Generalized Alder-Type Partition Inequalities

Liam Armstrong[*]
Oregon State University
Corvallis, OR, U.S.A.
armstrli@oregonstate.edu

Bryan Ducasse[†]
University of Central Florida
Orlando, FL, U.S.A.
bducasse77@knights.ucf.edu

Thomas Meyer[‡]
Amherst College
Amherst, MA, U.S.A.
tmeyer23@amherst.edu

Holly Swisher[§]
Department of Mathematics
Oregon State University
Corvallis, OR, U.S.A.
swisherh@oregonstate.edu

Submitted: Oct 14, 2022; Accepted: May 12, 2023; Published: Jun 2, 2023

Abstract

In 2020, Kang and Park conjectured a "level 2" Alder-type partition inequality which encompasses the second Rogers-Ramanujan Identity. Duncan, Khunger, the fourth author, and Tamura proved Kang and Park's conjecture for all but finitely many cases utilizing a "shift" inequality and conjectured a further, weaker generalization that would extend both Alder's (now proven) as well as Kang and Park's conjecture to general level. Utilizing a modified shift inequality, Inagaki and Tamura have recently proven that the Kang and Park conjecture holds for level 3 in all but finitely many cases. They further conjectured a stronger shift inequality which would imply a general level result for all but finitely many cases. Here, we prove their conjecture for large enough n, generalize the result for an arbitrary shift, and discuss the implications for Alder-type partition inequalities.

Mathematics Subject Classifications: 05A17, 05A20, 11P81, 11P84

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, that sum to n. Let $p(n \mid \text{condition})$ count the number of partitions of n that satisfy

[*]Supported by NSF grant DMS-2101906.
[†]Supported by NSF grant DMS-2101906.
[‡]Supported by NSF grant DMS-2101906.
[§]Supported by NSF grant DMS-2101906.

the specified condition, and define

$$\begin{aligned}
q_d^{(a)}(n) &:= p(n \,|\, \text{parts} \geqslant a \text{ and differ by at least } d),\\
Q_d^{(a)}(n) &:= p(n \,|\, \text{parts} \equiv \pm a \pmod{d+3}),\\
\Delta_d^{(a)}(n) &:= q_d^{(a)}(n) - Q_d^{(a)}(n).
\end{aligned}$$

Euler's well-known partition identity, which states that the number of partitions of n into distinct parts equals those into odd parts, can be written as $\Delta_1^{(1)}(n) = 0$. Moreover, the celebrated first and second Rogers-Ramanujan identities, written here in terms of q-Pochhammer notation[1],

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} &= \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},\\
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} &= \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty},
\end{aligned}$$

are interpreted in terms of partitions as $\Delta_2^{(1)}(n) = 0$ and $\Delta_2^{(2)}(n) = 0$, respectively.

Schur [10] proved that the number of partitions of n into parts differing by at least 3, where no two consecutive multiples of 3 appear, equals the number of partitions of n into parts congruent to $\pm 1 \pmod 6$, which yields that $\Delta_3^{(1)}(n) \geqslant 0$. Lehmer [9] and Alder [1] proved that such a pattern of identities can not continue by showing that no other such partition identities can exist. However, in 1956 Alder [2] conjectured a different type of generalization. Namely, that for all $n, d \geqslant 1$,

$$\Delta_d^{(1)}(n) \geqslant 0. \tag{1}$$

In 1971, Andrews [4] proved (1) when $d = 2^k - 1$ and $k \geqslant 4$, and in 2004, Yee [11, 12] proved (1) for $d \geqslant 32$ and $d = 7$, both using q-series and combinatorial methods. Then in 2011, Alfes, Jameson, and Lemke Oliver [3] used asymptotic methods and detailed computer programming to prove the remaining cases of $4 \leqslant d \leqslant 30$ with $d \neq 7, 15$.

It is natural to ask whether (1) can be generalized to $a = 2$ in order to encapsulate the second Rogers-Ramanujan identity, or perhaps even be generalized to arbitrary a.

In 2020, after observing that $\Delta_d^{(2)}(n) \geqslant 0$ does not hold for all $n, d \geqslant 1$, Kang and Park [8] defined

$$\begin{aligned}
Q_d^{(a,-)}(n) &:= p(n \,|\, \text{parts} \equiv \pm a \pmod{d+3}, \text{ excluding the part } d+3-a),\\
\Delta_d^{(a,-)}(n) &:= q_d^{(a)}(n) - Q_d^{(a,-)}(n),
\end{aligned}$$

and conjectured that for all $n, d \geqslant 1$,

$$\Delta_d^{(2,-)}(n) \geqslant 0. \tag{2}$$

[1] $(a;q)_0 := 1$ and $(a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k)$ for $1 \leqslant n \leqslant \infty$

Kang and Park [8] proved (2) when n is even, $d = 2^k - 2$, and $k \geqslant 5$ or $k = 2$. Then in 2021, Duncan, Khunger, the fourth author, and Tamura [5] proved (2) for all $d \geqslant 62$. Exploring the question for larger a, they conjectured that for all $n, d \geqslant 1$,

$$\Delta_d^{(3,-)}(n) \geqslant 0, \tag{3}$$

but found that when $a \geqslant 4$, the removal of one additional part appears to be both necessary and sufficient to obtain such a result for all $n, d \geqslant 1$. Letting

$$Q_d^{(a,-,-)}(n) := p(n \mid \text{parts} \equiv \pm a \pmod{d+3}, \text{ excluding the parts } a \text{ and } d+3-a),$$
$$\Delta_d^{(a,-,-)}(n) := q_d^{(a)}(n) - Q_d^{(a,-,-)}(n),$$

Duncan et al. [5] conjectured that if $a, d \geqslant 1$ such that $1 \leqslant a \leqslant d+2$, then for all $n \geqslant 1$,

$$\Delta_d^{(a,-,-)}(n) \geqslant 0. \tag{4}$$

Recently, Inagaki and Tamura [6] proved (3) for $d \geqslant 187$ and $d = 1, 2, 91, 92, 93$, and further proved that $\Delta_d^{(4,-)}(n) \geqslant 0$ for $d \geqslant 249$ and $121 \leqslant d \leqslant 124$ as a corollary to a result for general a for certain residue classes of d. Inagaki and Tamura [6] were also able to prove the general conjecture (4) of Duncan et al. [5] for sufficiently large d with respect to a, namely when $\lceil \frac{d}{a} \rceil \geqslant 2^{a+3} - 1$.

The proof of (2) for $d \geqslant 62$ by Duncan et al. [5] utilized a particular shift identity. Namely, they showed that if $d \geqslant 31$ or $d = 15$, then for $n \geqslant 1$,

$$q_d^{(1)}(n) \geqslant Q_{d-2}^{(1,-)}(n). \tag{5}$$

The proof of (3) for $d \geqslant 187$ or $d = 1, 2, 91, 92, 93$ by Inagaki and Tamura [6] utilized a stronger shift identity that holds for large enough n with respect to d. Namely, they showed that if $d \geqslant 63$ or $d = 31$, then for $n \geqslant d+2$,

$$q_d^{(1)}(n) \geqslant Q_{d-3}^{(1,-)}(n). \tag{6}$$

Given a choice of a, it is natural to ask for which $n, d \geqslant 1$,

$$\Delta_d^{(a,-)}(n) \geqslant 0. \tag{7}$$

Inagaki and Tamura [6] posed the following shift identity conjecture, which they further determined can be used to obtain answers to (7) and a vast improvement on the bounds for (4).

Conjecture 1 (Inagaki, Tamura [6], 2022)**.** Let $d \geqslant 12$ and $n \geqslant d+2$. Then

$$q_d^{(1)}(n) - Q_{d-4}^{(1,-)}(n) \geqslant 0.$$

In this paper, we prove a generalized shift identity. We have the following theorem.

Theorem 2. *If $N \geqslant 2$, $d \geqslant \max\{63, 46N-79\}$, and $n \geqslant d+2$, then*

$$q_d^{(1)}(n) \geqslant Q_{d-N}^{(1,-)}(n).$$

As an immediate corollary of Theorem 2 we obtain Conjecture 1 when $d \geqslant 105$.

Corollary 3. *For $d \geqslant 105$, and $n \geqslant d+2$,*

$$q_d^{(1)}(n) \geqslant Q_{d-4}^{(1,-)}(n).$$

Moreover, using the methods of Inagaki and Tamura [6] Corollary 3 can be applied to obtain a more complete answer to (7) as well as stronger bounds for (4).

Theorem 4. *Let $a \geqslant 1$ and $d \not\equiv -3 \pmod{a}$ such that $\lceil \frac{d}{a} \rceil \geqslant 105$. Then for all $n \geqslant 1$,*

$$\Delta_d^{(a,-)}(n) \geqslant 0.$$

Moreover, for $d \equiv -3 \pmod{a}$ then $\Delta_d^{(a,-)}(n) \geqslant 0$ for all $n \neq d+a+3$.

As a corollary of Theorem 4 we obtain the following, which proves conjecture (4) of Duncan et al. [5] for $\lceil \frac{d}{a} \rceil \geqslant 105$. We note that this bound is lower than that given by Inagaki and Tamura [6, Thm. 1.8] when $a \geqslant 4$, and is significantly lower as a grows.

Corollary 5. *For all $a, d \geqslant 1$ such that $\lceil \frac{d}{a} \rceil \geqslant 105$, and $n \geqslant 1$,*

$$\Delta_d^{(a,-,-)}(n) \geqslant 0.$$

We now outline the rest of the paper. In Section 2, we state a fundamental result of Andrews [4] and discuss some notation and lemmas used in the proofs of Theorems 2, 4, and 5. In Section 3, we prove Theorem 2, and in Section 4, we use Corollary 3 to prove Theorem 4 and Corollary 5. We conclude with additional remarks and discussion.

2 Preliminaries

For a nonempty set $A \subseteq \mathbb{N}$, define $\rho(A; n)$ to count the number of partitions of n with parts in A. The following theorem of Andrews [4] gives a way to compare the number of partitions of n with parts coming from different sets.

Theorem 6 (Andrews [4], 1971)**.** *Let $S = \{x_i\}_{i=1}^{\infty}$ and $T = \{y_i\}_{i=1}^{\infty}$ be two strictly increasing sequences of positive integers such that $y_1 = 1$ and $x_i \geqslant y_i$ for all i. Then*

$$\rho(T; n) \geqslant \rho(S; n).$$

For fixed $d \geqslant 1$, define r to be the greatest integer such that

$$2^r - 1 \leqslant d. \tag{8}$$

Further define for integers $d, s \geqslant 1$

$$T_{s,d} := \{y \in \mathbb{N} \mid y \equiv 1, d+2, \dots, d+2^{s-1} \pmod{2d}\}. \tag{9}$$

Table 1: Elements of $T_{s,d}$ in increasing order by rows for $s \leqslant r$.

1	$d+2$	$\cdots$	$d+2^{s-1}$
$2d+1$	$3d+2$	$\cdots$	$3d+2^{s-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(2j-2)d+1$	$(2j-1)d+2$	$\cdots$	$(2j-1)d+2^{s-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Lemma 7. *Let $d \geqslant 1$ and $1 \leqslant a \leqslant b \leqslant r$, with r as in (8). Then $\rho(T_{a,d}; n) \leqslant \rho(T_{b,d}; n)$.*

Proof. When $s \leqslant r$, we have $2^{s-1} - 1 < d$ which implies that $(2k-1)d + 2^{s-1} < 2kd + 1$ for all $k \geqslant 1$. Thus Table 1 shows the elements of $T_{s,d}$ listed in increasing order when read left to right.

Let y_i^s denote the i^{th} smallest element of $T_{s,d}$. Observe that when $1 \leqslant a \leqslant b \leqslant r$ we must have that $y_i^a \geqslant y_i^b$ for all i, since the number of columns in Table 1, and thus the index of the elements in the first column, is weakly increasing when $s = a$ is replaced by $s = b$. Thus, by Theorem 6, we conclude that $\rho(T_{a,d}; n) \leqslant \rho(T_{b,d}; n)$. □

Previous work of Andrews [4] and Yee [12] on Alder's conjecture gives the following lower bound for $q_d^{(1)}(n)$ for sufficiently large d and n.

Lemma 8 (Andrews [4], Yee [12])**.** *Let $d \geqslant 63$ and $n \geqslant 5d$. Then $q_d^{(1)}(n) \geqslant \rho(T_{5,d}; n)$.*

Proof. Recall for fixed $d \geqslant 1$, r is defined as in (8). When $d > 2^r - 1$ for $r \geqslant 5$ and $n \geqslant 4d + 2^r$, work of Yee [[12], Lemmas 2.2 and 2.7] gives that

$$q_d^{(1)}(n) \geqslant \mathcal{G}_d^{(1)}(n),$$

where

$$\sum_{k \geqslant 0} \mathcal{G}_d^{(1)}(n) q^n = \frac{(-q^{d+2^{r-1}}; q^{2d})_\infty}{(q; q^{2d})_\infty (q^{d+2}; q^{2d})_\infty \cdots (d^{d+2^{r-2}}; q^{2d})_\infty}.$$

From this generating function it follows that $\mathcal{G}_d^{(1)}(n)$ counts the number of partitions of n into distinct parts congruent to $d + 2^{r-1}$ modulo $2d$ and unrestricted parts from the set $T_{r-1,d}$ as defined in (9). Thus it follows that

$$q_d^{(1)}(n) \geqslant \mathcal{G}_d^{(1)}(n) \geqslant \rho(T_{r-1,d}; n).$$

From our hypotheses $d \geqslant 63$, so $r \geqslant 6$. Hence by Lemma 7, we have when $d > 2^r - 1$ that

$$q_d^{(1)}(n) \geqslant \rho(T_{5,d}; n).$$

When $d = 2^r - 1$ for $r \geqslant 4$, work of Andrews [[4], Theorem 1 and discussion], gives

$$q_d^{(1)}(n) \geqslant \mathcal{L}_d(n),$$

where

$$\sum_{n\geqslant 0} \mathcal{L}_d(n) q^n = \frac{1}{(q;q^{2d})_\infty (d^{d+2};q^{2d})_\infty \cdots (q^{d+2^{r-1}};q^{2d})_\infty}.$$

From this generating function it follows that $\mathcal{L}_d(n) = \rho(T_{r,d};n)$. Thus with our hypotheses, and Lemma 7, it follows that when $d = 2^r - 1$, $q_d^{(1)}(n) \geqslant \rho(T_{5,d};n)$. □

Let

$$S_d^N := \{x \in \mathbb{N} \mid x \equiv \pm 1 \pmod{d - N + 3}\} \setminus \{d - N + 2\},$$

so that we have by definition

$$Q_{d-N}^{(1,-)}(n) = \rho(S_d^N;n). \tag{10}$$

We write x_i^N and y_i to denote the i^{th} smallest elements of S_d^N and $T_{5,d}$, respectively.

If $x_i^N \geqslant y_i$ for all i, then Theorem 2 would follow easily from Theorem 6 and Lemma 8. While this is not the case, the inequality does hold for all but the index $i = 2$, as shown in the following lemma.

Lemma 9. *If $N \geqslant 2$ and $d \geqslant \max\{31, 6N-17\}$, then $x_i^N - y_i \geqslant 0$ for all $i \geqslant 3$. Moreover, we have that*

$$\min_{i\geqslant 3}\{x_i^N - y_i\} = \min\{d - 2N - 1, d - 6N + 17\}.$$

Proof. Fix $d \geqslant 1$. We first show that we can reduce the indices modulo 10 in our comparison. By definition of S_d^N, we see that for $i \geqslant 3$, $x_i^N = \lceil \frac{i}{2} \rceil (d - N + 3) + (-1)^i$, so it follows that $x_{i+10}^N = x_i^N + 5d - 5N + 15$. Since $d \geqslant 31$ we have that $r \geqslant 5$. Thus recalling Table 1, we can write $y_{i+10} = y_i + 4d$ for all $i \geqslant 1$. Thus for $i \geqslant 3$, we have

$$x_{i+10}^N - y_{i+10} = (x_i^N - y_i) + (d - 5N + 15) \geqslant x_i^N - y_i, \tag{11}$$

since $d \geqslant \max\{31, 6N - 17\} \geqslant 5N - 15$ when $N \geqslant 2$.

Thus, it suffices to show $x_i^N - y_i \geqslant 0$ for the indices $3 \leqslant i \leqslant 12$. By direct computation,

$$\begin{aligned}
x_3^N - y_3 &= d - 2N + 1,\\
x_4^N - y_4 &= d - 2N - 1,\\
x_5^N - y_5 &= 2d - 3N - 8,\\
x_6^N - y_6 &= d - 3N + 9,\\
x_7^N - y_7 &= d - 4N + 9,\\
x_8^N - y_8 &= d - 4N + 9,\\
x_9^N - y_9 &= 2d - 5N + 6,\\
x_{10}^N - y_{10} &= 2d - 5N,\\
x_{11}^N - y_{11} &= 2d - 6N + 16,\\
x_{12}^N - y_{12} &= d - 6N + 17,
\end{aligned}$$

so that $x_i^N - y_i \geqslant 0$ when

$$d \geqslant \max\{31, 5N-15, 2N-1, 2N+1, \frac{3N+8}{2}, 3N-9, \ldots, 3N-8, 6N-17\}.$$

Among these terms, 31 is maximal when $N \leqslant 8$ and $6N-17$ is maximal for $N \geqslant 8$, so that $x_i^N - y_i \geqslant 0$ for $d \geqslant \max\{31, 6N-17\}$. Moreover from (11) we have that

$$\min_{i \geqslant 3}\{x_i^N - y_i\} = \min_{3 \leqslant i \leqslant 12}\{x_i^N - y_i\}.$$

By direct computation we see that among the terms $x_i^N - y_i$ for $3 \leqslant i \leqslant 12$ listed above, $d - 2N - 1$ is minimal when $N \leqslant 4$ and $d - 6N + 17$ is minimal when $N \geqslant 5$. Thus

$$\min_{i \geqslant 3}\{x_i^N - y_i\} = \begin{cases} d - 2N - 1 & N \leqslant 4 \\ d - 6N + 17 & N \geqslant 5. \end{cases} \qquad \square$$

For fixed $d, n \geqslant 1$, write S^N to denote the set of partitions of n with parts in S_d^N so that $|S^N| = \rho(S_d^N; n)$. For $\lambda \in S^N$, let p_i denote the number of times x_i^N occurs as a part in λ, and define

$$\alpha = \alpha(\lambda) := \sum_{i \geqslant 3}(x_i^N - y_i)p_i. \tag{12}$$

The following lemma gives a lower bound on the number of parts that are equal to $x_2^N = d - N + 4$ for certain partitions $\lambda \in S^N$. It is imperative to our proof of Theorem 2.

Lemma 10. *Let $N \geqslant 2$, $d \geqslant \max\{31, 9N-13, 13N-31\}$, $n \geqslant 7d+14$, and $\lambda \in S^N$ such that $p_1 + \alpha < (N-2)p_2$. Then $p_2 \geqslant 8$.*

Proof. Suppose $p_2 \leqslant 7$. We first observe that if $\alpha \neq 0$, then there exists some $i \geqslant 3$ such that $p_i \neq 0$. By Lemma 9 and our bounds on d it follows that

$$\alpha \geqslant \min\{d - 2N - 1, d - 6N + 17\} \geqslant 7N - 14.$$

But then

$$p_1 + \alpha \geqslant 7N - 14 \geqslant (N-2)p_2,$$

which contradicts our hypothesis on p_1.

However, if $\alpha = 0$, then $p_i = 0$ for all $i \geqslant 3$, and $p_1 < 7N - 14$, so

$$n = p_1 + p_2(d - N + 4) < (7N - 14) + 7(d - N + 4) = 7d + 14,$$

which contradicts our hypothesis on n. Thus we must have $p_2 \geqslant 8$ as desired. $\square$

We conclude this section with a few results that will be used in Section 4. The first two are lemmas from work of Duncan et al. [5] which give key inequalities in our proof of Theorem 4.

Lemma 11 (Duncan et al. [5], 2021)**.** *Let $a, d \geqslant 1$, and let $n \geqslant d+2a$. Then*

$$q_d^{(a)}(n) \geqslant q^{(1)}_{\lceil \frac{d}{a} \rceil}\left(\left\lceil \frac{n}{a} \right\rceil\right).$$

Lemma 12 (Duncan et al. [5], 2021)**.** *Let $a, d, n \geqslant 1$ be such that $a \mid (d+3)$. Then*

$$Q_d^{(a,-)}(an) = Q^{(1,-)}_{\frac{d+3}{a}-3}(n).$$

Inagaki and Tamura [6] expanded Theorem 6 to allow for partitions of different integers, which enables us to prove another key inequality in our proof of Theorem 4.

Lemma 13 (Inagaki and Tamura [6])**.** *Let $a \geqslant 1$, and let $S = \{x_i\}_{i=1}^{\infty}$ and $T = \{y_i\}_{i=1}^{\infty}$ be two strictly increasing sequences of positive integers such that $y_1 = a$ and $a \mid y_i$, $x_i \geqslant y_i$ for all $i \geqslant 1$. Then for all $n \geqslant 1$,*

$$\rho(T; n + \hat{n}_a) \geqslant \rho(S; n),$$

where $\hat{n}_a$ denotes the least nonnegative integer such that $a \mid (n + \hat{n}_a)$.

3 Proof of Theorem 2

In this section, we modify the work of Inagaki and Tamura [6] and use results from Andrews [4] and Yee [12] to prove Theorem 2. As our primary method works only when $n \geqslant 7d+14$, we first consider the case when $d+2 \leqslant n \leqslant 7d+13$ below.

Lemma 14. *Let $N \geqslant 2$ and $d \geqslant \max\{63, 46N-79\}$. Then for all $d+2 \leqslant n \leqslant 7d+13$,*

$$q_d^{(1)}(n) \geqslant Q_{d-N}^{(1,-)}(n).$$

Proof. Observe that $q_d^{(1)}(n)$ and $Q_{d-N}^{(1,-)}(n)$ are both weakly increasing functions since every partition of n counted by $q_d^{(1)}(n)$ or $Q_{d-N}^{(1,-)}(n)$, respectively, injects to a partition of $n+1$ counted by $q_d^{(1)}(n+1)$ or $Q_{d-N}^{(1,-)}(n+1)$, respectively by adding 1 to the largest part or adding a part of size 1, respectively. Thus, if $q_d^{(1)}(k_1) \geqslant Q_{d-N}^{(1,-)}(k_2)$ for integers $k_1 \leqslant k_2$, it follows that $q_d^{(1)}(n) \geqslant Q_{d-N}^{(1,-)}(n)$ for all $k_1 \leqslant n \leqslant k_2$. By our hypotheses on d, it follows that $d+2 \leqslant 2d-2N+4$, $2d-2N+5 \leqslant 5d-5N+16$, and $5d-5N+17 \leqslant 7d+13$. Thus it suffices to prove the following three inequalities.

$$q_d^{(1)}(d+2) \geqslant Q_{d-N}^{(1,-)}(2d-2N+4), \tag{13}$$

$$q_d^{(1)}(2d-2N+5) \geqslant Q_{d-N}^{(1,-)}(5d-5N+16), \tag{14}$$

$$q_d^{(1)}(5d-5N+17) \geqslant Q_{d-N}^{(1,-)}(7d+13). \tag{15}$$

Note that the partition n itself is always counted by $q_d^{(1)}(n)$, and for any $1 \leqslant k \leqslant \left\lfloor \frac{n-d}{2} \right\rfloor$, the partition $(n-k)+k$ is counted by $q_d^{(1)}(n)$ since then $(n-k)-k \geqslant d$. Thus, for any $d, n \geqslant 1$,

$$q_d^{(1)}(n) \geqslant \max\left\{1, \left\lfloor \frac{n-d}{2} \right\rfloor + 1\right\}. \tag{16}$$

We first prove (13). Observe that any partition counted by $Q_{d-N}^{(1,-)}(2d-2N+4)$ can only use the parts $x_1^N = 1$ and $x_2^N = d-N+4$ since $x_3^N > 2d-2N+4$. There is exactly one such partition with largest part x_1^N, and one with largest part x_2^N. Thus $Q_{d-N}^{(1,-)}(2d-2N+4) = 2$. Using (16) we obtain that $q_d^{(1)}(d+2) \geqslant 2$ which gives (13).

We next prove (14). Since $x_{10}^N = 5d-5N+16$, any partition that is counted by $Q_{d-N}^{(1,-)}(5d-5N+16)$ can only use the parts x_i^N with $1 \leqslant i \leqslant 10$. Using the fact that $d \geqslant \max\{63, 46N-79\}$, one can calculate that the number of partitions of $5d-5N+16$ with largest part x_i^N as i ranges from 1 to 10 is 1, 4, 5, 6, 5, 3, 2, 1, 1, 1, respectively. Thus $Q_{d-N}^{(1,-)}(5d-5N+16) = 29$. Since $d \geqslant \max\{63, 46N-79\}$, it follows that $d-2N+5 \geqslant 56$, and thus (16) gives that

$$q_d^{(1)}(2d-2N+5) \geqslant \left\lfloor \frac{d-2N+5}{2} \right\rfloor + 1 \geqslant 29,$$

which yields (14).

We now prove (15). Since $d \geqslant \max\{63, 46N-79\}$, it follows that $x_{15}^N > 7d+13$. Thus any partition counted by $Q_{d-N}^{(1,-)}(7d+13)$ can only use the parts x_i^N with $1 \leqslant i \leqslant 14$. Using the fact that $d \geqslant \max\{63, 46N-79\}$, one can calculate that the number of partitions of $7d+13$ with largest part x_i^N as i ranges from 1 to 14, is at most[2] 1, 7, 12, 20, 16, 18, 10, 10, 5, 5, 2, 2, 1, 1, respectively. Thus $Q_{d-N}^{(1,-)}(7d+13) \leqslant 110$. Since $d \geqslant \max\{63, 46N-79\}$, it follows that $4d-5N+17 \geqslant 218$, and thus (16) gives that

$$q_d^{(1)}(5d-5N+17) \geqslant \left\lfloor \frac{4d-5N+17}{2} \right\rfloor + 1 \geqslant 110,$$

which yields (15). □

We now complete the proof of Theorem 2 with the following lemma.

Lemma 15. *Let $N \geqslant 2$ and $d \geqslant \max\{63, 46N-79\}$. Then for all $n \geqslant 7d+14$,*

$$q_d^{(1)}(n) \geqslant Q_{d-N}^{(1,-)}(n).$$

Proof. We first note that our bound on d allows us to apply Lemma 8, so we have the inequality $q_d^{(1)}(n) \geqslant \rho(T_{5,d}; n)$, and thus by (10) it suffices to show

$$\rho(T_{5,d}; n) \geqslant \rho(S_d^N; n). \tag{17}$$

Recall that for fixed d and n we write S^N to denote the set of partitions of n with parts in S_d^N, and for $\lambda \in S^N$, we let p_i denote the number of times x_i^N occurs as a part

[2]Some variance can occur for certain choices of d and N.

in λ. Furthermore write T to denote the set of partitions of n with parts in $T_{5,d}$, and for $\mu \in T$, let q_i denote the number of times y_i occurs as a part in μ. Then $|S^N| = \rho(S_d^N; n)$ and $|T| = \rho(T_{5,d}; n)$, so to prove (17), it suffices to construct an injection $\varphi^N : S^N \hookrightarrow T$.

We decompose S^N into the subsets

$$\begin{aligned} S_1^N &:= \{\lambda \in S^N \mid p_1 + \alpha \geqslant (N-2)p_2\}, \\ S_2^N &:= \{\lambda \in S^N \mid p_1 + \alpha < (N-2)p_2\}, \end{aligned} \tag{18}$$

and we further partition S_2^N for integers $\beta \geqslant 0$ by

$$S_{(2,\beta)}^N := \left\{\lambda \in S_2^N \mid \beta = \left\lfloor \frac{p_1 + p_5}{d - N - 1} \right\rfloor \right\}. \tag{19}$$

By inspection, it is clear that S^N is the disjoint union of the sets S_1^N and $S_{(2,\beta)}^N$ for all $\beta \geqslant 0$. Thus we can construct φ^N piecewise by constructing injections $\varphi_1^N : S_1^N \hookrightarrow T$ and $\varphi_{(2,\beta)}^N : S_{(2,\beta)}^N \hookrightarrow T$ for each $\beta \geqslant 0$ that have mutually disjoint images. To describe such maps, given $\lambda \in S^N$, we define its image in T by specifying the q_i associated to the image in terms of the p_i associated to λ. Also, recall by (12) that

$$\alpha = \alpha(\lambda) := \sum_{i \geqslant 3} (x_i^N - y_i) p_i.$$

Define $\varphi_1^N : S_1^N \to T$ by

$$q_i = \begin{cases} p_1 + \alpha - (N-2)p_2, & \text{if } i = 1 \\ p_i, & \text{if } i \geqslant 2. \end{cases}$$

We first show φ_1^N is well defined. Given $\lambda \in S_1^N$, we have by definition of S_1^N that $p_1 + \alpha \geqslant (N-2)p_2$. Thus each $q_i \geqslant 0$ so that $\varphi_1^N(\lambda)$ is indeed a partition into parts from $T_{5,d}$. Furthermore, we see that $\varphi_1^N(\lambda)$ is a partition of n, i.e., $\varphi_1^N(\lambda) \in T$, as

$$\begin{aligned} \sum_{i \geqslant 1} q_i y_i &= (p_1 + \alpha - (N-2)p_2) + p_2(d+2) + \sum_{i \geqslant 3} p_i y_i \\ &= p_1 + (d - N + 4)p_2 + \sum_{i \geqslant 3} p_i x_i^N = \sum_{i \geqslant 1} p_i x_i^N = n. \end{aligned} \tag{20}$$

To see that φ_1^N is injective, suppose $\lambda, \lambda' \in S_1^N$ such that $\varphi_1^N(\lambda) = \varphi_1^N(\lambda')$. Let p_i' and q_i' denote the number of times x_i^N and y_i occur in λ' and $\varphi_1^N(\lambda')$, respectively, and let $\alpha' = \sum_{i \geqslant 3} (x_i^N - y_i) p_i'$. Then $q_i = q_i'$ for all i implies that $p_i = p_i'$ for all $i \geqslant 2$ and $p_1 + \alpha - (N-2)p_2 = p_1' + \alpha' - (N-2)p_2'$. Since $p_i = p_i'$ for all $i \geqslant 2$ implies $\alpha = \alpha'$, we have $p_1 = p_1'$ and hence that $\lambda = \lambda'$. So $\varphi_1^N : S_1^N \hookrightarrow T$ as desired.

Next, for fixed $\beta \geqslant 0$, given $\lambda \in S_{(2,\beta)}^N$, let

$$\varepsilon = \varepsilon(\lambda) := \begin{cases} 0 & \text{if } p_2 \text{ is even,} \\ 1 & \text{if } p_2 \text{ is odd.} \end{cases}$$

Then define $\varphi^N_{(2,\beta)} : S^N_{(2,\beta)} \to T$ by

$$q_i = \begin{cases} p_1 + \alpha + \frac{(p_2+\varepsilon)(d-2N-8)}{2} + 28\beta + (26+N)\varepsilon, & \text{if } i = 1 \\ 2\beta + \varepsilon, & \text{if } i = 2 \\ p_5 + \frac{p_2+\varepsilon}{2} - 2\beta - 2\varepsilon, & \text{if } i = 5 \\ p_i, & \text{if } i \neq 1, 2, 5, \end{cases}$$

To see that $\varphi^N_{(2,\beta)}$ is well defined, we first observe that since $d \geqslant \max\{63, 46N - 79\}$, we have easily that $q_i \geqslant 0$ for all $i \neq 5$. To prove $q_5 \geqslant 0$, it suffices to show that $p_2 - 3\varepsilon \geqslant 4\beta$. By the definitions (19), (12), (18), as well as $d \geqslant \max\{63, 46N - 79\}$, it follows that

$$4\beta \leqslant 4\left(\frac{p_1 + p_5}{d - N - 1}\right) \leqslant 4\left(\frac{p_1 + \alpha}{d - N - 1}\right) < \frac{4(N-2)p_2}{d - N - 1} \leqslant \frac{p_2}{2}.$$

Moreover, the hypotheses of Lemma 10 are satisfied, so $p_2 \geqslant 8$. Thus,

$$4\beta < \frac{p_2}{2} = p_2 - \frac{p_2}{2} < p_2 - 3 \leqslant p_2 - 3\varepsilon.$$

Thus each $q_i \geqslant 0$ so that $\varphi^N_{(2,\beta)}(\lambda)$ is indeed a partition into parts from $T_{5,d}$. Furthermore, we see that $\varphi^N_{(2,\beta)}(\lambda)$ is a partition of n, i.e., $\varphi^N_{(2,\beta)}(\lambda) \in T$, as

$$\begin{aligned} \sum_{i \geqslant 1} q_i y_i = &\left(p_1 + \alpha + \frac{(p_2+\varepsilon)(d-2N-8)}{2} + 28\beta + (26+N)\varepsilon\right) + (2\beta+\varepsilon)(d+2) \\ &+ \left(p_5 + \frac{p_2+\varepsilon}{2} - 2\beta - 2\varepsilon\right)(d+16) + \sum_{i \neq 1,2,5} p_i y_i \\ = &\, p_1 + \frac{(p_2+\varepsilon)(2d-2N+8)}{2} + (-d+N-4)\varepsilon + \sum_{i \geqslant 3} p_i x_i^N \\ = &\, p_1 + p_2(d-N+4) + \sum_{i \geqslant 3} p_i x_i^N = \sum_{i \geqslant 1} p_i x_i^N = n. \end{aligned}$$

To see that $\varphi^N_{(2,\beta)}$ is injective, suppose $\lambda, \lambda' \in S^N_{(2,\beta)}$ such that $\varphi^N_{(2,\beta)}(\lambda) = \varphi^N_{(2,\beta)}(\lambda')$. As in the previous case, let p'_i and q'_i denote the number of times x_i^N and y_i occur in λ' and $\varphi^N_{(2,\beta)}(\lambda')$, respectively, $\alpha' = \sum_{i \geqslant 3}(x_i^N - y_i)p'_i$, and also let ε' denote the residue of p'_2 modulo 2. Then $q_i = q'_i$ for all i implies that $p_i = p'_i$ for all $i \neq 1, 2, 5$ and $\varepsilon = \varepsilon'$. From $q_1 = q'_1$ and $q_5 = q'_5$, we obtain that

$$p_1 + (2d - 3N - 8)p_5 + \frac{p_2(d-2N-8)}{2} = p'_1 + (2d - 3N - 8)p'_5 + \frac{p'_2(d-2N-8)}{2}, \quad (21)$$

$$p_5 + \frac{p_2}{2} = p'_5 + \frac{p'_2}{2}. \quad (22)$$

Multiplying (22) by $(d - 2N - 8)$ and subtracting this from (21) gives

$$p_1 + (d-N)p_5 = p'_1 + (d-N)p'_5. \quad (23)$$

From (19), we see that $p_1 + p_5 = \beta(d-N-1) + m$ and $p_1' + p_5' = \beta(d-N-1) + m'$, where $0 \leqslant m, m' < d-N-1$. Thus subtracting yields

$$(p_1 - p_1') + (p_5 - p_5') = m - m'. \tag{24}$$

Combining (24) and (23) gives

$$m' - m = (d-N-1)(p_5' - p_5). \tag{25}$$

Since $0 \leqslant m, m' < d-N-1$, (25) implies that $m = m'$ and thus $p_5 = p_5'$. Thus from (22) it follows that $p_2 = p_2'$, so (21) yields that $p_1 = p_1'$, and hence $\lambda = \lambda'$. So $\varphi^N_{(2,\beta)} : S^N_{(2,\beta)} \hookrightarrow T$ as desired.

It remains to show that the images of all of the φ^N_1 and $\varphi^N_{(2,\beta)}$ are distinct. First observe that if $\beta \neq \beta'$, $\lambda \in S^N_{(2,\beta)}$, and $\lambda' \in S^N_{(2,\beta')}$, then $\varphi^N_{(2,\beta)}(\lambda) \neq \varphi^N_{(2,\beta')}(\lambda')$ since $q_2 \neq q_2'$.

Now fix $\beta \geqslant 0$, and suppose toward contradiction that $\lambda \in S^N_{(2,\beta)}$ and $\lambda' \in S^N_1$ such that $\varphi^N_{(2,\beta)}(\lambda) = \varphi^N_1(\lambda')$. Then $q_i = q_i'$ for all i immediately gives that $p_i = p_i'$ for all $i \neq 1,2,5$ and

$$\begin{aligned} p_1' + \alpha' - (N-2)p_2' &= p_1 + \alpha + \frac{(p_2+\varepsilon)(d-2N-8)}{2} + 28\beta + (26+N)\varepsilon, \\ p_2' &= 2\beta + \varepsilon, \\ p_5' &= p_5 + \frac{p_2+\varepsilon}{2} - 2\beta - 2\varepsilon, \end{aligned}$$

which yield that

$$\begin{aligned} &p_1' + (2d-3N-8)p_5' = \\ &\qquad p_1 + (2d-3N-8)p_5 + \frac{(p_2+\varepsilon)(d-2N-8)}{2} + (2N+24)(\beta+\varepsilon), \end{aligned} \tag{26}$$

$$p_5' = p_5 + \frac{p_2+\varepsilon}{2} - 2(\beta+\varepsilon). \tag{27}$$

Multiplying (27) by $(2d-3N-8)$ and subtracting this from (26) gives

$$p_1' = p_1 + \frac{(p_2+\varepsilon)(N-d)}{2} + (4d-4N+8)\beta + (4d-4N+8)\varepsilon. \tag{28}$$

From (18) and (19) we have

$$\begin{aligned} p_1 &\leqslant p_1 + \alpha < (N-2)p_2, \\ \beta &\leqslant \frac{p_1+p_5}{d-N-1} \leqslant \frac{p_1+\alpha}{d-N-1} < \frac{(N-2)p_2}{d-N-1}. \end{aligned} \tag{29}$$

Thus, (28) and (29) yield that

$$p_1' < (N-2)p_2 + \frac{(p_2+\varepsilon)(N-d)}{2} + (4d-4N+8)\left(\frac{(N-2)p_2}{d-N-1}+\varepsilon\right)$$
$$= \frac{(-d^2+(12N-19)d-11N^2+33N-28)p_2}{2d-2N-2}$$
$$+ \frac{(7d^2+d(-14N+9)+7N^2-9N-16)\varepsilon}{2d-2N-2}. \quad (30)$$

Since the hypotheses of Lemma 10 are satisfied, we have that $p_2 \geqslant 8$. If $p_2 = 8$, then $\varepsilon = 0$ and (30) becomes

$$p_1' < \frac{-4d^2+(48N-76)d-44N^2+132N-112}{d-N-1}.$$

Since $d \geqslant \max\{63, 46N-79\}$, the denominator above is always positive. But when $d > \frac{12N-19+\sqrt{100N^2-324N+249}}{2}$, the numerator is negative, which would yield a contradiction since $p_1' \geqslant 0$. Since $100N^2-324N+249 < (10N-16)^2$, it thus suffices to show that $d \geqslant 11N-17$, which follows easily from the fact that $d \geqslant \max\{63, 46N-79\}$. Thus we have a contradiction in the case when $p_2 = 8$.

Suppose $p_2 \geqslant 9$. Since $d \geqslant \max\{63, 46N-79\}$, for all $N \geqslant 2$ we have

$$-d^2+d(12N-19)-11N^2+33N-28 \leqslant 0,$$
$$7d^2+d(-14N+9)+7N^2-9N-16 \geqslant 0.$$

Thus (30) yields that

$$p_1' \leqslant \frac{-d^2+(47N-81)d-46N^2+144N-134}{d-N-1}.$$

As above, when $d > \frac{47N-81+\sqrt{2025N^2-7038N+6025}}{2}$ the right hand side is negative which contradicts the nonnegativity of p_1'. Since $2025N^2-7038N+6025 < (45N-78)^2$, it suffices to show that $d \geqslant 46N-79$, which is immediate from our bound $d \geqslant \max\{63, 46N-79\}$. Thus we have a contradiction in the case when $p_2 \geqslant 9$, and have shown $\varphi^N_{(2,\beta)}(\lambda) \neq \varphi^N_1(\lambda')$ for any $\lambda \in S^N_{(2,\beta)}$ and $\lambda' \in S^N_1$.

Thus considered together, φ^N_1 and $\varphi^N_{(2,\beta)}$ for each $\beta \geqslant 0$ form a piecewise injective map $\varphi^N : S^N \hookrightarrow T$, which gives our desired inequality. □

4 Proof of Theorem 4 and Corollary 5

We now demonstrate that the methods of Inagaki and Tamura [6] together with Corollary 3 yield the generalized Kang-Park type result given in Theorem 4.

Proof of Theorem 4. We first suppose that $n \geqslant d+2a$. Write $\hat{n}_a$ and $\hat{d}_a$ to denote the least nonnegative residue of $-n$ and $-d$ modulo a, respectively, so that $\lceil \frac{n}{a} \rceil = \frac{n+\hat{n}_a}{a}$ and $\lceil \frac{d}{a} \rceil = \frac{d+\hat{d}_a}{a}$. Then using Lemma 11, Corollary 3, and Lemma 12, we obtain

$$q_d^{(a)}(n) \geqslant q^{(1)}_{\frac{d+\hat{d}_a}{a}}\left(\frac{n+\hat{n}_a}{a}\right) \geqslant Q^{(1,-)}_{\frac{d+\hat{d}_a}{a}-4}\left(\frac{n+\hat{n}_a}{a}\right) = Q^{(a,-)}_{d+\hat{d}_a-a-3}(n+\hat{n}_a).$$

Thus it remains to show that

$$Q^{(a,-)}_{d+\hat{d}_a-a-3}(n+\hat{n}_a) \geqslant Q_d^{(a,-)}(n). \tag{31}$$

Define

$$\begin{aligned} S &:= \{x \in \mathbb{N} \mid x \equiv \pm a \pmod{d+3}\} \setminus \{d+3-a\}, \\ T &:= \{x \in \mathbb{N} \mid x \equiv \pm a \pmod{d+\hat{d}_a-a}\} \setminus \{d+\hat{d}_a-2a\}, \end{aligned}$$

and observe that $Q_d^{(a,-)}(n) = \rho(S;n)$ and $Q^{(a,-)}_{d+\hat{d}_a-a-3}(n+\hat{n}_a) = \rho(T;n+\hat{n}_a)$. Letting x_i and y_i denote the i^{th} smallest elements of S and T, respectively, we have that $x_1 = y_1 = a$, and

$$\begin{aligned} x_{2i} &= i(d+3)+a, & y_{2i} &= i(d+\hat{d}_a-a)+a, & &\text{for } i \geqslant 1, \\ x_{2i-1} &= i(d+3)-a, & y_{2i-1} &= i(d+\hat{d}_a-a)-a, & &\text{for } i \geqslant 2. \end{aligned}$$

Clearly $a \mid y_i$ for all $i \geqslant 1$, and moreover, $x_i \geqslant y_i$ for all $i \geqslant 1$ since $0 \leqslant \hat{d}_a < a$. Thus by Lemma 13, we have (31) as desired.

We now consider $1 \leqslant n \leqslant d+2a-1$. As in the proof of Lemma 14, we observe that $q_d^{(a)}(n)$ is a weakly increasing function, however $Q_d^{(a,-)}(n)$ is not.

If $1 \leqslant n \leqslant a-1$, then $q_d^{(a)}(n) = 0 = Q_d^{(a,-)}(n)$. Also, $q_d^{(a)}(a) = 1$ and $Q_d^{(a,-)}(n) \leqslant 1$ for all $a \leqslant n \leqslant d+a+2$ since a is the only available part. Thus it remains to consider when $d+a+3 \leqslant n \leqslant d+2a-1$, which only occurs for $a \geqslant 4$.

By our hypothesis that $\lceil \frac{d}{a} \rceil \geqslant 105$, it follows that $d+2a-1 < 2d-a+6$. Thus the only available parts for a partition counted by $Q_d^{(a,-)}(n)$ when $d+a+3 \leqslant n \leqslant d+2a-1$ are a and $d+a+3$. Furthermore, the part $d+a+3$ can occur at most once since $2d+2a+6 > d+2a-1$. Thus a partition counted by $Q_d^{(a,-)}(n)$ when $d+a+3 \leqslant n \leqslant d+2a-1$ is either a sum of parts of size a, which can only occur when $n \equiv 0 \pmod{a}$, or $d+a+3$ plus a sum of parts of size a, which can only occur when $n \equiv d+3 \pmod{a}$. Thus $Q_d^{(a,-)}(n) \leqslant 1 \leqslant q_d^{(a)}(n)$ except when $d \equiv -3 \pmod{a}$ and $n \equiv 0 \pmod{a}$ simultaneously. But if $d = ka-3$ for $k \geqslant 1$, then $(k+1)a \leqslant n \leqslant (k+2)a-4$, so the only exception occurs when $n = d+a+3$. $\square$

We now prove Corollary 5.

Proof of Corollary 5. By definition, $\Delta_d^{(a,-,-)}(n) \geqslant \Delta_d^{(a,-)}(n)$, since there are fewer parts available for partitions counted by $\Delta_d^{(a,-,-)}(n)$. Thus by Theorem 4, it follows that $\Delta_d^{(a,-,-)}(n) \geqslant 0$ for any $a, d \geqslant 1$ such that $\lceil \frac{d}{a} \rceil \geqslant 105$ and $n \geqslant 1$, except possibly when $d \equiv -3 \pmod{a}$ and $n = d+a+3$. However in these cases, observe that

$Q_d^{(a,-,-)}(d+a+3) = 1$, since $d+a+3$ is the only available part by definition. Also, $q_d^{(a)}(d+a+3) \geqslant 1$ since $d+a+3$ is a partition counted by $q_d^{(a)}(d+a+3)$. Thus $q_d^{(a)}(n) \geqslant Q_d^{(a,-,-)}(n)$ in all of our considered cases. □

5 Concluding Remarks

By work of Kang and Kim[3] [7, Thm. 1.1] and the fact that $Q_d^{(a)}(n) \geqslant Q_d^{(a,-)}(n)$, it follows that when $\gcd(a, d-N) = 1$,

$$\lim_{n\to\infty} (q_d^{(a)}(n) - Q_{d-N}^{(a,-)}(n)) = \infty,$$

for all $N < d+3-\lfloor \frac{\pi^2}{3A_d} \rfloor$, where $A_d = \frac{d}{2}\log^2 \alpha_d + \sum_{r=1}^{\infty} r^{-2}\alpha_d^{rd}$, with α_d the unique real root of $x^d + x - 1$ in the interval $(0,1)$. Thus it may be possible to generalize Theorem 2 to an inequality of the form $q_d^{(a)}(n) \geqslant Q_{d-N}^{(a,-)}(n)$ for more general a.

Acknowledgements

We thank Ryota Inagaki and Ryan Tamura for their helpful correspondence and interesting idea.

References

[1] Henry L. Alder. The nonexistence of certain identities in the theory of partitions and compositions. *Bulletin of the American Mathematical Society*, 54(8):712 – 722, 1948.

[2] Henry L. Alder. Research problems, no. 4. *Bull. Amer. Math. Soc.*, 62(1):76, 1956.

[3] Claudia Alfes, Marie Jameson, and Robert J. Lemke Oliver. Proof of the Alder-Andrews conjecture. *Proceedings of the American Mathematical Society*, 139(1):63–78, 2011.

[4] George E. Andrews. *The theory of partitions.* Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. Encyclopedia of Mathematics and its Applications, Vol. 2.

[5] Adriana L. Duncan, Simran Khunger, Holly Swisher, and Ryan Tamura. Generalizations of Alder's conjecture via a conjecture of Kang and Park. *Res. Number Theory*, 7(1):Paper No. 11, 26, 2021.

[6] Ryota Inagaki and Ryan Tamura. On generalizations of a conjecture of Kang and Park. arXiv:2206.04842, 2022.

[7] Soon-Yi Kang and Young Kim. Bounds for d-distinct partitions. *Hardy-Ramanujan Journal*, Volume 43 - Special Commemorative volume in honour of Srinivasa Ramanujan - 2020, May 2021.

[3]Note that Kang and Kim use different notation that what we are using here.

[8] Soon-Yi Kang and Eun Young Park. An analogue of Alder-Andrews conjecture generalizing the 2nd Rogers-Ramanujan identity. *Discrete Mathematics*, 343(7):111882, 2020.

[9] D. H. Lehmer. Two nonexistence theorems on partitions. *Bull. Amer. Math. Soc.*, 52:538–544, 1946.

[10] Issai Schur. *Gesammelte Abhandlungen. Band III.* Springer-Verlag, Berlin-New York, 1973. reprint containing original reference published in 1926.

[11] Ae Ja Yee. Partitions with difference conditions and Alder's conjecture. *Proceedings of the National Academy of Sciences - PNAS*, 101(47):16417–16418, 2004.

[12] Ae Ja Yee. Alder's conjecture. *J. Reine Angew. Math.*, 616:67–88, 2008.