Turán Numbers and Anti-Ramsey Numbers for Short Cycles in Complete 3-Partite Graphs

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Abstract

We call a 4-cycle in K_{n_1,n_2,n_3} multipartite, denoted by C_4^{multi}, if it contains at least one vertex in each part of K_{n_1,n_2,n_3}. The Turán number $\text{ex}(K_{n_1,n_2,n_3}, C_4^{\text{multi}})$

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(respectively, $\mathrm{ex}(K_{n_1,n_2,n_3},\{C_3,C_4^{\mathrm{multi}}\})$) is the maximum number of edges in a graph $G\subseteq K_{n_1,n_2,n_3}$ such that G contains no C_4^{multi} (respectively, G contains neither C_3 nor C_4^{multi}). We call an edge-colored C_4^{multi} rainbow if all four edges of it have different colors. The anti-Ramsey number $\mathrm{ar}(K_{n_1,n_2,n_3},C_4^{\mathrm{multi}})$ is the maximum number of colors in an edge-colored K_{n_1,n_2,n_3} with no rainbow C_4^{multi}. In this paper, we determine that $\mathrm{ex}(K_{n_1,n_2,n_3},C_4^{\mathrm{multi}})=n_1n_2+2n_3$ and $\mathrm{ar}(K_{n_1,n_2,n_3},C_4^{\mathrm{multi}})=\mathrm{ex}(K_{n_1,n_2,n_3},\{C_3,C_4^{\mathrm{multi}}\})+1=n_1n_2+n_3+1$, where $n_1\geqslant n_2\geqslant n_3\geqslant 1$.

Mathematics Subject Classifications: 05C15, 05C35, 05C38

1 Introduction

We consider only nonempty simple graphs. Let G be such a graph, the vertex and edge set of G is denoted by $V(G)$ and $E(G)$, the number of vertices and edges in G by $\nu(G)$ and $e(G)$, respectively. We denote the neighborhood of v in G by $N_G(v)$, and the degree of a vertex v in G by $d_G(v)$, the size of $N_G(v)$. Let U_1, U_2 be vertex sets, denote by $e_G(U_1,U_2)$ the number of edges between U_1 and U_2 in G. We write $d(v)$ instead of $d_G(v)$, $N(v)$ instead of $N_G(v)$ and $e(U_1,U_2)$ instead of $e_G(U_1,U_2)$ if the underlying graph G is clear.

Given a graph family $\mathcal{F}$, we call a graph H an $\mathcal{F}$-*free* graph, if H contains no graph in $\mathcal{F}$ as a subgraph. The *Turán number* $\mathrm{ex}(G,\mathcal{F})$ for a graph family $\mathcal{F}$ in G is the maximum number of edges in a graph $H\subseteq G$ which is $\mathcal{F}$-free. If $\mathcal{F}=\{F\}$, then we denote $\mathrm{ex}(G,\mathcal{F})$ by $\mathrm{ex}(G,F)$.

An old result of Bollobás, Erdős and Szemerédi [3] showed that $\mathrm{ex}(K_{n_1,n_2,n_3},C_3)=n_1n_2+n_1n_3$ for $n_1\geqslant n_2\geqslant n_3\geqslant 1$ (also see [4, 2, 5]). Lv, Lu and Fang [8, 9] constructed balanced 3-partite graphs which are C_4-free and $\{C_3,C_4\}$-free respectively and showed that $\mathrm{ex}(K_{n,n,n},C_4)=(\frac{3}{\sqrt{2}}+o(1))n^{3/2}$ and $\mathrm{ex}(K_{n,n,n},\{C_3,C_4\})\geqslant(1.82+o(1))n^{3/2}$.

For further discussion, we need the definitions of the multipartite subgraphs and a function $f(n_1,n_2,\dots,n_r)$.

Definition 1. [7] Let $r\geqslant 3$ and G be an r-partite graph with vertex partition $V_1,\dots,V_r$, we call a subgraph H of G *multipartite*, if there are at least three distinct parts V_i,V_j,V_k such that $V(H)\cap V_i\neq\emptyset, V(H)\cap V_j\neq\emptyset$ and $V(H)\cap V_k\neq\emptyset$. In particular, we denote a multipartite H by H^{multi} (see Figure 3 for an example of a C_4^{multi} in a 3-partite graph).

For $r\geqslant 3$ and $n_1\geqslant n_2\geqslant\cdots\geqslant n_r\geqslant 1$, let

$$f(n_1,n_2,\dots,n_r)=\begin{cases} n_1n_2+n_3n_4+\cdots+n_{r-2}n_{r-1}+n_r+\frac{r-1}{2}-1, & r \text{ is odd;}\\ n_1n_2+n_3n_4+\cdots+n_{r-1}n_r+\frac{r}{2}-1, & r \text{ is even.}\end{cases}$$

Fang, Győri, Li and Xiao [7] recently showed that if $G\subseteq K_{n_1,n_2,\dots,n_r}$ and $e(G)\geqslant f(n_1,n_2,\dots,n_r)+1$, then G contains a multipartite cycle. Furthermore, they proposed the following conjecture.

Conjecture 2. [7] For $r\geqslant 3$ and $n_1\geqslant n_2\geqslant\cdots\geqslant n_r\geqslant 1$, if $G\subset K_{n_1,n_2,\dots,n_r}$ and $e(G)\geqslant f(n_1,n_2,\dots,n_r)+1$, then G contains a multipartite cycle C^{multi} of length at most $\frac{3}{2}r$.

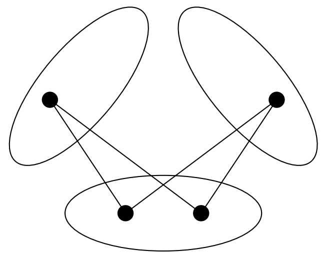


Figure 1: A C_4^{multi} in a 3-partite graph.

In this paper, we study the Turán numbers of C_4^{multi} and $\{C_3, C_4^{\text{multi}}\}$ in complete 3-partite graphs and obtain the following results.

Theorem 3. *For* $n_1 \geqslant n_2 \geqslant n_3 \geqslant 1$, $\text{ex}(K_{n_1,n_2,n_3}, C_4^{\text{multi}}) = n_1n_2 + 2n_3$.

Theorem 4. *For* $n_1 \geqslant n_2 \geqslant n_3 \geqslant 1$, $\text{ex}(K_{n_1,n_2,n_3}, \{C_3, C_4^{\text{multi}}\}) = n_1n_2 + n_3$.

Notice that Theorem 4 confirms Conjecture 2 for the case when $r = 3$.

A subgraph of an edge-colored graph is *rainbow*, if all of its edges have different colors. For graphs G and H, the *anti-Ramsey number* $\text{ar}(G, H)$ is the maximum number of colors in an edge-colored G with no rainbow copy of H. Erdős, Simonovits and Sós [6] first studied the anti-Ramsey number in the case when the host graph G is a complete graph K_n and showed the close relationship between it and the Turán number. In this paper, we consider the anti-Ramsey number of C_4^{multi} in complete 3-partite graphs.

Theorem 5. *For* $n_1 \geqslant n_2 \geqslant n_3 \geqslant 1$, $\text{ar}(K_{n_1,n_2,n_3}, C_4^{\text{multi}}) = n_1n_2 + n_3 + 1$.

We prove Theorems 3 and 4 in Section 2 and Theorem 5 in Section 3, respectively. We always denote the vertex partition of K_{n_1,n_2,n_3} by V_1, V_2 and V_3, where $|V_i| = n_i$, $1 \leqslant i \leqslant 3$.

2 The Turán numbers of C_4^{multi} and $\{C_3, C_4^{\text{multi}}\}$

In this section, we first give the following lemma which will play an important role in our proof.

Lemma 6. *Let G be a 3-partite graph with vertex partition X, Y and Z, such that for all $x \in X$, $N(x) \cap Y \neq \emptyset$ and $N(x) \cap Z \neq \emptyset$.*
(i) If G is C_4^{multi}-free, then $e(G) \leqslant |Y||Z| + 2|X|$;
(ii) If G is $\{C_3, C_4^{\text{multi}}\}$-free, then $e(G) \leqslant |Y||Z| + |X|$.

Proof. (i) Since G is C_4^{multi}-free, $G[N(x)]$ is $K_{1,2}$-free for each $x \in X$. Therefore,

$$e(G[N(x)]) = e\big(N(x) \cap Y, N(x) \cap Z\big) \leqslant \min\big\{|N(x) \cap Y|, |N(x) \cap Z|\big\}. \quad (1)$$

For $x \in X$, we let e_x be the number of missing edges of G between $N(x) \cap Y$ and $N(x) \cap Z$. By (1), we have

$$\begin{aligned} e_x &= |N(x)\cap Y| \cdot |N(x)\cap Z| - e\big(N(x)\cap Y, N(x)\cap Z\big) \\ &\geqslant |N(x)\cap Y| \cdot |N(x)\cap Z| - \min\big\{|N(x)\cap Y|, |N(x)\cap Z|\big\} \\ &\geqslant |N(x)\cap Y| + |N(x)\cap Z| - 2, \end{aligned} \tag{2}$$

where the last inequality holds since $|N(x) \cap Y| \geqslant 1$ and $|N(x) \cap Z| \geqslant 1$ for all $x \in X$.

By (2), we get

$$\sum_{x\in X} e_x \geqslant \sum_{x\in X} \big(|N(x)\cap Y| + |N(x)\cap Z| - 2\big) = e(X,Y) + e(X,Z) - 2|X|. \tag{3}$$

Notice that for any two distinct vertices $x_1, x_2 \in X$, they cannot have common neighbors in both Y and Z at the same time, otherwise we find a copy of C_4^{multi} in G. Thus each missing edge between Y and Z is calculated at most once in the sum $\sum_{x\in X} e_x$. Hence the number of missing edges between Y and Z is at least $\sum_{x\in X} e_x$. Then we have

$$e(Y,Z) \leqslant |Y||Z| - \sum_{x\in X} e_x \leqslant |Y||Z| - (e(X,Y) + e(X,Z) - 2|X|). \tag{4}$$

By (4), we get

$$e(G) = e(X,Y) + e(X,Z) + e(Y,Z) \leqslant |Y||Z| + 2|X|.$$

(ii) Since G is C_3-free, for each $x \in X$,

$$e\big(N(x)\cap Y, N(x)\cap Z\big) = 0. \tag{5}$$

Since for each $x \in X$, $|N(x) \cap Y| \geqslant 1$ and $|N(x) \cap Z| \geqslant 1$ hold, by (5), the number of missing edges between $N(x) \cap Y$ and $N(x) \cap Z$ is $|N(x) \cap Y| \cdot |N(x) \cap Z|$. Notice that for any two distinct vertices $x_1, x_2 \in X$, they cannot have common neighbors in both Y and Z at the same time, otherwise we find a copy of C_4^{multi} in G. Hence, the number of missing edges between Y and Z is at least $\sum_{x\in X} |N(x) \cap Y| \cdot |N(x) \cap Z|$. Thus,

$$\begin{aligned} e(Y,Z) &\leqslant |Y||Z| - \sum_{x\in X} |N(x)\cap Y| \cdot |N(x)\cap Z| \\ &\leqslant |Y||Z| - \sum_{x\in X} (|N(x)\cap Y| + |N(x)\cap Z| - 1) \\ &= |Y||Z| + |X| - e(X,Y) - e(X,Z), \end{aligned} \tag{6}$$

the second inequality holds since $|N(x) \cap Y| \geqslant 1$ and $|N(x) \cap Z| \geqslant 1$ for $x \in X$.

By (6), we have $e(G) = e(Y,Z) + e(X,Y) + e(X,Z) \leqslant |Y||Z| + |X|$. □

Now we are ready to prove Theorems 3 and 4.

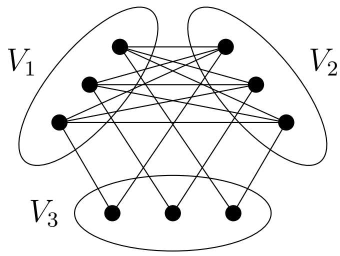


Figure 2: An example of C_4^{multi}-free graph with $n_1n_2 + 2n_3$ edges.

Proof of Theorem 3. Let $G \subseteq K_{n_1,n_2,n_3}$ be a graph, such that V_1 and V_2 are completely joined, V_1 (respectively, V_2) and V_3 are joined by an n_3-matching, see Figure 2. Clearly, G is C_4^{multi}-free and $e(G) = n_1n_2 + 2n_3$. Therefore, $\text{ex}(K_{n_1,n_2,n_3}, C_4^{\text{multi}}) \geqslant n_1n_2 + 2n_3$.

Let $G \subseteq K_{n_1,n_2,n_3}$ such that G is C_4^{multi}-free, now we are going to prove that $e(G) \leqslant n_1n_2 + 2n_3$ by induction on $n_1 + n_2 + n_3$.

For the base case $n_3 = 1$, let $V_3 = \{v\}$, we consider the following four subcases:
(i) $N(v) \cap V_1 \neq \emptyset$ and $N(v) \cap V_2 \neq \emptyset$. By Lemma 6, we have $e(G) \leqslant n_1n_2 + 2$.
(ii) $N(v) \cap V_1 \neq \emptyset$ and $N(v) \cap V_2 = \emptyset$.

For any vertex $x \in V_2$, we have $e(x, N(v)) \leqslant 1$, otherwise there is a C_4^{multi}. Hence, $e(V_2, N(v)) = \sum_{x \in V_2} e(x, N(v)) \leqslant n_2$. Therefore,

$$\begin{aligned} e(G) &= e(V_3, N(v)) + e(V_2, N(v)) + e(V_1 \setminus N(v), V_2) \\ &\leqslant d(v) + n_2 + \Big(n_1 - d(v)\Big)n_2 \\ &\leqslant n_1n_2 + 1. \end{aligned}$$

(iii) $N(v) \cap V_1 = \emptyset$ and $N(v) \cap V_2 \neq \emptyset$.

For any vertex $x \in V_1$, we have $e(x, N(v)) \leqslant 1$, otherwise there is a C_4^{multi}. Hence, $e(V_1, N(v)) = \sum_{x \in V_1} e(x, N(v)) \leqslant n_1$. Therefore,

$$\begin{aligned} e(G) &= e(V_3, N(v)) + e(V_1, N(v)) + e(V_2 \setminus N(v), V_1) \\ &\leqslant d(v) + n_1 + (n_2 - d(v))n_1 \\ &\leqslant n_1n_2 + 1. \end{aligned}$$

(iv) $N(v) \cap V_1 = \emptyset$ and $N(v) \cap V_2 = \emptyset$. We have $e(G) = e(V_1, V_2) \leqslant n_1n_2$.

Now let $n_3 \geqslant 2$, and assume that the statement is true for order less than $n_1 + n_2 + n_3$. We distinguish the three cases depending on the equality of the numbers n_1, n_2, n_3.
Case 1. $n_1 = n_2 = n_3 = n \geqslant 2$.

If there exists one part, say V_1, such that $N(v) \cap V_2 \neq \emptyset$ and $N(v) \cap V_3 \neq \emptyset$, for all $v \in V_1$, then by Lemma 6, we have $e(G) \leqslant |V_2||V_3| + 2|V_1| = n^2 + 2n$.

Thus, we may assume that for all $i \in [3] = \{1, 2, 3\}$, there exist a vertex $v \in V_i$ and $j \in [3] \setminus \{i\}$ such that $N(v) \cap V_j = \emptyset$. We divide it into two subcases.
Case 1.1. There exist two parts, say V_1 and V_2, such that $N(v_1) \cap V_2 = \emptyset$ and $N(v_2) \cap V_1 = \emptyset$ for some vertices $v_1 \in V_1$ and $v_2 \in V_2$.

Since G is C_4^{multi}-free, $d(v_1) + d(v_2) \leqslant |V_3| + 1 = n + 1$. Without loss of generality, let $v_3 \in V_3$ be the vertex such that $N(v_3) \cap V_1 = \emptyset$. Then the number of edges incident with $\{v_1, v_2, v_3\}$ in G is at most $d(v_1) + d(v_2) + n - 1 \leqslant 2n$. By the induction hypothesis, $e(G - \{v_1, v_2, v_3\}) \leqslant (n-1)^2 + 2(n-1)$. Thus, $e(G) \leqslant (n-1)^2 + 2(n-1) + 2n \leqslant n^2 + 2n$.

Case 1.2. There exist vertices $v_1 \in V_1, v_2 \in V_2$ and $v_3 \in V_3$ such that either $N(v_1) \cap V_2 = \emptyset, N(v_2) \cap V_3 = \emptyset, N(v_3) \cap V_1 = \emptyset$ or $N(v_1) \cap V_3 = \emptyset, N(v_3) \cap V_2 = \emptyset, N(v_2) \cap V_1 = \emptyset$ holds.

Without loss of generality, we assume that $N(v_1) \cap V_2 = \emptyset, N(v_2) \cap V_3 = \emptyset, N(v_3) \cap V_1 = \emptyset$. If $d(v_1) + d(v_2) + d(v_3) \leqslant 2n + 1$, then by the induction hypothesis, we have

$$\begin{aligned} e(G) &\leqslant e(G - \{v_1, v_2, v_3\}) + d(v_1) + d(v_2) + d(v_3) \\ &\leqslant (n-1)^2 + 2(n-1) + 2n + 1 \\ &\leqslant n^2 + 2n. \end{aligned}$$

Now we assume that $d(v_1) + d(v_2) + d(v_3) \geqslant 2n + 2$, hence, $d(v_1) \geqslant 1, d(v_2) \geqslant 1, d(v_3) \geqslant 1$. Since G is C_4^{multi}-free, each vertex in $V_1 \setminus \{v_1\}$ can have at most one neighbor in $N(v_3)$, we have $e(V_1 \setminus \{v_1\}, N(v_3)) \leqslant n - 1$. Similarly, we have $e(V_3 \setminus \{v_3\}, N(v_2)) \leqslant n - 1$ and $e(V_2 \setminus \{v_2\}, N(v_1)) \leqslant n - 1$.

Therefore,

$$e(V_1, V_2) = e(V_1 \setminus \{v_1\}, V_2 \setminus N(v_3)) + e(V_1 \setminus \{v_1\}, N(v_3)) \leqslant (n - d(v_3))(n-1) + (n-1),$$

$$e(V_1, V_3) = e(V_3 \setminus \{v_3\}, V_1 \setminus N(v_2)) + e(V_3 \setminus \{v_3\}, N(v_2)) \leqslant (n - d(v_2))(n-1) + (n-1),$$

$$e(V_2, V_3) = e(V_2 \setminus \{v_2\}, V_3 \setminus N(v_1)) + e(V_2 \setminus \{v_2\}, N(v_1)) \leqslant (n - d(v_1))(n-1) + (n-1).$$

Thus,

$$\begin{aligned} e(G) &= e(V_1, V_2) + e(V_1, V_3) + e(V_2, V_3) \\ &\leqslant \big(3n - (d(v_1) + d(v_2) + d(v_3))\big)(n-1) + 3(n-1) \\ &\leqslant \big(3n - (2n+2)\big)(n-1) + 3(n-1) \\ &\leqslant n^2 - 1. \end{aligned}$$

Case 2. $n_1 > n_2 = n_3 = n \geqslant 2$.

If there exists one vertex $v_0 \in V_1$ such that $d(v_0) \leqslant n$, then by the induction hypothesis, we have $e(G) = e(G - v_0) + d(v_0) \leqslant (n_1 - 1)n + 2n + n \leqslant n_1 n + 2n$. Otherwise, we have $d(v) \geqslant n + 1$ for all vertices $v \in V_1$. Hence, $N(v) \cap V_2 \neq \emptyset$ and $N(v) \cap V_3 \neq \emptyset$ hold for all $v \in V_1$. By Lemma 6, we get $e(G) \leqslant n^2 + 2n_1 \leqslant n_1 n + 2n$.

Case 3. $n_1 \geqslant n_2 > n_3 \geqslant 2$.

If there exists one vertex $v_0 \in V_2$ such that $d(v_0) \leqslant n_1$, by the induction hypothesis, we have $e(G) = e(G - v_0) + d(v_0) \leqslant n_1(n_2 - 1) + 2n_3 + n_1 \leqslant n_1 n_2 + 2n_3$. Otherwise, we have $d(v) \geqslant n_1 + 1$ for all vertices $v \in V_2$. Hence, $N(v) \cap V_1 \neq \emptyset$ and $N(v) \cap V_3 \neq \emptyset$ for all $v \in V_2$. By Lemma 6, we get $e(G) \leqslant n_1 n_3 + 2n_2 \leqslant n_1 n_2 + 2n_3$. □

Proof of Theorem 4. Let $G \subseteq K_{n_1,n_2,n_3}$ be a graph, such that V_1 and V_2 are completely joined, V_1 and V_3 are joined by an n_3-matching and there is no edge between V_2 and

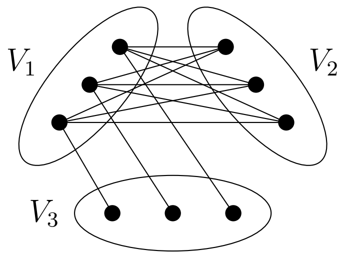


Figure 3: An example of $\{C_3, C_4^{\text{multi}}\}$-free graph with $n_1n_2 + n_3$ edges.

V_3, see Figure 3. Clearly, G is $\{C_3, C_4^{\text{multi}}\}$-free and $e(G) = n_1n_2 + n_3$. Therefore, $\text{ex}(K_{n_1,n_2,n_3}, \{C_3, C_4^{\text{multi}}\}) \geqslant n_1n_2 + n_3$.

Let $G \subseteq K_{n_1,n_2,n_3}$ such that G is $\{C_3, C_4^{\text{multi}}\}$-free, now we can prove $e(G) \leqslant n_1n_2 + n_3$ by induction on $n_1 + n_2 + n_3$ in the same way as we did in the proof of Theorem 3, just the coefficients in the computation change a bit. For sake of brevity, we skip the details of the proof. □

3 The anti-Ramsey number of C_4^{multi}

In this section, we study the anti-Ramsey number of C_4^{multi} in the complete 3-partite graphs. Given an edge-coloring c of G, we denote the color of an edge e by $c(e)$. For a subgraph H of G, we denote $C(H) = \{c(e)|e \in E(H)\}$. We call a spanning subgraph of an edge-colored graph a *representing subgraph*, if it contains exactly one edge of each color.

Given graphs G_1 and G_2, we use $G_1 \wedge G_2$ to denote the graph consisting of G_1 and G_2 sharing exactly one common vertex. We call a multipartite C_6 in a 3-partite graph non-cyclic if there exists a vertex v in C_6 such that the two neigborhoods in C_6 of v belong to the same part. Let $\mathcal{F}$ be a graph family which consists of C_4^{multi} (see graph G_1 in Figure 4), $C_3 \wedge C_3$ (see graph G_2 in Figure 4), the non-cyclic C_6^{multi} (see graphs G_3, G_4 in Figure 4) and $C_3 \wedge C_5$ (see graphs G_5, G_6, G_7 in Figure 4) and the C_8^{multi} which contains at least two vertex-disjoint non-multipartite P_3 (see graph G_8 in Figure 4).

To find a rainbow C_4^{multi} in the edge-colored complete 3-partite graphs, we follow the idea of Alon [1] and prove the lemma as follows..

Lemma 7. *Let $n_1 \geqslant n_2 \geqslant n_3 \geqslant 1$. For an edge-colored K_{n_1,n_2,n_3}, if there is a rainbow copy of some graph in $\mathcal{F}$, then there is a rainbow copy of C_4^{multi}.*

Proof. We separate the proof into three cases.
Case 1. An edge-colored K_{n_1,n_2,n_3} contains a rainbow copy of G_2, G_3 or G_4.

Suppose there is a rainbow copy of G_2 in K_{n_1,n_2,n_3} (see Figure 5), then whatever the color of v_1w_2 is, at least one of $v_1uv_2w_2v_1$ and $v_1w_2uw_1v_1$ is a rainbow C_4^{multi}. Similarly, with the help of the red edge that is showed in G_3 and G_4 (see Figure 5), there are two C_4^{multi}'s whose edge-intersection is the red edge, so one of the two C_4^{multi}'s must be rainbow.
Case 2. An edge-colored K_{n_1,n_2,n_3} contains a rainbow copy of G_5.

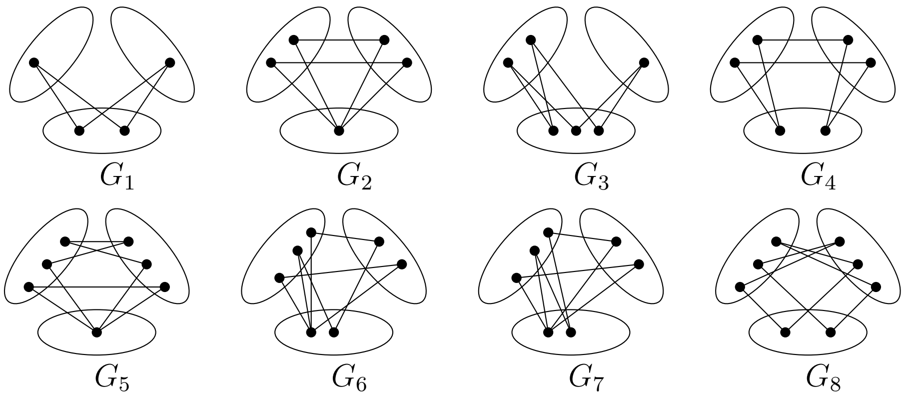


Figure 4: $\mathcal{F} = \{G_1\} \cup \{G_2\} \cup \{G_3, G_4\} \cup \{G_5, G_6, G_7\} \cup \{G_8\}$.

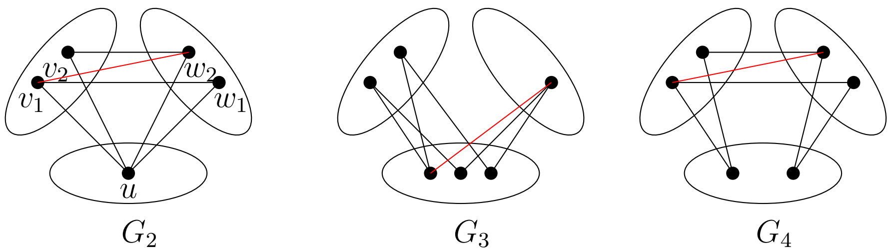
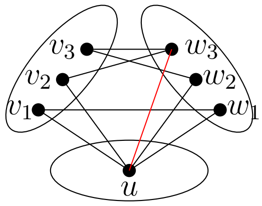


Figure 5: Illustration of Case 1.

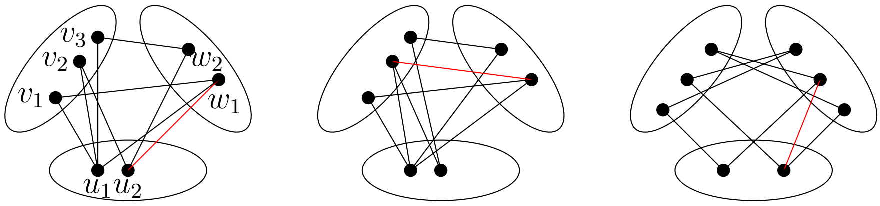
Suppose there is a rainbow copy of G_5 in K_{n_1,n_2,n_3} (see Figure 6). If $v_3w_3uw_2v_3$ is not rainbow, then uw_3 shares the same color with one of v_3w_3, v_3w_2 and uw_2. Hence, $uv_2w_3u \cup uv_1w_2u$ is a rainbow copy of G_2, by Case 1, we can find a rainbow copy of C_4^{multi}.



G_5

Figure 6: Illustration of Case 2.

Case 3. An edge-colored K_{n_1,n_2,n_3} contains a rainbow copy of G_6, G_7 or G_8.



G_6 G_7 G_8

Figure 7: Illustration of Case 3.

Suppose there is a rainbow copy of G_6 in K_{n_1,n_2,n_3} (see Figure 7). If $v_2u_1w_1u_2v_2$ is not rainbow, then u_2w_1 shares the same color with one of v_2u_1, u_1w_1 and u_2v_2. Hence, $v_1u_1v_3w_2u_2w_1v_1$ is a rainbow copy of G_4, by Case 1, we can find a rainbow copy of C_4^{multi}. Similarly, with the help of the red edge that is showed in G_7 and G_8 (see Figure 7), one can always find a rainbow copy of C_4^{multi} if there is a rainbow copy of G_7 or G_8. □

Now we are able to prove Theorem 5.

Proof of Theorem 5. **Lower bound:** We color the edges of K_{n_1,n_2,n_3} as follows. First, color all edges between V_1 and V_2 rainbow. Second, for each vertex $v \in V_3$, color all the edges between v and V_1 with one new distinct color. Finally, assign a new color to all edges between V_2 and V_3. In such way, we use exactly $n_1n_2 + n_3 + 1$ colors, and there is no rainbow C_4^{multi}.

Upper bound: We prove the upper bound by induction on $n_1 + n_2 + n_3$. By Theorem 3, we have $\text{ar}(K_{n_1,n_2,1}, C_4^{\text{multi}}) \leqslant \text{ex}(K_{n_1,n_2,1}, C_4^{\text{multi}}) = n_1n_2 + 2$, the conclusion holds for $n_3 = 1$. Let $n_3 \geqslant 2$, suppose the conclusion holds for all integers less than $n_1+n_2+n_3$. We suppose there exists an $(n_1n_2 + n_3 + 2)$-edge-coloring c of K_{n_1,n_2,n_3} such that there is no rainbow C_4^{multi} in it. We take a representing subgraph G.

Claim 8. *G contains two vertex-disjoint triangles.*

Proof of Claim 8. Recall that Theorem 4 says that $\mathrm{ex}(K_{n_1,n_2,n_3}, \{C_3, C_4^{\text{multi}}\}) = n_1n_2+n_3$. Since $e(G) = n_1n_2 + n_3 + 2$ and G contains no C_4^{multi}, G contains at least two triangles T_1 and T_2. If $|V(T_1) \cap V(T_2)| = 2$, then $T_1 \cup T_2$ contains a C_4^{multi}, a contradiction. If $|V(T_1) \cap V(T_2)| = 1$, then $T_1 \cup T_2$ is a copy of $C_3 \wedge C_3$. By Lemma 7, we can find a rainbow C_4^{multi}, a contradiction. Thus, T_1 and T_2 are vertex-disjoint. □

Let the two vertex-disjoint triangles be $T_1 = x_1y_1z_1x_1$ and $T_2 = x_2y_2z_2x_2$, where $\{x_1, x_2\} \subseteq V_1$, $\{y_1, y_2\} \subseteq V_2$ and $\{z_1, z_2\} \subseteq V_3$. Denote $V_0 = \{x_1, x_2, y_1, y_2, z_1, z_2\}$ and $U = (V_1 \cup V_2 \cup V_3) \setminus V_0$.

Claim 9. $e(G[V_0]) \leqslant 7$.

Proof of Claim 9. If $e(G[V_0]) \geqslant 8$, then $e(V(T_1), V(T_2)) \geqslant 2$. Without loss of generality, assume that $x_1y_2 \in E(G)$, we claim that $x_1z_2, x_2z_1, y_1z_2, y_2z_1 \notin E(G)$, otherwise $x_1y_2x_2z_2x_1$, $x_1y_2x_2z_1x_1$, $x_1y_2z_2y_1x_1$ or $x_1y_2z_1y_1x_1$ would be a rainbow C_4^{multi}. Thus, we have $x_2y_1 \in E(G)$. We claim that $c(y_1z_2) = c(y_2z_2)$, otherwise at least one of $\{x_1y_1z_2y_2x_1, x_2y_1z_2y_2x_2\}$ is a rainbow C_4^{multi}. Thus, $G[V_0] - y_2z_2 + y_1z_2$ is rainbow and contains a $C_3 \wedge C_3$. By Lemma 7, we find a rainbow C_4^{multi}, a contradiction. □

If $U = \emptyset$, that is $n_1 = n_2 = n_3 = 2$, then $8 = e(G) = e(G[V_0]) \leqslant 7$, by Claim 9, a contradiction. Thus we may assume that $U \neq \emptyset$.

Claim 10. *For all $v \in U$, $e(v, V_0) \leqslant 2$.*

Proof of Claim 10. If there is a vertex $v \in U$, such that $e_G(v, V_0) \geqslant 3$, then $G[V_0 \cup \{v\}]$ contains a C_4^{multi}, a contradiction. □

Claim 11. $n_3 \geqslant 3$.

Proof of Claim 11. Suppose $n_3 = 2$. Since $U \neq \emptyset$, we have $n_1 \geqslant 3 = n_3 + 1$. If there is a vertex $v \in V_1$ such that $d(v) \leqslant n_2$, then $e(G-v) = n_1n_2+n_3+2-d(v) \geqslant (n_1-1)n_2+n_3+2$. By the induction hypothesis, we have

$$|C(K_{n_1,n_2,n_3} - v)| \geqslant e(G - v) \geqslant (n_1 - 1)n_2 + n_3 + 2 = \mathrm{ar}(K_{n_1-1,n_2,n_3}, C_4^{\text{multi}}) + 1,$$

thus $K_{n_1,n_2,n_3} - v$ contains a rainbow C_4^{multi}, a contradiction. Thus we assume that $d(v) \geqslant n_2 + 1$ for all $v \in V_1$. By Claim 8, we have $e(V_2, V_3) \geqslant 2$. Hence, we have

$$e(G) = e(V_1, V_2 \cup V_3) + e(V_2, V_3) = \sum_{v \in V_1} d(v) + e(V_2, V_3) \geqslant n_1(n_2 + 1) + 2 = n_1n_2 + n_1 + 2,$$

and this contradicts to the fact that $e(G) = n_1n_2 + n_3 + 2$. □

Claim 12. $e(G[V_0]) + e(V_0, U) \geqslant 2n_1 + 2n_2 - 1$.

Proof of Claim 12. If $e(G[V_0]) + e(V_0, U) \leqslant 2n_1 + 2n_2 - 2$, then

$$e(G[U]) = e(G) - (e(G[V_0]) + e(V_0, U)) \geqslant n_1 n_2 + n_3 + 2 - (2n_1 + 2n_2 - 2)$$
$$= (n_1 - 2)(n_2 - 2) + (n_3 - 2) + 2.$$

By Claim 11, $n_3 - 2 \geqslant 1$. By the induction hypothesis, we have

$$|C(K_{n_1,n_2,n_3} - V_0)| \geqslant e(G[U]) \geqslant (n_1 - 2)(n_2 - 2) + (n_3 - 2) + 2$$
$$= \text{ar}(K_{n_1-2,n_2-2,n_3-2}, C_4^{\text{multi}}) + 1,$$

thus $K_{n_1,n_2,n_3} - V_0$ contains a rainbow C_4^{multi}, a contradiction. □

Denote $U_0 = \{v \in U : e(v, V_0) = 2\}$. By Claim 10, we have $e(U, V_0) \leqslant |U_0| + |U|$. By Claim 9, we just need to consider the following two cases.

Case 1. $e(G[V_0]) = 7$.

Without loss of generality, let $x_1 z_2$ be the unique edge of $G[V_0]$ between T_1 and T_2. By Claim 12, we have $e(U, V_0) \geqslant 2n_1 + 2n_2 - 1 - e(G[V_0]) = 2n_1 + 2n_2 - 8$. Since $|U| = n_1 + n_2 + n_3 - 6$ and $e(U, V_0) \leqslant |U_0| + |U|$, we have $|U_0| \geqslant n_1 + n_2 - n_3 - 2 \geqslant 1$. Take a vertice $v \in U_0$, we consider the following two subcases to show that $G[V_0 \cup \{v\}]$ contains one rainbow copy of some graph in $\mathcal{F}$ (see Figure 4). By Lemma 7, there is a rainbow C_4^{multi}, a contradiction.

Case 1.1 $v \in V_1 \cup V_3$.

Without loss of generality, we may assume that $v \in V_1$, the orange edges in $G[V_0 \cup \{v\}]$ (see Figure 8) forms a copy of some graph in $\mathcal{F}$ (see Figure 4).

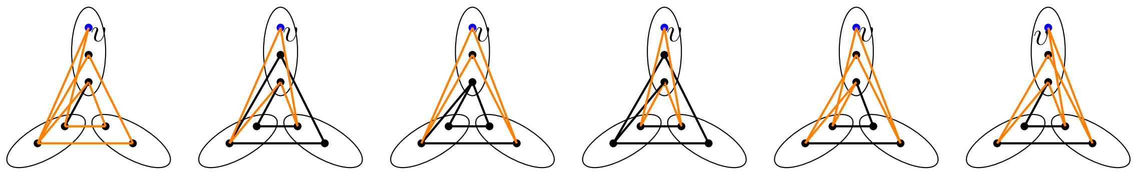


Figure 8: Illustration of Case 1.1.

Case 1.2 $v \in V_2$.

The orange edges in $G[V_0 \cup \{v\}]$ (see Figure 9) forms a copy of some graph in $\mathcal{F}$ (see Figure 4).

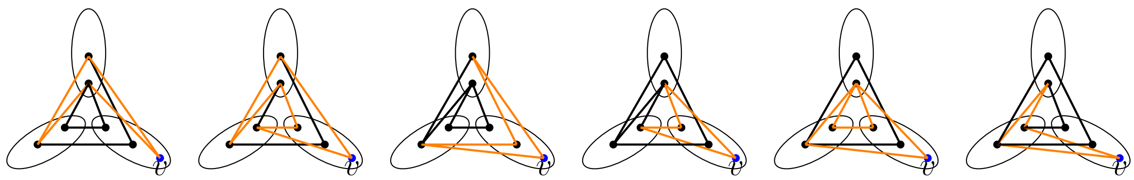


Figure 9: Illustration of Case 1.2.

Case 2. $e(G[V_0]) = 6$.

By Claim 12, we have $e(U, V_0) \geqslant 2n_1 + 2n_2 - 1 - e(G[V_0]) = 2n_1 + 2n_2 - 7$. Since $|U| = n_1 + n_2 + n_3 - 6$ and $e(U, V_0) \leqslant |U_0| + |U|$, we have $|U_0| \geqslant n_1 + n_2 - n_3 - 1 \geqslant n_1 - 1 > n_1 - 2$. Thus, U_0 contains at least two vertices v_1 and v_2 which come from distinct parts. Without loss of generality, assume that $v_1 \in V_1$ and $v_2 \in V_2$. We consider the following three subcases to show that $G[V_0 \cup \{v_1, v_2\}]$ contains one rainbow copy of some graph in $\mathcal{F}$ (see Figure 4). By Lemma 7, there exists a rainbow C_4^{multi}, a contradiction.

Case 2.1 $N(v_1) \cap V_0 \subset V_3$.

The orange edges in $G[V_0 \cup \{v_1, v_2\}]$ (see Figure 10) forms a copy of some graph in $\mathcal{F}$ (see Figure 4).

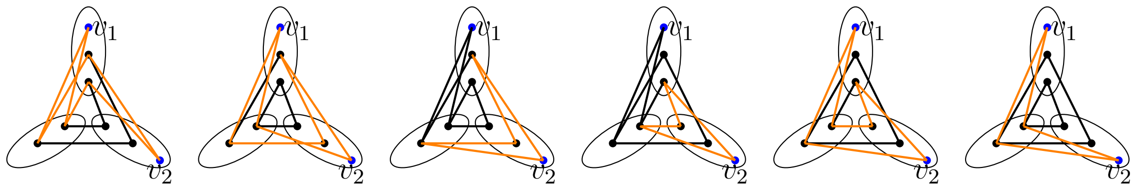


Figure 10: Illustration of Case 2.1.

Case 2.2 $|N(v_1) \cap V_0 \cap V_2| = |N(v_1) \cap V_0 \cap V_3| = 1$.

If $N(v_1) \cap V_0 \subset V(T_1)$ or $N(v_1) \cap V_0 \subset V(T_2)$, then $G[V(T_1) \cup \{v_1\}]$ or $G[V(T_2) \cup \{v_1\}]$ contains a C_4^{multi}. Thus, we assume that $|N(v_1) \cap V(T_1)| = |N(v_1) \cap V(T_2)| = 1$, the orange edges in $G[V_0 \cup \{v_1, v_2\}]$ (see Figure 11) forms a copy of some graph in $\mathcal{F}$ (see Figure 4).

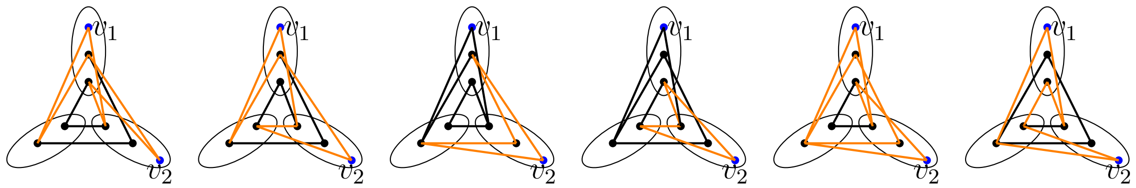


Figure 11: Illustration of Case 2.2.

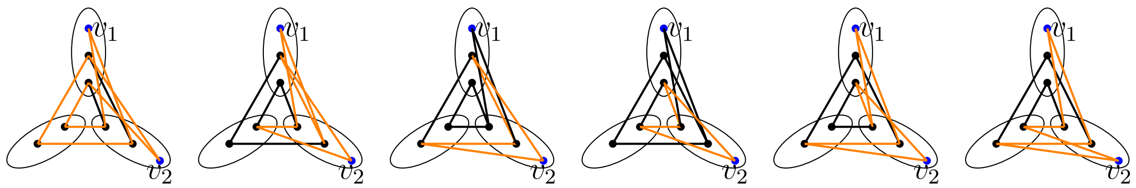


Figure 12: Illustration of Case 2.3.

Case 2.3 $N(v_1) \cap V_0 \subset V_3$.

The orange edges in $G[V_0 \cup \{v_1, v_2\}]$ (see Figure 12) forms a copy of some graph in $\mathcal{F}$ (see Figure 4). □

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