

Induced Subgraphs and Path Decompositions

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Abstract

A graph H is an induced subgraph of a graph G if a graph isomorphic to H can be obtained from G by deleting vertices. Recently, there has been significant interest in understanding the unavoidable induced subgraphs for graphs of large treewidth. Motivated by this work, we consider the analogous problem for pathwidth: what are the unavoidable induced subgraphs for graphs of large pathwidth? While resolving this question in the general setting looks challenging, we prove various results for sparse graphs. In particular, we show that every graph with bounded maximum degree and sufficiently large pathwidth contains a subdivision of a large complete binary tree or the line graph of a subdivision of a large complete binary tree as an induced subgraph. Similarly, we show that every graph excluding a fixed minor and with sufficiently large pathwidth contains a subdivision of a large complete binary tree or the line graph of a subdivision of a large complete binary tree as an induced subgraph. Finally, we present a characterisation for when a hereditary class defined by a finite set of forbidden induced subgraphs has bounded pathwidth.

Mathematics Subject Classifications: 05C83, 05C76

1 Introduction

Pathwidth and treewidth are fundamental parameters in structural and algorithmic graph theory [11, 18, 25]. A paramount theme has been to understand the substructures of graphs with large pathwidth or large treewidth. Under the graph minor and subgraph relations, these substructures are well understood. For pathwidth, the excluded forest minor theorem [26] implies that graphs with sufficiently large pathwidth contain a subdivision of a large complete binary tree as a subgraph. For treewidth, the excluded grid minor

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theorem [27] implies that graphs with sufficiently large treewidth contain a subdivision of a large wall as a subgraph.

In recent years, there has been substantial interest in understanding the substructures for graphs with large treewidth with respect to induced subgraphs [1–8, 13, 19, 21–23, 29]. For example, it was recently shown that graphs with bounded degree and sufficiently large treewidth contain a subdivision of a large wall or the line graph of a subdivision of a large wall as an induced subgraph [21]. Motivated by this work, we consider the analogous problem for pathwidth: what are the unavoidable induced subgraphs for graphs with large pathwidth? Due to the excluded forest minor theorem [26], obvious candidates are subdivisions of large complete binary trees and line graphs¹ of subdivisions of large complete binary trees. While determining the other candidates in the general setting looks challenging (see Section 4 for a discussion on this), we show that these graphs suffice in the bounded degree setting (Section 3.2) as well as for K_n -minor-free graphs (Section 3.3). Let T_k denote the complete binary tree of height k .

Theorem 1. *There is a function f such that every graph G with maximum degree Δ and pathwidth at least $f(k, \Delta)$ contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph.*

Theorem 2. *For every fixed $n \in \mathbb{N}$, there is a function f such that every K_n -minor-free graph G with pathwidth at least $f(k)$ contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph.*

In addition, we characterise when a hereditary graph class defined by a finite set of forbidden induced subgraphs has bounded pathwidth. Let \mathcal{G}_S be the class of graphs that contain no graph in S as an induced subgraph. We call $K_{1,3}$ a *claw*. A *fork* is either a subdivision of a claw or a path, and a *semi-fork* is the line graph of a fork. A *tripod* is a forest where each component is a fork and a *semi-tripod* is a graph where each component is a semi-fork. We prove the following characterisation for when \mathcal{G}_S has bounded pathwidth.

Theorem 3. *\mathcal{G}_S has bounded pathwidth if and only if S includes a complete graph, a complete bipartite graph, a tripod and a semi-tripod.*

See [24] for other results concerning induced subgraphs and path decompositions.

2 Preliminaries

For undefined terms and notations, see the textbook by Diestel [16]. A *class* of graphs \mathcal{G} is a set of graphs that is closed under isomorphism. \mathcal{G} is *hereditary* if it is closed under induced subgraphs. For a graph G and vertex $v \in V(G)$, let $N(v)$ denote the set of vertices in $V(G)$ adjacent to v and let $N[v]$ denote $N(v) \cup \{v\}$.

¹The *line graph* $L(G)$ of a graph G has $V(L(G)) = E(G)$ where two vertices in $L(G)$ are adjacent if they are incident to a common vertex in G .

Let G and H be graphs and let $r \geq 0$ be an integer. H is a *minor* of G if a graph isomorphic to H can be obtained from G by vertex deletion, edge deletion, and edge contraction. If H is isomorphic to a graph obtained from G by vertex deletion and edge contraction, then H is an *induced minor* of G . A (*minor*) *model* $(X_v : v \in V(H))$ of H in G is a collection of non-empty subsets of $V(G)$ such that (i) $G[X_v]$ is a connected subgraph of G ; (ii) $X_u \cap X_v = \emptyset$ for all distinct $u, v \in V(H)$; and (iii) $G[X_u \cup X_v]$ is connected for every edge $uv \in E(H)$. If, in addition, $G[X_u \cup X_v]$ is disconnected whenever $uv \notin E(H)$, then $(X_v : v \in V(H))$ is an *induced minor model* of H . It is folklore that H is a minor of G if and only if G contains a model of H and that H is an induced minor of G if and only if G contains an induced minor model of H . If there exists a model $(X_v : v \in V(H))$ of H in G such that $G[X_v]$ has radius at most r for all $v \in V(H)$, then H is an *r -shallow minor* of G . A graph G' is a *subdivision* of G if G' can be obtained from G by replacing each edge uv of G by a path of length at least one with end-vertices u and v whose internal vertices are new vertices private to that path. We call the vertices $V(G) \subseteq V(G')$ the *original vertices* in G' .

For a tree T , a *T -decomposition* of a graph G is a collection $\mathcal{W} = (W_x : x \in V(T))$ of subsets of $V(G)$ indexed by the nodes of T such that (i) for every edge $vw \in E(G)$, there exists a node $x \in V(T)$ with $v, w \in W_x$; and (ii) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in W_x\}$ induces a subtree of T . The *width* of \mathcal{W} is $\max\{|W_x| : x \in V(T)\} - 1$. A *tree-decomposition* is a T -decomposition for any tree T . A *path-decomposition* is a P -decomposition for any path P . The *treewidth* $\text{tw}(G)$ of a graph G is the minimum width of a tree-decomposition of G and the *pathwidth* $\text{pw}(G)$ of a graph G is the minimum width of a path-decomposition of G . Clearly $\text{tw}(G) \leq \text{pw}(G)$ for all graphs G . Treewidth and pathwidth are the standard measures of how similar a graph is to a tree and to a path respectively. A fundamental result for pathwidth is the excluded forest minor theorem.

Theorem 4 ([10, 26]). *For every forest F , every graph with pathwidth at least $|V(F)| - 1$ contains F as a minor.*

Note that Theorem 4 implies that every graph with pathwidth at least $|V(T_k)| - 1$ contains a subdivision of T_k as a subgraph since T_k has maximum degree 3.

3 Results

3.1 From Induced Minors to Induced Subgraphs

To prove Theorems 1 and 2, we first construct an induced minor of a large complete binary tree in our graph. We then use the following lemma to find our desired induced subgraphs.

Lemma 5. *Every graph G that contains T_{8k} as an induced minor contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph.*

The rest of this subsection is dedicated to proving Lemma 5. A *net-graph* is a semi-fork obtained from a triangle by appending disjoint paths of length 1 at each vertex of the triangle. For a graph G and a vertex $v \in V(G)$ with degree 3 and neighbours a, b, c , a *net graph replacement* at v is the graph G' , with $V(G') = V(G) \setminus \{v\} \cup \{x, y, z\}$ where x, y, z are new vertices and $E(H) = E(G - v) \cup \{xy, yz, zx, xa, yb, zc\}$. A *wattle* \tilde{T}_k is obtained from a subdivision of T_k by picking a (possibly empty) subset X of the degree 3-vertices and performing net graph replacements at each vertex in X . The following lemma from Aboulker et al. [1] will be useful in constructing an induced wattle \tilde{T}_k from a large induced minor.

Lemma 6 ([1]). *Let G be a connected graph whose vertex set is partitioned into connected sets $A, B, C, \{a\}, \{b\}, \{c\}$ and S . Suppose that every edge of G has either both ends in one of the sets, or is from $\{a\}$ to A , from $\{b\}$ to B , from $\{c\}$ to C , or from S to $A \cup B \cup C$. Then a, b, c are the degree one vertices of some induced fork or semi-fork in G .*

For a rooted tree (T, r) , a vertex $u \in V(T)$ is an *ancestor* of $v \in V(T)$ (and v is a *descendant* of u) if u is a vertex on the (v, r) -path in T . If $p_v \in V(T)$ is the neighbour of v on the (v, r) -path in T , then p_v is the *parent* of v (and v is a *child* of p_v). For a rooted complete binary tree (T_k, r) , every vertex $u \in V(T_k)$ with degree at least 2 has a *left child* ℓ_v and a *right child* c_v . A vertex $v \in V(T_k)$ is a *left (right) descendant* of u if it is a descendant of the left (right) child of u .

Lemma 7. *If a graph G contains T_{4k} as an induced minor, then G contains a wattle \tilde{T}_k as an induced subgraph.*

Proof. Let $(X_v : v \in V(T_{4k}))$ be an induced minor model of (T_{4k}, r) in G . Let V_0, V_1, \dots, V_{4k} be a BFS-layering of T_{4k} where $V_0 = \{r\}$.² For a vertex $v \in V(T_{4k})$ with degree 3, let $P_v := (w_{v,0}, w_{v,1}, \dots, w_{v,m_v})$ be a vertex-minimal path in $G[X_v]$ such that $N_G(w_{v,0}) \cap X_{p_v}$ and $N_G(w_{v,m_v}) \cap X_{\ell_v}$ are non-empty. By minimality, $w_{v,0}$ is the only vertex in $V(P_v)$ adjacent to vertices in X_{p_v} and w_{v,m_v} is the only vertex in $V(P_v)$ adjacent to vertices in X_{ℓ_v} .

We prove the following claim by induction on $k \geq 0$: If a graph G contains T_{4k} as an induced minor, then G contains a wattle \tilde{T}_k as an induced subgraph whose leaves are contained in $\{w_{v,0} : v \in V_{4k}\}$ and every vertex in $V(\tilde{T}_k) \cap (\bigcup(X_v : v \in V_{4k}))$ is a leaf.

For $k = 0$, the claim holds trivially by letting $V(\tilde{T}_k)$ be an arbitrary vertex in X_r . For $k = 1$, let $a, b \in V_4$ respectively be a left and right descendant of r in T_{4k} . By taking \tilde{T}_k to be a vertex minimal $(w_{a,0}, w_{b,0})$ -path whose internal vertices are contained in $\bigcup(X_v : v \in V_1 \cup V_2 \cup V_3)$, we are done.

Now suppose the claim holds for $k - 1$. By induction, T_{4k-4} contains a wattle \tilde{T}_{k-1} as an induced subgraph whose leaves are contained in $\{w_{v,0} : v \in V_{4k-4}\}$ and every vertex in $V(\tilde{T}_{k-1}) \cap (\bigcup(X_v : v \in V_{4k-4}))$ is a leaf. Now consider a leaf $w_{v,0} \in V(\tilde{T}_{k-1})$. First append the path P_v to \tilde{T}_{k-1} . Let $s \in V_{4k-2}$ be a left descendant of v and let $b, c \in V_{4k}$

² V_0, V_1, \dots , is a *BFS-layering* of a graph G if $V_0 = \{r\}$ for some $r \in V(G)$ and $V_i = \{v \in V(G) : \text{dist}_G(v, r) = i\}$ for all $i \geq 1$.

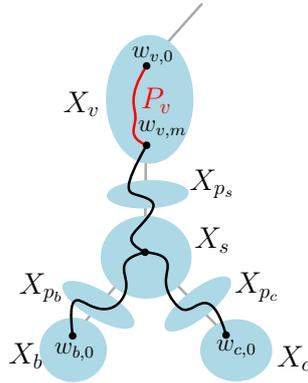


Figure 1: Extending the wattle \tilde{T}_k .

respectively be left and right descendants of s . Let $A = X_{p_s}$, $B = X_{p_b}$, $C = X_{p_c}$ and $S = X_s$. Since $(X_v: v \in V(T_{4k}))$ is an induced minor, we may apply Lemma 6 on $G[\{w_{v,m_v}, w_{b,0}, w_{c,0}\} \cup A \cup B \cup C \cup S]$ to obtain an induced fork or semi-fork with end points $w_{v,m_v}, w_{b,0}, w_{c,0}$. Add this induced fork or semi-fork to \tilde{T}_{k-1} and repeat for all leaves in \tilde{T}_{k-1} to obtain an induced wattle \tilde{T}_k that satisfies the induction hypothesis (see Figure 1). \square

For a rooted tree (T, r) , we say that a rooted subtree (T', r') is *vertical* if r' is an ancestor (with respect to (T, r)) of every vertex in $V(T')$. We use the following lemma to clean up our wattle \tilde{T}_k .

Lemma 8. *For every red-blue colouring of (T_{2k}, r) , there exists a subdivision of a vertical (T_k, r') whose original vertices are monochromatic.*

Proof. We prove the following by induction on k : for every red-blue colouring of (T_k, r) , there exists a subdivision of a vertical (T_h, r') whose original vertices are red and a subdivision of a vertical (T_j, r'') whose original vertices are blue such that $h + j \geq k$.

For $k = 0$, the claim is trivial. Now suppose the claim holds for $k - 1$. Let (T_{k-1}^1, r^1) and (T_{k-2}^2, r^2) be the components of $T_k - r$. By induction, for each $i \in \{1, 2\}$ there exists a subdivision of a vertical (T_{h_i}, r'_i) in (T_{k-1}^i, r^i) whose original vertices are red and a subdivision of a vertical (T_{j_i}, r''_i) in (T_{k-1}^i, r^i) whose original vertices are blue such that $h_i + j_i \geq k - 1$. If $\max\{h_1, h_2\} + \max\{j_1, j_2\} \geq k$ then we are done. Otherwise, $h_1 = h_2$, $j_1 = j_2$ and $h_1 + j_1 = k - 1$. If r is coloured red, then together with the (r'_1, r'_2) -path in T_k (which goes through r), we have a vertical red (T_{h_1+1}, r) . If r is coloured blue, then together with the (r''_1, r''_2) -path in T_k , we have a vertical blue (T_{j_1+1}, r) , as required. \square

Lemma 9. *Every wattle \tilde{T}_{2k} contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph.*

Proof. Let T'_{2k} be an auxiliary copy of T_{2k} obtained from the wattle \tilde{T}_{2k} by first contracting each triangle into a red vertex then contracting each subdivided path to an edge and colouring the remaining vertices blue. By Lemma 8, T'_{2k} contains a subdivision T'_k of T_k

whose original vertices are monochromatic. If the original vertices of T'_k are red, then the wattle \tilde{T}_{2k} contains the line graph of a subdivision of T_k as an induced subgraph (where each triangle in the line graph corresponds to an original red vertex in T'_k). Otherwise, the original vertices of T'_k are blue, and thus the wattle \tilde{T}_{2k} contains a subdivision of T_k as an induced subgraph (where the original vertices correspond to the original blue vertices in T'_k). \square

Lemma 5 immediately follows from Lemmas 7 and 9.

3.2 From Bounded Degree to Induced Minors

We now show that graphs with bounded degree and sufficiently large pathwidth contain a large complete binary tree as an induced minor. Theorem 1 immediately follows from the next theorem together with Lemma 5.

Theorem 10. *There is a function f such that every graph with maximum degree Δ and pathwidth at least $f(k, \Delta)$ contains T_k as an induced minor.*

To prove Theorem 10, we use sparsifiable graphs which is a new technique introduced by Korhonen [21]. For a graph G , a vertex $v \in V(G)$ is *sparsifiable* if it satisfies one of the following conditions:

1. v has degree at most 2;
2. v has degree 3 and all of its neighbours have degree at most 2;
3. v has degree 3, one of its neighbours has degree at most 2, and the two other neighbours form a triangle with v .

We say that G is *sparsifiable* if all of its vertices are sparsifiable. Such graphs are useful since minors and induced minors are roughly equivalent in this setting. More precisely, Korhonen [21] showed that if a sparsifiable graph G contains a graph H with minimum degree at least 3 as a minor, then G contains H as an induced minor. We prove a slightly stronger version of this result that relaxes the minimum degree condition for H .

Lemma 11. *Let H and H^+ be graphs such that H is an induced subgraph of H^+ and each vertex in $V(H)$ has degree at least 3 in H^+ . If a sparsifiable graph G contains H^+ as a minor, then G contains H as an induced minor.*

Proof. Since G is sparsifiable, it has maximum degree at most 3. For a model $(X_v : v \in V(H^+))$ of H^+ in G , we say that an edge $ab \in E(G)$ is *H -violating* if there are vertices $u, v \in V(H)$ such that $a \in X_u$, $b \in X_v$ and $uv \notin E(H)$. Choose $(X_v : v \in V(H^+))$ such that the number of H -violating edges is minimised. We claim that there are no H -violating edges, which implies that $(X_v : v \in V(H))$ is an induced minor of H in G .

For the sake of contradiction, suppose $ab \in E(G)$ is an H -violating edge with $a \in X_u$ and $b \in X_v$. Since a has degree at most 3 in G , u has degree at least 3 in H^+ and $uv \notin E(H^+)$, it follows that a has a neighbour in X_u . Likewise, b has a neighbour in X_v . Now if a has degree at most 2 in G , then $G[X_u \setminus \{a\}]$ is connected, so we may replace X_u by $X_u \setminus \{a\}$ to obtain a model of H^+ with strictly less H -violating edges, a contradiction. Thus a must have degree 3 in G and b likewise must also have degree 3 in G .

Now since G is sparsifiable and a and b are adjacent with degree 3, they have a common neighbour c in G . If $c \notin \bigcup(X_w : w \in N_{H^+}[u])$, then $G[X_u \setminus \{a\}]$ is connected, and so we may replace X_u by $X_u \setminus \{a\}$ to obtain a model of H^+ with strictly less H -violating edges, a contradiction. By symmetry, a contradiction also occurs if $c \notin \bigcup(X_w : w \in N_{H^+}[v])$. Since $uv \notin E(H^+)$ it remains to consider the case when there exists $w \in V(H^+) \setminus \{u, v\}$ such that $c \in X_w$ and $uw, vw \in E(H^+)$. Since $w \notin \{u, v\}$, it follows that $G[X_u \setminus \{a\}]$ is connected and contains a neighbour of a and $G[X_v \setminus \{b\}]$ is connected and contains a neighbour of b . As such, by replacing X_u by $X_u \setminus \{a\}$, X_v by $X_v \setminus \{b\}$ and X_w by $X_w \cup \{a, b\}$, we obtain a model of H^+ with strictly less H -violating edges, a contradiction. \square

A *distance-5 independent set* $\mathcal{I} \subseteq V(G)$ in a graph G is a set of vertices such that the distance between any pair of vertices in \mathcal{I} is at least 5. For a distance-5 independent set \mathcal{I} in a graph G , let $\mathcal{I}(G)$ be the 2-shallow minor of G obtained by contracting each of the balls of radius 2 that are centred at vertices in \mathcal{I} with corresponding model $(X_v : v \in V(\mathcal{I}(G)))$. Observe that if G has maximum degree Δ , then $|X_v| \leq \Delta^2 + 1$ for all $v \in V(\mathcal{I}(G))$ and so $\text{pw}(\mathcal{I}(G)) \geq (\text{pw}(G) + 1)/(\Delta^2 + 1) - 1$ (see Theorem 21 in [12] for an implicit proof). We use the following lemma implicitly proved by Korhonen [21].

Lemma 12 ([21]). *Let G be a graph and $\mathcal{I} \subseteq V(G)$ be a distance-5 independent set. Then for any subgraph H' of $\mathcal{I}(G)$ with maximum degree 3, there exists an induced subgraph $G[S]$ of G such that $G[S]$ contains H' as a minor and every vertex in $\mathcal{I} \cap S$ is sparsifiable in $G[S]$.*

Lemma 13. *There exists a constant δ such that every graph G with $\text{pw}(G) \geq 2k^2 \log^\delta(k)$ contains a subgraph H with maximum degree 3 and $\text{pw}(H) \geq k$.*

Proof. By a result of Chekuri and Chuzhoy [14], if $\text{tw}(G) \geq k \log^\delta k$ for some constant δ , then G contains a subgraph H with maximum degree 3 and treewidth at least k . Since pathwidth is bounded from below by treewidth, we are done. So assume that $\text{tw}(G) < k \log^\delta k$. In which case, by a result of Groenland, Joret, Nadara, and Walczak [17], G contains a subdivision of the complete binary tree T_{2k} as a subgraph which has the desired pathwidth [28]. \square

Let T_k^+ be the tree obtained from T_{k+1} by adding a leaf vertex adjacent to the root. Observe that T_k is an induced subtree of T_k^+ where each vertex in $V(T_k)$ has degree at least 3 in T_k^+ . We are now ready to prove Theorem 10.

Proof of Theorem 10. Let $g(\Delta^4 + 1) := 2^{k+2} - 1$, $g(i) := (2(g(i+1))^2 \log^\delta(g(i+1)) + 1)(\Delta^2 + 1) - 1$ and $f(k, \Delta) := g(0)$ where δ is from Lemma 13. Let G be a graph with maximum degree Δ and $\text{pw}(G) \geq f(k, \Delta)$. We first construct a sparsifiable induced subgraph of G with large pathwidth. Using a greedy algorithm, partition $V(G)$ into $\Delta^4 + 1$ distance-5 independent sets $\mathcal{I}_0, \dots, \mathcal{I}_{\Delta^4}$. Initialise $i := 0$ and $G_i := G$. We construct G_{i+1} as an induced subgraph of G_i such that every vertex in $\mathcal{I}_i \cap V(G_{i+1})$ is sparsifiable in G_{i+1} and $\text{pw}(G_{i+1}) \geq g(i+1)$. Since G_i has maximum degree at most Δ , $\text{pw}(\mathcal{I}_i(G_i)) \geq (\text{pw}(G_i) + 1)/(\Delta^2 + 1) - 1 \geq 2(g(i+1))^2 \log^\delta(g(i+1))$. By Lemma 13, $\mathcal{I}_i(G_i)$ contains a subgraph H_i with maximum degree 3 where $\text{pw}(H_i) \geq g(i+1)$. By

Lemma 12, there exists an induced subgraph $G_i[S_i]$ of G_i that contains H_i as a minor and every vertex in $\mathcal{I}_i \cap S_i$ is sparsifiable in $G_i[S_i]$. As pathwidth is closed under minors, $\text{pw}(G_i[S_i]) \geq \text{pw}(H_i) \geq g(i+1)$. Set $G_{i+1} := G_i[S_i]$.

Now consider $\tilde{G} := G_{d^4+1}$. By the above procedure, \tilde{G} is a sparsifiable induced subgraph of G with pathwidth at least $g(\Delta^4 + 1) = 2^{k+2} - 1 = |V(T_k^+)| - 1$. By Theorem 4, \tilde{G} contains T_k^+ as a minor. Therefore, by Lemma 11, \tilde{G} contains T_k as an induced minor and hence G contains T_k as an induced minor. \square

3.3 From Minor-Free to Induced Minors

Let \mathcal{T}_k be the rooted tree of height k in which every non-leaf node has k children and every path from the root to a leaf has k edges. We prove the following result for K_n -minor-free graphs.

Theorem 14. *There is a function f such that every K_n -minor-free graph G with pathwidth at least $f(k, n)$ contains \mathcal{T}_k as an induced minor.*

Since T_k is an induced subgraph of \mathcal{T}_k , Lemma 5 and theorem 14 imply Theorem 2. Theorem 14 quickly follows from the following Ramsey-type result due to Kierstead and Penrice [20] that was recently re-proven by Atminas and Lozin [9].

Lemma 15 ([9, 20]). *There is a function g such that any graph that contains $\mathcal{T}_{g(n)}$ as a subgraph contains K_n , $K_{n,n}$ or \mathcal{T}_n as an induced subgraph.*

Proof of Theorem 14. Let $f(k, n) := |V(\mathcal{T}_{g(\max\{k, n\})})| - 1$ where g is from Lemma 15.³ By Theorem 4, G contains a minor model $(X_v : v \in V(\mathcal{T}_{g(\max\{k, n\})}))$ of $\mathcal{T}_{g(\max\{k, n\})}$. Let G' be the induced minor of G obtained from contracting each of the X_v 's. Since G' contains $\mathcal{T}_{g(\max\{k, n\})}$ as a subgraph, it follows by Lemma 15 that G' contains K_n , $K_{n,n}$ or \mathcal{T}_k as an induced subgraph. Since G excludes K_n as a minor, G' does not contain K_n or $K_{n,n}$ as subgraphs. Hence G' contains \mathcal{T}_k as an induced subgraph and thus G contains \mathcal{T}_k as an induced minor. \square

3.4 Finitely Many Forbidden Induced Subgraphs

Recall that \mathcal{G}_S is the hereditary graph class defined by a finite set S of forbidden induced subgraphs. We now characterise when \mathcal{G}_S has bounded pathwidth. Lozin and Razgon [22] showed that \mathcal{G}_S has bounded treewidth if and only if S includes a complete graph, a complete bipartite graph, a tripod and a semi-tripod. We strengthen this result to show that \mathcal{G}_S in fact has bounded pathwidth if and only if S includes a complete graph, a complete bipartite graph, a tripod and a semi-tripod.

Theorem 3. *\mathcal{G}_S has bounded pathwidth if and only if S includes a complete graph, a complete bipartite graph, a tripod and a semi-tripod.*

³Note that $|V(\mathcal{T}_k)| = (k^{k+1} - 1)/(k - 1)$ whenever $k \geq 2$.

Proof. If S does not include a complete graph, a complete bipartite graph, a tripod or a semi-tripod, then by the observations of Lozin and Razgon [22], \mathcal{G}_S has unbounded treewidth and thus has unbounded pathwidth.

Now assume S contains a complete graph, a complete bipartite graph, a tripod and a semi-tripod. Observe that for every tripod (semi-tripod), there exists $k \in \mathbb{N}$ such that every (line graph of a) subdivision of T_k contains the tripod (semi-tripod) as an induced subgraph. For the sake of contradiction, suppose \mathcal{G}_S has unbounded pathwidth. By the results of Lozin and Razgon [22], there exists $w \in \mathbb{N}$ such that every graph in \mathcal{G}_S has treewidth at most w . Thus every graph in \mathcal{G}_S is K_{w+2} -minor-free. Since \mathcal{G}_S has unbounded pathwidth, Theorem 2 implies that for every $k \in \mathbb{N}$, there exists a graph $G_k \in \mathcal{G}_S$ that contains a subdivision of T_k or the line graph of a subdivision of T_k as an induced subgraph. Therefore, for k sufficiently large, G_k contains the tripod or the semi-tripod in S as an induced subgraph, a contradiction. \square

4 Conclusion

We raise the following open problem: what are the unavoidable induced subgraphs for graphs with large pathwidth in the general setting? Apart from subdivisions of large complete binary trees and line graphs of subdivisions of large complete binary trees, the other obvious candidates are complete graphs and complete bipartite graphs. While it is tempting to conjecture that this list is exhaustive, Pohoata [24] gave a non-trivial family of graphs with unbounded pathwidth that avoids these graphs as induced subgraphs (see also [15]). In particular, this graph family forbids $\{K_3, K_{2,2}\} \cup \mathcal{S}(T_5)$ as induced subgraphs (see Figure 2).

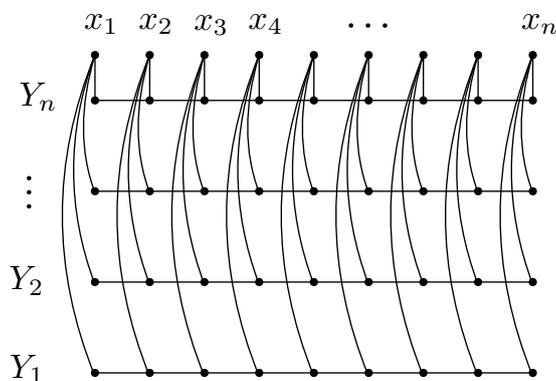


Figure 2: Pohoata's construction.

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