear subspace, and we set

$$l(D) = \dim_k L(D).$$

For a divisor D, we use $\deg(D)$ to denote the sum of the $\mathbb{Z}$-coefficients in D. For $f \in K \setminus \{0\}$, f has the same number of zeroes and poles, counted with multiplicity, i.e., $\deg((f)) = 0$. It follows that $l(D) = 0$ when $\deg(D) < 0$. *Riemann's theorem* says that if $g \in \mathbb{Z}_{\geqslant 0}$ is the *genus of* X, for any divisor D with $\deg(D)$ sufficiently large, we have

$$l(D) = \deg(D) + 1 - g.$$

[1]Here $\mathrm{ord}_v(h)$ is (1) 0 if $h(v)$ is finite and non-zero, (2) the multiplicity of the zero at v if $h(v) = 0$, and (3) minus the multiplicity of the pole at v if $h(v) = \infty$.

Hence for any points $P_1, \dots, P_n \in X$ we have

$$f(\mathbf{d}) \stackrel{\text{def}}{=} l(d_1 P_1 + \cdots + d_n P_n) \tag{15}$$

is a Riemann function. The Riemann-Roch formula states that

$$l(D) = l(\omega - D) + \deg(D) + 1 - g$$

where ω is the *canonical divisor*, i.e., the divisor associated to any 1-form.

The following example shows that the Riemann functions described above are not generally periodic.

Example 14. Let K be an elliptic curve, i.e., a curve of genus $g = 0$, and P_1, P_2 two points of the curve. The Riemann-Roch theorem implies that $f(\mathbf{d}) = 0$ if $\deg(\mathbf{d}) < 0$ and $f(\mathbf{d}) = \deg(\mathbf{d}) - 1$ if $\deg(\mathbf{d}) > 0$. This determines $f(\mathbf{d})$ for all $\mathbf{d} = (d_1, d_2)$ except when $d_2 = -d_1$, and in this case, for each $d_1 \in \mathbb{Z}$ we have that $f(d_1, -d_1)$ is either 0 or 1. However, if D is of degree 0, then any f with $(f) \geqslant -D$ must satsisfy $(f) = D$, since (f) has degree 0. Furthermore, for elliptic curves it is well-known that for any P_1 and $n \in \mathbb{N}$, there at most n^2 points P_2 such that for some f, $(f) = n(P_2 - P_1)$ (exactly n^2 if the characteristic of k is relatively prime to n); this is due to the group law(s) one can define on elliptic curves; see, for example, [13], Example 4.8.1 of Chapter IV. It follows that for any point P_1, for all but countably many P_2 one has $f(d_1, -d_1) = \ell(d_1 P_1 - d_1 P_2) = 0$ for all $d_1 \in \mathbb{Z}$. By contrast, for any n relatively prime to the characteristic of k, there are n^2 points P_2 such that $f(d_1, -d_1) = 0$ for d_1 divisible by n.

By contrast, the Riemann functions described in this section are all slowly growing, since for any divisor $D = d_1 P_1 + \cdots + d_n P_n$ and point P_i we have

$$\mathcal{L}(D) \subset \mathcal{L}(D + P_i)$$

(so $f(\mathbf{d} + \mathbf{e}_i) \geqslant f(\mathbf{d})$ holds), and if the inclusion is proper, then there exists a $g \in \mathcal{L}(D + P_i)$ that does not lie in $\mathcal{L}(D)$; it follows that $\mathrm{ord}_{P_i}(G) = -d_i - 1$, and then for any other $h \in \mathcal{L}(D + P_i) \setminus \mathcal{L}(D)$ we have $\mathrm{ord}_{P_i}(h - cg) \leqslant -d_i$ for some $c \in k$; but then $h - cg \in \mathcal{L}(D)$, and hence the quotient $\mathcal{L}(D + P_i)/\mathcal{L}(D)$ is at most one-dimensional, and hence $f(\mathbf{d} + \mathbf{e}_i) \leqslant f(\mathbf{d}) + 1$.

When we discuss theorem Theorem 38, we will see that the Riemann functions in this subsection have one interesting property, which results from the following fact: if $D = d_1 P_1 + \cdots + d_n P_n$ is any divisor, and P_1, P_2 are (distinct) points, then we claim that

$$\ell(D + P_1) = \ell(D + P_2) = \ell(D + P_1 + P_2) \quad \Rightarrow \quad \ell(D) = \ell(D + P_1). \tag{16}$$

To see why, by the slowly growing property, we see that if the above does not hold, then $\ell(D + P_1) = \ell(D) + 1$; it follows that there is a function h_1 that lies in $\mathcal{L}(D + P_1)$ but not in $\mathcal{L}(D)$, and hence $\mathrm{ord}_{P_1}(h_1) = -d_1 - 1$ but $\mathrm{ord}_{P_2}(h_1) \geqslant -d_2$; similarly there is an h_2 with $\mathrm{ord}_{P_2}(h_2) = -d_2 - 1$ but $\mathrm{ord}_{P_1}(h_2) \geqslant -d_1$. It follows that for all $c_1, c_2 \in k$, $c_1 h_1 + c_2 h_2$ cannot lie in $\mathcal{L}(D)$ unless $c_1 = c_2 = 0$, since otherwise $c_1 h_1 + c_2 h_2$ has order $-d_1 - 1$ at P_1 or order $-d_2 - 1$ at P_2 (or both). Hence $\mathcal{L}(D + P_1 + P_2)/\mathcal{L}(D)$ is (at least) two-dimensional (but exactly two-dimensional since f above are slowly growing), so $\ell(D + P_1 + P_2) = \ell(D + P_1) + 1$, contradicting (16).

2.7 Riemann Functions from other Riemann Functions

Example 15. If for some $k, n \in \mathbb{N}$, $f_1, \ldots, f_{2k+1}$ are Riemann functions, then so is

$$f_1 - f_2 + f_3 - \cdots - f_{2k} + f_{2k+1}.$$

If $f_1, \ldots, f_{2k+1}$ are all periodic, and p is the least common multiple of all the periods of the f_i, then f is p-periodic. By contrast, if $f_1, \ldots, f_{2k+1}$ are all slowly growing, then f above need not be slowly growing (e.g., take $f_1(\mathbf{d}) = f_3(\mathbf{d}) = \max(0, \deg(\mathbf{d}))$, and take $f_2(\mathbf{d}) = \max(10, \deg(\mathbf{d})) - 10$, so that $f(0, -1) = 0$ but $f(0, 0) = 2$).

The following construction takes a Riemann function on n variables and generates many different Riemann functions on a smaller number of variables.

Example 16. Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be any Riemann function with $f(\mathbf{d}) = \deg(\mathbf{d}) + C$ for $\deg(\mathbf{d})$ sufficiently large. Then for any distinct $i, j \in [n]$ and $\mathbf{d} \in \mathbb{Z}^n$, the function $f_{i,j,\mathbf{d}}\colon \mathbb{Z}^2 \to \mathbb{Z}$ given as

$$f_{i,j,\mathbf{d}}(a_i, a_j) = f\bigl(\mathbf{d} + a_i\mathbf{e}_i + a_j\mathbf{e}_j\bigr) \tag{17}$$

is a Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$, and for $a_i + a_j$ large we have

$$f_{i,j,\mathbf{d}}(a_i, a_j) = a_i + a_j + C', \quad \text{where} \quad C' = \deg(\mathbf{d}) + C. \tag{18}$$

We call $f_{i,j,\mathbf{d}}$ a *two-variable restriction* of f; we may similarly restrict f to one variable or three or more variables, and any such restriction is clearly a Riemann function.

In [11], the restriction to two variables is the most important. [It turns out that in [11], it is important that that C' depends only on $\mathbf{d}$ and not on i, j.]

Clearly if f is either periodic or slowly growing, then the same holds for any restriction of f as above.

3 The Weight of a Riemann Function, and Generalized Riemann Functions

In this section we define the *weight* of a Riemann function, a notion central to this article.

Since a Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$ eventually equals $d_1 + d_2 + C$, one may consider that one possible generalization of this notion for a function $\mathbb{Z}^3 \to \mathbb{Z}$ might be a function that eventually equals a polynomial of degree two in d_1, d_2, d_3. In fact, most everything we say about Riemann functions hold for a much larger class of functions $\mathbb{Z}^n \to \mathbb{Z}$ which we call *generalized Riemann functions*; this includes all polynomials of $d_1, \ldots, d_n$ of degree $n - 1$, but many more functions.

3.1 Weights and Möbius Inversion

If $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is initially zero, then there is a unique initially zero $W \in \mathbb{Z}^n \to \mathbb{Z}$ for which

$$f(\mathbf{d}) = \sum_{\mathbf{d}' \leqslant \mathbf{d}} W(\mathbf{d}'), \tag{19}$$

since we can determine $W(\mathbf{d})$ inductively on $\deg(\mathbf{d})$ set

$$W(\mathbf{d}) = f(\mathbf{d}) - \sum_{\mathbf{d}' \leqslant \mathbf{d},\ \mathbf{d}' \neq \mathbf{d}} W(\mathbf{d}'). \tag{20}$$

Recall from (2) the notation $\mathbf{e}_I$ for $I \subset [n]$.

Proposition 17. *Consider the operator* $\mathfrak{m}$ *on functions* $f\colon \mathbb{Z}^n \to \mathbb{Z}$ *defined via*

$$(\mathfrak{m}f)(\mathbf{d}) = \sum_{I \subset [n]} (-1)^{|I|} f(\mathbf{d} - \mathbf{e}_I), \tag{21}$$

and the operator on functions $W\colon \mathbb{Z}^n \to \mathbb{Z}$ *that are initially zero given by*

$$(\mathfrak{s}W)(\mathbf{d}) = \sum_{\mathbf{d}' \leqslant \mathbf{d}} W(\mathbf{d}'), \tag{22}$$

Then if f *is any initially zero function, and* W *is given by the equation* $f = \mathfrak{s}W$ *(i.e.,* W *is defined inductively by* (20)*), then* $W = \mathfrak{m}f$*.*

The above can be viewed as the Möbius inversion formula for the partial order $\leqslant$ on $\mathbb{Z}^n$.

Proof. We have $f(\mathbf{d}) = 0$ whenever $\deg(\mathbf{d}) \leqslant b$ for some b, and then (21) shows that $(\mathfrak{m}f)(\mathbf{d}) = 0$ for $\deg(\mathbf{d}) \leqslant b$ as well. Since there is a unique initially zero W with $\mathfrak{s}W = f$, it suffices to show that $\mathfrak{s}\mathfrak{m}f = f$. Since f is initially zero, for any $\mathbf{d} \in \mathbb{Z}^n$ write $(\mathfrak{s}\mathfrak{m}f)(\mathbf{d})$ as

$$(\mathfrak{s}\mathfrak{m}f)(\mathbf{d}) = \sum_{\mathbf{d}' \leqslant \mathbf{d}} \sum_{I \subset [n]} (-1)^{|I|} f(\mathbf{d} - \mathbf{e}_I)$$

which is a double sum of finitely many terms since f is initially zero; hence we may rearrange terms, set $\mathbf{d}'' = \mathbf{d} - \mathbf{e}_I$ and write this double sum as

$$\sum_{\mathbf{d}'' \leqslant \mathbf{d}} f(\mathbf{d}'')\, a_{\mathbf{d}''}, \quad \text{where} \quad a_{\mathbf{d}''} = \sum_{I \text{ s.t. } \mathbf{d}'' + \mathbf{e}_I \leqslant \mathbf{d}} (-1)^{|I|};$$

to compute $a_{\mathbf{d}''}$, setting $J = \{j \in [n] \mid d''_j < d_j\}$, we have

$$\sum_{I \text{ s.t. } \mathbf{d}'' + \mathbf{e}_I \leqslant \mathbf{d}} (-1)^{|I|} = \sum_{I \subset J} (-1)^{|I|}$$

which equals 1 if $J = \emptyset$ and otherwise equals 0. It follows that $a_{\mathbf{d}} = 1$, and for $\mathbf{d}'' \neq \mathbf{d}$, we have $a_{\mathbf{d}''} = 0$. ☐

Definition 18. Throughout this article we reserve the symbols $\mathfrak{m}, \mathfrak{s}$ for their meanings in (22) and (21). If f, W are initially zero functions $\mathbb{Z}^n \to \mathbb{Z}$ with $f = \mathfrak{s}W$, we say that f *counts* W and that W is the *weight* of f.

3.2 Weights of Riemann Functions $\mathbb{Z}^2 \to \mathbb{Z}$

We will be especially interested in Riemann functions $\mathbb{Z}^2 \to \mathbb{Z}$ and their weights $W = \mathfrak{m}f$.

Theorem 19. *Let $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ be an arbitrary function, and let $W = \mathfrak{m}f$. Then the following are equivalent:*

1. *f is a Riemann function;*
2. *W is initially and eventually zero, and W satisfies*

$$\forall d_1 \in \mathbb{Z}, \qquad \sum_{d_2=-\infty}^{\infty} W(d_1, d_2) = 1, \tag{23}$$

and

$$\forall d_2 \in \mathbb{Z}, \qquad \sum_{d_1=-\infty}^{\infty} W(d_1, d_2) = 1. \tag{24}$$

Proof. If f is a Riemann function, then $f(\mathbf{d})$ is initially zero and eventually equal to d_1+d_2+C for some C. Since $f(\mathbf{d}) = 0$ for $\deg(\mathbf{d}) \leqslant a$ for some $a \in \mathbb{Z}$, we have $W(\mathbf{d}) = 0$ for $\deg(\mathbf{d}) \leqslant a$. Since $f(\mathbf{d}) = d_1+d_2+C$ for $\deg(\mathbf{d}) \geqslant b$ for some constants C, b, we easily see that for $\deg(\mathbf{d}) \geqslant b+2$ we have $f(\mathbf{d}') = d_1' + d_2' + C$ for all $\mathbf{d}'$ with $d_i - 1 \leqslant d_i' \leqslant d_i$ for $i = 1, 2$, and hence $W(\mathbf{d}) = 0$ for all such $\mathbf{d}$. Hence W is initially and eventually 0.

Since $f(d_1, d_2) = d_1 + d_2 + C$ for $d_1 + d_2 \geqslant b$, for fixed d_1 we have that $d_1 + d_2 \geqslant b+1$ implies that

$$f(d_1, d_2) - f(d_1 - 1, d_2) = (d_1 + d_2 + C) - (d_1 - 1 + d_2 + C) = 1. \tag{25}$$

Since W is initially zero and $W = \mathfrak{m}f$, we have $f = \mathfrak{s}W$, and hence for $d_1 + d_2 \geqslant b+1$ we have

$$f(d_1, d_2) - f(d_1 - 1, d_2) = \sum_{\mathbf{d}' \leqslant (d_1, d_2)} W(\mathbf{d}') - \sum_{\mathbf{d}' \leqslant (d_1 - 1, d_2)} W(\mathbf{d}') = \sum_{d_2' \leqslant d_2} W(d_1, d_2'). \tag{26}$$

In view of (25) we have

$$\sum_{d_2' \leqslant d_2} W(d_1, d_2') = 1$$

provided that $d_2 \geqslant b + 1 - d_1$. Hence we have (23) (and $W(d_1, d_2)$ is only nonzero for $a \leqslant d_1 + d_2 \leqslant b+2$, so this sum is really a finite sum). Similarly we have (24).

Conversely, say that W is initially and eventually zero and satisfies (23) and (24). Since W is initially zero and $W = \mathfrak{m}f$, we have $f = \mathfrak{s}W$ is initially zero. To show that f is a Riemann function it suffices to show that for $\deg(\mathbf{d})$ sufficiently large we have $f(\mathbf{d}) = \deg(\mathbf{d}) + C$ for some constant, C; we now do so.

Since W is eventually zero, $W(d_1, d_2) = 0$ for $d_1 + d_2 \geqslant B$ for some B. Hence for any d_1 we have

$$\sum_{d_2'=-\infty}^{\infty} W(d_1, d_2') = 1 = \sum_{d_2' \leqslant d_2} W(d_1, d_2')$$

for $d_2 \geqslant B - d_1$; in view of (26) we have

$$d_1 + d_2 \geqslant B \quad \Rightarrow \quad f(d_1, d_2) - f(d_1, d_2 - 1) = 1. \tag{27}$$

Similarly we have

$$d_1 + d_2 \geqslant B \quad \Rightarrow \quad f(d_1, d_2) - f(d_1 - 1, d_2) = 1.$$

In particular for $d_1 + d_2 \geqslant B$ we have

$$f(d_1 - 1, d_2) = f(d_1, d_2) - 1 = f(d_1, d_2 - 1).$$

Setting $\mathbf{d}' = (d_1 - 1, d_2)$, we have that if $\deg(\mathbf{d}') \geqslant B - 1$, then

$$f(\mathbf{d}') = f\big(\mathbf{d}' + (1, -1)\big).$$

It follows that for any $s \in \mathbb{Z}$,

$$f(\mathbf{d}') = f\big(\mathbf{d}' + s(1, -1)\big),$$

and hence for $d_1 + d_2 \geqslant B - 1$ we have

$$f(d_1, d_2) = f(0, d_1 + d_2).$$

But for $\deg(\mathbf{d}') \geqslant B - 1$, (27) implies that

$$f\big(\mathbf{d}' + (0, 1)\big) = f(\mathbf{d}') + 1,$$

and hence by induction we have that for any $s \in \mathbb{N}$,

$$f\big(\mathbf{d}' + s(0, 1)\big) = f(\mathbf{d}') + s.$$

Hence for $\deg(\mathbf{d}) \geqslant B - 1$ we have

$$f(d_1, d_2) = f(0, d_1 + d_2) = f(0, B - 1) + d_1 + d_2 - (B - 1) = \deg(\mathbf{d}) + C,$$

where $C = f(0, B - 1) - (B - 1)$. Hence f is a Riemann function. $\square$

Viewing W as a two-dimensional infinite array of numbers indexed in $\mathbb{Z} \times \mathbb{Z}$, the above theorem says that $W : \mathbb{Z}^2 \to \mathbb{Z}$ is the weight of a Riemann function iff W is initially and eventually zero, and all its "row sums" (23) and all its "column sums" (24) equal one.

3.3 Modular Functions and Generalized Riemann Functions

Some of what we do in this article applies to a class of functions that is much wider than Riemann functions; in this subsection we define this class of functions, which we call *generalized Riemann functions*.

Definition 20. A function $h\colon \mathbb{Z}^n \to \mathbb{Z}$ is *modular* if $\mathfrak{m}h = 0$ is the zero function, i.e., $h \in \ker \mathfrak{m}$.

For example, for any $C \in \mathbb{Z}$ the function $h(\mathbf{d}) = \deg(\mathbf{d}) + C$ is modular.

Definition 21. We say that $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is a *generalized Riemann function* if

1. f is initially zero, and
2. f eventually equals a modular function, i.e., for some $h \in \ker \mathfrak{m}$ we have $f(\mathbf{d}) = h(\mathbf{d})$ for $\deg(\mathbf{d})$ sufficiently large.

Example 22. Since for any $C \in \mathbb{Z}$, $h(\mathbf{d}) = \deg(\mathbf{d}) + C$ is modular, it follows that any Riemann function is a generalized Riemann function.

Let us give some more examples of modular functions. At times it is convenient to write $\mathfrak{m}$ using the "downward shift operators," $\mathfrak{t}_i$ for $i \in [n]$, where $\mathfrak{t}_i$ is the operator on functions $\mathbb{Z}^n \to \mathbb{Z}$ given by

$$(\mathfrak{t}_i f)(\mathbf{d}) = f(\mathbf{d} - \mathbf{e}_i); \tag{28}$$

one easily verifies that the $\mathfrak{t}_i$ commute with one another, and that

$$\mathfrak{m} = (1 - \mathfrak{t}_1) \cdots (1 - \mathfrak{t}_n), \tag{29}$$

(where 1 is the identity operator).

Example 23. If $h = h(\mathbf{d})$ is independent of its i-th variable, then $(1 - \mathfrak{t}_i)h = 0$, and then (29) implies that $\mathfrak{m}f = 0$. Hence a function $f = f(\mathbf{d})$ that is independent of one of its variables is modular, and therefore any sum of such functions is also modular. The converse turns out to be true: see Theorem 25 below.

Example 24. If $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is a polynomial (with integer coefficients) of degree at most $n-1$, then f is a sum of functions, each of which is independent of at least one of its variables. Hence such an f is modular. Of course, for $n \geqslant 2$, the function $f(x_1, \ldots, x_n) = 2^{|x_1 x_2 \ldots x_{n-1}|}$ is not a polynomial, but is independent of x_n and therefore modular.

We now characterize modular functions in two different ways.

In Example 23, we saw that for a function $h = h(\mathbf{d})$ to be modular, it is sufficient that it be a sum of functions each of which is independent of one of its variables. Our first characterization of modular functions shows that this condition is also necessary.

Theorem 25. *A function $h\colon \mathbb{Z}^n \to \mathbb{Z}$ is modular iff it can be written as a sum of functions each of which depends on only $n-1$ of its n variables.*

We postpone its proof to Section 7. The following description of modular functions will be needed when we discuss what we call *Riemann-Roch formulas* (see Definition 28).

Theorem 26. *If $a \in \mathbb{Z}$, $n \in \mathbb{N}$, and h is any integer-valued function defined on $\mathbf{d} \in \mathbb{Z}^n$ with $a \leqslant \deg(\mathbf{d}) \leqslant a+n-1$, then h has a unique extension to a modular function $\mathbb{Z}^n \to \mathbb{Z}$.*

We also postpone the proof of this theorem to Section 7.

According to this theorem, if h_1, h_2 are two modular functions, then h_1 and h_2 are equal whenever they are eventually equal (i.e., whenever $h_1(\mathbf{d}) = h_2(\mathbf{d})$ for $\deg(\mathbf{d})$ sufficiently large). In particular, if $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is a generalized Riemann function, then the modular function h that is eventually equal to f is uniquely determined.

3.4 The Weight of the Baker-Norine Rank and Other Functions Initially Equal to -1

Recall that

$$\mathfrak{m} = (1 - \mathfrak{t}_1) \cdots (1 - \mathfrak{t}_n).$$

It follows that for any $f\colon \mathbb{Z}^n \to \mathbb{Z}$ and a constant $C \in \mathbb{Z}$, if $g\colon \mathbb{Z}^n \to \mathbb{Z}$ is defined as $g(\mathbf{d}) = f(\mathbf{d}) + C$, then

$$\big((1 - \mathfrak{t}_n)g\big)(\mathbf{d}) = f(\mathbf{d}) + C - f(\mathbf{d} - \mathbf{e}_n) - C = \big((1 - \mathfrak{t}_n)f\big)(\mathbf{d}).$$

It follows that $\mathfrak{m}$ of a function remains the same upon adding a constant to the function.

Since the Baker-Norine rank and many similar functions are initially equal to -1, we make the following convention.

Definition 27. If $r\colon \mathbb{Z}^n \to \mathbb{Z}$ is a function that is initially equal to -1, by the *weight* of r we mean the function $\mathfrak{m}r$, which clearly equals $\mathfrak{m}f$ with $f = 1 + r$.

When computing the weight of Baker-Norine type functions, we often use the more suggestive r_{BN} rather than the Riemann function $f = 1 + r_{\mathrm{BN}}$.

4 Riemann-Roch Formulas and Self-Duality

In this section we express Riemann-Roch formulas more simply in terms of the weight of the Riemann function.

Definition 28. Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be a generalized Riemann function, and h the modular function eventually equal to f. For $\mathbf{K} \in \mathbb{Z}^n$, the $\mathbf{K}$-*dual of* f, denoted $f^\wedge_{\mathbf{K}}$, refers to the function $\mathbb{Z}^n \to \mathbb{Z}$ given by

$$f^\wedge_{\mathbf{K}}(\mathbf{d}) = f(\mathbf{K} - \mathbf{d}) - h(\mathbf{K} - \mathbf{d}). \tag{30}$$

We equivalently write

$$f(\mathbf{d}) - f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = h(\mathbf{d}) \tag{31}$$

and refer to this equation as a *generalized Riemann-Roch formula*.

In particular, if f is a Riemann function with offset C, then $h(\mathbf{d}) = \deg(\mathbf{d}) + C$, and (31) means that

$$f(\mathbf{d}) - f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + C. \tag{32}$$

The usual Riemann-Roch formulas—the classical one and the Baker-Norine formula—are cases where $f^\wedge_{\mathbf{K}} = f$ equals f for some $f, \mathbf{K}$. Hence the above definition is very loose: it says that for any generalized Riemann function, f, and any $\mathbf{K} \in \mathbb{Z}^n$, there is always a "generalized Riemann-Roch formula;" we refer to the special cases where $f = f^\wedge_{\mathbf{K}}$ for some $\mathbf{K}$ as *self-duality* in Definition 31 below.

In Subsection 1.1 we explained some reasons we work with generalized Riemann-Roch formulas; briefly, these reasons are: (1) requiring self-duality would eliminate many interesting Riemann functions, such as the general ones considered by [2], and likely some interesting generalized Riemann functions; and (2) self-duality does not behave well under fixing some of the variables of a Riemann function and considering the resulting restriction.

We now give remarks, a theorem, and examples regarding generalized Riemann-Roch formulas.

Definition 29. If $W\colon \mathbb{Z}^n \to \mathbb{Z}$ is any function and $\mathbf{L} \in \mathbb{Z}^n$, the $\mathbf{L}$-*dual weight of* W, denoted $W^*_{\mathbf{L}}$ refers to the function given by

$$W^*_{\mathbf{L}}(\mathbf{d}) = W(\mathbf{L} - \mathbf{d}).$$

It is immediate that $(W^*_{\mathbf{L}})^*_{\mathbf{L}} = W$.

Theorem 30. *Let* $f\colon \mathbb{Z}^n \to \mathbb{Z}$ *be a generalized Riemann function, and* $W = \mathfrak{m}f$. *Let* $\mathbf{K} \in \mathbb{Z}^n$ *and let* $\mathbf{L} = \mathbf{K} + \mathbf{1}$.

1. *we have*
$$\mathfrak{m}\bigl(f^\wedge_{\mathbf{K}}\bigr) = (-1)^n W^*_{\mathbf{L}} = (-1)^n (\mathfrak{m}f)^*_{\mathbf{L}}. \tag{33}$$

2. $f^\wedge_{\mathbf{K}}$ *is a generalized Riemann function, and a Riemann function if* f *is.*

3. $(f^\wedge_{\mathbf{K}})^\wedge_{\mathbf{K}} = f$.

4. $f^\wedge_{\mathbf{K}} = f$ *iff* $W^*_{\mathbf{L}} = (-1)^n W$.

Proof. Proof of (1): applying $\mathfrak{m}$ to $f^\wedge_{\mathbf{K}}(\mathbf{d})$ we have

$$\bigl(\mathfrak{m}\bigl(f^\wedge_{\mathbf{K}}\bigr)\bigr)(\mathbf{d}) = \sum_{I \subset [n]} (-1)^{|I|} f^\wedge_{\mathbf{K}}(\mathbf{d} - \mathbf{e}_I) \tag{34}$$

which, in view of (30), equals

$$\sum_{I \subset [n]} (-1)^{|I|} \Bigl(f(\mathbf{K} - \mathbf{d} + \mathbf{e}_I) - h(\mathbf{K} - \mathbf{d} + \mathbf{e}_I) \Bigr). \tag{35}$$

Substituting $J = [n] \setminus I$, for any $g\colon \mathbb{Z}^n \to \mathbb{Z}$ we can write

$$\sum_{I \subset [n]} (-1)^{|I|} g(\mathbf{K} - \mathbf{d} + \mathbf{e}_I) = \sum_{J \subset [n]} (-1)^{n-|J|} g(\mathbf{K} - \mathbf{d} + \mathbf{1} - \mathbf{e}_J)$$

$$= (-1)^n \sum_{J\subset[n]} (-1)^{|J|} g(\mathbf{K}-\mathbf{d}+\mathbf{1}-\mathbf{e}_J) = (-1)^n (\mathfrak{m}g)(\mathbf{K}-\mathbf{d}+\mathbf{1}) = (-1)^n (\mathfrak{m}g)^*_{\mathbf{L}}(\mathbf{d}).$$

Taking $g = f - h$, and using $\mathfrak{m}f = W$ and $\mathfrak{m}h = 0$, we have (35) equals $(-1)^n W^*_{\mathbf{L}}(\mathbf{d})$, and since this also equals (30) we get (33).

Proof of (2): f is a generalized Riemann function iff $W = \mathfrak{m}f$ is initially and eventually zero, which is equivalent to $W^*_{\mathbf{L}}$ being initially and eventually zero; hence f is a generalized Riemann function iff $f^{\wedge}_{\mathbf{K}}$ is. Moreover, f is a Riemann function iff in addition (31) has $h(\mathbf{d}) = \deg(\mathbf{d}) + C$; in this case (32) with $\mathbf{d}$ replaced with $\mathbf{K}-\mathbf{d}$ is equivalent to

$$f(K-\mathbf{d}) - f^{\wedge}_{\mathbf{K}}(\mathbf{d}) = h(K-\mathbf{d})$$

for all $\mathbf{d}$, which reversing the sign gives

$$f^{\wedge}_{\mathbf{K}}(\mathbf{d}) - f(\mathbf{K}-\mathbf{d}) = -h(\mathbf{K}-\mathbf{d}) = -\deg(\mathbf{K}-\mathbf{d}) + C = \deg(\mathbf{d}) + C',$$

where $C' = C - \deg(\mathbf{K})$.

Proof of (3): we may write (33) as

$$f^{\wedge}_{\mathbf{K}} = \mathfrak{s}(-1)^n (\mathfrak{m}f)^*_{\mathbf{L}},$$

and hence, replacing f with $f^{\wedge}_{\mathbf{K}}$ in this last equation, we have

$$(f^{\wedge}_{\mathbf{K}})^{\wedge}_{\mathbf{K}} = \mathfrak{s}(-1)^n (\mathfrak{m}f^{\wedge}_{\mathbf{K}})^*_{\mathbf{L}} = \mathfrak{s}(-1)^n \big((-1)^n W^*_{\mathbf{L}}\big)^*_{\mathbf{L}} = \mathfrak{s}W = f.$$

Proof of (4): Since both $f^{\wedge}_{\mathbf{K}}$ and f are initially zero, $f^{\wedge}_{\mathbf{K}} = f$ iff $\mathfrak{m}f^{\wedge}_{\mathbf{K}} = \mathfrak{m}f$, and by (33) this is equivalent to $(-1)^n W^*_{\mathbf{L}} = W$. □

Definition 31. We say that a generalized Riemann function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is *self-dual* if for some $\mathbf{K} \in \mathbb{Z}^n$, $f^{\wedge}_{\mathbf{K}} = f$.

In view of (4) of Theorem 30, f is self-dual iff $W = \mathfrak{m}f$ satisfies $W^*_{\mathbf{L}} = (-1)^n W$ for some $\mathbf{L} \in \mathbb{Z}^n$.

Let us remark on the uniqueness of $\mathbf{K}$ in the above definition, and of $\mathbf{L}$ with $W^*_{\mathbf{L}} = (-1)^n W$.

Definition 32. For a function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ we say f *is invariant under translation by* $\mathbf{T}$ if for all $\mathbf{d} \in \mathbb{Z}^n$, $f(\mathbf{d}+\mathbf{T}) = f(\mathbf{d})$. We define the set of *invariant translations of* f to be

$$\mathrm{Trans}(f) = \{\mathbf{T} \in \mathbb{Z}^n \mid f \text{ is invariant under translation by } \mathbf{T}\}.$$

We easily see that $\mathrm{Trans}(f)$ is a lattice in $\mathbb{Z}^n$, i.e., if $\mathbf{T}_1, \mathbf{T}_2$ are in $\mathrm{Trans}(f)$, then so are $\mathbf{T}_1 + \mathbf{T}_2$ and $-\mathbf{T}_1$.

Proposition 33. *Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be initially zero, and $W = \mathfrak{m}f$. Then:*

1. *if f is not identically zero, then* $\mathrm{Trans}(f) \subset \mathbb{Z}^n_{\deg=0}$ *(hence this holds if f is a Riemann function).*

2. *For any* $\mathbf{T}$*, and any* $g\colon \mathbb{Z}^n \to \mathbb{Z}$*, let* g_T *be the function given by* $g_T(\mathbf{d}) = g(\mathbf{d}+\mathbf{T})$*. Then the operator* $g \mapsto g_T$ *commutes with the operator* $\mathfrak{m}$*.*

3. *The above operator* $g \mapsto g_T$ *restricts to an operator on functions that are initially zero, and on these functions this operator commutes with* $\mathfrak{s}$*.*

4. *We have*
$$\mathrm{Trans}(f) = \mathrm{Trans}(W). \tag{36}$$

5. *For any* $\mathbf{L}_1, \mathbf{L}_2$ *we have*
$$W^*_{\mathbf{L}_1} = W^*_{\mathbf{L}_2} \quad \iff \quad \mathbf{L}_2 - \mathbf{L}_1 \in \mathrm{Trans}(W). \tag{37}$$

6. *if* f *is a Riemann function, then for any* $\mathbf{K}_1, \mathbf{K}_2 \in \mathbb{Z}^n$*,*
$$f^\wedge_{\mathbf{K}_1} = f^\wedge_{\mathbf{K}_2} \quad \iff \quad \mathbf{K}_1 - \mathbf{K}_2 \in \mathrm{Trans}(f).$$

Proof. (1): let $\mathbf{T} \in \mathrm{Trans}(f)$. Then for all $s \in \mathbb{Z}$ and all $\mathbf{d} \in \mathbb{Z}^n$, $f(\mathbf{d}+s\mathbf{T}) = f(\mathbf{d})$; if $\deg(\mathbf{T}) \neq 0$, then for any $\mathbf{d}$, the degree of $\mathbf{d}+s\mathbf{T}$ can be made arbitrary small for $\pm s$ sufficiently large, in which case $f(\mathbf{d}) = f(\mathbf{d}+s\mathbf{T}) = 0$.

(2): in view of (21), $\mathfrak{m}(g_T)$ and $(\mathfrak{m}g)_T$ both take $\mathbf{d}$ to
$$\sum_{I \subset [n]} (-1)^{|I|} f(\mathbf{d}+\mathbf{T}-\mathbf{e}_I).$$

(3): clearly g_T is initially zero if g is. In view of (22),
$$\big(\mathfrak{s}(g_T)\big)(\mathbf{d}) = \sum_{\mathbf{d}' \leqslant \mathbf{d}} g_T(\mathbf{d}') = \sum_{\mathbf{d}' \leqslant \mathbf{d}} g(\mathbf{d}'+\mathbf{T}) = \sum_{\mathbf{d}' \leqslant \mathbf{d}+\mathbf{T}} g(\mathbf{d}') = \big((\mathfrak{s}g)_T\big)(\mathbf{d}).$$

(4): we have $\mathbf{T} \in \mathrm{Trans}(f)$ implies $f = f_T$ which implies
$$W = \mathfrak{m}f = \mathfrak{m}(f_{\mathbf{T}}) = (\mathfrak{m}f)_{\mathbf{T}} = W_{\mathbf{T}},$$
and hence $\mathbf{T} \in \mathrm{Trans}(W)$. Similarly $\mathbf{T} \in \mathrm{Trans}(W)$ implies
$$f = \mathfrak{s}W = \mathfrak{s}(W_{\mathbf{T}}) = (\mathfrak{s}W)_{\mathbf{T}} = f_{\mathbf{T}},$$
and hence $\mathbf{T} \in \mathrm{Trans}(f)$.

(5): if $\mathbf{L}_1 \in \mathbb{Z}^n$, the map $U \mapsto U^*_{\mathbf{L}_1}$ is defined for any function $U\colon \mathbb{Z}^n \to \mathbb{Z}$, and applying this map twice gives U. Hence if U_1, U_2 are arbitrary functions $\mathbb{Z}^n \to \mathbb{Z}$, then $U_1 = U_2$ iff $(U_1)^*_{\mathbf{L}_1} = (U_2)^*_{\mathbf{L}_1}$. Hence $W^*_{\mathbf{L}_1} = W^*_{\mathbf{L}_2}$ iff
$$\big(W^*_{\mathbf{L}_1}\big)^*_{\mathbf{L}_1} = \big(W^*_{\mathbf{L}_2}\big)^*_{\mathbf{L}_1}. \tag{38}$$
But the left-hand-side of (38) is just W, and the right-hand-side is the function
$$\big(W^*_{\mathbf{L}_2}\big)^*_{\mathbf{L}_1}(\mathbf{d}) = W^*_{\mathbf{L}_2}(\mathbf{L}_1 - \mathbf{d}) = W(\mathbf{L}_2 - \mathbf{L}_1 + \mathbf{d}).$$

Hence $W^*_{\mathbf{L}_1} = W^*_{\mathbf{L}_2}$ iff $\mathbf{L}_2 - \mathbf{L}_1 \in \mathrm{Trans}(W)$.

(6): by part (4) of Theorem 30, we have $f^{\wedge}_{\mathbf{K}_1} = f^{\wedge}_{\mathbf{K}_2}$ iff $W^*_{\mathbf{L}_1} = W^*_{\mathbf{L}_2}$ where $\mathbf{L}_i = \mathbf{K}_i + \mathbf{1}$. By (37), the latter holds iff $\mathbf{L}_2 - \mathbf{L}_1 \in \mathrm{Trans}(W)$. This holds, in view of (36), iff $\mathbf{L}_2 - \mathbf{L}_1 \in \mathrm{Trans}(f)$. Since $\mathbf{K}_2 - \mathbf{K}_1 = \mathbf{L}_2 - \mathbf{L}_1$,

$$\mathbf{L}_2 - \mathbf{L}_1 \in \mathrm{Trans}(f) \quad \iff \quad \mathbf{K}_2 - \mathbf{K}_1 \in \mathrm{Trans}(f). \qquad \square$$

Hence the above proposition implies that a Riemann function, f, and its weight $W = \mathfrak{m}f$ have the same set of invariant translations, all of which lie in $\mathbb{Z}^n_{\deg=0}$, and moreover, when f is self-dual then the $\mathbf{K}$ in Definition 31 (and $\mathbf{L} = \mathbf{K} + \mathbf{1}$) are unique up to a translation by an element of $\mathrm{Trans}(f) = \mathrm{Trans}(W)$.

Remark 34. We remark that the condition $(-1)^n W^*_{\mathbf{L}} = W$ seems to have more direct symmetry than the equivalent condition $f^{\wedge}_{\mathbf{K}} = f$; furthermore, in the examples of the W that we compute in Sections 5 and 6, the W are very sparse (i.e., mostly 0), and so verifying $(-1)^n W^*_{\mathbf{L}} = W$ seems simpler.

Of course, the classical or Graph Riemann-Roch formulas, in terms of our Definition 31, are assertions that self-duality holds in these cases.

Example 35. The Baker-Norine [6] Graph Riemann-Roch theorem for a graph, $G = (V_G, E_G)$, with $V = \{v_1, \ldots, v_n\}$ can be stated as

$$r_{\mathrm{BN},G}(\mathbf{d}) - r_{\mathrm{BN},G}(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + 1 - g,$$

where $g = |E_G| - |V_G| + 1$ and $\mathbf{K} = \mathbf{K}_G = \sum_i \mathbf{e}_i(\deg_G(v_i) - 2)$. Since $f = r_{\mathrm{BN},G} + 1$ is the associated Riemann function, the left-hand-side above also equals $f(\mathbf{d}) - f^{\wedge}_K(\mathbf{K} - \mathbf{d})$, and hence $f = f^{\wedge}_K$ is self-dual.

Example 36. Amini and Manjunath [2] give conditions for f as in (3) with $\mathcal{N}$ as in Example 8 to satisfy self-duality; see Theorem 1.4 of [2]. However, to us these Riemann functions seem interesting to study whether or not self-duality holds.

5 The Weight of Slowly Growing Riemann Functions of Two Variables and of the Baker-Norine Rank of Graphs on Two Vertices

Recall that Theorem 19 characterized the weights of Riemann functions in terms of their "row sums" and "column sums" (namely that the all must equal 1). The point of this section is to characterize the weights of some successively more specific classes of Riemann functions. Let us state the main results; the proofs will be provided in later subsections of this section.

Theorem 37. *Let $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ be a Riemann function, and $W = \mathfrak{m}f$ its weight. Then the following are equivalent:*

1. *f is slowly growing; and*

2. *the following two conditions hold:*

   (a) *for each $d_1 \in \mathbb{Z}$, the non-zero elements of the "d_1-th row of W," i.e., of the sequence*

   $$\ldots, W(d_1,-1), W(d_1,0), W(d_1,1), \ldots,$$

   is an alternating sequence of ± 1, beginning and ending in $+1$, i.e., for some $k \in \mathbb{N}$ and integers $i_1 < i_2 < \cdots < i_{2k-1}$ (depending on d_1) we have

   $$W(d_1,\ell) = \begin{cases} 1 & \text{if } \ell = i_j \text{ for some odd } j; \\ -1 & \text{if } \ell = i_j \text{ for some even } j; \text{ and} \\ 0 & \text{otherwise;} \end{cases}$$

   and

   (b) *similarly for each $d_2 \in \mathbb{Z}$ and the sequence*

   $$\ldots, W(-1,d_2), W(0,d_2), W(1,d_2), \ldots$$

Theorem 37 will be proved in Subsection 5.1.

In Figure 1 we illustrate an example of the weight of a slowly growing function; notice that each row and each column has its non-zero values being $1, -1, 1$ in sequence (two blue dots, with a green dot between the two blues).

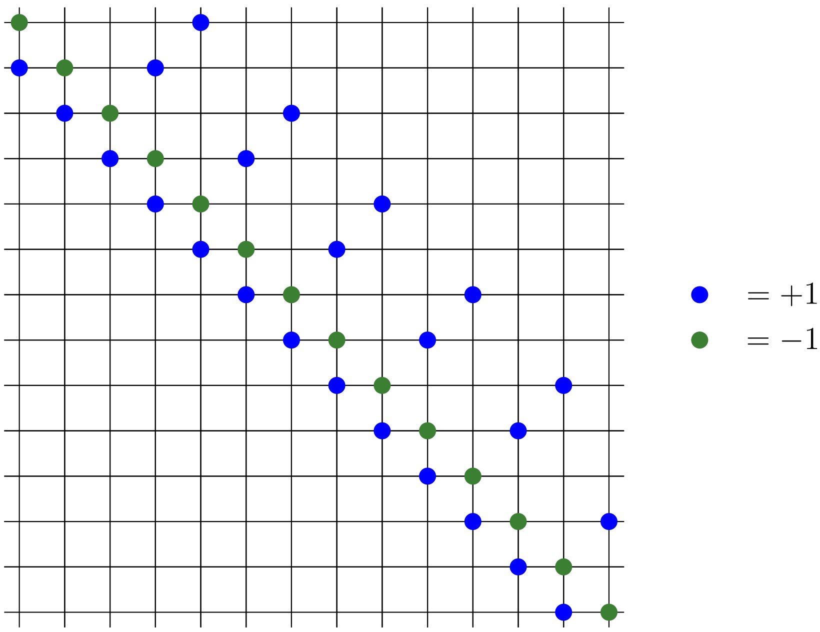


Figure 1: Example of a Weight of a Slowly Growing Function.

We will use Theorem 37 to prove the following theorem.

Theorem 38. *Let $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ be a slowly growing Riemann function. Then the following are equivalent:*

1. *the values of W are all non-negative (hence equal to* 0 *or* 1*);*

2. *there is a bijection $\pi\colon \mathbb{Z} \to \mathbb{Z}$ such that $\pi(x) + x$ is bounded over all $x \in \mathbb{Z}$, and for all $\mathbf{d} \in \mathbb{Z}^2$,*
$$W\big(d_1, d_2\big) = \begin{cases} 1 & \text{if } d_2 = \pi(d_1), \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

3. *f is* supermodular *in the sense that for all $\mathbf{d}$,*
$$f(\mathbf{d} - \mathbf{e}_1 - \mathbf{e}_2) + f(\mathbf{d}) \geqslant f(\mathbf{d} - \mathbf{e}_1) + f(\mathbf{d} - \mathbf{e}_2);$$

4. *for any $\mathbf{d} \in \mathbb{Z}^2$,*
$$f(\mathbf{d}) = f(\mathbf{d} - \mathbf{e}_1) = f(\mathbf{d} - \mathbf{e}_2) \quad \Rightarrow \quad f(\mathbf{d} - \mathbf{e}_1 - \mathbf{e}_2) = f(\mathbf{d});$$

Theorem 38 will be proved in Subsection 5.2.

Definition 39. We say that $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ is a *perfect matching* if W satisfies condition (2) of Theorem 38, i.e., if there is a bijection $\pi\colon \mathbb{Z} \to \mathbb{Z}$ such that $\pi(x) + x$ is bounded over all $x \in \mathbb{Z}$, and for all $\mathbf{d} \in \mathbb{Z}^2$, (39) holds.

We remark that in Figure 1, the set of green dots form a perfect matching, as do the set of blue dots below (or to the left) of the green dots, as do the set of blue dots above (or to the right) of the green dots.

Notice that the weight of a slowly growing function can have -1 appearing in some rows and columns, but not in all rows and columns, as illustrated by Figure 2.

We remark that if W is a perfect matching, π is as in (39), and $f = \mathfrak{s}W$ the associated Riemann function, then f has period r iff f is invariant by translation under $(-r, r) \in \mathbb{Z}^2$; by Proposition 33, this holds iff W in invariant by translation under $(-r, r) \in \mathbb{Z}^2$, and clearly this holds iff $\pi(i + r) = \pi(i) - r$ for all $i \in \mathbb{Z}$.

Definition 40. We say that a bijection $\pi\colon \mathbb{Z} \to \mathbb{Z}$ is *r-skew symmetric* for an $r \in \mathbb{Z}$ if for all $i \in \mathbb{Z}$, $\pi(i + r) = \pi(i) - r$.

Once we have proven the above two theorems, it will be easy to determine the Baker-Norine rank of any graph with two vertices (joined by some number of edges but without self-loops).

Theorem 41. *Let G be a graph on two vertices, v_1, v_2 with $r \geqslant 1$ edges joining v_1 and v_2. Let $r_{\rm BN}\colon \mathbb{Z}^2 \to \mathbb{Z}$ be the Baker-Norine rank, let $f = 1 + r_{\rm BN}$, i.e., f is as in* (3) *in Definition 4. Then $\mathbf{d}$ is in the image of the Laplacian iff $\mathbf{d}$ is an integral multiple of $(r, -r)$. Let $W = \mathfrak{m}f$ be the weight of f. Then*
$$W(0, 0) = W(1, 1) = \ldots = W(r - 1, r - 1) = 1;$$
furthermore $W(\mathbf{d}) = 1$ if $\mathbf{d}$ is equivalent to one of (i, i) with $i = 0, \ldots, r-1$, and otherwise $W(\mathbf{d}) = 0$.

In Figure 3 we illustrate the $r = 3$ case of Theorem 41.

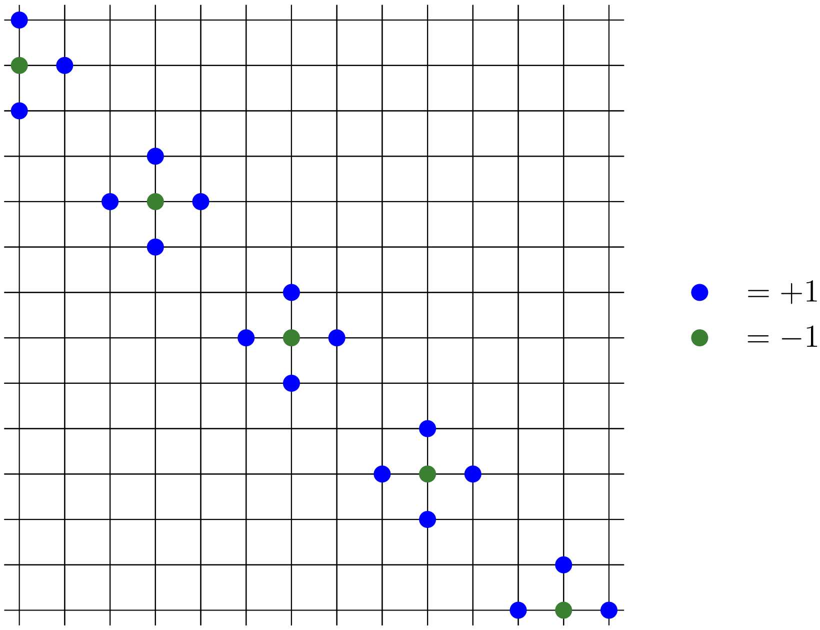


Figure 2: Another Example of a Weight of a Slowly Growing Function.

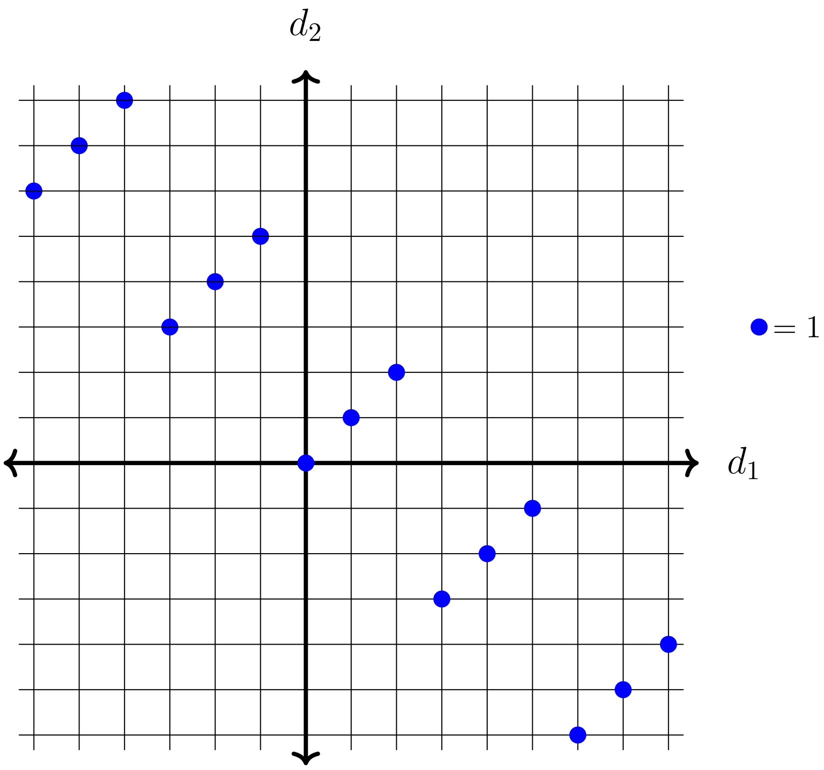


Figure 3: Theorem 41 in the Case $r = 3$.

5.1 Proof of Theorem 37

Proof of Theorem 37. Fix $d_1 \in \mathbb{Z}$. According to (26) we have

$$f(d_1,d_2) - f(d_1-1,d_2) = \sum_{d_2' \leqslant d_2} W(d_1,d_2'), \tag{40}$$

and hence for all d_2 we have

$$\sum_{d_2' \leqslant d_2} W(d_1,d_2') = 0,1 \tag{41}$$

(we use $=0,1$ to mean "equals either 0 or 1"). Since W is initially zero, we have $W(d_1,d_2')=0$ for d_2' sufficiently small and sufficiently large. Hence there are only finitely many values of d_2' such that $W(d_1,d_2') \neq 0$; let these values of d_2' be $i_1 < \cdots < i_\ell$. Since $W(d_1,d_2')=0$ for $d_2' < i_1$, we have

$$\sum_{d_2' \leqslant i_1} W(d_1,d_2') = W(d_1,i_1),$$

and since $W(d_1,i_1) \neq 0$, (41) implies that $W(d_1,i_1)=1$. Similarly, we have

$$\sum_{d_2' \leqslant i_2} W(d_1,d_2') = W(d_1,i_1) + W(d_1,i_2) = 1 + W(d_1,i_2),$$

and so (41) implies that $W(d_1,i_2) = -1$. Similarly, by induction on r we show that $W(d_1,i_r)$ is 1 for r odd, and -1 for r even. Since f is a Riemann function, (23) holds (which just follows from the fact that $f(d_1,d_2)-f(d_1-1,d_2)=1$ for d_2 sufficiently large); hence

$$1 = \sum_{d_2=-\infty}^{\infty} W(d_1,d_2) = W(d_1,i_1) + \cdots + W(d_1,i_\ell),$$

and we see that ℓ must be odd.

This proves that if f is slowly growing, then (2a) of Theorem 37 holds. Similarly (2b) holds.

Conversely, say that f is a Riemann function and condition (2a) holds. Then for each d_1,d_2, if s is the largest integer such that $i_s \leqslant d_2$, then

$$\sum_{d_2' \leqslant d_2} W(d_1,d_2') = i_1 + \cdots + i_s = 0,1.$$

It follows from (40) that have

$$\forall d_1,d_2 \in \mathbb{Z}, \quad f(d_1,d_2) - f(d_1-1,d_2) = 0,1.$$

Similarly condition (2b) implies that

$$\forall d_1,d_2 \in \mathbb{Z}, \quad f(d_1,d_2) - f(d_1-1,d_2) = 0,1.$$

These two conditions on f are clearly equivalent to f being slowly growing. □

5.2 Proof of Theorem 38

To prove Theorem 38, we first prove the following lemma that is interesting in its own right. Notice that the notion of *slowly growing* makes sense for any function $f\colon \mathbb{Z}^2 \to \mathbb{Z}$, i.e., that for all $\mathbf{d} \in \mathbb{Z}^2$ and all $i \in [2]$,

$$0 \leqslant f(\mathbf{d}+\mathbf{e}_i) - f(\mathbf{d}) \leqslant 1$$

(i.e., this makes sense even if f is not a Riemann function or not even initially zero).

Lemma 42. *Let $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ be any slowly growing function. For any $\mathbf{d} \in \mathbb{Z}^2$, let $a = f(\mathbf{d})$. Then*

$$W(\mathbf{d}) = 1 \quad \iff \quad a-1 = f(\mathbf{d}-\mathbf{e}_1) = f(\mathbf{d}-\mathbf{e}_1) = f(\mathbf{d}-\mathbf{e}_1-\mathbf{e}_2), \tag{42}$$

and

$$W(\mathbf{d}) = -1 \quad \iff \quad a = f(\mathbf{d}-\mathbf{e}_1) = f(\mathbf{d}-\mathbf{e}_1) = f(\mathbf{d}-\mathbf{e}_1-\mathbf{e}_2)+1, \tag{43}$$

Proof. Since f is slowly growing, $f(\mathbf{d}-\mathbf{e}_1-\mathbf{e}_2)$ equals one of $a, a-1, a-2$. We now analyze these three cases: if it equals a, then $f(\mathbf{d}-\mathbf{e}_i) = a$ for $i = 1, 2$, and hence $W(\mathbf{d}) = 0$; if it equals $a-2$, then $f(\mathbf{d}-\mathbf{e}_i) = a-1$ for $i = 1, 2$, and again $W(\mathbf{d}) = 0$. Hence we can only have $W(\mathbf{d}) \neq 0$ if $f(\mathbf{d}-\mathbf{e}_1-\mathbf{e}_2) = a-1$, and in this case we see that if one of $f(\mathbf{d}-\mathbf{e}_1), f(\mathbf{d}-\mathbf{e}_2)$ equals a and the other $a-1$, then $W(\mathbf{d}) = 0$ Hence if $W(\mathbf{d}) \neq 0$, then $f(\mathbf{d}-\mathbf{e}_1) = f(\mathbf{d}-\mathbf{e}_2)$ is one of $a-1$ or a, and these two cases are those described in, respectively, (42) and (43). □

Proof of Theorem 38. (1) ⇒ (2): let us define the bijection π: by Theorem 37, if W never equals -1, then for each d_1 there is a unique i_1; such that $W(d_1, i_1) = 1$; let $\pi(d_1) = i_1$. Theorem 37 also implies that for each d_2 there is a unique i_1 such that $W(i_1, d_2) = 1$, and therefore π is a bijection. Since $W(d_1, d_2) = 0$ if $d_1 + d_2$ is sufficiently small or sufficiently large, and since $W(x, \pi(x)) = 1$, it follows that $x+\pi(x)$ is bounded from below and above.

(2) ⇒ (1): immediate from (39).

(1) $\iff$ (3): immediate from the formula

$$W(\mathbf{d}) = f(\mathbf{d}) - f(\mathbf{d}-\mathbf{e}_1) - f(\mathbf{d}-\mathbf{e}_2) + f(\mathbf{d}-\mathbf{e}_1-\mathbf{e}_2).$$

(1) $\iff$ (4): follows easily from (43) of Lemma 42. □

5.3 Proof of Theorem 41

Proof of Theorem 41. The rows of the Laplacian of G are $(r, -r)$ and $(-r, r)$, and hence the image, L, of the Laplacian equals the integer multiples of $(r, -r)$. According to Corollary 7, f is slowly growing (and so is $r_{\mathrm{BN}} = f-1$).

First let us prove that W takes on only non-negative values: indeed by

1. if $f(\mathbf{d}) = 0$, then $f(\mathbf{d}') = 0$ for $\mathbf{d}' \leqslant \mathbf{d}$ and hence $W(\mathbf{d}) = 0$;

2. if $f(\mathbf{d}) \geqslant 1$, then by (9) of Proposition 6 (and the fact that $\mathcal{N}$ in Definition 4 is a downset), we have $f(\mathbf{d}) = f(\mathbf{d} - \mathbf{e}_i) + 1$ for some $i = 1, 2$. Hence, in Lemma 42, (43) implies that $W(\mathbf{d}) \geqslant 0$.

It follows that W is a perfect matching, and hence W is given by (39) for some perfect matching π; since f is r-periodic, it suffices to determine $\pi(i)$ for $i = 0, 1, \ldots, r-1$. Let us do so by finding some values of f.

First note that for $0 \leqslant i \leqslant r-1$, $(i, -1)$ is not equivalent to an effective divisor, for otherwise for some $m \in \mathbb{Z}$ we would have $(i, -1) \geqslant m(r, -r)$, which implies both $m \leqslant i/r < 1$ and $m \geqslant 1/r > 0$, which is impossible since there is no integer r that satisfies $r < 1$ and $r > 0$. Hence $f(i, -1) = 0$, and similarly $f(-1, i) = 0$. Figure 4 illustrates these value in blue for $r = 4$.

$f(-1,3)=0$				$f(3,3)\geqslant 4$
$f(-1,2)=0$			$f(2,2)\geqslant 3$	
$f(-1,1)=0$		$f(1,1)\geqslant 2$		
$f(-1,0)=0$	$f(0,0)\geqslant 1$			
$f(-1,-1)=0$	$f(0,-1)=0$	$f(1,-1)=0$	$f(2,-1)=0$	$f(3,-1)=0$

Figure 4: Some $f = r_{\mathrm{BN}} + 1$ Values, Two Vertices Joined by 4 Edges.

Next we claim that for $i \geqslant 0$ we have $f(i, i) \geqslant i + 1$: indeed, if $\|(i, i) - \mathbf{d}'\|_{L^1} \leqslant i$, then $(i, i) - \mathbf{d}'$ has non-negative components, and hence $(i, i) - \mathbf{d}' \notin \mathcal{N}$ with $\mathcal{N}$ as in Definition 4. Hence $\rho_{L^1}((i, i), \mathcal{N}) \geqslant i + 1$. These conclusions are illustrated in Figure 4 in green for $r = 4$.

For $0 \leqslant i \leqslant r-1$, since $f(i, -1) = 0$ and $f(i, i) \geqslant i + 1$, the fact that f is slowly growing implies that $f(i, j) = j + 1$ for $0 \leqslant j \leqslant i$. Similarly, for such i, j with $0 \leqslant i \leqslant j$, $f(i, j) = i + 1$. These conclusions are illustrated in Figure 5 in black for $r = 4$.

$f(-1,3)=0$	$f(0,3)=1$	$f(1,3)=2$	$f(2,3)=3$	$f(3,3)=4$
$f(-1,2)=0$	$f(0,2)=1$	$f(1,2)=2$	$f(2,2)=3$	$f(3,2)=3$
$f(-1,1)=0$	$f(0,1)=1$	$f(1,1)=2$	$f(2,1)=2$	$f(3,1)=2$
$f(-1,0)=0$	$f(0,0)=1$	$f(1,0)=1$	$f(2,0)=1$	$f(3,0)=1$
$f(-1,-1)=0$	$f(0,-1)=0$	$f(1,-1)=0$	$f(2,-1)=0$	$f(3,-1)=0$

Figure 5: Some $f = r_{\mathrm{BN}} + 1$ Value Inferred from Figure 4.

Using this, it follows that for $i = 0, \dots, r-1$ we have

$$W(i,i) = f(i,i) - 2f(i,i-1) + f(i-1,i-1) = i - 2(i-1) + i - 1 = 1.$$

It follows that $\pi(i) = i$ for $0 \leqslant i \leqslant r-1$, and the theorem follows. □

Notice that this computation proves the Riemann-Roch formula in this case: this computation shows that $W = W^*_{\mathbf{L}}$ for $L = (r-1, r-1)$. Hence $f = f^{\wedge}_{\mathbf{K}}$ for $\mathbf{K} = (r-2, r-2)$, and therefore

$$f(\mathbf{d}) - f(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + C$$

for some C. Taking $\mathbf{d} = 0$ and using $f(0,0) = 1$ we get

$$1 - f(\mathbf{K}) = C,$$

and taking $\mathbf{d} = \mathbf{K}$ we get

$$f(\mathbf{K}) - 1 = \deg(\mathbf{K}) + C = 2(r-2) + C;$$

adding these last two equations, the $f(\mathbf{K})$ cancels and we get $0 = 2(r-2) + 2C$, and so $C = 2 - r$ is the offset. Hence

$$f(\mathbf{d}) - f(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) - r + 2.$$

6 The Weight of the Riemann-Roch Rank of Complete Graphs

The point of this subsection is to give a self-contained computation of the remarkably simple and sparse weight function of the Baker-Norine rank for complete graphs.

Our proof uses many standard ideas in the graph Riemann-Roch literature [6, 4, 2, 9], but also one rather ingenious idea of Cori and Le Borgne [9].

6.1 Proof Overview and Computer-Aided Computations

The folloing subsection is not essential to the rest of the article. In this subsection we make some general remarks about computing the Baker-Norine rank of a graph, G, and then we make some specific remarks about our to computer-aided computation on complete graphs, K_n, on n vertices. We also remark on the surprising results these computations gave for $n \leqslant 6$.

This section also serves to give an overview of the proof of the main theorems in this section, and to motivate the definitions we give. However, the reader can skip this subsection entirely, and will find precise definition and theorems starting in Subsection 6.2.

6.1.1 Remarks about Computing the Baker-Norine Rank for a General Graph, G

Let us begin with some general remarks on algorithms to compute the Baker-Norine rank, $r_{\rm BN}$, of a general graph and resulting weight, W.

Let G be a graph on n-vertices ordered $v_1,\ldots,v_n$. To compute the Baker-Norine function, $r_{\rm BN}$ of a graph (and the resulting weight, W), we note that $r_{\rm BN}(\mathbf{d})=-1$ if $\deg(\mathbf{d})<0$; hence it suffices to compute $r_{\rm BN}(\mathbf{d})$ on $\mathbb{Z}^n_{\deg=0}$, then on $\mathbb{Z}^n_{\deg=1}$, then $\mathbb{Z}^n_{\deg=2}$, etc. Since $r_{\rm BN}$ and W are invariant under the image of the Laplacian, Δ_G, it suffices to determine the value of $r_{\rm BN}$ on a set of representatives of

$$\mathrm{Pic}_i(G)=\mathbb{Z}^n_{\deg=i}/\mathrm{Image}(\Delta_G)$$

for $i=0,1,\ldots$. To do so, it is natural to: find a set of "convenient coordinates" for $\mathrm{Pic}_0(G)=\mathbb{Z}^n_{\deg=0}/\mathrm{Image}(\Delta_G)$, meaning a set $\mathcal{B}$ and a bijection $\iota\colon\mathcal{B}\to\mathrm{Pic}_0(G)$ such that the computations below are easy to do for $i=0,1,\ldots$, namely:

1. for all $\mathbf{b}\in\mathcal{B}$, determine if $\iota(\mathbf{b})+i\mathbf{e}_n$ is not equivalent to an effective divisor, i.e., if $r_{\rm BN}(\iota(\mathbf{b})+i\mathbf{e}_n)=-1$; and

2. for all other $\mathbf{b}\in\mathcal{B}$ we compute $r_{\rm BN}(\mathbf{b}+i\mathbf{e}_n)$ via the formula

$$r_{\rm BN}(b+i\mathbf{e}_n)=1+\min_{j\in[n]}r_{\rm BN}(\mathbf{b}+i\mathbf{e}_n-\mathbf{e}_j);$$

hence we need a reasonably fast algorithm to determine the element of $\mathcal{B}$ that is equivalent to $\iota^{-1}(\mathbf{b}+\mathbf{e}_n-\mathbf{e}_j)$. [We are finished when $i\geqslant\deg(\mathbf{L})$ where $\mathbf{L}=\mathbf{K}+\mathbf{1}$ where $\mathbf{K}=\mathbf{K}_G$ is the Baker-Norine canonical divisor, and hence when $i\geqslant 2(|E|-|V|)+|V|=2|E|-|V|$; we may use $W=(-1)^nW^*_{\mathbf{L}}$ to finish when $i\geqslant|E|+(1-|V|)/2$.]

Of course, one can replace $\mathbf{e}_n$ above by any of $\mathbf{e}_1,\ldots,\mathbf{e}_{n-1}$, or, more generally, any element of $\mathbb{Z}^n$ of degree 1; our choice of $\mathbf{e}_n$ is convenient for the representatives of $\mathcal{B}$ below.

6.1.2 Remarks about the Baker-Norine Rank for Complete Graphs

At this point we turn our attention to implementing such computations on the complete graph, K_n, on the set of n vertices $[n]=\{1,\ldots,n\}$, i.e., that has one edge for each pair of distinct vertices, and therefore has $\binom{n}{2}$ vertices.

It turns out that there is a very convenient choice of coordinates $(\iota,\mathcal{B})$ for $\mathrm{Pic}_0(G)$ when $G=K_n$. We describe these coordinates in two steps. First, in Lemma 50, we show for each element of $\mathbf{a}\in\mathbb{Z}^n$ there exists a unique element is equivalent to $\mathbf{a}$ (i.e., modulo $L=\mathrm{Image}_{K_n}$) in

$$\mathcal{A}\overset{\rm def}{=}\{0,\ldots,n-1\}^{n-2}\times\{0\}\times\mathbb{Z}=\bigl\{\mathbf{a}\in\mathbb{Z}^n\mid a_1,\ldots,a_{n-2}\in\{0,\ldots,n-1\},a_{n-1}=0\bigr\}.$$

Proving existence is easy from the fact that $\mathbf{1}-\mathbf{e}_n$ and $n(\mathbf{e}_i-\mathbf{e}_n)$ lie in the image of Δ_{K_n}: first we apply multiples of $\mathbf{1}-\mathbf{e}_n$ to get the $n-1$ component to 0, and then apply multiples of $n(\mathbf{e}_i - \mathbf{e}_n)$ with $1 \leqslant i \leqslant n-2$ to get the first $n-2$ components between 0 an $n-1$; the uniqueness argument is more involved, but fairly straightforward in view of the the well-known fact that $|\operatorname{Pic}_0(K_n)| = n^{n-2}$.

[The "Algorithm" of Section 2.1 of [9], takes an element $f \in \mathbb{Z}^n$ and produces an equivalent g with $g_1 = 0$ and $0 \leqslant g_i \leqslant n-1$ for $i = 2, \ldots, n-1$; hence the set $\mathcal{A}$ appears there — at least implicitly (after exchanging the 1 and $n-1$ components) — in Section 2.1 of [9]; there they use the language of "configuation topplings" for adding a combination of rows of Δ_{K_n} to get equivalent configurations: for example, the fact that $\mathbf{1} - n\mathbf{e}_n = (1, \ldots, 1, -n)$ is in the image of Δ_{K_n} corresponds to "toppling at the vertex n," and the same for $n(\mathbf{e}_i - \mathbf{e}_n)$ corresponds to "toppling at n minus the toppling at i."]

It turns out that there is a remarkably simple way to determine, for $\mathbf{a} \in \mathcal{A}$, the value of

$$r_{\mathrm{BN}}(\mathbf{a}) - r_{\mathrm{BN}}(\mathbf{a} - \mathbf{e}_{n-1})$$

(which is either 0 or 1). This is given in Theorem 54, namely, the value is 1 iff

$$a_1 + \cdots + a_{n-2} \leqslant \deg(\mathbf{a}). \tag{44}$$

The proof involves a rather ingenious and elegant observation of [9], and a "counting method" that we describe in Subsection 6.1.4 below and formalize in the special case of the Baker-Norine rank function of a general graph, G as Lemma 53 below (in this article we apply this only for $G = K_n$). Since

$$\big((1 - \mathfrak{t}_{n-1})\, r_{\mathrm{BN}}\big)(\mathbf{a}) = r_{\mathrm{BN}}(\mathbf{a}) - r_{\mathrm{BN}}(\mathbf{a} - \mathbf{e}_{n-1})$$

where $\mathfrak{t}_{n-1}$ is as in (28). It immediately follows that for $\mathbf{a} \in \mathcal{A}$ we have

$$\big((1 - \mathfrak{t}_n)(1 - \mathfrak{t}_{n-1})\, r_{\mathrm{BN}}\big)(\mathbf{a}) = \begin{cases} 1 & \text{if } a_1 + \cdots + a_{n-2} = \deg(\mathbf{a}), \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{45}$$

The last task is to apply $(1 - \mathfrak{t}_1) \cdots (1 - \mathfrak{t}_{n-2})$ to the above formula, to compute $W(\mathbf{a})$. The problem in doing this is that it seems very awkward to apply $(1 - \mathfrak{t}_1) \cdots (1 - \mathfrak{t}_{n-2})$ to expressions involving $\mathbf{a} \in \mathcal{A}$; the reason is that the most natural way to apply $(1 - \mathfrak{t}_1) \cdots (1 - \mathfrak{t}_{n-2})$ to a function $g \colon \mathbb{Z}^n \to \mathbb{Z}$ is to write

$$\big((1 - \mathfrak{t}_1) \cdots (1 - \mathfrak{t}_{n-2})\, g\big)(\mathbf{a}) = \sum_{J \subset [n-2]} g(\mathbf{a} - \mathbf{e}_J)(-1)^{|J|},$$

where

$$\mathbf{e}_J = \sum_{j \in J} \mathbf{e}_j.$$

This means that for each $\mathbf{a} \in \mathcal{A}$, and each J above, we need to find the vector in $\mathcal{A}$ that is equivalent to $\mathbf{a} - \mathbf{e}_J$. This seems a bit difficult to do in an organized fashion, at least if we "coordinatize" $\mathcal{A}$ by its $n-1$ components $a_1, \ldots, a_{n-2}$ and a_n (we omit a_{n-1} since it is zero on all elements of $\mathcal{A}$), thereby associating each element of $\mathcal{A}$ with an element of

$\{0,\ldots,n-1\}^{n-2}\times\mathbb{Z}$. The broader problem is that this identification does not directly give convenient coordinates for $\mathrm{Pic}(K_n)$ for the following reason: to determine the addition law in $\mathrm{Pic}(K_n)$, one needs to determine, for $\mathbf{a},\mathbf{a}'\in\mathcal{A}$, the unique $\mathbf{a}''\in\mathcal{A}$ equivalent to $\mathbf{a}+\mathbf{a}'$; although for $1\leqslant i\leqslant n-2$, a''_i is simply $a_i+a'_i$ modulo n, we have

$$a''_n=a_n+a'_n-n\Big|\big\{i\leqslant n-2 \mid a_i+a'_i\geqslant n\big\}\Big|. \tag{46}$$

In other words, the addition law on $\mathcal{A}$ induced by the law on Pic corresponds to addition in a semidirect product $(\mathbb{Z}/n\mathbb{Z})^{n-2}\ltimes\mathbb{Z}$. (involving the rather inconvenient formula (46)).

To compute $(1-\mathfrak{t}_1)\cdots(1-\mathfrak{t}_{n-2})$ applied to (45), we choose coordinates a second set of coordinates on $\mathcal{A}$, which allows us to conveniently determine the values in $\mathcal{A}$ equivalent to $\mathbf{a}-\mathbf{e}_J$ for all $\mathbf{a}\in\mathcal{A}$ and $J\subset[n-2]$. Our second coordinates gives an isomorphism between $\mathrm{Pic}(K_n)$ and $(\mathbb{Z}/n\mathbb{Z})^{n-2}\times\mathbb{Z}$ (with the product, rather than the semidirect product) as follows: we set $\mathcal{B}=\{0,\ldots,n-1\}^{n-2}$, we define $\iota\colon\mathcal{B}\to\mathrm{Pic}_0$ via

$$\iota\mathbf{b}=\big(b_1,\ldots,b_{n-2},0,-b_1-\cdots-b_{n-2}\big)\in\mathbb{Z}^n_{\deg=0}.$$

In order to avoid writing ι all the time, for $(\mathbf{b},i)\in\mathcal{B}\times\mathbb{Z}$ we set

$$\langle\mathbf{b},i\rangle=\iota(\mathbf{b})+i\mathbf{e}_n, \tag{47}$$

which equals

$$\big(b_1,\ldots,b_{n-2},0,i-b_1-\cdots-b_{n-2}\big)\in\mathbb{Z}^n_{\deg=i}.$$

Hence we leave the first $n-1$ coordinates as is in $\mathcal{A}$, but we form $\langle\mathbf{b},i\rangle$ to have degree i. In this way

$$\langle\mathbf{b},i\rangle+\langle\mathbf{b}',i'\rangle$$

has degree $i+i'$, has $(n-1)$-th coordinate 0, and has the first $n-2$ coordinates given by addition in $(\mathbb{Z}/n\mathbb{Z})^{n-2}$; hence the addition law in Pic in the second coordinates $(\mathbf{b},i)$, is just addition on $(\mathbb{Z}/n\mathbb{Z})^{n-2}\times\mathbb{Z}$. For computing the Baker-Norine rank function of K_n, it is convenient that

$$\mathbf{e}_n\sim\langle(0,\ldots,0),1\rangle \quad\text{and}\quad \forall i\in[n-2],\quad \mathbf{e}_i\sim\langle\mathbf{e}_i,1\rangle;$$

the only minor inconvenience is that

$$\mathbf{e}_{n-1}\sim\langle(-1,\ldots,-1),1\rangle,$$

which makes the computation r_{BN} and $W=\mathfrak{m}\,r_{\mathrm{BN}}$ more subtle; this seems to be the price of passing from the coordinates suggested by $\mathcal{A}$ (involving the semidirect product and (46)) to the coordinates of $\mathcal{B}\times\mathbb{Z}$.

In essence, the second coordinates parameterize an element of $\mathbf{a}\in\mathcal{A}$ by the values of a_i modulo n for $i\leqslant n-2$, and by $\deg(\mathbf{a})$. Hence the last "coordinate" in our first coordinates for $\mathbf{a}\in\mathcal{A}$, namely a_n, is replaced with $\deg(\mathbf{a})$, which serves to "straighten" the last coordinate in our second coordinate system, and sets up an isomorphism between $\mathrm{Pic}(K_n)$ and $(\mathbb{Z}/n\mathbb{Z})^{n-2}\times\mathbb{Z}$, rather than the semidirect product of our first coordinates. Furthermore, since (44) and (45) are expressed in terms of $a_1,\ldots,a_n$ and $\deg(\mathbf{a})$, the second coordinates are simpler to apply.

6.1.3 Computer Experiments Involving Complete Graphs and Proofs

The main theorems in this section were proven based on the suggested patterns we observed in computer experiments on the Baker-Norine rank for the complete graph, K_n, on n vertices with $n \leqslant 6$ (using the the coordinates (47) above); these showed some remarkable patterns that we now describe. The point of this subsubsection is to explain that although the pattern we observed for $W = \mathfrak{m}\, r_{\mathrm{BN}}$ was very simple, we performed more experiments to look for patterns that would allow us to rigorously prove our observations regarding $W = \mathfrak{m}\, r_{\mathrm{BN}}$.

First, we computed $W = \mathfrak{m}\, r_{\mathrm{BN}}$ for the complete graph were very sparse, i.e., mostly 0's, and the non-zero values of W followed a simple pattern:

$$W(\langle \mathbf{b}, i\rangle) = \begin{cases} (-1)^\ell \binom{n-2}{\ell} & \text{if } \mathbf{b} = \mathbf{0} \text{ and } i = n\ell \text{ for some } \ell = 0, \ldots, n-2, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We will prove this holds rigorously, for all $n \in \mathbb{Z}$, in Theorem 67. Unfortunately, if a computer suggests a simple pattern for W, it is not clear how to rigorously prove this.

So second, since $W = \mathfrak{m}\, r_{\mathrm{BN}}$, and since

$$\mathfrak{m} = (1 - \mathfrak{t}_1) \cdots (1 - \mathfrak{t}_n)$$

(recall that $\mathfrak{t}_i$ is the "downward shift operator" given in (28)), we hoped that applying a single $1 - \mathfrak{t}_i$ to r_{BN}, or a small number of them, we would observe a simple pattern that one might be able to prove rigorously: presumably would be much easier to prove some pattern about $(1 - \mathfrak{t}_i)\, r_{\mathrm{BN}}$ than to prove something about $\mathfrak{m}\, r_{\mathrm{BN}}$. After some unsuccessful experiments, we discovered that $(1 - \mathfrak{t}_{n-1}) r_{\mathrm{BN}}$ had a remarkably simple pattern, namely that for $n \leqslant 6$,

$$(1 - \mathfrak{t}_{n-1}) r_{\mathrm{BN}}\big(\langle \mathbf{b}, i\rangle\big) = \begin{cases} 1 & \text{if } b_1 + \cdots + b_{n-2} \leqslant i \\ 0 & \text{otherwise.} \end{cases} \tag{48}$$

We will rigorously prove that this holds in Theorem 54. Once we prove this, it easily follows that:

$$(1 - \mathfrak{t}_n)(1 - \mathfrak{t}_{n-1}) r_{\mathrm{BN}}\big(\langle \mathbf{b}, i\rangle\big) = \begin{cases} 1 & \text{if } b_1 + \cdots + b_{n-2} = i \\ 0 & \text{otherwise.} \end{cases} \tag{49}$$

From there it is not too hard to apply $(1 - \mathfrak{t}_1) \cdots (1 - \mathfrak{t}_{n-2})$ to both sides of (49) to obtain the result about $W = \mathfrak{m}\, r_{\mathrm{BN}}$.

Let us make some general comments on how we prove (48): since r_{BN} is slowly growing, we have that for each $\mathbf{d}$,

$$\big((1 - \mathfrak{t}_{n-1}) r_{\mathrm{BN}}\big)(\mathbf{d}) = r_{\mathrm{BN}}(\mathbf{d}) - r_{\mathrm{BN}}(\mathbf{d} - \mathbf{e}_{n-1})$$

is either 0 or 1. For a general graph, G, it is not easy to determine for which $\mathbf{d}$ this value is 1, and for which it is 0. Our proof for $G = K_n$ works as follows: first, we prove that this value is 1 when $\mathbf{d} = \langle \mathbf{b}, i\rangle$ with $b_1 + \cdots + b_{n-2} \geqslant i$; this relies on a remarkable observation of [9] on r_{BN} for complete graphs. Second, we show that the remaining values of $(1 - \mathfrak{t}_{n-1}) r_{\mathrm{BN}}$ must be 0, based on a "counting method" which we now describe.

6.1.4 A Counting Method for Slowly Growing Riemann Functions

To clarify the "counting method" we use, it is helpful to explain the method more abstractly and generally. In Subsection 6.5, we will specialize this discussion to the Baker-Norine rank; there we give a more formal statement, namely Lemma 53, and supply a more detailed proof.

We remark that if $r_{\rm BN} = f-1$, then $(1-\mathfrak{t}_{n-1})r_{\rm BN} = (1-\mathfrak{t}_{n-1})f$ (and similarly for any two functions that differ by an additive constant). Hence as soon as we apply $1-\mathfrak{t}_{n-1}$ or, similarly, the "difference operator" $1-\mathfrak{t}_i$ for any $i\in[n]$, we may use $r_{\rm BN}$ and $f = r_{\rm BN}+1$ interchangeably.

Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be a slowly growing Riemann function with offset C such that $f(\mathbf{d}) = 0$ for $\deg(\mathbf{d}) < 0$ (we make this assumption for notational convenience; one can eliminate this assumption at the cost of more complicated notation). Since f is slowly growing, for any $\mathbf{d} \in \mathbb{Z}^n$ we have

$$f(\mathbf{d}) - f(\mathbf{d}-\mathbf{e}_{n-1}) = 0,1 \tag{50}$$

(i.e., equals either 0 or 1). Our "counting method" says, roughly speaking, that if f is invariant under translation by a set L with $\mathbb{Z}^n_{\deg=0}/L$ finite, and if we can prove that $=1$ holds in (50) for "enough $\mathbf{d}$" in a set of representatives for $\mathbb{Z}^n_{\deg=0}/L$, then $=0$ must hold in (50) for the other representatives. Let us describe this in more detail.

For any $\mathbf{d}\in\mathbb{Z}^n$ and $\ell\in\mathbb{N}$ we have

$$\Big(f(\mathbf{d})-f(\mathbf{d}-\mathbf{e}_{n-1})\Big)+\Big(f(\mathbf{d}+\mathbf{e}_{n-1})-f(\mathbf{d})\Big)+\cdots+\Big(f(\mathbf{d}+\ell\mathbf{e}_{n-1})-f(\mathbf{d}+(\ell-1)\mathbf{e}_{n-1})\Big)$$
$$= f(\mathbf{d}+\ell\mathbf{e}_{n-1}) - f(\mathbf{d}-\mathbf{e}_{n-1});$$

hence for any $\mathbf{d}\in\mathbb{Z}^n_{\deg=0}$, for ℓ sufficiently large,

$$\sum_{i=0}^{\ell}\big((1-\mathfrak{t}_{n-1})r_{\rm BN}\big)(\mathbf{d}+i\mathbf{e}_{n-1}) = f(\mathbf{d}+\ell\mathbf{e}_{n-1}) - f(\mathbf{d}-\mathbf{e}_{n-1}) = \ell + C \tag{51}$$

($\mathbf{e}_{n-1}$ can be replaced with any $\mathbf{e}_i$; in this section, our application involves $\mathbf{e}_{n-1}$).

Next assume that f is invariant by a lattice of translations $L\subset\mathbb{Z}^n_{\deg=0}$ with L of rank $n-1$, so that $\mathbb{Z}^n_{\deg=0}/L$ is a finite set; let $\mathcal{P}$ be any set of representatives of the classes of $\mathbb{Z}^n$ modulo L, and let $\mathcal{P}_i = \mathcal{P}\cap\mathbb{Z}^n_{\deg=i}$. Let M_i denote the number of $\mathbf{p}\in\mathcal{P}_i$ such that

$$\big((1-\mathfrak{t}_{n-1})r_{\rm BN}\big)(\mathbf{p}) = 1 \tag{52}$$

(therefore $M_i = 0$ for $i<0$ and $M_i = |\mathcal{P}_i| = |\mathcal{P}_0| = |\mathbb{Z}^n_{\deg=0}/L|$ for i sufficiently large). Then summing (51) over all $\mathbf{d}\in\mathcal{P}_0$ and exchanging summation we have

$$\text{for } \ell \text{ sufficiently large,}\qquad \sum_{i=0}^{\ell} M_i = |\mathcal{P}_0|(\ell+C). \tag{53}$$

Now for all $i\in\mathbb{Z}_{\geqslant 0}$, assume that we have subsets $\mathcal{P}'_i\subset\mathcal{P}_i$ satisfying

$$\mathbf{p}\in\mathcal{P}'_i \quad\Rightarrow\quad \big((1-\mathfrak{t}_{n-1})f\big)(\mathbf{p}) = 1. \tag{54}$$

Then we can draw the following conclusions:

1. for all $i \in \mathbb{Z}$, $|\mathcal{P}'_i| \leqslant M_i$;

2. if for some i we have $|\mathcal{P}'_i| = M_i$, then $\mathcal{P}'_i$ consists of all $\mathbf{p} \in \mathcal{P}_i$ such that (52) holds, and hence for any such i we have

$$\mathbf{p} \in \mathcal{P}_i \setminus \mathcal{P}'_i \quad \Rightarrow \quad \big((1 - \mathfrak{t}_{n-1}) r_{\mathrm{BN}}\big)(\mathbf{p}) = 0; \tag{55}$$

3. for all $\ell \in \mathbb{N}$ we have

$$\sum_{i=0}^{\ell} \big|\mathcal{P}'_i\big| \leqslant \sum_{i=0}^{\ell} M_i, \tag{56}$$

   and therefore

$$\text{for } \ell \text{ sufficiently large,} \qquad \sum_{i=0}^{\ell} \big|\mathcal{P}'_i\big| \leqslant |\mathcal{P}_0|(\ell + C); \tag{57}$$

   and

4. therefore, if for all ℓ sufficiently large we have

$$\sum_{i=0}^{\ell} \big|\mathcal{P}'_i\big| = |\mathcal{P}_0|(\ell + C), \tag{58}$$

   then $|\mathcal{P}'_i| = M_i$ for all $i \geqslant 0$, and hence for all $i \geqslant 0$, (55) holds.

In summary, if for all $i \geqslant 0$ we can find $\mathcal{P}'_i \subset \mathcal{P}_i$ such that (54) and (58), then we have (55).

One can also formulate the analogous statement for subsets $\mathcal{P}''_i \subset \mathcal{P}_i$ such that

$$\mathbf{p} \in \mathcal{P}''_i \quad \Rightarrow \quad \big((1 - \mathfrak{t}_{n-1}) f\big)(\mathbf{p} + i\mathbf{e}_{n-1}) = 0,$$

and for all ℓ sufficiently large

$$\sum_{i=0}^{\ell} |\mathcal{P}''_i| = (1 - C)|\mathcal{P}_0|.$$

Then it follows that

$$\mathbf{p} \in \mathcal{P}_i \setminus \mathcal{P}''_i \quad \Rightarrow \quad \big((1 - \mathfrak{t}_{n-1}) r_{\mathrm{BN}}\big)(\mathbf{p}) = 1.$$

Remark 43. (This remark is independent of the rest of this article.) Let us also mention that the telescoping sum (51) is related to what are called "Weierstrass gaps" and "Weierstrass points" in the context of Riemann surfaces or algebraic curves. Let us describe such ideas in our context of a slowly growing Riemann functions $f\colon \mathbb{Z}^n \to \mathbb{Z}$ where $f(\mathbf{d}) = 0$

for $\deg(\mathbf{d}) < 0$: namely, one notices that for $\ell \in \mathbb{N}$ sufficiently large one has that for any $j \in [n]$,

$$\sum_{i=0}^{\ell}\Big(f(i\mathbf{e}_j) - f\big((i-1)\mathbf{e}_j\big)\Big) = f(\ell\mathbf{e}_j) - f(-\mathbf{e}_j) = \ell + C.$$

Hence the set of $i \geqslant 0$ for which

$$f(i\mathbf{e}_j) - f\big((i-1)\mathbf{e}_j\big) = 0 \tag{59}$$

is a finite set of size equal to $1 - C$. Such values i are viewed as "gaps" for $\mathbf{e}_j$, which is associated to a point P_j on a curve in examples in (15) of Subsection 2.6; for points P_j on a curve, a value of i such that (59) means that there is no function with a pole of order i at P_j (and no other poles), and such i are called "Weierstrass gaps" for P_j. Hence our "counting method" involves with a telescoping sum that is related to (although not directly analogous to) the idea of "Weierstrass gaps."

The rest of this section is devoted to proving that the above stated formulas for W and the formulas (48) and (49) hold.

6.2 Maximal Decrease

The following is a standard tool used in studying the graph Riemann-Roch rank, used by Baker-Norine [6] and many subsequent papers. It is valid in the general setting of (3), i.e., $f(\mathbf{d}) = \rho_{L^1}(\mathbf{d}, \mathcal{N})$, when $\mathcal{N}$ is a downset.

Recall from Definition 5 that $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is *slowly growing* if

$$\forall j \in [n],\ \forall \mathbf{d} \in \mathbb{Z}^n, \quad f(\mathbf{d}) \leqslant f(\mathbf{d} + \mathbf{e}_j) \leqslant f(\mathbf{d}) + 1. \tag{60}$$

If so, an easy induction argument (on $\deg(\mathbf{d} - \mathbf{d}')$) shows that if $\mathbf{d}', \mathbf{d} \in \mathbb{Z}^n$ with $\mathbf{d}' \leqslant \mathbf{d}$, then

$$f(\mathbf{d}) \leqslant f(\mathbf{d}') + \deg(\mathbf{d} - \mathbf{d}'),$$

and hence

$$f(\mathbf{d}') \geqslant f(\mathbf{d}) - \deg(\mathbf{d} - \mathbf{d}'). \tag{61}$$

Definition 44. Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be slowly growing ((60)). Let $\mathbf{d}', \mathbf{d} \in \mathbb{Z}^n$ with $\mathbf{d}' \leqslant \mathbf{d}$. We say that f *is maximally decreasing from* $\mathbf{d}$ *to* $\mathbf{d}'$ if equality holds in (61), i.e.,

$$f(\mathbf{d}) = f(\mathbf{d}') + \deg(\mathbf{d} - \mathbf{d}').$$

The following is Lemma 5 of [9], but is used in most papers we have seen involving the Baker-Norine rank, e.g., [6, 4, 2].

Proposition 45. *Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be slowly growing, and let $\mathbf{d}'', \mathbf{d}', \mathbf{d} \in \mathbb{Z}^n$ with $\mathbf{d}'' \leqslant \mathbf{d}' \leqslant \mathbf{d}$. Then f is maximally decreasing from $\mathbf{d}$ to $\mathbf{d}''$ iff it is maximally decreasing from both $\mathbf{d}$ to $\mathbf{d}'$ and from $\mathbf{d}'$ to $\mathbf{d}''$.*

The proof is immediate from the fact that the two inequalities

$$f(\mathbf{d}) - f(\mathbf{d}') \leqslant \deg(\mathbf{d} - \mathbf{d}'),$$
$$f(\mathbf{d}') - f(\mathbf{d}'') \leqslant \deg(\mathbf{d}' - \mathbf{d}'')$$

both hold with equality iff their sum does, and their sum is

$$f(\mathbf{d}) - f(\mathbf{d}'') \leqslant \deg(\mathbf{d} - \mathbf{d}').$$

The next observation, again common in papers dealing with the Baker-Norine rank (again, such as [6, 4, 2]) explains the interest in maximal decrease.

Lemma 46. *Let $\mathcal{N} \subset \mathbb{Z}^n$ be a non-empty downset, and let f be given as in* (3)*, i.e., $f(\mathbf{d}) = \rho_{L^1}(\mathbf{d}, \mathcal{N})$. Then there exists a $\mathbf{d}' \leqslant \mathbf{d}$ with $f(\mathbf{d}') = 0$, i.e., $\mathbf{d}' \in \mathcal{N}$ such that f is maximally decreasing from $\mathbf{d}$ to $\mathbf{d}'$.*

Proof. Let $\mathbf{d}' \in \mathcal{N}$ satisfy $\rho_{L^1}(\mathbf{d}, \mathbf{d}') = \rho_{L^1}(\mathbf{d}, \mathcal{N})$. Proposition 6 implies that $\mathbf{d}' \leqslant \mathbf{d}$; hence $f(\mathbf{d}) - f(\mathbf{d}') = f(\mathbf{d}) = \deg(\mathbf{d} - \mathbf{d}')$, so f is maximally decreasing from $\mathbf{d}$ to $\mathbf{d}'$. □

We remark that the converse is true: if f is maximally decreasing from $\mathbf{d}$ to $\mathbf{d}'$ with $\mathbf{d}' \in \mathcal{N}$, then $f(\mathbf{d}) = \deg(\mathbf{d} - \mathbf{d}')$; hence knowledge of $f(\mathbf{d})$ is equivalent to knowing a $\mathbf{d}' \in \mathcal{N}$ of maximal decrease of f from $\mathbf{d}$ to $\mathbf{d}'$.

6.3 A Generalization of a Fundamental Lemma of Cori and Le Borgne

Next we give an elegant and rather ingenious observation of [9] (half of the proof of Proposition 10 there) that we state as Lemma 47 below and is the starting point of their (and our) study the Baker-Norine rank for the complete graph. We state their observation in slightly more general terms: Proposition 10 of [9] requires f there (f is the analog of $\mathbf{a}$ below) to be a parking function, but we note that the same fact (and similar proof) holds in the more general case that $\mathbf{a}$ below satisfies $\mathbf{a} \geqslant 0$ and $a_{n-1} = 0$ (without assuming that $\mathbf{a}$ is a parking function).

Lemma 47. *Fix $n \in \mathbb{N}$, and let $K_n = (V, E)$ be the complete graph on vertex set $V = [n]$, i.e., E consists of exactly one edge joining any two distinct vertices (hence $|E| = \binom{n}{2}$). Consider the Baker-Norine rank $r_{\rm BN} \colon \mathbb{Z}^n \to \mathbb{Z}$ on K_n. If $\mathbf{a} \geqslant \mathbf{0}$ then*

$$a_{n-1} = 0 \quad \Rightarrow \quad r_{\rm BN}(\mathbf{a}) = r_{\rm BN}(\mathbf{a} - \mathbf{e}_{n-1}) + 1. \tag{62}$$

Of course, by symmetry (62) holds with both occurrences of $n - 1$ replaced by any $j \in [n]$.

Proof. Since $\mathbf{a} \geqslant \mathbf{0}$, $r_{\rm BN}(\mathbf{a}) \geqslant 0$. In view of Lemma 46 and Definition 4, we have that $r_{\rm BN}$ is maximally decreasing from $\mathbf{a}$ to some $\widetilde{\mathbf{a}} \leqslant \mathbf{a}$ with $r_{\rm BN}(\widetilde{\mathbf{a}}) = -1$. Let $\mathbf{b} = \mathbf{a} - \widetilde{\mathbf{a}} \geqslant \mathbf{0}$. Since $r_{\rm BN}(\mathbf{a} - \mathbf{b}) = -1$, we must have $a_j - b_j \leqslant -1$ for some $j \in [n]$; fix any such j. Then $b_j \geqslant a_j + 1 \geqslant 1$; setting $\mathbf{a}' = \mathbf{a} - b_j \mathbf{e}_j$ we have

$$\mathbf{a} - \mathbf{b} \leqslant \mathbf{a}' \leqslant \mathbf{a},$$

and hence r_{BN} is maximally decreasing from $\mathbf{a}$ to $\mathbf{a}'$, and so $r_{\mathrm{BN}}(\mathbf{a}') = r_{\mathrm{BN}}(\mathbf{a}) - \deg(\mathbf{a}-\mathbf{a}')$. But the vector

$$\mathbf{a}'' = \mathbf{a} - a_j \mathbf{e}_j - (b_j - a_j)\mathbf{e}_{n-1} \leqslant \mathbf{a}, \tag{63}$$

is merely the vector $\mathbf{a}'$ followed by an exchange of the $(n-1)$-th and j-th coordinates (if $j = n-1$, then $\mathbf{a}'' = \mathbf{a}'$). Hence, by symmetry, $\mathbf{a}'', \mathbf{a}'$ have the same degree and same value of r_{BN}; but $\mathbf{a}'' \leqslant \mathbf{a}$, and hence f is also maximally decreasing from $\mathbf{a}$ to $\mathbf{a}''$. Since $b_j - a_j \geqslant 1$, (63) implies

$$\mathbf{a}'' \leqslant \mathbf{a} - \mathbf{e}_{n-1} \leqslant \mathbf{a};$$

since f is maximally decreasing from $\mathbf{a}$ to $\mathbf{a}''$, f is maximally decreasing from $\mathbf{a}$ to $\mathbf{a} - \mathbf{e}_{n-1}$ as well, and hence (62) holds. □

Remark 48. If $n, m \in \mathbb{N}$, we use $K_n^m = (V, E)$ to denote the graph with $V = [n]$ and m edges between any two vertices (so $|E| = m\binom{n}{2}$ and $K_n^1 = K_n$). Then $r_{\mathrm{BN},K_n^m}(\mathbf{d})$ is again a symmetric function of its variables $(d_1, \ldots, d_n) = \mathbf{d}$, and the same argument shows that for any $b \in \mathbb{Z}_{\geqslant 0}$, $\mathbf{a} \geqslant b\mathbf{1}$ and $a_{n-1} = b$ implies that $f(\mathbf{d}) = f(\mathbf{d} - \mathbf{e}_{n-1}) + 1$. We believe it is possible to use this observation, specifically for $b = m$, to give an analog of Theorem 67 below regarding K_n^m.

6.4 The First Coordinates for Pic, D'après Cori-Le Borgne

Let us recall some more standard graph Riemann-Roch terminology (see, e.g., [6, 9]), and then give our first set of coordinates for the *Picard group* of a graph. These coordinates are those found in the Algorithm at the end of Section 2.1 of [9].

Recall $\mathbb{Z}^n_{\deg=i}$ consists of the elements of $\mathbb{Z}^n$ of degree i. Recall [6] the *Picard group* of a graph, G, with n vertices $v_1, \ldots, v_n$ is defined as

$$\mathrm{Pic}(G) = \mathbb{Z}^n / \mathrm{Image}(\Delta_G);$$

since $\mathrm{Image}(\Delta_G)$ consists entirely of vectors of degree 0, $\mathrm{Pic}(G)$ is the union over $i \in \mathbb{Z}$ of

$$\mathrm{Pic}_i(G) = \mathbb{Z}^n_{\deg=i} / \mathrm{Image}(\Delta_G). \tag{64}$$

It is known that for all i, $|\mathrm{Pic}_i(G)|$ equals $(1/n)\det'(\Delta_G)$, where $\det'$ denotes the product of the nonzero eigenvalues of Δ_G (and Kirchoff's theorem says that this is the number of unrooted spanning trees of G). For $G = K_n$ it is a standard fact that this number of trees is n^{n-2}, i.e.,

$$|\mathrm{Pic}_i(K_n)| = n^{n-2} \tag{65}$$

(indeed, one easily sees that Δ_{K_n} has the eigenvalue n with multiplicity $n-1$, and hence $(1/n)\det'(\Delta_{K_n}) = (1/n)n^{n-1} = n^{n-2}$).

Next we pick a convenient set of representatives for each class in $\mathbb{Z}^n/\mathrm{Image}(\Delta_{K_n})$.

Notation 49. For any $n \in \mathbb{N}$, we let

$$\mathcal{A} = \mathcal{A}(n) = \{\mathbf{a} \in \mathbb{Z}^n \mid a_1, \ldots, a_{n-2} \in \{0, \ldots, n-1\}, a_{n-1} = 0\} \tag{66}$$

$$= \{0, \dots, n-1\}^{n-2} \times \{0\} \times \mathbb{Z}$$

(we usually simply write $\mathcal{A}$ since n will be understood and fixed); in addition, for $i \in \mathbb{Z}$, we use $\mathcal{A}_{\deg=i}$ to denote the set

$$\mathcal{A}_{\deg=i} \stackrel{\text{def}}{=} \mathcal{A} \cap \mathbb{Z}^n_{\deg=i} = \{\mathbf{a} \in \mathcal{A} \mid \deg(\mathbf{a}) = i\}.$$

In the above notation, note that

$$\mathbf{a} \in \mathcal{A}_{\deg=i} \quad \iff \quad a_n = i - a_1 - \cdots - a_{n-2}$$

and hence

$$\mathbf{a} \in \mathcal{A}_{\deg=i} \;\Rightarrow \Big(a_n \geqslant 0 \iff a_1 + \cdots + a_{n-2} \leqslant i\Big) \tag{67}$$

$$\mathbf{a} \in \mathcal{A}_{\deg=i} \;\Rightarrow \Big(a_n = 0 \iff a_1 + \cdots + a_{n-2} = i\Big) \tag{68}$$

Lemma 50. *Fix $n \in \mathbb{N}$, and let $K_n = (V, E)$ be the complete graph on vertex set $V = [n]$. Then for all $\mathbf{d} \in \mathbb{Z}^n$ there exists a unique $\mathbf{a} \in \mathcal{A} = \mathcal{A}(n)$ with $\mathbf{d} \sim \mathbf{a}$ (i.e., $\mathbf{d} - \mathbf{a} \in$ $\mathrm{Image}(\Delta_{K_n})$), given by: for $j \in [n-2]$, $a_j = (d_j - d_{n-1}) \bmod n$, i.e., a_j is the element of $\{0, \dots, n-1\}$ congruent to $d_j - d_{n-1}$ modulo n, $a_{n-1} = 0$, and $a_n = \deg(\mathbf{d}) - a_1 - \cdots - a_{n-2}$.*

Proof. Existence is implicit in "Algorithm" at the end of Section 2.1 of [9] and is easy: since the i-th column of Δ_{K_n} is $n\mathbf{e}_i - \mathbf{1}$, the image of Δ_{K_n} contains

$$\mathbf{1} - n\mathbf{e}_n = (1, \dots, 1, -(n-1)), \quad \text{and} \quad \forall i \in [n], (\mathbf{e}_i - \mathbf{1}) - (n\mathbf{e}_n - \mathbf{1}) = n(\mathbf{e}_i - \mathbf{e}_n);$$

For any $\mathbf{d}$ we get an equivalent vector with $(n-1)$-th coordinate 0 by subtracting multiples of $(1, \dots, 1, 1-n)$; then we find an equivalent vector with the first $n-2$ coordinates between 0 and $n-1$ by subtracting multiples of $n(\mathbf{e}_j - \mathbf{e}_n)$ for $j \in [n-2]$.

Note that the above algorithm determines a map $\mu \colon \mathbb{Z}^n \to \mathcal{A}$ that such that

$$\forall \mathbf{d} \in \mathbb{Z}^n, \quad \mathbf{d} \sim \mu(\mathbf{d}), \tag{69}$$

i.e., $\mathbf{d}$ and $\mu(\mathbf{d})$ are equivalent modulo $\mathrm{Image}(K_n)$.

To prove that each $\mathbf{d}$ is equivalent to a unique element of $\mathcal{A}$, we need to show that if $\mathbf{a}, \mathbf{a}' \in \mathcal{A}$ are equivalent, i.e., $\mathbf{a} - \mathbf{a}' \in \mathrm{Image}(\Delta_{K_n})$, then we must have $\mathbf{a} = \mathbf{a}'$. Note that if $\mathbf{a}, \mathbf{a}'$ are equivalent, then they have the same degree and hence both lie in $\mathcal{A}_{\deg=i}$ for the same i. Hence it suffices to show that each element of $\mathcal{A}_{\deg=i}$ is in a distinct class of $\mathrm{Pic}_i(K_n)$. Let us rephrase this condition.

Note that since $\mathcal{A}_{\deg=i} \subset \mathbb{Z}^n_{\deg=i}$, the quotient map

$$\mathbb{Z}^n_{\deg=i} \to \mathbb{Z}^n_{\deg=i} / \mathrm{Image}(\Delta_{K_n}) = \mathrm{Pic}_i(K_n)$$

restricts to a map

$$\nu_i \colon \mathcal{A}_{\deg=i} \to \mathrm{Pic}_i(K_n).$$

To show that each element of $\mathcal{A}_{\deg=i}$ is in its own class of $\mathrm{Pic}_i(K_n)$ simply means that ν_i is injective. Let us prove this.

So fix an $i \in \mathbb{Z}$. Choosing a set of representatives, $\mathcal{P}_i \subset \mathbb{Z}^n_{\deg=i}$ for Pic_i; in view of (69), μ restricted to $\mathcal{P}_i$ gives a map of sets $\mu|_{\mathcal{P}_i} \colon \mathcal{P}_i \to \mathcal{A}_{\deg=i}$ that takes each element in the domain to a vector equivalent to it; hence this gives a map of sets $\mu_i \colon \mathrm{Pic}_i \to \mathcal{A}_{\deg=i}$ such that μ_i takes each $p \in \mathrm{Pic}_i$ to an element that lies in p. It follows that the map $\nu_i \mu_i$ is the identity map on Pic_i.

But we easily see that $\mathcal{A}_{\deg=i}$ has size n^{n-2}, since if $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{A}_{\deg=i}$ then $a_1, \ldots, a_{n-2} \in \{0, \ldots, n-1\}$, and any $a_1, \ldots, a_{n-2} \in \{0, \ldots, n-1\}$ determine the values of a_{n-1}, a_n, namely

$$a_{n-1} = 0, \quad a_n = i - a_1 - \cdots - a_{n-2}.$$

Since $\nu_i \mu_i$ is the identity map on Pic_i, and this map factors through the set $\mathcal{A}_{\deg=i}$ of the same size, both ν_i and μ_i must be bijections. Hence ν_i is an injection, which proves the desired uniqueness property. □

Remark 51. A referee for this paper has pointed out that the proof of uniqueness above seems to be a special case of the uniqueness of recurrent configurations in toppling equivalence classes in the sandpile model.

Here is how we often use the above theorem.

Corollary 52. *Fix an $n \in \mathbb{N}$. For each $i \in \mathbb{Z}$, $\mathcal{A}_{\deg=i}$ is a set of representatives of the classes $\mathrm{Pic}_i(K_n)$ in $\mathbb{Z}^n_{\deg=i}$. Similarly, for any $\mathbf{d} \in \mathbb{Z}^n$, as $\mathbf{a}$ ranges over $\mathcal{A}_{\deg=i}$, $\mathbf{a} - \mathbf{d}$ ranges over a set of representatives of $\mathcal{A}_{\deg=i'}$ where $i' = i - \deg(\mathbf{d})$.*

6.5 A Counting Method for the Baker-Norine Rank

In the subsection after this one we will determine the function $(1 - \mathfrak{t}_{n-1}) r_{\mathrm{BN}}$ for complete graphs. We will use the following method; we state it in terms of the Baker-Norine rank for a general graph, G; it can be stated in terms of any slowly growing Riemann function with a sufficiently large set of invariant translations; see Subsubsection 6.1.4 for this more general formulation.

Lemma 53. *Let G be a graph on n vertices, and let $\mathcal{A} \subset \mathbb{Z}^n$ be a set of representatives for $\mathrm{Pic}(G) = \mathbb{Z}^n / \Delta_G$. For each $i \in \mathbb{Z}$, let $\mathcal{A}_{\deg=i} = \mathcal{A} \cap \mathbb{Z}^n_{\deg=i}$, whose cardinality therefore equals $|\,\mathrm{Pic}_i(G)| = |\,\mathrm{Pic}_0(G)| = |\mathbb{Z}^n_{\deg=0}/L|$ where $L = \mathrm{Image}(\Delta_G)$. For each $i \in \mathbb{Z}_{\geqslant 0}$, let $\mathcal{A}'_i \subset \mathcal{A}_{\deg=i}$ be subsets such that*

$$\mathbf{a} \in \mathcal{A}'_i \quad \Rightarrow \quad r_{\mathrm{BN}}(\mathbf{a}) = r_{\mathrm{BN}}(\mathbf{a} - \mathbf{e}_{n-1}) + 1.$$

Then for all $\ell \in \mathbb{Z}$ sufficiently large we have

$$\sum_{i=0}^{\ell} |\mathcal{A}'_i| \leqslant \big|\mathrm{Pic}_0(G)\big| (\ell + 1 - g), \tag{70}$$

where $g = 1 - \chi(G)$ where χ is the Euler characteristic (i.e., $\chi(G) = |V_G| - |E_G|$). Furthermore, if equality holds in (70) *for all ℓ sufficiently large, then for all $i \in \mathbb{Z}_{\geqslant 0}$,*

$$\mathbf{a} \in \mathcal{A}_{\deg=i} \setminus \mathcal{A}'_i \quad \Rightarrow \quad r_{\rm BN}(\mathbf{a}) = r_{\rm BN}(\mathbf{a} - \mathbf{e}_{n-1}). \tag{71}$$

In other words, if equality holds in (70) *for all ℓ sufficiently large, then for all $i \in \mathbb{Z}$ we have*

$$\forall \mathbf{a} \in \mathcal{A}_{\deg=i}, \quad \Bigl(\mathbf{a} \in \mathcal{A}'_i \iff r_{\rm BN}(\mathbf{a}) = r_{\rm BN}(\mathbf{a} - \mathbf{e}_{n-1}) + 1\Bigr),$$

and

$$\forall \mathbf{a} \in \mathcal{A}_{\deg=i}, \quad \Bigl(\mathbf{a} \notin \mathcal{A}'_i \iff r_{\rm BN}(\mathbf{a}) = r_{\rm BN}(\mathbf{a} - \mathbf{e}_{n-1})\Bigr).$$

Of course, by symmetry we can replace all occurrences of $\mathbf{e}_{n-1}$ with $\mathbf{e}_j$ for any $j \in [n]$; in the subsection after this we apply the lemma above with $\mathbf{e}_{n-1}$.

Proof. For all $\mathbf{d} \in \mathbb{Z}^n$ and $\ell \in \mathbb{N}$, we have

$$\Bigl(f(\mathbf{d}) - f(\mathbf{d} - \mathbf{e}_{n-1})\Bigr) + \Bigl(f(\mathbf{d} + \mathbf{e}_{n-1}) - f(\mathbf{d})\Bigr) + \cdots + \Bigl(f(\mathbf{d} + \ell\mathbf{e}_{n-1}) - f(\mathbf{d} + (\ell-1)\mathbf{e}_{n-1})\Bigr)$$
$$= f(\mathbf{d} + \ell\mathbf{e}_{n-1}) - f(\mathbf{d} - \mathbf{e}_{n-1}).$$

Hence for all $\mathbf{a} \in \mathcal{A}_{\deg=0}$ we have

$$\sum_{i=0}^{\ell}\Bigl(r_{\rm BN}\bigl(\mathbf{a}+i\mathbf{e}_{n-1}\bigr) - r_{\rm BN}\bigl(\mathbf{a}+(i-1)\mathbf{e}_{n-1}\bigr)\Bigr) = r_{\rm BN}(\mathbf{a}+\ell\mathbf{e}_{n-1}) - r_{\rm BN}(\mathbf{a}-\mathbf{e}_{n-1}) = r_{\rm BN}(\mathbf{a}+\ell\mathbf{e}_{n-1}) + 1$$

which for ℓ sufficiently large equals $\ell + 1 - g$. Hence for sufficiently large ℓ we have

$$\sum_{\mathbf{a} \in \mathcal{A}_{\deg=0}} \sum_{i=0}^{\ell}\Bigl(r_{\rm BN}\bigl(\mathbf{a} + i\mathbf{e}_{n-1}\bigr) - r_{\rm BN}\bigl(\mathbf{a} + (i-1)\mathbf{e}_{n-1}\bigr)\Bigr) = |\mathcal{A}_{\deg=0}|\,(\ell + 1 - g),$$

and interchanging these finite summations we have

$$\sum_{i=0}^{\ell} \sum_{\mathbf{a} \in \mathcal{A}_{\deg=0}} \Bigl(r_{\rm BN}\bigl(\mathbf{a} + i\mathbf{e}_{n-1}\bigr) - r_{\rm BN}\bigl(\mathbf{a} + (i-1)\mathbf{e}_{n-1}\bigr)\Bigr) = |\mathcal{A}_{\deg=0}|\,(\ell + 1 - g).$$

But for any i, as $\mathbf{a}$ ranges over a set of $\mathrm{Pic}_0(G)$ representatives, $\mathbf{a} + i\mathbf{e}_{n-1}$ ranges over a set of $\mathrm{Pic}_i(G)$ representatives, and hence the equation above may be rewritten as

$$\sum_{i=0}^{\ell} \sum_{\mathbf{a} \in \mathcal{A}_{\deg=i}} \Bigl(r_{\rm BN}\bigl(\mathbf{a}\bigr) - r_{\rm BN}\bigl(\mathbf{a} - \mathbf{e}_{n-1}\bigr)\Bigr) = |\mathcal{A}_{\deg=0}|\,(\ell + 1 - g).$$

Since $r_{\rm BN}\bigl(\mathbf{a}\bigr) - r_{\rm BN}\bigl(\mathbf{a} - \mathbf{e}_{n-1}\bigr)$ is either 0 or 1, we may write the above as

$$\sum_{i=0}^{\ell}\Bigl|\Bigl\{\mathbf{a} \in \mathcal{A}_{\deg=i} \mid r_{\rm BN}\bigl(\mathbf{a}\bigr) - r_{\rm BN}\bigl(\mathbf{a} - \mathbf{e}_{n-1}\bigr) = 1\Bigr\}\Bigr| = |\mathcal{A}_{\deg=0}|\,(\ell + 1 - g). \tag{72}$$

But by assumption, for each i

$$\mathcal{A}'_i \subset \bigl\{\mathbf{a} \in \mathcal{A}_{\deg=i} \mid r_{\rm BN}\bigl(\mathbf{a}\bigr) - r_{\rm BN}\bigl(\mathbf{a} - \mathbf{e}_{n-1}\bigr) = 1\bigr\}. \tag{73}$$

Hence we have

$$\sum_{i=0}^{\ell} |\mathcal{A}'_i| \leqslant \sum_{i=0}^{\ell} \Bigl|\bigl\{\mathbf{a} \in \mathcal{A}_{\deg=i} \mid r_{\rm BN}\bigl(\mathbf{a}\bigr) - r_{\rm BN}\bigl(\mathbf{a} - \mathbf{e}_{n-1}\bigr) = 1\bigr\}\Bigr| = |\mathcal{A}_{\deg=0}|(\ell + 1 - g) \tag{74}$$

and the inequality here holds summand for summand. Hence (70) follows; if equality holds there, then (74) holds with equality. Hence (74) holds with equality summand for summand, i.e., for all i between 0 and ℓ we have

$$|\mathcal{A}'_i| = \Bigl|\bigl\{\mathbf{a} \in \mathcal{A}_{\deg=i} \mid r_{\rm BN}\bigl(\mathbf{a}\bigr) - r_{\rm BN}\bigl(\mathbf{a} - \mathbf{e}_{n-1}\bigr) = 1\bigr\}\Bigr|,$$

and hence (73) holds with equality, i.e.,

$$\mathcal{A}'_i = \bigl\{\mathbf{a} \in \mathcal{A}_{\deg=i} \mid r_{\rm BN}\bigl(\mathbf{a}\bigr) - r_{\rm BN}\bigl(\mathbf{a} - \mathbf{e}_{n-1}\bigr) = 1\bigr\};$$

hence (71) holds.

The conclusions after (71) are clear in view of (71) and (74). □

We remark that there is a similar lemma where the $\mathcal{A}'_i$ are replaced by $\mathcal{A}''_i \subset \mathcal{A}_{\deg=i}$ such that $\mathbf{a} \in \mathcal{A}''_i$ implies $r_{\rm BN}\bigl(\mathbf{a}\bigr) = r_{\rm BN}\bigl(\mathbf{a} - \mathbf{e}_{n-1}\bigr)$, and for all ℓ sufficiently large; in this case

$$\sum_{i=0}^{\ell} |\mathcal{A}''_i| \leqslant |\operatorname{Pic}_0(G)|\, g,$$

and if equality holds for all ℓ sufficiently large, then for all i we have

$$\mathbf{a} \in \mathcal{A}_{\deg=i} \setminus \mathcal{A}''_i \quad \Rightarrow \quad r_{\rm BN}\bigl(\mathbf{a}\bigr) - r_{\rm BN}\bigl(\mathbf{a} - \mathbf{e}_{n-1}\bigr) = 1.$$

Lemma 53 and the similar lemma above can be rephrased in intuitive terms as follows: to determine which $\mathbf{a} \in \mathcal{A}$ have $r_{\rm BN}\bigl(\mathbf{a}\bigr) - r_{\rm BN}\bigl(\mathbf{a} - \mathbf{e}_{n-1}\bigr)$ equal to 0 or 1, it suffices to determine either (1) a subset $\mathcal{A}'_i \subset \mathcal{A}_{\deg=i}$ for each $i \in \mathbb{Z}_{\geqslant 0}$ where this value is 1 and where $\sum_{i=0}^{\ell} |\mathcal{A}'_i|$ is the "maximum possible count" of this sum, i.e., $|\operatorname{Pic}_0(G)|(\ell + 1 - g)$, or (2) a subset $\mathcal{A}''$ where the value is 0, and where $\sum_{i=0}^{\ell} |\mathcal{A}''_i|$ is the "maximum possible count" for this sum, i.e., $|\operatorname{Pic}_0(G)|g$.

6.6 An Intermediate Weight Calculation: $(1 - \mathbf{t}_{n-1})r_{\rm BN}$

In this section we prove that the pattern we noticed in computer-aided calculation for small values of n can be proved to hold for all n.

Theorem 54. *Fix $n \in \mathbb{N}$, and let $K_n = (V, E)$ be the complete graph on vertex set $V = [n]$. Consider the Baker-Norine rank $r_{\mathrm{BN}} \colon \mathbb{Z}^n \to \mathbb{Z}$ on K_n. For any $\mathbf{a} \in \mathcal{A}_{\deg=i}$,*

$$a_1 + \cdots + a_{n-2} \leqslant i \iff a_n \geqslant 0 \iff r_{\mathrm{BN}}(\mathbf{a}) = r_{\mathrm{BN}}(\mathbf{a} - \mathbf{e}_{n-1}) + 1, \tag{75}$$

or, equivalently,

$$a_1 + \cdots + a_{n-2} > i \iff a_n < 0 \iff r_{\mathrm{BN}}(\mathbf{a}) = r_{\mathrm{BN}}(\mathbf{a} - \mathbf{e}_{n-1}). \tag{76}$$

We remark that (75) generalizes Proposition 10 of [9].

Proof. For all $\mathbf{a} \in \mathcal{A}$, $\mathbf{a} \geqslant \mathbf{0}$ iff $a_n \geqslant 0$, since all other coordinates of $\mathbf{a}$ are non-negative. For $\mathbf{a} \in \mathcal{A}_{\deg=i}$, in view of (67) we get

$$\mathbf{a} \geqslant \mathbf{0} \iff a_n \geqslant 0 \iff a_1 + \cdots + a_{n-2} \leqslant i.$$

Hence Lemma 47 implies that for $\mathbf{a} \in \mathcal{A}_{\deg=i}$,

$$a_1 + \cdots + a_{n-2} \leqslant i \quad \Rightarrow \quad r_{\mathrm{BN}}(\mathbf{a}) = r_{\mathrm{BN}}(\mathbf{a} - \mathbf{e}_{n-1}) + 1. \tag{77}$$

So let

$$\mathcal{A}'_i = \{\mathbf{a} \in \mathcal{A}_{\deg=i} \mid a_1 + \cdots + a_{n-2} \leqslant i\}.$$

According to Lemma 53, to prove (75) and therefore the theorem, it suffices to show that for all ℓ sufficiently large we have

$$\sum_{i=0}^{\ell} |\mathcal{A}'_i| = |\operatorname{Pic}_0(K_n)|(\ell + 1 - g). \tag{78}$$

Let us do so.

Since for $\mathbf{a} \in \mathcal{A}$ we have that $a_1, \ldots, a_{n-2}$ are between 0 and $n-1$, we have $a_1 + \cdots + a_{n-2}$ is between 0 and $(n-1)(n-2)$, and hence $|\mathcal{A}'_i| = 0$ for $i < 0$ and $|\mathcal{A}'_i| = |\mathcal{A}_{\deg=i}| = n^{n-2}$ for $i \geqslant (n-1)(n-2)$.

Next let us prove the following "symmetry:' for all $i \in \mathbb{Z}$, we have

$$|\mathcal{A}'_i| + |\mathcal{A}'_{i'}| = |\mathcal{A}_{\deg=i}| = n^{n-2}, \quad \text{where } i' = (n-1)(n-2) - i - 1. \tag{79}$$

(The special case $i < 0$ corresponds to $i' \geqslant (n-1)(n-2)$, both of which were discussed above.) To prove this, fix i and let $i' = (n-1)(n-2) - i - 1$. Since the map $x \mapsto (n-1) - x$ is a bijection from $\{0, \ldots, n-1\}$ to itself, the map $f \colon \mathcal{A}_{\deg=i} \to \mathbb{Z}^n$ given by

$$\begin{aligned} &f\big((a_1, \ldots, a_{n-2}, 0, i - a_1 - \cdots - a_{n-2})\big) \\ &\stackrel{\text{def}}{=} \big(n - 1 - a_1, \ldots, n - 1 - a_{n-2}, 0, i' - (n - 1 - a_1) - \cdots - (n - 1 - a_{n-2})\big)\big) \end{aligned}$$

is a bijection $\mathcal{A}_{\deg=i} \to \mathcal{A}_{\deg=i'}$ (this is true for all $i, i' \in \mathbb{Z}$, without assuming any relation between i and i'). Furthermore for all $\mathbf{a} \in \mathcal{A}_{\deg=i}$ we have

$$\begin{aligned} \mathbf{a} \in \mathcal{A}'_i &\iff a_1 + \cdots + a_{n-2} \leqslant i \\ &\iff (n - 1 - a_1) + \cdots + (n - 1 - a_{n-2}) \geqslant (n-1)(n-2) - i \\ &\iff (n - 1 - a_1) + \cdots + (n - 1 - a_{n-2}) > i' \iff f(\mathbf{a}) \notin \mathcal{A}_{i'}. \end{aligned}$$

Hence for all $\mathbf{a} \in \mathcal{A}_{\deg=i}$, the bijection $f\colon \mathcal{A}_{\deg=i} \to \mathcal{A}_{\deg=i'}$ has the property that $\mathbf{a} \in \mathcal{A}'_i$ iff $f(\mathbf{a}) \notin \mathcal{A}'_{i'}$. Hence (79) holds.

It follows that the average value of $|\mathcal{A}'_i|$ for i bewteen 0 and $(n-1)(n-2)-1$ is $(1/2)n^{n-2} = (1/2)|\operatorname{Pic}_0(K_n)|$, and hence

$$\sum_{i=0}^{(n-1)(n-2)-1} |\mathcal{A}'_i| = (1/2)(n-1)(n-2)|\operatorname{Pic}_0(K_n)| = \binom{n-1}{2}|\operatorname{Pic}_0(K_n)|;$$

since $|\mathcal{A}'_i| = |\operatorname{Pic}_0(K_n)|$ for all $i \geqslant (n-1)(n-2)$, it follows that for all $\ell \geqslant (n-1)(n-2)-1$ we have

$$\sum_{i=0}^{\ell} |\mathcal{A}'_i| = \left(\binom{n-1}{2} + \bigl(\ell-(n-1)(n-2)+1\bigr)\right)|\operatorname{Pic}_0(K_n)| = \left(\ell - \binom{n-1}{2} + 1\right)|\operatorname{Pic}_0(K_n)|.$$

Since the genus, g, of K_n is $1 + \binom{n}{2} - n = \binom{n-1}{2}$, we may rewrite the above as

$$\sum_{i=0}^{\ell} |\mathcal{A}'_i| = (\ell + 1 - g)|\operatorname{Pic}_0(K_n)|,$$

which verifies (78). □

6.7 A New Rank Formula for the Complete Graph and an Algorithm

Cori and Le Borgne [9] (after Proposition 6, bottom of page 9 and in [10],Proposition 13) describe an $O(n)$ algorithm that computes $r_{\mathrm{BN}}(\mathbf{d})$ for the complete graph K_n. Also, they show that when $\mathbf{d}$ is a *sorted parking configuration*, meaning that $0 \leqslant d_i < i$ for $i < n$ and $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_{n-1}$ (and d_n is unconstrained), they show (see Theorem 12 [10]) that setting

$$q = \lfloor (d_n + 1)/(n-1) \rfloor, \quad r = (d_n + 1) \bmod (n-1)$$

one has

$$r_{\mathrm{BN}}(\mathbf{d}) = -1 + \sum_{i=1}^{n} \max\Bigl(0, q - i + 1 + d_i + \chi\bigl(i \leqslant r\bigr)\Bigr), \tag{80}$$

where $\chi(P)$ is 1 if P is true, and 0 if P is false.

Here we give another formula for the rank, perhaps related to the above formula; by contrast, our formula holds for $\mathbf{a} \in \mathcal{A}$, but easily generalizes to all $\mathbf{d} \in \mathbb{Z}^n$. The formula is a corollary to Theorem 54. After giving this corollary, we will give an $O(n)$ time algorithm to evalute it (which is not clear from the corollary alone), and we will compare it to (80).

6.7.1 A New Rank Formula for Complete Graphs

Corollary 55. *Let $n \in \mathbb{Z}$, and $\mathcal{A}$ be as in* (66)*. For any $\mathbf{a} \in \mathcal{A}$ we have*

$$r_{\mathrm{BN},K_n}(\mathbf{a}) = -1 + \left|\left\{ i = 0, \ldots, \deg(\mathbf{a}) \;\middle|\; \sum_{j=1}^{n-2} \bigl((a_j + i) \bmod n\bigr) \leqslant \deg(\mathbf{a}) - i \right\}\right|. \tag{81}$$

More generally, for any $\mathbf{d} \in \mathbb{Z}^n$ *we have*

$$r_{\mathrm{BN},K_n}(\mathbf{d}) = -1+\left|\left\{i=0,\dots,\deg(\mathbf{d}) \;\middle|\; \sum_{j=1}^{n-2}\big((d_j-d_{n-1}+i) \bmod n\big) \leqslant \deg(\mathbf{d})-i\right\}\right|. \quad (82)$$

Proof. Let $\mathbf{a} \in \mathcal{A}$; if $\deg(\mathbf{a}) < 0$, then (81) is clear since both sides equal -1; so assume $\deg(\mathbf{a}) \geqslant 0$. Since $\mathbf{a} - (\deg(\mathbf{a}) + 1)e_{n-1}$ has negative degree, we have

$$\sum_{i=0}^{\deg(\mathbf{a})} \Big(r_{\mathrm{BN}}(\mathbf{a} - i\mathbf{e}_{n-1}) - r_{\mathrm{BN}}(\mathbf{a} - (i+1)\mathbf{e}_{n-1})\Big) = r_{\mathrm{BN}}(\mathbf{a}) - (-1). \quad (83)$$

According to Theorem 54, for a fixed i,

$$r_{\mathrm{BN}}(\mathbf{a} - i\mathbf{e}_{n-1}) - r_{\mathrm{BN}}(\mathbf{a} - (i+1)\mathbf{e}_{n-1})$$

equals 1 or 0 according to whether or not the unique $\mathbf{a}' \in \mathcal{A}$ that is equivalent to $\mathbf{a}-i\mathbf{e}_{n-1}$ satisfies

$$a_1' + \cdots + a_{n-2}' \leqslant \deg(\mathbf{a}'). \quad (84)$$

So to compute this unique $\mathbf{a}' \in \mathcal{A}$ equivalent to $\mathbf{a} - i\mathbf{e}_{n-1}$, notice that according to Lemma 50, since the $(n-1)$-th component of $\mathbf{a} - i\mathbf{e}_{n-1}$ is $-i$, $\mathbf{a}'$ is given as

$$\forall j \in [n-2], \quad a_j' = (a_j + i) \bmod n,$$

and $(a_{n-1}' = 0)$ and $\deg(\mathbf{a}') = \deg(\mathbf{a}) - i$. Hence (84) holds iff

$$\sum_{j=1}^{n-2}\big((a_j+i) \bmod n\big) \leqslant \deg(\mathbf{a}) - i.$$

Hence, in view of (83) we have (81).

To prove (82), we note that any $\mathbf{d} \in \mathbb{Z}^n$ is equivalent to $\mathbf{a} \in \mathcal{A}$, where

$$a_j = (d_j - d_{n-1}) \bmod n$$

for $j \leqslant n-2$, and $\deg(\mathbf{a}) = \deg(\mathbf{d})$. □

Example 56. Let $n = 4$, and let $\mathbf{a} = (3, 3, 0, -5) \in \mathcal{A}_{\deg=1}$. Let us evaluate the set

$$S = \left\{i = 0,\dots,\deg(\mathbf{a}) \;\middle|\; \sum_{j=1}^{n-2}\big((a_j+i) \bmod n\big) \leqslant \deg(\mathbf{a}) - i\right\}$$

Since $\deg(\mathbf{a}) = 1$, we need to consider $i = 0, 1$ in the above set. Note that $a_1 = a_2 = 3$. For $i = 0$, we have

$$\sum_{j=1}^{n-2}\big((a_j+i) \bmod n\big) = \sum_{j=1}^{2}\big((3+0) \bmod 4\big) = \sum_{j=1}^{2} 3 = 6.$$

Since $6 \leqslant 1 - 0$ is false, $i = 0$ is not in S. For $i = 1$, we have

$$\sum_{j=1}^{n-2} \big((a_j + i) \bmod n\big) = \sum_{j=1}^{2} \big((3+1) \bmod 4\big) = \sum_{j=1}^{2} \big(4 \bmod 4\big) = \sum_{j=1}^{2} 0 = 0.$$

Since $0 \leqslant 1 - 1$, $i = 1$ lies in S. Hence the set S above equals $\{1\}$, so $|S| = 1$. Hence

$$r_{\mathrm{BN},K_n}(\mathbf{a}) = -1 + |S| = 0.$$

Example 57. As a check to the above formula, let us compute $r_{\mathrm{BN},K_4}(\mathbf{a})$ with $\mathbf{a} = (3,3,0,-5)$ in a more standard fashion. First note that $\mathbf{a} = (3,3,0,-5)$ is equivalent to

$$(3,3,0,-5) + 2(-1,-1,-1,3) + (-1,-1,3,-1) = (0,0,1,0).$$

Hence $r_{\mathrm{BN},K_4}(\mathbf{a}) = r_{\mathrm{BN},K_4}((0,0,1,0)) \geqslant 0$. It will follow that $r_{\mathrm{BN},K_4}(\mathbf{a}) = 0$ provided that we can show that $(0,0,1,-1)$ is not equivalent to an effective divisor. For this we use a standard observation about degree 0 divisors: the only degree zero effective divisor is $(0,0,0,0)$, and therefore a degree zero divisor $\mathbf{d} \in \mathbb{Z}^4$ is equivalent to an effective divisor iff there are integers $c_1, \ldots, c_4$ such that

$$\mathbf{d} + c_1(3,-1,-1,-1) + c_2(-1,3,-1,-1) + c_3(-1,-1,3,-1) + c_4(-1,-1,-1,3) = (0,0,0,0).$$

If so, one can take $c_4 = 0$, since $(-1,-1,-1,3)$ is an integer linear combination of the first three columns of Δ_{K_4}. Hence, for $\mathbf{a} = (3,3,0,-5)$, it suffices to check whether or not

$$(0,0,1,-1) + c_1(3,-1,-1,-1) + c_2(-1,3,-1,-1) + c_3(-1,-1,3,-1) = (0,0,0,0)$$

has an integer solution c_1, c_2, c_3. However, this system has a unique solution, since otherwise Δ_{K_4} would be of rank at most 2, so $\det'(\Delta_{K_4})$ would be zero, which it is not. Now we easily check that $c_1 = c_2 = -1/4$ and $c_3 = -1/2$ satisfies this equation. Hence $(0,0,-1,1)$ is not equivalent to an effective divisor, and hence

$$r_{\mathrm{BN},K_4}\big((3,3,0,-5)\big) = r_{\mathrm{BN},K_4}\big((0,0,1,0)\big) = 0.$$

[Another way — which is shorter but more ad hoc — to see that $(0,0,1,-1)$ is not in the image of Δ_{K_4} is that if it were, then by symmetry $\mathbf{e}_i - \mathbf{e}_j$ would be in this image for any $i, j \in [4]$, and hence the image, L, of Δ_{K_4} would be all of $\mathbb{Z}^4_{\deg=0}$; but we know that $\mathrm{Pic}_0(K_4) = \mathbb{Z}^4_{\deg=0}/L$ is of size $4^{4-2} = 16$.]

Remark 58. Since $a_{n-1} = 0$ for all $\mathbf{a} \in \mathcal{A}$, one can write the condition

$$\sum_{j=1}^{n-2} \big((a_j + i) \bmod n\big) \leqslant \deg(\mathbf{a}) - i$$

in a more symmetrical looking fashion:

$$\sum_{j=1}^{n-1} \big((a_j + i) \bmod n\big) \leqslant \deg(\mathbf{a}).$$

Remark 59. In the proof of Corollary 55 we are making use of the fact that if $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is any function that is initially equal to a constant, C, then

$$f(\mathbf{d}) = C + \Big(\big((1-\mathsf{t}_{n-1}) + (1-\mathsf{t}_{n-1})\mathsf{t}_{n-1} + (1-\mathsf{t}_{n-1})\mathsf{t}_{n-1}^2 + \cdots\big)f\Big)(\mathbf{d})$$

where the right-hand-side represents a finite sum, since for any fixed $\mathbf{d}$, for sufficiently large $m \in \mathbb{N}$ we have

$$\big((1-\mathsf{t}_{n-1})\mathsf{t}_{n-1}^m f\big)(\mathbf{d}) = 0.$$

Corollary 55 applies this principle to the case of $f = r_{\mathrm{BN},K_n}$, specifically in (83). One can similarly write, for any $i \in [n]$,

$$(1-\mathsf{t}_i)^{-1} = 1 + \mathsf{t}_i + \mathsf{t}_i^2 + \cdots$$

with the right-hand-side representing a finite sum when applied to an initially vanishing function f at any given value $\mathbf{d}$. It follows that if f, f' are initially zero, then

$$(1-\mathsf{t}_i)f = h \quad \iff \quad f = (1 + \mathsf{t}_i + \mathsf{t}_i^2 + \cdots)h. \tag{85}$$

At times one of the two conditions above is easier to show that the other, at times not. For example, Theorem 54 above gives us a formula for $f = (1-\mathsf{t}_{n-1})r_{\mathrm{BN}}$ over $\mathbf{a} \in \mathcal{A}$; in Theorem 65 we determine $h = (1-\mathsf{t}_n)f$, but it is just as easy to apply either side of (85) with $i = n$. On the other hand, to compute the weight of r_{BN} in Theorem 67, setting h as above and computing

$$W = (1-\mathsf{t}_1)\cdots(1-\mathsf{t}_{n-2})h$$

seems much easier that verifying the (essentially) equivalent

$$h = (1 + \mathsf{t}_1 + \mathsf{t}_1^2 + \cdots)\cdots(1 + \mathsf{t}_{n-2} + \mathsf{t}_{n-2}^2 + \cdots)W.$$

6.7.2 A Linear Time Algorithm for the New Formula

Next we give a linear time algorithm to compute r_{BN} of the complete graph based on (81) or (82) in Corollary 55. After doing so we will state the algorithm as a theorem.

First, for simplicity, take an arbitrary $\mathbf{d} \in \mathbb{Z}^n$ and note that the equivalent $\mathbf{a} \in \mathcal{A}$ has $a_i = (d_i - d_{n-1}) \bmod n$ for $i \leqslant n-2$ and $\deg(\mathbf{a}) = \deg(\mathbf{d})$. Hence it suffices to show how to compute (81) with $\mathbf{a} \in \mathcal{A}$. (This is just the point made in the "Algorithm" at the end of Section 2.1 of [9].)

Setting

$$g(i) = \sum_{j=1}^{n-2}\big((a_j + i) \bmod n\big)$$

we have that $g(i+n) = g(i)$ for all i, and

$$g(i) = -m_i n + \sum_{j=1}^{n-2} a_j, \tag{86}$$

where m_i is the number of $j \in [n-2]$ such that $a_j + i \geqslant n$, i.e., with $a_j \geqslant n - i$.

Next, we claim that we can compute $m_0, \dots, m_{n-1}$ in linear time: indeed, by a single pass through $a_1, \dots, a_{n-2}$, one can count for each $k = 1, \dots, n-1$ the number,

$$m'_k = \bigl|\{j \in [n-2] \mid a_j = k\}\bigr|,$$

i.e., the number of j for which $a_j = k$; then one computes $m_0, \dots, m_{n-1}$ by setting $m_0 = 0$ and for $k = 1, \dots, n-1$ setting $m_k = m'_{n-k} + m_{k-1}$.

Once we compute $m_0, \dots, m_{n-1}$, we can compute $g(0), \dots, g(n-1)$ in linear time by computing $\sum_{j=1}^{n-2} a_j$ (once) and then applying (86) for each $i = 0, \dots, n-1$.

Now note that for $k = \{0, \dots, n-1\}$, we have that for any $i \in \{0, \dots, \deg(\mathbf{a})\}$ with $i \bmod n = k$, we have $g(i) = g(k)$, and hence the condition

$$\sum_{j=1}^{n-2} \bigl((a_j + i) \bmod n\bigr) \leqslant \deg(\mathbf{a}) - i$$

is equivalent to

$$i + g(k) \leqslant \deg(\mathbf{a}),$$

and hence the number of such i, for k fixed, is

$$\Bigl\lfloor \bigl(\deg(\mathbf{a}) - g(k) + n\bigr)/n \Bigr\rfloor.$$

Hence one can write

$$r_{\mathrm{BN}}(\mathbf{a}) = -1 + \sum_{k=0}^{n-1} \Bigl\lfloor \bigl(\deg(\mathbf{a}) - g(k) + n\bigr)/n \Bigr\rfloor,$$

which completes an $O(n)$ time algorithm to compute r_{BN}.

Let us organize the above steps into a formal algorithm.

Theorem 60. *Let $n \in \mathbb{Z}$ and $\mathbf{d} \in \mathbb{Z}^n$. The following algorithm correctly computes $r_{\mathrm{BN},K_n}(\mathbf{d})$.*

1. *Let $\mathbf{a}$ be given as follows: for $i \in [n-2]$, let $a_i = d_i - d_{n-1} \bmod n$; let $a_{n-1} = 0$, and let a_n be chosen so that $\deg(\mathbf{a}) = \deg(\mathbf{d})$ (i.e., $a_n = d_1 + \cdots + d_n - a_1 - \cdots - a_{n-1}$). Then $\mathbf{a} \in \mathcal{A}$ and is equivalent to $\mathbf{d}$.*

2. *For any $\mathbf{a} \in \mathcal{A}$ we compute $r_{\mathrm{BN},K_n}(\mathbf{a})$ as follows:*

   (a) *For $k = 1, \dots, n-1$, in a single pass through $a_1, \dots, a_{n-2}$ compute*

   $$m'_k = \bigl|\{j \in [n-2] \mid a_j = k\}\bigr|$$

   (so $m'_1, \dots, m'_{n-1}$ is a list initially set to zero, and exactly one of these variables is augmented each time we read a single a_i with $a_i \neq 0$);

(b) *we let* $m_0 = 0$*, and successively compute* $m_1, \dots, m_{n-1}$ *via the formula* $m_k = m_{k-1} + m'_{n-k}$*; (hence* m_i *is the number of* $j = 1, \dots, n-2$ *such that* $a_j \geqslant n-i$*;)*

(c) *compute* $s = \sum_{j=0}^{n-2} a_i$*, and for* $i = 0, \dots, n-1$ *let* $g(i) = -m_0 n + s$*;*

(d) *we have*

$$r_{\mathrm{BN}}(\mathbf{a}) = -1 + \sum_{k=0}^{n-1} \Big\lfloor \big(\deg(\mathbf{a}) - g(k) + n\big)/n \Big\rfloor.$$

6.7.3 Comparison of the New Formula to the Cori-Le Borgne Formula

The formula of Cori and Le Borgne (80) and the new formula Corollary 55 look rather different, and seem to be based on different principles.

The formula of Cori and Le Borgne requires $\mathbf{d}$ to be a *sorted parking configuration*, which (for K_n) is equivalent to requiring — after sorting $d_1, \dots, d_{n-1}$ to $0 = d_1 \leqslant \cdots \leqslant d_{n-1}$ with $0 \leqslant d_i < i$, and the d_n is unconstrained. Once we bring an arbitrary element of $\mathbb{Z}^n$ to an equivalent sorted parking configuration— which can be done in linear time (see Section 3.4 and Proposition 13 of [10]) — their formula (80) can clearly be evaluated in linear time.

By contrast, our formula in Corollary 55 requires some care if we wish to evaluate it in linear time. However, this formula is valid for all elements of $\mathcal{A}$, and it is computationally simple to take an arbitrary element of $\mathbb{Z}^n$ and find an equivalent one in $\mathbf{a} \in \mathcal{A}$. Moreover, $\mathcal{A}$ can be viewed as a much larger set than those of sorted parking configurations, since a sorted parking configuration $\mathbf{d}$ has $d_1 = 0$, and hence $(d_2, d_3, \dots, d_{n-1}, d_1, d_n)$ is an element of $\mathcal{A}$. By constrast, an element $\mathbf{a} \in \mathcal{A}$ with, for example, $a_1, \dots, a_{n-2} \geqslant 2$ does not become a sorted parking configuration after mere sorting alone, so in this sense $\mathcal{A}$ is a much larger set than the set of sorted parking configurations.

One could directly compare the above two formulas by taking a sorted parking configuration $\mathbf{d}$, and comparing (80) to the formula obtained by taking $\mathbf{a} = (d_2, d_3, \dots, d_{n-1}, d_1, d_n)$, therefore an element of $\mathcal{A}$, in (81). For example, the theory of parking configurations implies that $r_{\mathrm{BN}}(\mathbf{d}) = -1$ if $d_n = -1$ and $r_{\mathrm{BN}}(\mathbf{d}) = 0$ if $d_n = 0$; in case $d_n = -1, 0$, this easily follows from the Cori-Le Borgne formula (80): indeed, $q = 0$ in both cases, and $r = d_n + 1$ for $d_n = -1, 0$, so the fact that $d_i \leqslant i - 1$ for sorted parking configurations implies that

$$\max\Big(0, q - i + 1 + d_i + \chi\big(i \leqslant r\big)\Big)$$

is 0 for $i \geqslant 1$, and for $i = 0$ it is 0 for $d_n = -1$, and 1 for $d_n = 0$. Hence (80) is verified in the cases $d_n = -1, 0$. One can also check these cases for the formula (81), but this (straightforward) calculation is more tedious: indeed, we need to check for which i bewteen 0 and $\deg(\mathbf{a})$ we have:

$$\sum_{j=1}^{n-2} \big((a_j + i) \bmod n\big) \leqslant \deg(\mathbf{a}) - i, \tag{87}$$

and so we wish to verify that for $a_n = -1$ there are no such i, and for $a_n = 0$ there is exactly one such i. The following calculation shows that for $a_n = -1$, (87) holds for no $i \geqslant 0$, and for $a_n = 0$ it holds for $i = 0$, but holds for no $i \geqslant 1$:

The case $i = 0$ For $i = 0$, since $\sum_{j=1}^{n-2} a_j = \deg(\mathbf{a}) - a_n$, (87) holds when $a_n = 0$ but does not hold when $a_n = -1$.

The case $i = n$ For $i = n$, (87) does not hold, since since $(a_j + n) \bmod n = a_j$, and hence, $i = n$,

$$\deg(\mathbf{a}) - i = \deg(\mathbf{a}) - n \leqslant \sum_{j=1}^{n-2} a_j - a_n - n < \sum_{j=1}^{n-2} a_j.$$

The case $1 \leqslant i \leqslant n-1$ consider the largest $p = 0, 1, \ldots, n-2$ such that $a_p \leqslant n-i-1$; then $p \geqslant n-i-1$ (since $a_j \leqslant j$ for all $j \in [n-2]$). It follows that

$$(a_j + i) \bmod n = \begin{cases} a_j + i & \text{for } j \leqslant p, \text{ and} \\ a_j + i - n & \text{for } p+1 \leqslant j \leqslant n-2. \end{cases}$$

Therefore there are exactly $(n-2-p)$ values of $j \in [n-2]$ with $(a_j + i) \bmod n = a_j + i - n$, and hence

$$\sum_{j=1}^{n-2} \big((a_j+i) \bmod n\big) = -n(n-2-p) + \sum_{j=1}^{n-2}(a_j+i) = -n(n-2-p) + (n-2)i + \sum_{j-1}^{n-2} a_j.$$

Since $p \geqslant n-i-1$ we have $n-2-p \leqslant i-1$, and hence

$$-n(n-2-p) + (n-2)i \geqslant -n(i-1) + (n-2)i = n-2i.$$

It follows that

$$\sum_{j=1}^{n-2} \big((a_j+i) \bmod n\big) \geqslant n-2i + \sum_{j-1}^{n-2} a_j = n-2i+\deg(\mathbf{a})-a_n = \deg(\mathbf{a})-i+(n-i-a_n) > \deg(\mathbf{a})-i,$$

and hence (87) fails to hold.

The case $i \geqslant n+1$ Since $a_j + i$ depends only on the value of i modulo n, and since the right-hand-side of (87) is strictly decreasing as i increases, since (87) fails to hold for $1 \leqslant i \leqslant n$, it fails to hold for all $i \geqslant n+1$.

We remark that since the formula of Cori and Le Borgne involves the theory of parking configurations, it is not surprising that our new formula is considerably less pleasant to use when verifying facts that are simple to express in terms of parking configurations.

6.7.4 A Further Remarks on Theorem 54 and Corollary 55

We wish to make one remark on the curious way in which Theorem 54, and therefore its corollary, Corollary 55, organizes combinatorial information regarding the Baker-Norine rank for K_n. Theorem 54 is proven using Lemma 53, whose proof begins by observing that for any $\mathbf{a} \in \mathcal{A}_{\deg=0}$ we have

$$\sum_{i=0}^{\ell} \Big(r_{\rm BN}\big(\mathbf{a} + i\mathbf{e}_{n-1}\big) - r_{\rm BN}\big(\mathbf{a} + (i-1)\mathbf{e}_{n-1}\big) \Big) = \ell + 1 - g$$

for ℓ sufficiently large; therefore, there are exactly g values of $i \geqslant 0$ for which

$$r_{\rm BN}(\mathbf{a} + i\mathbf{e}_{n-1}) = r_{\rm BN}\big(\mathbf{a} + (i-1)\mathbf{e}_{n-1}\big) \tag{88}$$

However, it is not at all obvious (to us) what these values g values of i are just from Theorem 54. Let us consider some examples from $n = 4$.

For $n = 4$, we have $g = \binom{n-1}{2} = 3$, and hence for each $a_1, a_2 = 0, \ldots, 3$, there are exactly $g = 3$ values of $i \geqslant 0$ such that (88) holds with $\mathbf{a} = (a_1, a_2, 0, -a_1 - a_2)$. According to Theorem 54, for $\mathbf{a} \in \mathcal{A}_{\deg=j}$, we have

$$r_{\rm BN}\big(\mathbf{a}\big) = r_{\rm BN}\big(\mathbf{a} - \mathbf{e}_{n-1}\big) \tag{89}$$

iff $a_1 + a_2 > j$. Since $\mathbf{e}_3 \sim (3, 3, 0, -5)$, it follows that for any $a_1, a_2 = 0, \ldots, 3$ there are $g = 3$ values of $i \geqslant 0$ such that

$$(3i + a_1 \bmod 4) + (3i + a_2 \bmod 4) > i. \tag{90}$$

For example, these values of i are:

1. for $a_1 = a_2 = 0$, $i = 1, 2, 5$;
2. for $a_1 = 0$, $a_2 = 1$, $i = 0, 1, 2$;
3. for $a_1 = a_2 = 1$, $i = 0, 2, 3$.

So although (89) holds for $\mathbf{a} \in \mathcal{A}_{\deg=j}$ iff $a_1 + a_2 > j$, which is a very simple condition, finding the $g = 3$ values of $i \geqslant 0$ for which (90) still seems a bit mysterious; in particular, the fact that (90) has exactly 3 solutions for $i \geqslant 0$ (for any fixed a_1, a_2 between 0 and 3, and therefore any $a_1, a_2 \in \mathbb{Z}$) does not seem obvious. Perhaps a simple proof of this fact would shed some light on how Theorem 54 organizes the Baker-Norine rank for K_n.

We remark that the values of i above for a given a_1, a_2 are related to the idea of "Weierstrass gaps," as in Remark 43.

6.8 The Second Coordinates for Pic

To complete our computation of the weight of $r_{\rm BN}$ of the complete graph, we use a new set of coordinates. As explained in Subsection 6.1, the second coordinates turn out to represent Pic as a product

$$\mathrm{Pic} = (\mathbb{Z}/n\mathbb{Z})^{n-2} \times \mathbb{Z}. \tag{91}$$

Notation 61. For any $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, we use

1. $\mathcal{B} = \mathcal{B}(n)$ to denote the set $\{0, \ldots, n-1\}^{n-2}$ (and usually we just write $\mathcal{B}$ since n will be fixed); and

2. for any $\mathbf{b} \in \mathcal{B}$ and $i \in \mathbb{Z}$, we use $\langle \mathbf{b}, i\rangle$ to denote

$$\langle \mathbf{b}, i\rangle = (b_1, \ldots, b_{n-2}, 0, i - b_1 - \cdots - b_{n-2}) \in \mathcal{A}_{\deg=i} \subset \mathbb{Z}^n_{\deg=i} \subset \mathbb{Z}^n. \tag{92}$$

3. if $\mathbf{c} \in \mathbb{Z}^{n-2}$, we use $\mathbf{c} \bmod n$ to denote the component-wise application of mod$n$, i.e.,

$$\mathbf{c} \bmod n = \big(c_1 \bmod n, \ldots, c_{n-2} \bmod n\big) \in \mathcal{B} = \{0, \ldots, n-1\}^{n-2}.$$

Definition 62. For fixed $n \in \mathbb{Z}$, we refer to $\mathcal{B} = \mathcal{B}(n)$ and the map $\mathcal{B} \times \mathbb{Z} \to \mathbb{Z}^n$ in (92) as the *second coordinates* of $\mathrm{Pic}(K_n)$ representatives.

Proposition 63. *Let $n \in \mathbb{N}$, and let notation be as in Notation 49 and 61. Consider the complete graph, K_n, and equivalence modulo* $\mathrm{Image}(\Delta_{K_n})$. *Then:*

1. *for each $\mathbf{b} \in \mathcal{B}$ and $i \in \mathbb{Z}$,*

$$\langle (b_1, \ldots, b_{n-2}), i\rangle = (a_1, \ldots, a_n),$$

where

$$a_1 = b_1,\ \ldots,\ a_{n-2} = b_{n-2},\ a_{n-1} = 0,$$

and

$$a_n = i - b_1 - \cdots - b_{n-2}.$$

2. *For all $i \in \mathbb{Z}$, the set $\mathcal{B} \times \{i\}$ is taken via $\langle\cdot,\cdot\rangle$ bijectively to $\mathcal{A}_{\deg=i}$, and hence to a set of representatives of* Pic_i.

3. *For all $i \in \mathbb{Z}$, each $\mathbf{d} \in \mathbb{Z}^n_{\deg=i}$ is equivalent to a unique element of the form $\langle \mathbf{b}, i\rangle$ with $\mathbf{b} \in \mathcal{B}$, namely with*

$$\mathbf{b} = \big(d_1 - d_{n-1}, \ldots, d_{n-2} - d_{n-1}\big) \bmod n,$$

where mod n *is the component-wise application of* mod n, *i.e.*, $b_i = (d_i - d_{n-1}) \bmod n \in \{0, \ldots, n-1\}$.

4. *For any* $\mathbf{b}, \mathbf{b}' \in \mathcal{B} = \{0, \ldots, n-1\}^{n-2}$ *and any* $i, i' \in \mathbb{Z}$*, we have*

$$\langle \mathbf{b}, i\rangle + \langle \mathbf{b}', i'\rangle \sim \langle (\mathbf{b} + \mathbf{b}') \bmod n, i + i'\rangle.$$

Similarly for subtraction, i.e., with $-$ *everywhere replacing* $+$.

Proof. (1) is immediate from the notation. (2) follows from (1). (3) follows from (1) and Lemma 50. (4) follows from(3). □

Example 64. Applying the above proposition, we see that

$$\mathbf{e}_1 \sim \langle \mathbf{e}_1, 1\rangle,\ \ldots, \mathbf{e}_{n-2} \sim \langle \mathbf{e}_{n-2}, 1\rangle,\ \mathbf{e}_{n-1} \sim \langle (n-1)\mathbf{1}, 1\rangle,\ \mathbf{e}_n \sim \langle \mathbf{0}, 1\rangle, \tag{93}$$

where we use $\mathbf{e}_i$ to denote the vector in $\mathbb{Z}^n$ or in $\mathbb{Z}^{n-2}$, as appropriate. Moreover, equality holds in all the above, except for $\mathbf{e}_{n-1}$, where

$$\mathbf{e}_{n-1} \sim \langle (n-1)\mathbf{1}, 1\rangle = \big(n-1, \ldots, n-1, 0, 1-(n-2)(n-1)\big).$$

6.9 Computation of $(1-\mathfrak{t}_n)(1-\mathfrak{t}_{n-1})r_{\mathrm{BN}}$

Theorem 65. *Fix* $n \in \mathbb{N}$*, and let* $K_n = (V, G)$ *be the complete graph on vertex set* $V = [n]$*, i.e.,* E *consists of exactly one edge joining any two distinct vertices. Consider the Baker-Norine rank* $r_{\mathrm{BN}} \colon \mathbb{Z}^n \to \mathbb{Z}$ *on* K_n.

1. *If* $\mathbf{a} \in \mathcal{A}_{\deg=i}$*, then*

$$(1-\mathfrak{t}_n)(1-\mathfrak{t}_{n-1})r_{\mathrm{BN},K_n}(\mathbf{a}) = \begin{cases} 1 & \text{if } a_1 + \cdots + a_{n-2} = i, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{94}$$

2. *For all* $\mathbf{b} \in \mathcal{B}$ *and* $i \in \mathbb{Z}$,

$$(1-\mathfrak{t}_n)(1-\mathfrak{t}_{n-1})r_{\mathrm{BN},K_n}(\langle \mathbf{b}, i\rangle) = \begin{cases} 1 & \text{if } b_1 + \cdots + b_{n-2} = i, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{95}$$

Proof. The left-hand-side of (94) equals

$$(1-\mathfrak{t}_n)(1-\mathfrak{t}_{n-1})r_{\mathrm{BN},K_n}(\mathbf{a}) = (1-\mathfrak{t}_{n-1})r_{\mathrm{BN},K_n}(\mathbf{a}) - (1-\mathfrak{t}_{n-1})r_{\mathrm{BN},K_n}(\mathbf{a}-\mathbf{e}_n).$$

Note that if $\mathbf{a} \in \mathcal{A}_{\deg=i}$, then

$$\mathbf{a} - \mathbf{e}_n = (a_1, \ldots, a_{n-2}, 0, i-1-a_1-\cdots-a_{n-2}) \in \mathcal{A}_{\deg=i-1}.$$

By Theorem 54, $(1-\mathfrak{t}_{n-1})r_{\mathrm{BN},K_n}(\mathbf{a})$ is 1 or 0 according to whether or not $a_1+\cdots+a_{n-2} \leqslant i$ or not, and similarly with $\mathbf{a}$ replaced by $\mathbf{a}-\mathbf{e}_n \in \mathcal{A}_{\deg=i-1}$, according to whether or not $a_1 + \cdots + a_{n-2} \leqslant i-1$. Hence we conclude (94).

(2) (i.e., (95)) follows immediately from (1) (i.e., (94)). □

When going through the weight calculations in the next two sections, it may be helpful to visualize consequences of Theorem 54 in the case $n = 4$, and to consider what (95) means in terms of the $\langle \mathbf{b}, i\rangle$ coordinates, namely that $b_1 + b_2 = i$; see Figure 6.

$i=0$	$i=1$	$i=2$
$\begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{array}$	$\begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{array}$	$\begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{array}$
$i=3$	$i=4$	$i=5$
$\begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \end{array}$	$\begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 \end{array}$	$\begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{array}$
$i=6$		
$\begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{array}$		

Figure 6: The non-zero values of of $(1-\mathfrak{t}_{n-1})(1-\mathfrak{t}_n)r_{\mathrm{BN}}(\langle b, i\rangle)$ for $n=4$, $\mathbf{b}=(b_1,b_2)\in\{0,1,2,3\}^2$, namely 1 if $b_1+b_2=i$, and 0 otherwise.

6.10 A Generalization of the Weight Calculation

To compute the weight of the Baker-Norine rank on K_n, we need to apply

$$(1-\mathfrak{t}_1)\cdots(1-\mathfrak{t}_{n-2}).$$

However, (95) implies that

$$(1-\mathfrak{t}_n)(1-\mathfrak{t}_{n-1})r_{\mathrm{BN},K_n}(\langle \mathbf{b},i\rangle)=g(b_1+\cdots+b_{n-2}-i),$$

for some function g (namely the "Dirac delta function at 0," i.e., the function that is 1 at 0 and otherwise 0). We find it conceptually simpler to prove a theorem that applies

$$(1-\mathfrak{t}_1)\cdots(1-\mathfrak{t}_{n-2})$$

to any function of $\langle b,i\rangle$ of the form

$$g(b_1+\cdots+b_{n-2}-i).$$

Here is the result.

It will be helpful to introduce the following "tensor" notation: if $J\subset[n-2]$, then set

$$\mathfrak{t}_J=\prod_{j\in J}\mathfrak{t}_j. \tag{96}$$

Proposition 66. *Let $h\colon \mathbb{Z}^n\to\mathbb{Z}$ be any function that is invariant under translation by the image of the Laplacian of the complete graph. Say that for all $(\mathbf{b},i)\in\mathcal{B}\times\mathbb{Z}$, $h(\langle\mathbf{b},i\rangle)=g(b_1+\cdots+b_{n-2}-i)$ for some function g, i.e., h depends only on the value of $b_1+\cdots+b_{n-2}-i$. Then*

1. *if $j\in[n-2]$ and $\mathbf{b}\in\mathcal{B}=\{0,\ldots,n-1\}^{n-2}$ has $b_j>0$, then for all $i\in\mathbb{Z}$ we have*

$$((1-\mathfrak{t}_j)h)(\langle b,i\rangle)=0; \tag{97}$$

2. *let $j\in[n-2]$ and $J'\subset[n-2]$ with $j\notin J'$; if $\mathbf{b}\in\mathcal{B}=\{0,\ldots,n-1\}^{n-2}$ has $b_j>0$, then for all $i\in\mathbb{Z}$ we have*

$$\big((1-\mathfrak{t}_j)\mathfrak{t}_{J'}h\big)(\langle b,i\rangle)=0 \tag{98}$$

   (using the "tensor" notation (96));

3. *if $\mathbf{b}\in\mathcal{B}$ with $\mathbf{b}\neq\mathbf{0}$ (hence $b_j>0$ for some $j\in[n-2]$),*

$$\big((1-\mathfrak{t}_1)\cdots(1-\mathfrak{t}_{n-2})h\big)(\langle\mathbf{b},i\rangle)=0; \tag{99}$$

   and

4. (in the remaining case, $\mathbf{b}=\mathbf{0}$*)*

$$\big((1-\mathfrak{t}_1)\cdots(1-\mathfrak{t}_{n-2})h\big)(\langle \mathbf{0},i\rangle)=\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}g(i-kn). \tag{100}$$

We remark that the proof below shows that claims (1) and (2) above hold, more generally, whenever

$$h(\langle \mathbf{b},i\rangle)=g(b_1,\ldots,b_{j-1},b_j-i,b_{j+1},\ldots,b_{n-2})$$

for some g, i.e., h is an arbitrary function, except that its dependence on b_j and i is only on b_j-i and the rest of the $b_{j'}$ with $j'\neq j$.

Proof. Our proof will constantly use (93).

Proof of (1): if $b_j>0$, then $\mathbf{b}-\mathbf{e}_j\in\mathcal{B}$, and hence

$$\langle \mathbf{b},i\rangle-\mathbf{e}_j=\langle \mathbf{b}-\mathbf{e}_j,i-1\rangle,$$

and hence

$$\begin{gathered}\big((1-\mathfrak{t}_j)h\big)(\langle b,i\rangle)=h(\langle \mathbf{b},i\rangle)-h(\langle \mathbf{b}-\mathbf{e}_j,i-1\rangle)\\ =g\big((b_1+\cdots+b_{n-2})-i\big)-g\big((b_1+\cdots+b_{n-2}-1)-(i-1)\big)=0.\end{gathered}$$

This gives (97).

Proof of (2): let

$$\mathbf{b}'=(\mathbf{b}-\mathbf{e}_{J'})\bmod n.$$

Since $j\notin J'$ we have $b'_j=b_j>0$, and hence $\mathbf{b}'-\mathbf{e}_j\in\mathcal{B}$. Hence

$$\begin{gathered}\big(\mathfrak{t}_J h\big)(\langle \mathbf{b},i\rangle)=h\big(\langle \mathbf{b}',i-|J'|\rangle\big)\\ \big(\mathfrak{t}_j\mathfrak{t}_{J'}h\big)(\langle \mathbf{b},i\rangle)=h\big(\langle \mathbf{b}'-\mathbf{e}_j,i-|J'|-1\rangle\big).\end{gathered}$$

Hence the same calculation as in the previous paragraph (with $\mathbf{b}'$ replacing $\mathbf{b}$ and $i-|J'|$ replacing i) gives (98).

Proof of (3): we have

$$(1-\mathfrak{t}_1)\cdots(1-\mathfrak{t}_{n-2})=\sum_{J'\subset[n-2]\setminus\{j\}}(-1)^{|J'|}(1-\mathfrak{t}_j)\mathfrak{t}_{J'},$$

and so (98) implies (99).

Proof of (4): for any $J\subset[n-2]$, using (93) we have

$$\langle \mathbf{0},i\rangle-\mathbf{e}_J\sim\langle (n-1)\mathbf{e}_J,i-|J|\rangle,$$

and hence

$$f\big(\langle \mathbf{0},i\rangle-\mathbf{e}_J\big)=f\big(\langle (n-1)\mathbf{e}_J,i-|J|\rangle\big)=g((n-1)|J|-i+|J|)=g(n|J|-i).$$

Since

$$(1-\mathfrak{t}_1)\cdots(1-\mathfrak{t}_{n-2}) = \sum_{J\subset[n-2]} (-1)^{|J|}\mathfrak{t}_J,$$

we get

$$\bigl((1-\mathfrak{t}_1)\cdots(1-\mathfrak{t}_{n-2})h\bigr)\bigl(\langle \mathbf{0}, i\rangle\bigr) = \sum_{J\subset[n-2]} (-1)^{|J|} g\bigl(n|J|-i\bigr)$$

and (100) follows. ☐

6.11 Computation of W

Theorem 67. *Fix $n\in\mathbb{N}$, and let $K_n=(V,E)$ be the complete graph on vertex set $V=[n]$. Consider the Baker-Norine rank $r_{\rm BN}\colon \mathbb{Z}^n\to\mathbb{Z}$ on K_n. The weight, $W=\mathfrak{m}(r_{{\rm BN},K_n})$, is given by*

$$W(\langle \mathbf{b}, i\rangle) = \begin{cases} (-1)^\ell\binom{n-2}{\ell} & \text{if } \mathbf{b}=\mathbf{0} \text{ and } i=n\ell \text{ for some } \ell=0,\ldots,n-2, \text{ and} \\ 0 & \text{otherwise.}\end{cases} \tag{101}$$

Proof. Setting

$$h(\langle \mathbf{b}, i\rangle) = \bigl((1-\mathfrak{t}_{n-1})(1-\mathfrak{t}_n)r_{\rm BN}\bigr)(\langle \mathbf{b}, i\rangle),$$

(95) shows that

$$h(\langle \mathbf{b}, i\rangle) = g(b_1+\cdots+b_{n-2}-i),$$

where $g(0)=1$ and elsewhere g vanishes. Since

$$W = (1-\mathfrak{t}_1)\cdots(1-\mathfrak{t}_{n-2})h,$$

we may apply Proposition 66 and conclude: (1) if $\mathbf{b}\in\mathcal{B}$ is nonzero, then (99) implies that

$$W(\langle \mathbf{b}, i\rangle) = 0,$$

and (2) if $\mathbf{b}=\mathbf{0}$, then

$$W(\langle \mathbf{0}, i\rangle) = \sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}g(nk-i).$$

Hence $W(\langle \mathbf{0}, i\rangle)=0$ unless i is of the form nk, with $0\leqslant k\leqslant n-2$, in which case

$$W(\langle \mathbf{0}, nk\rangle) = (-1)^k\binom{n-2}{k}.$$

☐

6.12 Remark on Theorem 67

Another important consequence of Theorem 67 is that, by symmetry, for any $\mathbf{d}\in\mathbb{Z}^n$, and any distinct $i,j\in[n]$ we have

$$\bigl((1-\mathfrak{t}_i)(1-\mathfrak{t}_j)W\bigr)(\mathbf{d}) \geqslant 0.$$

In [11] this will imply that when we can model $f=1+r_{{\rm BN},K_n}$ as Euler characteristics of a family of sheaves in a sense explained there.

7 Fundamental Domains and the Proofs of Theorems 25 and 26

In this section we prove the Theorems 25 and 26. We do so with a tool that we call a *cubism* of $\mathbb{Z}^n$. However, Theorems 25 has a more direct proof without using cubisms, so we first give the direct proof. In fact, the direct proof will motivate the definition of a cubism.

7.1 Proof of Theorem 25 Without Reference to Cubisms

Before proving this theorem, we prove a lemma; let us give some examples to illustrate the lemma and the notation we will use.

Say that $f(d_1, d_2)$ is defined whenever either $d_1 = 0$ or $d_2 = 0$ (or both). The lemma below will extend f to a function defined on all of $\mathbb{Z}^2 \to \mathbb{Z}$ that can be written as a sum of functions, each independent of either d_1, d_2 or both. To do so, first let $g_1(d_1, d_2) = f(d_1, 0)$, which relies on knowning f only when $d_2 = 0$; however, note that g_1 is a function defined on all of $\mathbb{Z}^2$. Similarly for $g_2(d_1, d_2) = f(0, d_2)$, and $g_\emptyset(d_1, d_2) = f(0, 0)$. Now set

$$h(d_1, d_2) = g_1(d_1, d_2) + g_2(d_1, d_2) - g_\emptyset(d_1, d_2), \tag{102}$$

which is defined on all of $\mathbb{Z} \times \mathbb{Z}$. We claim that $h(d_1, d_2) = f(d_1, d_2)$ whenever $d_1 = 0$ or $d_2 = 0$: for example, if $d_1 = 0$, then

$$h(0, d_2) = g_1(0, d_2) + g_2(0, d_2) - g_\emptyset(0, d_2) = f(0, 0) + f(0, d_2) - f(0, 0) = f(0, d_2).$$

By symmetry the same holds if $d_2 = 0$. Hence (102) is a formula, reminiscent of inclusion-exclusion, that extends f to all of $\mathbb{Z}^2$; the right-hand-side of (102) is a linear combination of functions g_1, g_2, g_{12}, each of which depends on at most one variable. Similarly for a function $f(d_1, d_2, d_3)$ that is defined whenever at least one of d_1, d_2, d_3 is zero, by defining $g_{12}(d_1, d_2, d_3) = f(d_1, d_2, 0)$, $g_1(d_1, d_2, d_3) = f(d_1, 0, 0)$, $g_\emptyset = f(0, 0, 0)$, and similarly g_{ij} and g_i for all i, j, and setting

$$h = g_{12} + g_{13} + g_{23} - g_1 - g_2 - g_3 + g_\emptyset.$$

To show that h coincides with f whenever at least one of d_1, d_2, d_3 equals 0, we see that

$$h(0, d_2, d_3) = g_{12}(0, d_2, d_3) + g_{13}(0, d_2, d_3) + g_{23}(0, d_2, d_3)$$
$$-g_1(0, d_2, d_3) - g_2(0, d_2, d_3) - g_3(0, d_2, d_3) + g_\emptyset(0, 0, 0),$$

and since

$$d_1 = 0 \quad \Rightarrow \quad g_{12} = g_2,\ g_{13} = g_3,\ g_1 = g_\emptyset,$$

we have $h(0, d_2, d_3) = g_{23}(0, d_2, d_3) = f(0, d_2, d_3)$. Hence $h = f$ when $d_1 = 0$, and by symmetry the same holds when $d_2 = 0$ and when $d_3 = 0$.

Lemma 68. *Let $n \in \mathbb{Z}$, and let $\mathcal{D}^n_{\mathrm{coord}} \subset \mathbb{Z}^n$ given by*

$$\mathcal{D}^n_{\mathrm{coord}} = \{\mathbf{d} \mid d_i = 0 \text{ for at least one } i \in [n]\}. \tag{103}$$

Then for any $f \colon \mathcal{D}^n_{\mathrm{coord}} \to \mathbb{Z}$, there exist functions $h_i \colon \mathbb{Z}^n \to \mathbb{Z}$ for each $i \in [n]$ such that

1. $h_i = h_i(\mathbf{d})$ *is independent of the* i*-th variable,* d_i*, and*

2.

$$\forall \mathbf{d} \in \mathcal{D}^n_{\mathrm{coord}}, \quad f(\mathbf{d}) = \sum_{i=1}^n h_i(\mathbf{d}). \tag{104}$$

Hence the function $\sum_i h_i$ above is an extension of f to all of $\mathbb{Z}^n$ such that each h_i is independent of its i-th variable.

Proof. For $\mathbf{d} \in \mathbb{Z}^n$ and $I \subset [n]$, introduce the notation

$$\mathbf{d}_I = \sum_{i \in I} d_i \mathbf{e}_i.$$

Consider the function $g \colon \mathbb{Z}^n \to \mathbb{Z}$ given by

$$g(\mathbf{d}) = \sum_{I \subset [n],\ I \neq [n]} f(\mathbf{d}_I)(-1)^{n-1-|I|} \tag{105}$$

(which makes sense, since $\mathbf{d}_I \in \mathcal{D}^n_{\mathrm{coord}}$ whenever $I \neq [n]$). We claim that $g = f$ when restricted to $\mathbf{d} \in \mathcal{D}^n_{\mathrm{coord}}$; by symmetry it suffices to check the case $d_n = 0$, whereupon the term $f(\mathbf{d}_I)$ with $n \notin I$ cancels the term corresponding to $I \cup n$, except for the single remaining term where $I = \{1, \ldots, n-1\}$. Hence for $d_n = 0$, $g(\mathbf{d}) = f(\mathbf{d})$, and, by symmetry, $g = f$ on all of $\mathcal{D}^n_{\mathrm{coord}}$.

Now we see that the right-hand-side (105) is of the desired form $\sum_i h_i$ as in the statement of the lemma, by setting

$$h_i = \sum_{i \notin I,\ 1, \ldots, i-1 \in I} f(\mathbf{d}_I)(-1)^{n-1-|I|};$$

since for each $I \subset [n]$ with $I \neq [n]$ there is a unique $i \in [n]$ such that $i \notin I$ but $1, \ldots, i-1 \in I$ (namely the lowest value of i not in I), we have $\sum_i h_i$ equals the right-hand-side (105). □

Theorem 69. *Let* $n \in \mathbb{N}$ *and* $\mathcal{D}^n_{\mathrm{coord}}$ *be as in* (103)*. Then any function* $f \colon \mathcal{D}^n_{\mathrm{coord}} \to \mathbb{Z}$ *has a unique extension to a modular function* $h \colon \mathbb{Z}^n \to \mathbb{Z}$*.*

Proof. The existence of the extension of h is guaranteed by Lemma 68. Let us prove uniqueness. By symmetry it suffices to show that the values of h on the set

$$\mathbb{N}^n = \{\mathbf{d} \mid d_i > 0 \text{ for all } i \in [n]\}$$

are uniquely determined. But if h is modular, then

$$h(\mathbf{d}) = \sum_{I \subset [n],\ I \neq \emptyset} (-1)^{|I|+1} h(\mathbf{d} - \mathbf{e}_I). \tag{106}$$

Now we prove by induction on m that for all $m \geqslant n$, if $\mathbf{d} \in \mathbb{N}^n$ and $\deg(\mathbf{d}) = m$, then $h(\mathbf{d})$ is uniquely determined. The base case is $m = n$, where the only element of degree n in $\mathbb{N}^n$ is $\mathbf{d} = \mathbf{1}$. But for each $I \subset [n]$ with $I \neq \emptyset$, $\mathbf{1} - \mathbf{e}_I \in \mathcal{D}^n_{\mathrm{coord}}$; hence (106) uniquely determines $h(\mathbf{1})$. To prove the inductive claim: let $\mathbf{d} \in \mathbb{N}^n$ with $\deg(\mathbf{d}) = m$; for all $I \subset [n]$ with $I \neq \emptyset$, $\mathbf{d} - \mathbf{e}_I \geqslant \mathbf{0}$ and $\mathbf{d} - \mathbf{e}_I$ and has degree less than m. Hence (106) determines $h(\mathbf{d})$ in terms of values of h that, by induction, have already been determined. □

Proof of Theorem 25. One direction is immediate; it suffices to show that any modular function, h, can be written as a sum of functions, each of which depends on only $n-1$ of its variables. So consider the restriction of h to $\mathcal{D}^n_{\mathrm{coord}}$; then this restriction determines a unique modular function, which must be h. But then Theorem 69 implies that $h = \sum_i h_i$, where each h_i is independent of its i-th variable. □

7.2 Fundamental Modular Domains

Let us restate what we proved in the previous subsection.

Definition 70. Let $\mathcal{D} \subset \mathbb{Z}^n$. We call $\mathcal{D}$ a *fundamental modular domain* (respectively *subfundamental*, *superfundamental*) if for every function $f\colon \mathcal{D} \to \mathbb{Z}$ there exists a unique (respectively, at least one, at most one) modular function $h\colon \mathbb{Z}^n \to \mathbb{Z}$ such that $f = h$ on $\mathcal{D}$.

We remark that our terminology results from the following almost immediate facts: a subset of a subfundamental modular domain is subfundamental, and a strict subset of a fundamental domain is not fundamental; similarly for supersets and superfundamental domains.

In the last subsection, Theorem 25 was proven via Theorem 69, which proved that $\mathcal{D}^n_{\mathrm{coord}}$ is a fundamental modular domain. Theorem 26 essentially states that for any $n \in \mathbb{N}$ and $a \in \mathbb{Z}$,

$$\mathcal{D} = \{\mathbf{d} \in \mathbb{Z}^n \mid a \leqslant \deg(\mathbf{d}) \leqslant a + n - 1\}$$

is a fundamental modular domain. We can prove both ideas by the method of a *cubism*, that we now explain.

7.3 Cubisms: Motivation, Definition, and Implication of Domain Fundamentality

The proof of Theorem 69 can be viewed as follows: we ordered the elements of $\mathbb{N}^n$ by a function

$$\mathrm{rank}(\mathbf{d}) = d_1 + \cdots + d_n - (n-1),$$

(so the minimum rank of an element of $\mathbb{Z}^n$ is 1), and proved by induction on $m \geqslant 1$ that there is a unique extension of a function $h\colon \mathcal{D}^n_{\mathrm{coord}} \to \mathbb{Z}$ to all points of rank at most m so that $(\mathfrak{m}h)(\mathbf{d}) = 0$ for all $\mathbf{d}$ of rank at most m. Let us generalize this idea.

Definition 71. For $\mathbf{d} \in \mathbb{Z}^n$, the $\mathbf{d}$-*cube* refers to the set

$$\mathrm{Cube}(\mathbf{d}) = \{\mathbf{d}' \in \mathbb{Z}^n \mid \mathbf{d} - \mathbf{1} \leqslant \mathbf{d}' \leqslant \mathbf{d}\}.$$

We refer to the set of all $\mathbf{d}$-cubes as the set of n-cubes. If $\mathcal{D} \subset \mathbb{Z}^n$, we say that function $r\colon \mathbb{Z}^n \to \mathbb{N}$ is a *cubism of* $\mathcal{D}$ if, setting

$$\mathcal{D}_m = \mathcal{D} \cup \bigcup_{r(\mathbf{d}) \leqslant m} \mathrm{Cube}(\mathbf{d}) \tag{107}$$

for $m \in \mathbb{Z}_{\geqslant 0}$ (hence $\mathcal{D}_0 = \mathcal{D}$), we have

1. if $m \geqslant 1$ and $r(\mathbf{d}) = r(\mathbf{d}') = m$, then

$$\mathrm{Cube}(\mathbf{d}) \cap \mathrm{Cube}(\mathbf{d}') \in \mathcal{D}_{m-1}, \tag{108}$$

   and

2. for all $m \geqslant 1$ and $\mathbf{d} \in \mathbb{Z}^n$ with $r(\mathbf{d}) = m$ we have

$$\left|\mathrm{Cube}(\mathbf{d}) \setminus \mathcal{D}_{m-1}\right| = 1. \tag{109}$$

In the last paragraph of this section we remark that in some cubisms it is more convenient to replace the partial ordering of the n-cubes induced by the function $r\colon \mathbb{Z}^n \to \mathbb{N}$ above with, more generally, a well-ordering or a partial ordering such that each subset has a minimal element.

Example 72. In Figure 7 we illustrate an example of a cubism of $\mathcal{D}$, with $\mathcal{D} = \mathcal{D}_{\mathrm{coord}}$ as above, suggested by the above proof of Theorem 69 and $n = 2$ (so the n-cubes are really squares).

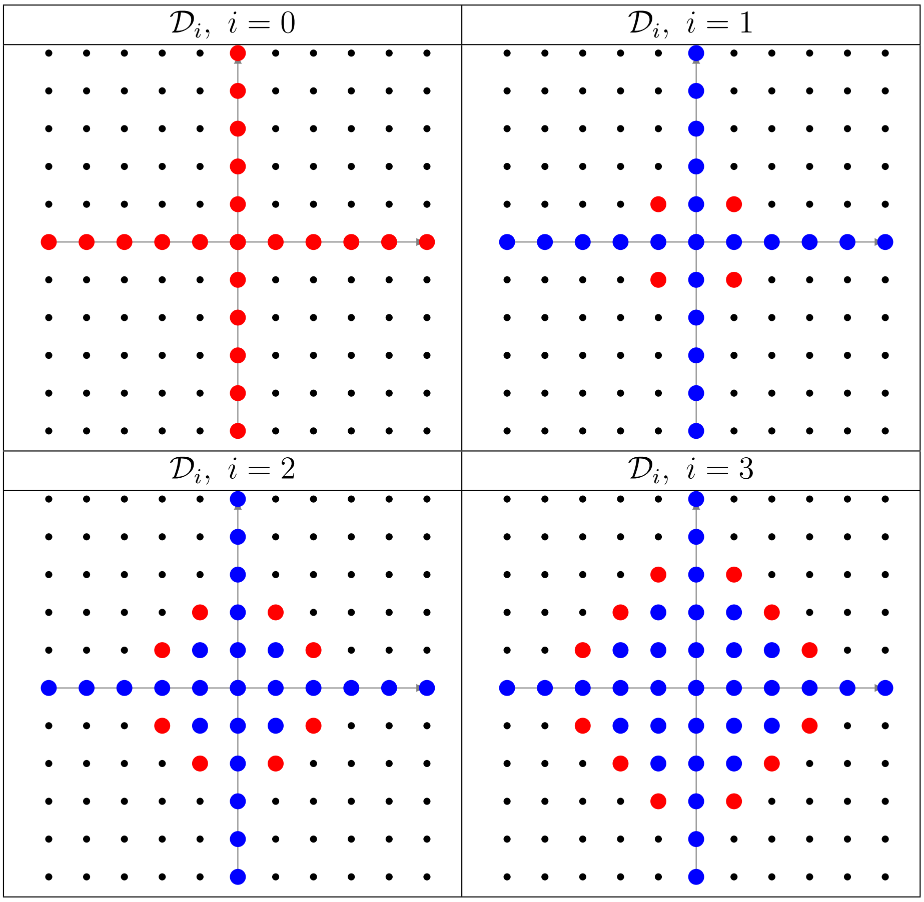
Proposition 73. *If $\mathcal{D} \subset \mathbb{Z}^n$ has a cubism, then $\mathcal{D}$ is fundamental.*

Proof. Fix a function $f\colon \mathcal{D} \to \mathbb{Z}$, and set $g_0 = f$.

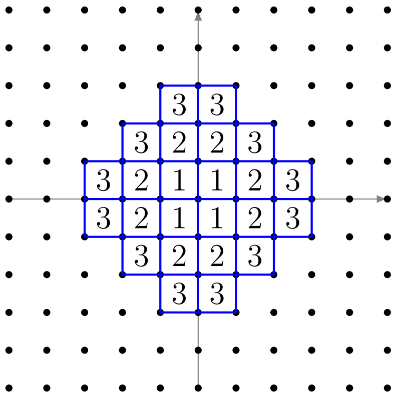
Let us prove by induction on $m \in \mathbb{N}$ that there is a unique function $\mathcal{D}_m \to \mathbb{Z}$ such that

1. $(\mathfrak{m} g_m)(\mathbf{d}) = 0$ for all $\mathbf{d}$ with $r(\mathbf{d}) \leqslant m$;
2. the restriction of g_m to $\mathcal{D}_{m-1}$ equals g_{m-1}; and
3. the value of g_m on each $\mathbf{c} \in \mathcal{D}_m \setminus \mathcal{D}_{m-1}$ is determined by the equation $(\mathfrak{m} g_m)(\mathbf{d}) = 0$ for a unique $\mathbf{d} \in \mathcal{D}_{m-1}$ such that $\mathbf{c} \in \mathrm{Cube}(\mathbf{d}) \setminus \mathcal{D}_{m-1}$, via the equation

$$-g_m(\mathbf{c})(-1)^{\deg(\mathbf{d}-\mathbf{c})} = \sum_{\mathbf{c}' \in \mathrm{Cube}(\mathbf{d}) \setminus \{\mathbf{c}\}} g_{m-1}(\mathbf{c}')(-1)^{\deg(\mathbf{d}-\mathbf{c}')}. \tag{110}$$



(a) New points $\mathcal{D}_i \setminus \mathcal{D}_{i-1}$ in red, old points, $\mathcal{D}_{i-i}$ in blue



(b) The cubism after 4 steps.

Figure 7: A cubism for $\mathcal{D}^n_{\text{coord}}$ with $n = 2$.

The base case $m = 1$ is argued almost exactly as the inductive claim from $m-1$ to m; so we will prove the base case $m = 1$, leaving in m everywhere.

For $m = 1$, we have that $\mathcal{D}_{m-1} = \mathcal{D}_0 = \mathcal{D}$, and (109) implies that for each $\mathbf{d}$ with $r(\mathbf{d}) = m$, there is a unique $\tilde{\mathbf{d}} \notin \mathcal{D}_{m-1}$ in $\mathrm{Cube}(\mathbf{d})$; the equation $(\mathfrak{m}g)(\mathbf{d}) = 0$ is equivalent to

$$\sum_{\mathbf{c} \in \mathrm{Cube}(\mathbf{d})} g_m(\mathbf{c})(-1)^{\deg(\mathbf{d}-\mathbf{c})} = 0. \tag{111}$$

This determines $g_m(\tilde{\mathbf{d}})$ via (110) with $\mathbf{c} = \tilde{\mathbf{d}}$, since all other $\mathbf{c} \in \mathrm{Cube}(\mathbf{d})$ in the sum (111) either lie in $\mathcal{D}$ or have rank at most $m-1$; (108) shows that for distinct $\mathbf{d}, \mathbf{d}'$ of rank m, the corresponding $\tilde{\mathbf{d}}, \tilde{\mathbf{d}}'$ are distinct, so that it is possible to set the value of g_m as required on all $\tilde{\mathbf{d}}$ that are the unique element of $\mathrm{Cube}(\mathbf{d}) \setminus \mathcal{D}_{m-1}$ for some $\mathbf{d}$ of rank m.

For the inductive step, we assume the claim holds for $m-1$, and we repeat the same argument above. This shows that $g_m \colon \mathcal{D}_m \to \mathbb{Z}$ exist for all m with the desired properties.

Now define $h \colon \mathbb{Z}^n \to \mathbb{Z}$ as follows: for any $\mathbf{d} \in \mathbb{Z}^n$, we have $\mathbf{d} \in \mathrm{Cube}(\mathbf{d}) \subset \mathcal{D}_m$, where $m = r(\mathbf{d})$; hence $g_m(\mathbf{d})$ is defined; set $h(\mathbf{d}) = g_m(\mathbf{d})$.

We claim that h above is modular: indeed, for any $\mathbf{d} \in \mathbb{Z}^n$, if $m = r(\mathbf{d})$, then $\mathfrak{m}g_m(\mathbf{d}) = 0$ and $\mathcal{D}_m$ contains $\mathrm{Cube}(\mathbf{d})$; since $g_{m+1}, g_{m+2}, \ldots$ are all extensions of g_m, we have $\mathfrak{m}h(\mathbf{d}) = \mathfrak{m}g_m(\mathbf{d}) = 0$.

Now we claim that h is the unique modular function $\mathbb{Z}^n \to \mathbb{Z}$ whose restriction to $\mathcal{D}$ is f: indeed, assume that h' is another such modular function, and that $h \neq h'$; then the definition of h implies that there exists an m such that g_m does not equal the restriction of h' to $\mathcal{D}_m$; consider the smallest such m. Since the restrictions of h and h' to $\mathcal{D}_0 = \mathcal{D}$ both equal f, we must have $m \geqslant 1$. It follows that $h(\mathbf{c}) \neq h'(\mathbf{c})$ for some $\mathbf{c} \in \mathcal{D}_m \setminus \mathcal{D}_{m-1}$ with $m \geqslant 1$; fix such a $\mathbf{c}$. By condition (3) on g_m (i.e., (110) and above), there is some $\mathbf{d}$ with $r(\mathbf{d}) = m$ for which $\mathbf{c}$ which is the unique element of $\mathrm{Cube}(\mathbf{d}) \setminus \mathcal{D}_{m-1}$. But since h, h' agree on g_{m-1}, we have

$$\begin{aligned}(\mathfrak{m}h')(\mathbf{d}) &= h'(\mathbf{c})(-1)^{\deg(\mathbf{d}-\mathbf{c})} + \sum_{\mathbf{c}' \in \mathrm{Cube}(\mathbf{d}) \setminus \{\mathbf{c}\}} g_{m-1}(\mathbf{c}')(-1)^{\deg(\mathbf{d}-\mathbf{c}')} \\ &\neq h(\mathbf{c})(-1)^{\deg(\mathbf{d}-\mathbf{c})} + \sum_{\mathbf{c}' \in \mathrm{Cube}(\mathbf{d}) \setminus \{\mathbf{c}\}} g_{m-1}(\mathbf{c}')(-1)^{\deg(\mathbf{d}-\mathbf{c}')} = 0,\end{aligned}$$

and hence $(\mathfrak{m}h')(\mathbf{d}) \neq 0$; hence h' is not modular. □

[Straying a bit, one could define a subcubism by replacing the $= 1$ in (109) by $\geqslant 1$, and the same proof shows that a $\mathcal{D}$ with a subcubism is subfundamental; similarly for supercubism and $\leqslant 1$.]

7.4 Second Proof of Theorem 69

The proof of Theorem 69 above can be viewed as giving a cubism (e.g., Figure 7 for $n = 2$). Let us formalize this.

Second proof of Theorem 69. For each $\mathbf{d} \in \mathbb{Z}^n$, let

$$r(\mathbf{d}) = |d_1| + \cdots + |d_n| + \bigl|\{i \in [n] \mid d_i \leqslant 0\}\bigr| - n + 1;$$

more intuitively, $r(\mathbf{d})$ is just the L^1 distance of the furthest point in $\mathrm{Cube}(\mathbf{d})$ to $\mathcal{D}^n_{\mathrm{coord}}$, since if all $d_i \geqslant 1$ then the furthest point is just $\mathbf{d}$, and $r(\mathbf{d})$ is just $d_1 + \cdots + d_n - n + 1$, and otherwise we need minor corrections for those $d_i \leqslant 0$. Now we claim that r is a cubism.

To show that r attains only positive integer values, we can write r as

$$r(\mathbf{d}) = 1 + \sum_{i=1}^{n} \max(d_i - 1, -d_i);$$

since $\max(d_i - 1, -d_i)$ is non-negative for any $d_i \in \mathbb{Z}$, r attains only positive values. We leave the verification of (1) and (2) in the definition of a cubism to the reader. □

We also remark that—unlike the above example—there is no need for $r^{-1}(\{m\})$ to be finite; in fact, the next example shows that it can be convenient for $r^{-1}\{m\}$ to be infinite.

7.5 Other Examples of Cubisms and the Proof of Theorem 26

Proof of Theorem 26. Let

$$\mathcal{D} = \{\mathbf{d} \mid a \leqslant \deg(\mathbf{d}) \leqslant a + n - 1\}.$$

Define $r\colon \mathbb{Z}^n \to \mathbb{N}$ as

$$r(\mathbf{d}) = \begin{cases} \deg(\mathbf{d}) - a + n + 1 & \text{if } \deg(\mathbf{d}) \geqslant a + n, \text{ and} \\ a + n - \deg(\mathbf{d}) & \text{if } \deg(\mathbf{d}) < a + n. \end{cases}$$

Setting $\mathcal{D}_0 = \mathcal{D}$ and, for $m \in \mathbb{N}$, $\mathcal{D}_m$ as in (107), we easily see that that if $r(\mathbf{d}) = m$ then $\mathrm{Cube}(\mathbf{d}) \setminus \mathcal{D}_{m-1}$ consists of a single point, namely $\mathbf{d}$ if $\deg(\mathbf{d}) \geqslant a + n$, and otherwise the single point $\mathbf{d} - \mathbf{1}$. We easily see that these single points are distinct as $\mathbf{d}$ varies over all $\mathbf{d} \notin \mathcal{D}$, and it follows that r is a cubism of $\mathcal{D}$. □

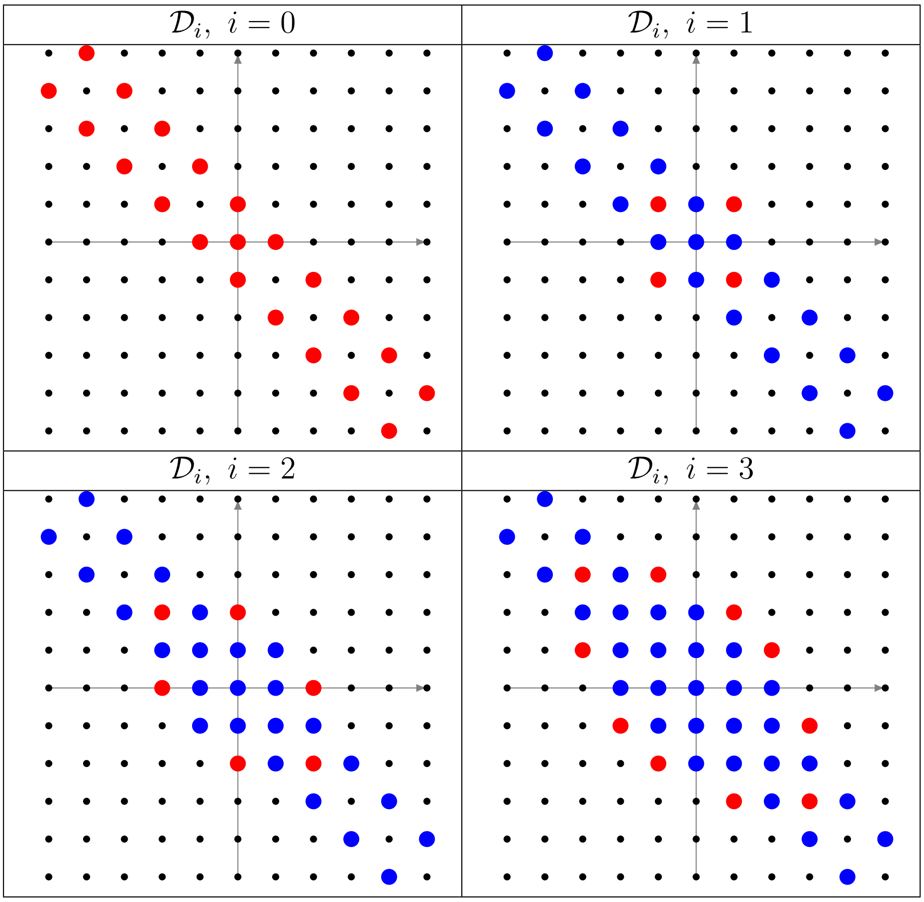
Example 74. One can show by a cubism argument that the set $\mathcal{D} \subset \mathbb{Z}^2$ given by

$$\{(0,0)\} \cup \{\mathbf{d} \in \mathbb{Z}^2 \mid \deg(\mathbf{d}) = \pm 1\}$$

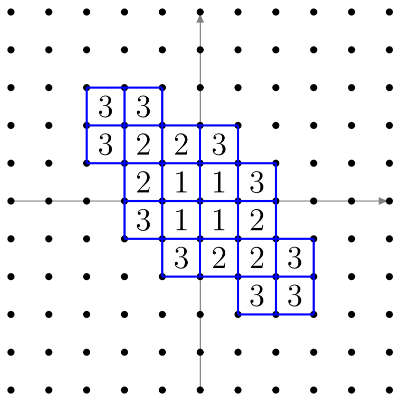
is fundamental, by defining $r(\mathbf{d})$ to be $|d_1|$ if $\deg(\mathbf{d}) = 1$ and otherwise $||\deg(\mathbf{d})| - 1|$; we depict this cubism in Figure 8. It follows that any subset of $\mathcal{D}$ is subfundamental (e.g., removing $(0,0)$), and any superset of $\mathcal{D}$ is superfundamental.

It is intriguing—but not relevant to this article—to consider the various other fundamental modular domains of $\mathbb{Z}^n$.

We also note that in Example 74, it may be simpler to first extend a function $\mathcal{D} \to \mathbb{Z}$ along all points of degree 0, whereupon the extension is defined on all points of degree between -1 and 1, and then further extend the function to all of $\mathbb{Z}^n$. In this case one can view the set of 2-cubes as a well-ordered set, where all points of degree 0 are ordered before all points of degrees not between -1 and 1. One can therefore define a more general cubism as any well-ordering of the n-cubes of $\mathbb{Z}^n$, or, more generally, any partial ordering such that each subset of n-cubes has a minimal element. The proofs of all theorems easily generalize to these more general notions of a cubism.



(a) New points in red, old points in blue



(b) The cubism after 4 steps.

Figure 8: A Cubism for Example 74.

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