Rooted Prism-Minors and Disjoint Cycles Containing a Specified Edge

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Abstract

Dirac and Lovász independently characterized the 3-connected graphs with no pair of vertex-disjoint cycles. Equivalently, they characterized all 3-connected graphs with no prism-minors. In this paper, we completely characterize the 3-connected graphs with no edge that is contained in the union of a pair of vertex-disjoint cycles. As applications, we answer the analogous questions for edge-disjoint cycles and for 4-connected graphs and we completely characterize the 3-connected graphs with no prism-minor using a specified edge.

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1 Introduction

Results about vertex-disjoint and edge-disjoint cycles have received extensive attention by researchers in graph theory. There are two main lines of research in this area. One line of research provides results that give a sufficient condition for a graph to contain a certain number of vertex-disjoint or edge-disjoint cycles (see, for example, [3, 4, 6, 8, 9, 10, 11, 12, 13, 22]). Another line of research provides results that classify graphs with no pairs of vertex-disjoint cycles (see, for example, [7] and [15]). We give some results of the latter type here. Such results are particularly useful in the study of graph structure.

All graphs in this paper contain no loops nor parallel edges and we think of an edge as the unordered pair of its endvertices. We use K_5^- to denote the graph obtained by deleting an edge from the complete graph on five vertices. All paths we consider are simple, *i.e.* they do not repeat vertices. The graphs $K'_{3,n}$, $K''_{3,n}$, and $K'''_{3,n}$ are obtained by adding, respectively, one, two, or three edges to a partite class of size three in the graph $K_{3,n}$. We denote by W_n the wheel with n spokes. The following result is independently due to Dirac [7] and Lovász [15]:

Theorem 1. *A* 3*-connected graph has no pair of vertex-disjoint cycles if and only if it is isomorphic to* W_n, K_5, K_5^-, $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, *or* $K'''_{3,n}$ *for some integer* n *exceeding two.*

If $e = uv$ is an edge of a graph G, we define the **contraction** of e in G as the graph G/e obtained by identifying u and v into a single vertex. Rigorously: $V(G/e) := (V(G) - \{u,v\}) \cup \{w\}$, where w is an element not in $V(G) - \{u,v\}$ and $E(G/e) := E(G - \{u,v\}) \cup \{xw : xu \in E(G-v) \text{ or } xv \in E(G-u)\}$. If $X = \{e_1, \ldots, e_n\}$, we denote by G/X the graph obtained by contracting the edges of X one by one (the order of the contractions does not matter).

We say that a graph H is a **minor** of a graph G if H is obtained from G by contracting edges, deleting edges and deleting vertices. An ***H*-minor** of a graph G is a minor of G that is isomorphic to the graph H. The **prism** is the graph obtained from two disjoint triangles by adding a perfect matching connecting the vertices of the different triangles. It follows from Menger's Theorem that Theorem 1 is equivalent to the following result:

Theorem 2. *A* 3*-connected graph has a prism-minor if and only if it is not isomorphic to* W_n, K_5, K_5^-, $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, *or* $K'''_{3,n}$ *for some integer* n *exceeding two.*

We generalize both Theorems 1 and 2. First we discuss the generalization of Theorem 2. We say that a minor H of a graph G **uses** an edge $uv \in E(G)$ if H has an edge xy such that each $z \in \{x,y\}$ either is in $\{u,v\}$ or is obtained by the identification of a set of vertices intersecting $\{u,v\}$ when making the contractions to obtain H. Some authors call this a minor **rooted** on uv. Determining when a class of graphs (resp. matroids) has a minor using specified edges (resp. elements) is often useful and important in the study of structure of graphs and matroids. For instance, Seymour [18] established a result on K_4-minors rooted on pairs of edges and derived results on disjoint paths on graphs. Results on the existence of $U_{2,4}$-minors rooted on one or two elements on matroids with $U_{2,4}$-minors (Bixby [2] and Seymour [17], retrospectively) are classic results in matroid theory; a property like this is called **roundedness** and has some variations, for instance, 3-connected graphs with a minor isomorphic to K_4, K_5^-e or K_5 have a minor rooted on the edge set of each triangle provided they have a minor isomorphic to these respective graphs [5, 19] and 3-connected graphs with $K_{3,3}$-minors have a $K_{3,3}$-minor rooted on each set of three edges incident to a common degree-3 vertex [21]. Our first main result is the following generalization of Theorem 2:

Theorem 3. *Suppose that* G *is a* 3*-connected graph on at least six vertices and* uv *is an edge of* G*. Then* G *has no prism-minor using* uv *if and only if*

(a) for some $n \geqslant 3$, $G \cong W_n$, $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$ *or* $K'''_{3,n}$ *or*

(b) G *has a vertex* w *such that* $\{u,v,w\}$ *is a vertex-cut of* G *and each connected component of* $G - \{u,v,w\}$ *is a tree with a unique vertex of* $N_G(w)$.

A proof of Theorem 3 will be given in Section 6. There are variants of Theorem 1, including [20, Theorem 1.2], [16, Theorem 1.1], and [14, Theorem 1.3]. Motivated by Theorem 1, we consider a much larger class of graphs. The graphs in this class may contain vertex-disjoint cycles, but they do not contain vertex-disjoint cycles whose union contains a specified edge. Equivalently, those are the graphs with an edge e with the property that, for each cycle C containing e, $G - V(C)$ is a forest. The full and detailed characterization is made in Theorem 8, in the end of this introduction. The statement of this theorem though requires a somewhat complicated definition, a structure we call a rope bridge. Although its statement is not intuitive at first sight, the description given by Theorem 8 is relatively easy to use and this theorem has practical applications and is of independent interest. Indeed, it is used to prove all other main results in this paper. A more succinct and intuitive description of the graphs with no pair of vertex-disjoint cycles whose union contains a specified edge is given in Theorem 6; although Theorem 6 might not be very suitable for producing rigorous proofs, it had its uses in the heuristic aspect and provided important insights for many results in this paper.

We also consider a more strict class within the class of graphs with no pair of vertex-disjoint cycles containing a specified edge, the class of graphs with no pair of edge-disjoint cycles containing a specified edge, which admits a simpler characterization. Theorems 4 and 5 completely characterizes the 3-connected and 2-connected graphs within this class.

Theorem 4. *If G is a 3-connected graph with an edge $e = u_1u_2$, then G contains no edge-disjoint cycles using e if and only if $G\backslash e$ has internally disjoint (u_1, u_2)-paths $\alpha = u_1, v_1, \ldots, v_n, u_2$ and $\beta = u_1, w_1, \ldots, w_n, u_2$, both not containing e, and there is a family $\mathcal{P}$ of pairwise disjoint pairs of consecutive elements of $\{1, \ldots, n\}$ such that*

(a) $V(G) = V(\alpha) \cup V(\beta)$ and

(b) $E(G) = \{v_iw_j, v_jw_i : \{i, j\} \in \mathcal{P}\} \cup \{v_kw_k : k$ is in no member of $\mathcal{P}\} \cup E(\alpha) \cup E(\beta) \cup \{e\}$.

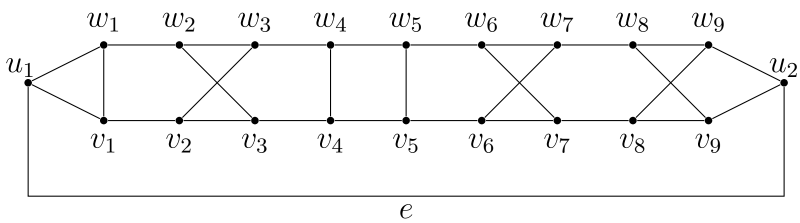

Figure 1: An example for Theorem 4 with $n = 9$ and $\mathcal{P} = \big\{\{2, 3\}, \{6, 7\}, \{8, 9\}\big\}$.

If $G_1, \ldots, G_n$ are (not necessarily disjoint) graphs, we define the **union** of these graphs, denoted by $G_1 \cup \cdots \cup G_n$, as the graph G such that $V(G) = V(G_1) \cup \cdots \cup V(G_n)$ and $E(G) = E(G_1) \cup \cdots \cup E(G_n)$. If G is a graph and e is an unordered pair of vertices of G we denote by $G + e$ the graph with same set of vertices as G and $E(G) \cup \{e\}$ as edge-set. The following theorem generalizes Theorem 4 for 2-connected graphs.

Theorem 5. *Suppose that G is a 2-connected graph and e is an edge of G. Then G has no pair of edge-disjoint cycles whose union contains e if and only if, for some integer $n \geqslant 1$, G has subgraphs $G_1, \ldots, G_n$ and an $(n+1)$-element set of vertices $U := \{u_0, \ldots, u_n\}$ such that all of the following assertions hold:*

(a) $G = (G_1 \cup \cdots \cup G_n) + e$ *and* $e \notin E(G_i)$ *for* $i = 1, \ldots, n$*;*

(b) $e = u_0u_n$*;*

(c) $u_0 \in G_1$ *and* $u_n \in G_n$*;*

(d) for $1 \leqslant i < j \leqslant n$*,* $V(G_i) \cap V(G_j) = \emptyset$ *if* $j > i + 1$ *and* $V(G_i) \cap V(G_{i+1}) = \{u_i\}$*; and*

(e) for each $i = 1, \ldots, n$*, one of the following assertions holds:*

  (e1) $G_i \cong K_2$ *with* $V(G_i) = \{u_{i-1}, u_i\}$*,*

  (e2) G_i *is a cycle, or*

  (e3) $G_i + u_{i-1}u_i$ *is a subdivision of a* 3*-connected graph with no pair of edge-disjoint cycles whose union contains* $u_{i-1}u_i$*.*

Now we state the results concerning the class of graphs with no pair of vertex-disjoint cycles containing a specified edge. The next theorem is a shorter form of Theorem 8.

Theorem 6. *Let* G *be a* 3*-connected graph with at least six vertices and* $e = u_1u_2$ *be an edge of* G*. Then* G *contains no pair of vertex-disjoint cycles whose union contains* e *if and only if one of the following assertions hold:*

(a) $G - \{u_1, u_2\}$ *is a cycle and both* u *and* v *have degree at least four;*

(b) G *has a vertex* w *such that* $\{u_1, u_2, w\}$ *is a vertex-cut of* G *and each connected component* $G - \{u_1, u_2, w\}$ *is a tree with a unique vertex of* $N_G(w)$*; or*

(c) G *is a* 3*-connected minor using using* e *of a graph as in one of Figures 2, 3, or 4.*

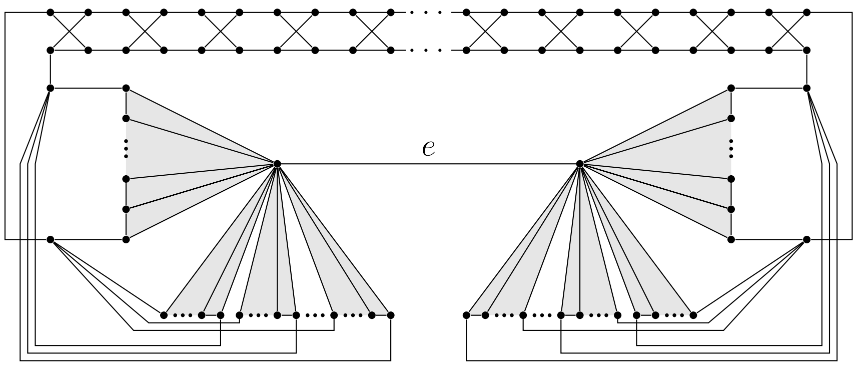

Figure 2: The first family of graphs relative to item (c) of Theorem 6. Fans are shaded in grey for better vizualization.

For 4-connected graphs, we have a strengthening of Theorem 6 as follows.

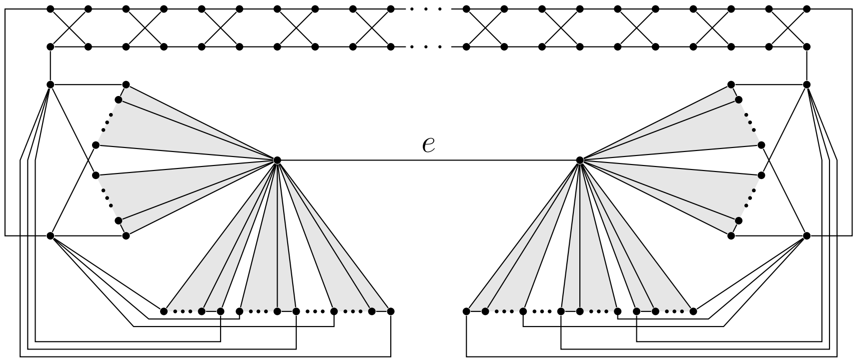

Figure 3: The second family of graphs relative to item (c) of Theorem 6. Fans are shaded in grey for better vizualization.

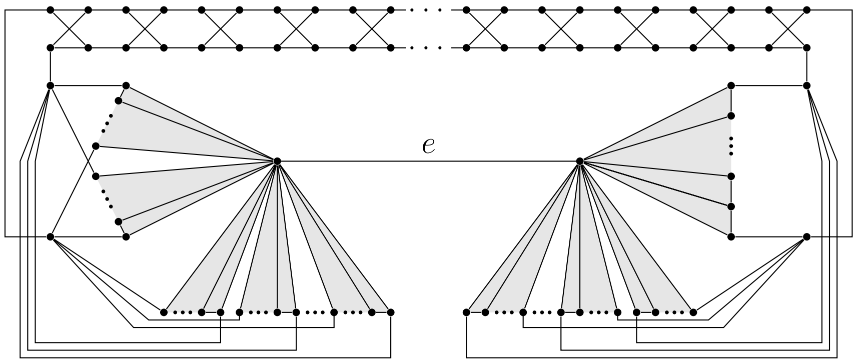

Figure 4: The third family of graphs relative to item (c) of Theorem 6. Fans are shaded in grey for better vizualization.

Theorem 7. *Let G be a 4-connected graph with a fixed edge uv. Then, either G contains a pair of vertex-disjoint cycles containing uv or $G - \{u, v\}$ is a cycle and both u and v are adjacent to all other vertices of G.*

For a path σ in a graph, we denote by $\mathrm{Int}(\sigma)$ the set of the non-extreme vertices of σ. For $A, B \subseteq V(G)$, an $\boldsymbol{(A, B)}$**-path** of G is a path with an extreme in A, the other extreme in B and no inner vertex in $A \cup B$.

Next we define the main structure used to describe the graphs in our fundamental and last main result. It is a apparently complex structure and it is hard to have an intuition on it on a first glimpse, but the properties that define this structure raises naturally when trying to characterize the graphs satisfying item (b) of Lemma 17, which appear when describing the graphs we are studying.

A graph R with distinct vertices u, x, and y is an $\boldsymbol{(u, x, y)}$**-rope bridge** with **ropes** ρ_x and ρ_y and **family of steps** $\boldsymbol{\mathcal{S}}$, if the following properties hold:

(RB0) $d_R(v) \geqslant 3$ for all $v \in V(R) - \{u, x, y\}$.

(RB1) ρ_x and ρ_y are paths from u to x and y respectively with $V(\rho_y) \cap V(\rho_x) = \{u\}$.

(RB2) $\mathcal{S}$ is a family of internally disjoint $(V(\rho_x) - u, V(\rho_y) - u)$-paths (whose members we call **steps**). We denote by x_α and y_α the extremities of the step α in ρ_x and ρ_y respectively. We say that a step α **crosses** a step β if for some $\{s, t\} = \{x, y\}$, s_α and s_β are distinct and appear in this order in ρ_s, while t_β and t_α are distinct and appear in this order in ρ_t.

(RB3) Each step crosses at most one other step.

(RB4) Each vertex in $V(R) - V(\rho_y \cup \rho_x)$ is in some step.

(RB5) Each edge not in a member of $\mathcal{S} \cup \{\rho_x, \rho_y\}$ is incident to u.

(RB6) Let $z \in \{x, y\}$ and $v \in \rho_z$. Suppose that two steps have extremities in $\mathrm{Int}(u, \rho_z, v)$. Then, $uv \notin E(R)$ and each step with extremity in v has no inner vertices.

Theorem 8. *Let G be a 3-connected graph with $|G| \geqslant 6$ and $e := u_1u_2 \in E(G)$. There is no pair of disjoint cycles of G whose union contains e if and only if one of the following assertions holds.*

(a) G has a vertex w such that $\{u_1, u_2, w\}$ is a vertex-cut of G and each connected component $G - \{u_1, u_2, w\}$ is a tree with a unique vertex of $N_G(w)$.

(b) $G - \{u_1, u_2\}$ is a cycle, $d_G(u_1), d_G(u_2) \geqslant 4$, $G \backslash e$ is 3-connected, and each 4-vertex-cut S^+ of G containing $\{u_1, u_2\}$ has the property that $G - S^+$ has a connected component with a neighbor of each vertex in $\{u_1, u_2\}$.

(c) $G - \{u_1, u_2\}$ is 2-connected and there is a 2-vertex-cut $\{x, y\}$ of $G \backslash e$. Moreover, for each such vertex-cut $\{x, y\}$ and for each $i \in \{1, 2\}$:

(c.1) there is an (u_i, x, y)-rope bridge R_i such that $V(R_1) \cap V(R_2) = \{x, y\}$, $G = (R_1 \cup R_2) + e$ and R_1 and R_2 are induced subgraphs of G; and

(c.2) if there is more than one connected component of $G - \{u_1, u_2, x, y\}$ meeting $N_G(u_i)$, then each such a connected component K is a path with one extreme adjacent to x and u_i, the other adjacent to y and u_i, and each inner vertex w satisfying $N_G(w) - V(K) = \{u_i\}$.

One last remark is that the analogous questions for disjoint cycles containing a specified vertex is reduced to Dirac's characterization, as we can see in the next proposition.

Proposition 9. *Let G be a 3-connected graph and $v \in V(G)$. If G has no pair of vertex-disjoint cycles using v, then G has no pair of vertex-disjoint cycles.*

Proof. Suppose that G and v contradict the proposition. Let C and D be vertex-disjoint cycles of G. So $v \notin V(C) \cup V(D)$. By Menger's Theorem, there are three $(v, V(C) \cup V(D))$-paths in G meeting only in v. We may assume that two of them have an endvertex in C. Note that the union of C with those two paths contains a cycle containing v and avoiding D, a contradiction. □

This paper is structured as described next. In section 2 we establish some terminologies and a few short lemmas. In section 3 we prove more specific properties of rope bridges. Sections 4, 5, and 6 are respectively dedicated to prove Theorems 8, 6 and 7; 4 and 5; and 3.

2 Terminologies and Preliminaries

In this section we establish most of the terminologies that will be used along the paper and establish a few preliminary results. The paths we consider are simple, *i.e.* they do not repeat vertices, we think of paths (and cycles) both as subgraphs and (cyclic) sequences of vertices. The number of vertices and edges in a graph G are denoted by $|G|$ and $\|G\|$ respectively. We denote by $V(e)$ the set of the endvertices of an edge e. For vertices $v_1, \dots, v_n$ and a subgraph H of a graph G, we say that a **cycle of the form** $v_1, \dots, v_k, H, v_{k+1}, \dots, v_n, v_1$ is a cycle that begins in v_1, follows through $v_1, \dots, v_k$, then, through a path of H, and, then, returns to v_1 through $v_{k+1}, \dots, v_n$. For a path γ, we denote by $\text{Int}(\gamma)$ the set of its inner vertices, *i.e.* its non-extreme vertices. For $A, B \subseteq V(G)$, an $\boldsymbol{(A,B)}$**-path** of G is a path with an extreme in A, the other extreme in B and no inner vertex in $A \cup B$. For $u, v \in V(G)$, we use the terms "(A,u)-path", "(u,A)-path" and "(u,v)-path" to abbreviate the respective terms "$(A,\{u\})$-path", "$(\{u\},A)$-path" and "$(\{u\},\{v\})$-path". For a subgraph H of G and $X \subseteq V(G) - V(H)$, we define the **sum** of H and X, denoted by $H + X$, as the graph obtained from H by adding the vertices of X and all edges of G linking the vertices of X with vertices of $V(H) \cup X$. Equivalently, $H + X$ is obtained from $G[V(H) \cup X]$ by deleting the edges in $E(G[V(H)]) - E(H)$ that are not incident to vertices of X. We denote by $V(e)$ the set of vertices incident to the edge e. For any set X and element x, we simplify the notations $X \cup \{x\}$ and $X - \{x\}$ by $X \cup x$ and $X - x$ respectively. The opperation of **splitting** a vertex in a graph, is the opposite of the contraction operation; pecisely, if when contracting an edge uv in a graph G, we identify u and v into a single vertex w to form G/uv, then we say that G is obained from G/uv by spliting w into u and v.

Let e be a fixed edge of a graph G. We say that G is an $\boldsymbol{e}$**-Dirac** graph if G contains no vertex-disjoint cycles using e. In such a case we say that e is a **Dirac** edge of G. An elementary observation about the class of e-Dirac graphs is that it is closed under minors that use e. Our classification is based on the study of a minimum-sized vertex-cut of G containing $V(e)$. The next lemma establishes that such a vertex-cut exists for e-Dirac graphs with an unique exception up to isomorphisms. We denote by $K_{3,3}^{1,1}$ the graph obtained from $K_{3,3}$ by adding two edges with no common incident vertex.

Lemma 10. *Let e be an edge in a 3-connected graph G and suppose that $V(e)$ is contained in no vertex-cut of G. Then $G - V(e)$ is complete. Moreover, if G is e-Dirac, then $G \cong K_{3,3}^{1,1}$ and $d_G(u) = d_G(v) = 3$.*

Proof. Suppose that $V(e)$ is in no vertex cut of G. If $u, v \in V(G) - V(e)$ and $uv \notin V(G)$, then $N_G(u) \cup V(e)$ is a vertex-cut of G separating u from v. Hence $G - V(E)$ is complete. This proves the first part of the lemma.

For the second part let us first prove that $|G| = 6$. Suppose that $|G| \geqslant 7$. Let $e = uv$. As G has minimum degree at least three, there are different vertices $x \in N_G(u)$ and $y \in N_G(v)$. As $G - V(e)$ is complete, it follows that u, v, y, x, u is a cycle of G avoiding $G - \{u, v, x, y\}$ which is complete with more than two vertices and, therefore, has a cycle. This implies that G is not e-Dirac. So $|G| = 6$. If u and v have a common neighbor w, then $G[\{u, v, w\}]$ and $G - \{u, v, w\}$ are disjoint cycles. So u and v have no common neighbor and now it is clear that $G \cong K_{3,3}^{1,1}$ and $d_G(u) = d_G(v) = 3$. □

We will usually denote a minimum-sized vertex-cut containing $V(e)$ by S^+ and also denote $S := S^+ - V(e)$. We will call the size of this vertex cut the ***e*-width** of G and denote it by $w(e, G)$ provided such a vertex-cut exists. Other; in this case., we will use the convention that $w(e, G) = 4$ as G will behave as graphs with $w(e, G) = 4$.

Lemma 11. *Let G be an e-Dirac graph and suppose that S^+ is a minimum-sized vertex-cut of G in respect to containing $V(e)$. Then each vertex of $S := S^+ - V(e)$ has a neighbor in each connected component of $G \backslash S^+$.*

Proof. If $x \in S$ has no neighbor in a connected component K of $G \backslash S^+$, then $S^+ - x$ separates K from the other connected components of $G \backslash S^+$, a contradiction to the minimality of $|S^+|$. □

Lemma 12. *Let G be an e-Dirac graph with more than five vertices. Each vertex of $V(G) - V(e)$ has at least $w(e, G) - 2$ neighbors in $G - V(e)$.*

Proof. The result is clear if $G \cong K_{3,3}^{1,1}$. So we may assume that G has a minimum-sized vertex cut S^+ in respect to containing $V(e)$. Define $S := S^+ - V(e)$. Now $|S^+| = w(e, G) = |S| + 2$.

Suppose for a contradiction that $u \in V(G) - V(E)$ and u has $n \leqslant |S| - 1$ neighbors in $G - V(e)$. Thus $X := N_G(u) \cup V(e)$ has $n + 2 \leqslant |S| + 1$ elements. As S^+ is a vertex-cut, then $|G| \geqslant |S^+| + 2 = |S| + 4$. Hence $V(G) - (X \cup u)$ has a vertex v. But $N_G(u) \subseteq X$. So, X separates u from v and, therefore, X is a vertex-cut containing $V(e)$. But $|X| = n + 2 \leqslant |S| + 1 < |S^+|$, a contradiction to the minimality of S^+. □

The next lemma has an elementary proof, which we omit.

Lemma 13. *Let G be an e-Dirac graph and let D be a cycle of G such that $e \in E(D)$ and H be a subgraph of G such that $H - V(D)$ is connected. Suppose that f is an edge of G incident to $x \in V(H) - V(D)$ but such that $f \notin E(H)$. Then, each $(x, V(H))$-path γ of G beginning with f intersects D. In particular, γ has an endvertex in $V(D)$ if $V(D) \subseteq V(H)$.*

Lemma 14. *Let H be a 3-connected graph, uv be an edge of H and X be a subset of $V(H)$ avoiding $\{u, v\}$ with at least 3 elements. Then, for some $x \in \{u, v\}$, there are $(X, \{u, v\})$-paths α, β, and γ of G such that $V(\alpha) \cap V(\gamma) = V(\beta) \cap V(\gamma) = \emptyset$ and $V(\alpha) \cap V(\beta) = \{x\}$.*

Proof. By Menger's Theorem there are $(X, \{u, v\})$-paths α_1, α_2 and α_3 such that $V(\alpha_i) \cap V(\alpha_j) \subseteq \{u, v\}$ for $1 \leqslant i < j \leqslant 3$. We may assume that $V(\alpha_i) \cap \{u, v\} = \{v\}$ for $i = 1, 2, 3$ because we are done otherwise. This implies that $d_H(v) \geqslant 4$. Consider a graph K with minimum degree at least 3 obtained from H by splitting v into vertices v_1 and v_2. It is well known and easy to check that K is 3-connected. By Menger's Theorem, K has three vertex-disjoint $(X, \{u, v_1, v_2\})$-paths. The desired paths are the corresponding paths in H. □

Lemma 15. *Let u be a vertex in a vertex-cut X of a graph H. If u has 3 neighbors in some connected component K of $H - X$, then $H[V(K) \cup u]$ has a cycle containing u with more than 3 vertices.*

Proof. Let x, y, z be distinct neighbors of u in K. As K is connected, there is an (x, y)-path and an (y, z)-path in K. If one of these paths has more than one edge, we are done. Otherwise, both have one edge and u, x, y, z, u is the cycle we seek. □

3 Rope Bridges

In this section we establish more specific properties of rope bridges that will be used on the proofs.

Lemma 16. *Suppose that R is an (u, x, y)-rope bridge with ropes ρ_x and ρ_y and that $w \in Int(\rho_y)$. Let t be the vertex that follows w in ρ_y, $\eta := t, \rho_y, y$ and $\sigma_1, \ldots, \sigma_k$ be the steps meeting η. Then, the graph R' obtained from $R - (V(\eta) \cup Int(\sigma_1), \ldots, Int(\sigma_k))$ by suppressing the degree-two vertices in $Int(\rho_x)$ is an (u, x, w)-rope bridge.*

Proof. Define $W = V(\eta \cup \mathrm{Int}(\sigma_1) \cup \cdots \cup \mathrm{Int}(\sigma_k))$. For some $F \subseteq E(\rho_x)$, up to isomorphisms, we have $R' = (R - W)/F$. Consider $\rho'_x := \rho_x/F$ and $\rho_w := \rho_y - V(\rho)$ as ropes for R'. When contracting the edges of F, preserve the labels of the vertices with degree exceeding two in $R - W$. Now consider as steps of R' the steps of R with exception of $\sigma_1, \ldots, \sigma_k$ (the steps we chose are indeed in R' because of our choice of labels). Now, its is easy to verify that R' with such ropes and steps inherits each one of the properties (RB0)-(RB6) from R. □

The next lemma makes evident the importance of rope-bridges in our main results. This lemma will play an important role when proving Theorem 8.

Lemma 17. *Let R be a connected graph with vertices u, x, y and paths ρ_x and ρ_y from u to x and y respectively, satisfying $V(\rho_y) \cap V(\rho_x) = \{u\}$. Suppose that $d_R(v) \geqslant 3$ for all $v \in V(R) - \{u, x, y\}$. Then, the following assertions are equivalent:*

(a) R is an (u, x, y)-rope bridge with ropes ρ_x and ρ_y.

(b) If $z \in \{x, y\}$ and C is a cycle of $R - \{u, z\}$, then R has no (u, z)-path disjoint from C.

Proof. Conditions (RB0) and (RB1) are given in the hypothesis. So, we have to prove that (b) is equivalent to (RB2)-(RB6) for some family of steps $\mathcal{S}$, which we will define ahead. Suppose that (b) holds.

First we prove that each $v \in V(R) - (V(\rho_x) \cup V(\rho_y))$ is in a $(V(\rho_x), V(\rho_y))$-path σ_v avoiding u. If v is in a cycle of $R - u$, then, by (b), this cycle must meet ρ_x and ρ_y and this implies the existence of σ_v. So, we may assume that v is in no cycle of $R - u$. As $d_{R-u}(v) \geqslant 2$, this implies that $R - \{u, v\}$ has different connected components K_1 and K_2, each one with a unique neighbor of v. If $\rho_x - u$ and $\rho_y - u$ are each one in a different component in $\{K_1, K_2\}$, then the existence of σ_v is straightforward. So, we may assume that K_1 avoids ρ_x and ρ_y. Let w_1 be the unique neighbor of v in K_1. As $d_{R-u}(w_1) \geqslant 2$, K_1 has more than one vertex. But, for all $w \in V(K_1) - w_1$, $d_{R-\{u,v\}}(w) \geqslant 2$. This implies that K_1 has a cycle, which must avoid ρ_x and ρ_y, contradicting (b). This proves the existence of σ_v.

Now, we define to be the elements of $\mathcal{S}$ the paths σ_v with $v \in V(R) - V(\rho_x \cup \rho_y)$ and the $(V(\rho_x) - u, V(\rho_y) - y)$-paths of length one. We will call *steps* the members of $\mathcal{S}$. For each step α, we denote by x_α and y_α the endvertices of α in ρ_x and ρ_y respectively.

To establish (RB2) we shall prove that the steps are internally disjoint. Indeed, suppose for a contradiction that a step α intersects a step β in an inner vertex v. As $\alpha \neq \beta$, we may assume that $x_\alpha, \alpha, v \neq x_\beta, \beta, v$. This implies that, $x_\alpha, \alpha, v, \beta, x_\beta, \rho_x, x_\alpha$ contains a cycle avoiding ρ_y, a contradiction to (b). Thus, the steps are internally disjoint and (RB2) holds.

Suppose that (RB3) does not hold. So, there is a step α crossing different steps β and γ. First we assume that x_α precedes both x_β and x_γ in ρ_x; in this case the cycle $x_\beta, \rho_x, x_\gamma, \gamma, y_\gamma, \rho_y, y_\beta, \beta, x_\beta$ avoids the path $u, \rho_x, x_\alpha, \alpha, y_\alpha, \rho_y, y$, a contradiction to (b). Now we assume that x_α succeeds x_β and x_γ in ρ_x; as α crosses β and γ, this implies that y_α precedes y_β and y_γ in ρ_y and we can use the same argument as in the preceding reversing the roles of the paths ρ_x and ρ_y to reach a contradiction. So we may assume without loosing generality that x_β, x_α, and x_γ appear in this order in ρ_x. Hence, y_γ, y_α, and y_α appear in this order in ρ_y. Now β crosses both α and γ and we may apply the same argument as in the first case reversing the roles of α and β. Now (RB3) holds.

By construction, each vertex of $V(R) - V(\rho_x \cup \rho_y)$ is in a step and we have (RB4).

To prove (RB5) suppose that $f = vw$ is an edge of $R - u$ not in $\rho_x \cup \rho_y$. We have to prove that f is in a step. If $\{v, w\} \subseteq V(\rho_x)$, then there is a cycle in $\rho_x + f$ avoiding ρ_y, contradicting (b). So, $\{v, w\} \nsubseteq V(\rho_x)$ and, analogously, $\{v, w\} \nsubseteq V(\rho_y)$. If $\{v, w\}$ meet both ρ_x and ρ_y then f is in a step and (RB5) holds; so assume the contrary. Now $\{v, w\} \nsubseteq V(\rho_x) \cup V(\rho_y)$ and we may assume that $v \notin V(\rho_x) \cup V(\rho_y)$. By (RB4), v is in the interior of a step σ. If w is in σ then either $f \in E(\sigma)$ and (RB5) holds or $\sigma + f$ has a cycle avoiding one of ρ_x or ρ_y, contradicting (b). So assume that w is not in σ. If w is in ρ_x or ρ_y, say the former, then $w, v, \sigma, x_\sigma, \rho_x, w$ is a cycle avoiding ρ_y, contradicting (b). Thus, w is in the interior of a step $\alpha \neq \sigma$. Now $w, v, \sigma, x_\sigma, \rho_x, x_\alpha, \alpha, w$ is a cycle avoiding ρ_y, a contradiction. So (RB5) holds.

Now, we prove (RB6). Let $z \in \{x, y\}$ and $v \in \rho_z$ and suppose that two steps α and β have extremities in $\mathrm{Int}(u, \rho_z, v)$. Say $z = x$. If $uv \in E(R)$, then the path u, v, ρ_x, x avoids the cycle $C := x_\alpha, \alpha, y_\alpha, \rho_y, y_\beta, \beta, x_\beta, \rho_x, x_\alpha$, a contradiction. So, $uv \notin E(R)$. If a step σ with extremity in v has an inner vertex w, then, as $d_R(w) \geqslant 3$, there is an edge incident to w not in σ, and by (RB5), this edge is uw. Now $u, w, \sigma, v, \rho_x, x$ avoids C, a contradiction. So, (RB6) holds and (b) implies (a).

Suppose that R is a graph for which (a) holds but (b) does not hold. Choose R with $|V(R)|$ as small as possible. Consider $z \in \{x, y\}$ such that there is a cycle C of $R - \{u, z\}$ and a (u, z)-path γ disjoint from C as short as possible. Say that $z = x$. Let v be the vertex of γ such that v, γ, x is contained in ρ_x and v, γ, x is as long as possible. It follows from (RB2), (RB4) and (RB5) that $G - V(\rho_x)$ and $G - V(\rho_y)$ have no cycles. Hence C meets ρ_x and ρ_y and this implies that C meets at least two steps. Those steps have their extremities in x preceding v in ρ_x. This implies that $v \neq u$ and, therefore, there is a vertex w preceding v in γ.

Let us check that $w \in V(\rho_y) - u$. As there are two steps arriving in $\mathrm{Int}(u, \rho_x, v)$ it follows by (RB6) that $uv \notin E(G)$ and $w \neq u$. It is also a consequence of (RB6) that all steps arriving in v have no inner vertices, so it suffices to check that vw is in a step to conclude that $w \in V(\rho_y) - u$. If this is not the case, then, by (RB5), wv is in ρ_x and w contradicts the choice of v such that v, γ, x is contained in ρ_x and v, γ, x is as long as possible. So wv is in a step and $w \in V(\rho_y) - u$.

If $w = y$, then u, γ, w violates the minimality of γ (for $z = y$), a contradiction. Let t be the vertex following w in ρ_y. If C meets $\eta := t, \rho_y, y$, then, at least two steps contained in C have endvertices in η, but these steps also have endvertices in $\mathrm{Int}(u, \rho_x, v)$ and, therefore, cross the step v, w, contradicting (RB3). So, C does not meet η. Let R' be obtained from R by deleting $V(\eta)$ and all inner vertices of steps with endpoints in η and, then, suppressing the degree-2 vertices. By Lemma 16, R' is an (u, x, w)-rope bridge with less vertices than R. But, the path induced by u, γ, w and the cycle induced by C in R' contradict (b). This is a contradiction to the minimality of $|V(R)|$. □

The rest of this section is dedicated to prove lemmas that will be used to prove Theorem 6.

We say that a step in a rope bridge is **short** if it has no inner vertices and **long** otherwise. A vertex v in a rope ρ of an (u, x, y)-rope bridge is **clean** if all steps arriving in v are short and $uv \notin E(G) - E(\rho)$. Note that (RB6) says that for the rope ρ containing v, v is clean if two steps arrive in $\mathrm{Int}(u, \rho, v)$. The following lemma follows directly from (RB0)-(RB6)

Lemma 18. *Let R be an (u, x, y)-rope bridge with ropes ρ_x and ρ_y. Write, for $z \in \{x, y\}$, $z_0 := u$ and $\rho_z := z_0, z_1, \ldots, z_{n(z)}$.*

(a) Suppose that, for $m \geqslant 2$ and $t \geqslant 1$, $\alpha_1, \ldots, \alpha_m$ are the steps arriving in x_t. Let $y_{k(i)} := y_{\alpha_i}$ and suppose $k(1) \leqslant \cdots \leqslant k(m)$. Let R_1 be the graph obtained from R by splitting x_t into vertices v and w in such a way that:

- *$x_{t-1}v \in E(R_1)$,*
- *either $x_t \neq x$ and $x_{t+1}w \in E(R_1)$ or $x_t = x$ and we consider $w = x$,*

- *for some $1 \leqslant l < m$, the paths corresponding to $\alpha_1, \dots, \alpha_l$ arrive in v and the ones corresponding to $\alpha_{l+1}, \dots, \alpha_m$ arrive in w in R_1 and*
- *$uw \notin E(R_1)$ and $uv \in E(R_1)$ if and only if $ux_t \in E(R_1)$.*

Suppose that all steps arriving in $x_{t+1}, \dots, x_{n(x)}$ are short. Then, R_1 is an (u, x, y)-rope bridge if one of the following assertions hold:

(a1) $\alpha_{l+1}, \dots, \alpha_m$ *are short, or*

(a2) $t = l = 1$.

Moreover, a similar construction with x and y playing swapped roles also results in an (u, x, y) rope bridge.

(b) Suppose that α is a step with endvertices in x_a and y_b, $x_a, \dots, x_{n(x)}, y_b, \dots, y_{n(y)}$ are all clean, α crosses no other step, and no other step has an endvertex in x_a or y_b. Let R_2 be the graph obtained from R by deleting the edge of α, splitting x_a into vertices v_1 and v_2 and y_b into w_1 and w_2, then, adding the edges v_1w_2 and v_2w_1 as steps in such a way that:

- $x_{a-1}v_1, y_{b-1}w_1 \in E(R_2)$,
- *either $x_a \neq x$ and $x_{a+1}v_2 \in E(R_2)$ or $x_a = x$ and we consider $v_2 = x$ and*
- *either $y_b \neq y$ and $y_{b+1}w_2 \in E(R_2)$ or $y_b = y$ and we consider $w_2 = y$.*

Then R_2 is an (u, x, y)-rope bridge.

Lemma 19. *If R is an (u, x, y)-rope bridge then, up to the labels of elements other than u, x and y, R is a minor of a graph as in Figures 5 or 6 (paths that contracts to the ropes are drawn in thick lines and fans are shaded in gray).*

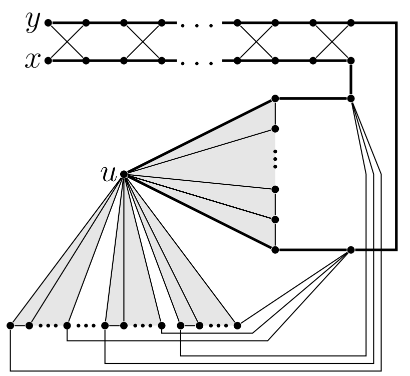

Figure 5:

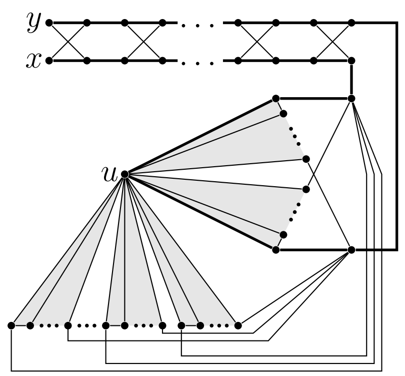

Figure 6:

Proof. Suppose that R' is a graph contradicting the lemma. Let R be a graph obtained from R' using the operations of Lemma 18 up to the point that they can no longer be performed. If we prove the lemma for R, it will also hold for R'. If all vertices in the ropes are clean the result is clear. So, we may assume that there is a non-clean vertex. Write $\rho_z = u, z_1, \dots, z_{n(z)}$ for each $z \in \{x, y\}$. Consider the smallest index $c(z)$ such that all vertices $z_{c(z)+1}, \dots, z_{n(z)}$ are clean. By (RB6), $1 \leqslant c(z) \leqslant 2$.

If $c(z) < t \leqslant n(z)$, there is a unique step arriving in z_t since operation (a) of Lemma 18 cannot be performed and (a1) would hold otherwise. In particular, the step arriving in z_t must cross another step since operation (b) of Lemma 18 also cannot be performed.

Also, there is a unique step α_z arriving in z_1, as we prove next. If $c(z) = 2$, this follows from (RB6). If $c(z) = 1$, this follows from the fact that item (a1) of Lemma 18 does not hold.

If there are no long steps, x_2 and y_2 are the unique possibly non-clean vertices and the result holds. So, we may assume that there is some long step.

If $c(x) = 1$, then x_1 is the unique non-clean vertex of ρ_x and, as a unique step arriving at x_1, α_x is the unique long step. If $\alpha_y = \alpha_x$, R is a minor of a graph as in Figure 5, otherwise it is a minor a graph as in Figure 6.

So, we may assume that $c(x) = 2$ and, analogously, that $c(y) = 2$. If $\alpha_x = \alpha_y$, then R is a minor of a graph as in Figure 5. Otherwise, as each step cross at most one other, α_x has an endvertex in y_2 and α_y has an endvertex in x_2. It follows that R is a minor of a graph as in Figure 6. $\square$

4 Proofs of Theorems 6, 7 and 8

In the end of this section we prove Theorems 8, 6, and 7 in this order. Next we establish some specific lemmas towards this purpose.

Lemma 20. *Suppose that G is an e-Dirac graph on more than six vertices and let $e := u_1u_2$. Suppose that $w(e, G) \geqslant 5$ and $d_G(u_1), d_G(u_2) \geqslant 4$. Let S^+ be a minimum-sized vertex-cut of G containing $V(e)$ and let $G_1, \dots, G_\kappa$ be the connected components of $G - S^+$.*

Suppose that there are $i \in \{1, 2\}$ and $k \in \{1, \dots, \kappa\}$ such that $|N_G(u_i) \cap V(G_k)| \geqslant 2$, then

(a) $\kappa = 2$ *and*

(b) $N_G(u_{3-i}) \subseteq V(G_{3-k}) \cup u_i$.

Proof. We begin with some definitions before proving items (a) and (b). Let us denote $S := S^+ - V(e)$. As $w(e, G) \geqslant 5$, $|S| \geqslant 3$. We may assume without loss of generality that $k = i = 1$. As $|N_G(u_1) \cap V(G_1)| \geqslant 2$, there is a cycle C in $G_1 + u_1$ meeting u_1. As $w(e, G) \geqslant 5$, $G - \{u_1, u_2\}$ is 3-connected. Next we will define three $(S, V(C) - u_1)$-paths α_1, α_2 and α_3 of $G - \{u_1, u_2\}$ considering the two possible cases itemized next:

- If $|C| = 3$, then, by Lemma 14 applied on the edge of $C - u_1$, we conclude that $G - \{u_1, u_2\}$ have $(S, V(C) - u_1)$-paths α_1, α_2 and α_3 such that α_3 does not intersect

α_1 nor α_2 and that α_1 and α_2 intersect only in their endvertex in $C - u_1$. For each $l \in \{1, 2, 3\}$, let a_l be the endvertex of α_l in $C - u_1$; note that $a_1 = a_2$ in this case.

- If $|C| \geqslant 4$, then Menger's Theorem implies that $G - \{u_1, u_2\}$ has three vertex-disjoint $(S, V(C) - u_1)$ paths α_1, α_2 and α_3, arriving in the respective vertices a_1, a_2 and a_3 of C. In this case, we pick all these labels in such a way that $\{a_1, a_3\}$ separates u_1 from a_2 in C, or, equivalently, u_1, a_1, a_2, a_3, u_1 appear in this order if we follow a specific cyclic ordering of C.

For each $l \in \{1, 2, 3\}$, we define v_l to be the endvertex of α_l in S. Next we define the following paths of C:

- β_1 is the (u_1, a_1)-path of $C - a_3$.
- β_{21} is the (a_1, a_2)-path of $C - u_1$. Note that if $|C| = 3$ then β_{21} is a trivial path with only one vertex a_1, which is equal to a_2 in this case.
- β_{23} is the (a_2, a_3)-path of $C - u_1$.
- β_3 is the (a_3, u_1)-path of $C - a_1$.

Note that a cyclic ordering of C is given by the path $u_1, \beta_1, a_1, \beta_{21}, a_2, \beta_{23}, a_3, \beta_3, u_1$. Moreover, $\|\beta_1\|, \|\beta_{23}\|, \|\beta_3\| \geqslant 1$ and $\|\beta_{21}\| \geqslant 1$ if and only if $|C| \geqslant 4$. Now we prove the items of the lemma.

Proof of item (a). Suppose for a contradiction that $\kappa \geqslant 3$.

First we will establish that there is no edge from u_2 to G_1. Suppose for a contradiction that such an edge exists. This implies that $G_1 + \{u_1, u_2\}$ has a cycle D containing e. Now for different vertices u and v in S, Lemma 11 implies that there are edges from both u and v to both G_2 and G_3 and, therefore, $(G_2 \cup G_3) + \{u, v\}$ has a cycle. This cycle must avoid D as $(G_2 \cup G_3) + \{u, v\}$ and $G_1 + \{u_1, u_2\}$ are vertex-disjoint. This contradicts the fact that G is e-Dirac. So, there is no edge from u_2 to G_1.

As $d_G(u_2) \geqslant 4$, we may assume that there is an edge linking u_2 and $G_2 + (S - \{v_1, v_2\})$, which is connected and contains v_3. By Lemma 11 again, both v_1 and v_2 have neighbors in G_3. Now, we have a cycle of the form

$$u_1, u_2, (G_2 + (S - \{v_1, v_2\})), v_3, \alpha_3, a_3, \beta_3, u_1$$

containing e and avoiding a cycle of the form

$$a_1, \beta_{21}, a_2, \alpha_2, v_2, G_3, v_1, \alpha_1, a_1,$$

a contradiction. So, $\kappa = 2$ and item (a) holds.

Proof of item (b). Now suppose that (b) fails. We make the choice for a counter-example to (b) and the structures we defined before proving (a) satisfying the following conditions:

(C1) The counter-example maximizes $|N_G(u_i) \cap V(G_k)|$. We still may assume without losing generality that $i = k = 1$.

(C2) Subject to (C1), we choose C such that $|C|$ is maximized.

(C3) Subject (C1) and (C2), we choose α_1, α_2, and α_3 in such a way that $\|\beta_1\|$ is minimized.

Define a subgraph X of G by $X := C \cup \alpha_1 \cup \alpha_2 \cup \alpha_3$. First we check:

(20.1). *If ζ is an $(u_2, V(X) - u_1)$-path of $G_1 + (S \cup u_2)$, then either*

(i) $|C| = 3$ and ζ has an endvertex in $V(\alpha_1) \cup V(\alpha_2)$ or

(ii) $|C| \geqslant 4$ and ζ has an endvertex in $V(\alpha_2)$.

Let u_2 and z be the endvertices of ζ. First we check that $z \in A := V(\alpha_1) \cup V(\beta_{21}) \cup V(\alpha_2)$. Suppose the contrary. As ζ is an $(u_2, V(X) - u_1)$-path and $A \subseteq V(X)$, ζ avoids A. As z is a vertex of $X - A$, a connected graph containing u_1, there is a cycle D containing e of the form $u_1, u_2, \zeta, z, X - A, u_1$. By Lemma 11, both v_2 and v_1 have neighbors in G_2. Hence, $G_2 + A$ is a graph with a cycle, which avoids D, contradicting that G is e-Dirac. So, $z \in A$.

If $|C| = 3$, $\|\beta_{21}\| = 0$ and $A = V(\alpha_1) \cup V(\alpha_2)$. So, (i) holds.

Now, assume that $|C| \geqslant 4$. As $z \in A$, analogously, $z \in B := V(\alpha_3) \cup V(\beta_{23}) \cup V(\alpha_2)$. Therefore, $z \in A \cap B = V(\alpha_2)$. So, (ii) holds. $\Diamond$

As (b) fails, u_2 has some neighbor in $G_1 \cup S$. By Lemma 11, each vertex of S has a neighbor in G_1 and, therefore, $G_1 + (S \cup u_2)$ is connected. This implies that $G_1 + (S \cup u_2)$ has an $(u_2, V(X) - u_1)$-path δ. We put the following condition when choosing δ:

(C4) Subject to (C1), (C2), and (C3), pick δ minimizing $|V(\delta) \cap S|$.

By (20.1), we may assume that δ has an endvertex d in α_2 (α_1 and α_2 play similar roles if $|C| = 3$). A consequence of (C4) is:

(20.2). *If u_2 has a neighbor in G_1 then $V(\delta) \cap S = \emptyset$.*

For each $i = 1, 3$ and each $(u_2, V(X) - u_1)$-path ζ of $G_1 + (S \cup u_2)$ with an endvertex z in α_2, we define the following cycle containing e:

$$C_{\zeta,i} := u_1, u_2, \zeta, z, \alpha_2, a_2, \beta_{2i}, a_i, \beta_i, u_1.$$

Now we prove:

(20.3). *v_1 has at least two neighbors in $V(G_1)$.*

Suppose the contrary. As δ is an $(u_2, V(X) - u_1)$-path arriving in $d \in V(\alpha_2)$, δ avoids $V(X) - d$; in particular δ avoids v_1 and this implies that $C_{\delta,1}$ avoids v_1.

If v_1 has two neighbors in $G_2 + (S - V(\delta))$, then by Lemma 11, $G_2 + (S - (V(\delta) \cup v_1))$ is connected, and, therefore, $G_2 + (S - V(\delta))$ has a cycle. But this cycle avoids $C_{\delta,1}$, contradicting that G is e-Dirac. Thus v_1 has at most one neighbor in $G_2 + (S - V(\delta))$.

By Lemma 12, v_1 has at least $w(e,G)-2 \geqslant 3$ neighbors in $G-\{u_1,u_2\} = (G_1 \cup G_2)+S$. As v_1 has at most one neighbor in each one of G_1 and $G_2+(S-V(\delta))$, it follows that v_1 has a neighbor v in $S \cap V(\delta)$.

Now, by (20.2), u_2 has no neighbors in G_1. As $d_G(u_2) \geqslant 4$, u_2 has a neighbor $w \in V(G_2) \cup (S-\{v,v_1\})$. By Lemma 11, either $w \in V(G_2)$ or w has a neighbor in G_2. By Lemma 11 again, v_3 has a neighbor in G_2. This implies that $G_2+\{w,v_3\}$ is connected. So, there is a cycle D containg e of the form

$$u_1, u_2, G_2+\{w,v_3\}, \alpha_3, a_3, \beta_3, u_1.$$

Since δ is an $(u_2, V(X)-u_1)$-path arriving in $d \in \alpha_2$, it follows that $v_3 \notin V(\delta)$. Thus D avoids the cycle

$$v_1, v, \delta, d, \alpha_2, a_2, \beta_{21}, a_1, \alpha_1, v_1.$$

Hence G is not e-Dirac, a contradiction. ♢

By (20.3), v_1 has a neighbor $v_1' \in V(G_1)$ such that $v_1v_1' \notin E(\alpha_1)$. Let σ be a $(v_1, V(X) \cap V(G_1))$-path of G_1+v_1 beginning with v_1v_1' and let s be the other endvertex of σ. Next we establish:

(20.4). *Let ζ be an $(u_2, V(X)-u_1)$-path of $G_1+(S \cup u_2)$ arriving in $z \in V(\alpha_2)$. Then σ intersects ζ.*

Assume the contrary. Let H_0 be the subgraph of G obtained from $G_2 \cup X$ by adding the edges that connects $\{v_1,v_2,v_3\}$ to G_2. Note that $H_0-V(C_{\zeta,1})$ is connected and, as the only edge of H_0 from v_1 to G_1 is in α_2, it follows that the edge v_1v_1' is not in H_0. Now we apply Lemma 13 for $D = C_{\zeta,1}$, $H = H_0$, $x = v_1$, and $f = v_1v_1'$ to conclude that σ intersects $C_{\zeta,1}$.

As σ is a $(v_1, V(X) \cap V(G_1))$-path, it follows that s is the unique common vertex of ζ and X.

If $s \in \mathrm{Int}(\beta_1)$, then σ, α_2 and α_3 contradict the choice of α_1, α_2 and α_3 minimizing $\|\beta_1\|$ required in (C3). So, $s \notin \mathrm{Int}(\beta_1)$ and σ avoids $\mathrm{Int}(\beta_1)$.

As $(H_0-\mathrm{Int}(\beta_1))-V(C_{\zeta,3})$ is connected, we may apply Lemma 13 for $D = C_{\zeta,3}$, $H = H_0-\mathrm{Int}(\beta_1)$, $f = v_1v_1'$, $x = v_1$ and $\gamma = \sigma$ to conclude that σ intersects $C_{\zeta,3}$.

Now σ intersects $C_{\zeta,j}$ for $j=1,3$. As σ avoids ζ, σ meets $V(C_{\zeta,j})-V(\zeta) \subseteq V(X)$. As σ is a $(v_1, V(X) \cap V(G_1))$-path, $s \in V(C_{\zeta,j})-V(\zeta)$. This implies that $s \in (V(C_{\zeta,1}) \cap V(C_{\zeta,3}))-V(\zeta) = V(z,\alpha_2,v_2)-z$.

Now we have the cycle $a_2, \alpha_2, s, \sigma, v_1, \alpha_1, a_1, \beta_{21}, a_2$ avoiding a cycle of the form

$$u_1, u_2, \zeta, z, \alpha_2, v_2, G_2, v_3, \alpha_3, a_3, \beta_3, u_1,$$

contradicting that G is e-Dirac. ♢

By (20.4) σ intersects δ in a vertex s', we define the cycle

$$B := v_1, \sigma, s', \delta, d, \alpha_2, a_2, \beta_{21} a_1, \alpha_1, v_1.$$

Note that B is a cycle of G_1+v meeting X only in $\alpha_1 \cup \alpha_2 \cup \beta_{21}$.

(20.5). $N_G(u_2) \subseteq V(G_1) \cup \{u_1, v_1\}$. *Moreover, δ is a path of $G_1 + u_2$.*

Suppose for a contradiction that v is a neighbor of u_2 in $V(G_2) \cup (S - v_1)$. By Lemma 11, $G_2 + v$ is connected and contains a neighbor of v_3. Now B avoids a cycle of the form $u_1, u_2, G_2 + \{v, v_3\}, \alpha_3, a_3, \beta_3, u_1$, contradicting that G is e-Dirac. This proves the first part of the claim. As $d_G(u_2) \geqslant 4$, u_2 has a neighbor in G_1 and now the second part of the claim follows from (20.2). ♢

(20.6). $|C| \geqslant 4$ *and* $a_1 \neq a_2$.

Suppose that $|C| \leqslant 3$. By (C2) and by Lemma 15, u_1 has at most two neighbors in G_1. Hence, u_1 has a neighbor $w_1 \in V(G_2) \cup S$.

Let us prove that u_2 also has a neighbor in $V(G_2) \cup S$. Suppose that this is false. BY (20.5), as $d_G(u_2) \geqslant 4$, it follows that u_2 has three neighbors in G_1. Now picking $(i, k) = (2, 1)$ would contradict the choice of $(i, k) = (1, 1)$ in order to satisfy the minimality required in condition (C1) since u_1 has at most two neighbors in G_1; this is a contradiction. Hence u_2 has a neighbor w_2 in $V(G_2) \cup S$.

As $|S| \geqslant 3$, there is $v \in S - \{w_1, w_2\}$. By Lemma 11 each one of v, w_1, and w_2 either is in G_2 or has a neighbor in G_2. So $G_2 + \{w_1, w_2\}$ is connected. If v has two different neighbors in G_1, then a cycle of $G_1 + v$ avoids a cycle of the form $u_1, u_2, G_2 + \{w_1, w_2\}, u_1$, a contradiction. So, by Lemma 12, v has two different neighbors in $G_2 + S$, which has a cycle, therefore; but this cycle avoids $C_{\delta,1}$ since δ is a path of $G_1 + u_2$ by (20.5), a contradiction to the fact that G is e-Dirac. Hence, $|C| \geqslant 4$. By the definition of a_1 and a_2, we have $a_1 \neq a_2$. This proves the claim. ♢

As G is 3-connected, $G - u_1$ is 2-connected and, by Menger's Theorem, has two $(u_2, V(X) - u_1)$-paths ζ_1 and ζ_2 that only meet in u_2. Let $l \in \{1, 2\}$. Let z_l be the endvertex of ζ_l other than u_2.

If σ intersects $\mathrm{Int}(\zeta_l)$ in a vertex s_l, then, as $a_1 \neq a_2$ by (20.6) and as $\mathrm{Int}(\zeta_l)$ avoids X because ζ_l is an $(u_2, V(X) - u_1)$-path, it follows, by Lemma 11 that the cycle

$$u_1, u_2, \zeta_l, s_l, \sigma, v_1, \alpha_1, a_1, \beta_1, u_1$$

avoids a cycle of the form

$$a_2, \alpha_2, v_2, G_2, v_3, \alpha_3, a_3, \beta_{23}, a_2,$$

a contradiction. Hence, σ avoids $\mathrm{Int}(\zeta_l)$.

If ζ_l meets $G_2 + (S - v_1)$, which is connected by Lemma 11, as σ avoids $\mathrm{Int}(\zeta)$ and $V(B) \subseteq V(X) \cup V(\sigma)$, it follows that B avoids a cycle of the form

$$u_1, u_2, \zeta_l, G_2 + (S - v_1), v_3, \alpha_3, a_3, \beta_3, u_1,$$

a contradiction. So, ζ_l is a path of $G_1 + \{v_1, u_2\}$. This implies that the endvertex z_l of ζ_l other than u_2 is in α_2 by (20.1) and (20.6). As σ avoids $\mathrm{Int}(\zeta_l)$, by (20.4), σ contains z_l. As σ is an $(v_1, V(X) \cap V(G_1))$-path and $z_l \in V(X)$, $s = z_l$. Now we have $z_1 = s = z_2$, a contradiction. This proves item (b) and finishes the proof of the lemma. □

Lemma 21. *Let G be an u_1u_2-Dirac graph with $d_G(u_1), d_G(u_2) \geqslant 4$. Then $w(u_1u_2, G) \leqslant 4$.*

Proof. Suppose the contrary. Let $e := u_1u_2$, let S^+ be a minimum-sized vertex-cut of G containing $V(e)$ and define $S := S^+ - V(e)$. As the lemma fails, $|S^+| \geqslant 5$ and $|S| \geqslant 3$. As $G \backslash S^+$ has more than one connected component, $|G| \geqslant 7$. Let $G_1, \ldots, G_\kappa$ be the connected components of $G - S^+$. Let us prove some assertions next.

(21.1). *Each one of u_1 and u_2 has at most one neighbor in each connected component of $G - S^+$.*

Suppose the contrary. We may assume without loosing generality that u_1 have two different neighbors in G_1. By Lemma 20, $\kappa = 2$ and $N_G(u_2) \subseteq V(G_2)$. But this implies that the hypothesis of Lemma 20 also holds for $i = k = 2$. Hence $N_G(u_1) \subseteq V(G_1)$. For $l = 1, 2$, as $d_G(u_l) \geqslant 4$, by Lemma 15, there is a cycle C_l of $G_l + u_l$ containing u_l with more than three vertices. As $w(e, G) \geqslant 5$ and G is 3-connected, it follows that $G - \{u_1, u_2\}$ is 3-connected. So, there are three vertex-disjoint $(C_1 - u_1, C_2 - u_2)$-paths in $G - \{u_1, u_2\}$ by Menger's Theorem. Together with the path u_1, u_2, we have four vertex-disjoint (C_1, C_2)-paths. Now it is easy to check that there are vertex-disjoint cycles covering these paths. Hence, we have two disjoint cycles with one of them containing e, a contradiction to the fact that G is e-Dirac. $\diamondsuit$

(21.2). *u_1 and u_2 have no common neighbor in S.*

If w contradicts (21.2), then u_1, u_2, w, u_1 is a cycle containing e that, for some distinct $x, y \in S - w$, avoids a cycle of the form x, G_1, y, G_2, x by Lemma 11. Hence G is not e-Dirac, a contradiction. $\diamondsuit$

(21.3). *$\kappa = 2$.*

Suppose the contrary. First we suppose that $u_1v \in E(G)$ for some $v \in S$.

If $N_G(u_2) \subseteq S \cup u_1$, then, as $d_G(u_2) \geqslant 4$, by (21.2), it follows that there is a 3-subset $\{x_1, x_2, x_3\}$ of $N_G(u_2) - \{u_1, v\}$. By Lemma 11, there are cycles of the forms $u_1, u_2, x_1, G_1, v, u_1$ and x_2, G_2, x_3, G_3, x_2 avoiding each other, a contradiction.

So u_2 has neighbors out of $S \cup u_1$ and we may assume that there is an edge from u_2 to G_1. By Lemma 11, for distinct $x, y \in S - v$, cycles of the forms u_1, u_2, G_1, v, u_1 and x, G_2, y, G_3, x avoid each other, a contradiction. Hence, u_1 has no neighbor in S and, analogously, neither has u_2.

If both u_1 and u_2 have neighbors in a common component of $G - S^+$, say G_1, then, by Lemma 11, we have, for distinct $x, y \in S$, cycles of the forms u_1, u_2, G_1, u_1 and x, G_2, y, G_3, x avoiding each other, a contradiction.

Therefore, u_1 and u_2 have no neighbors in S nor in a same connected component of $G - S^+$. By (21.1), $\kappa \geqslant 4$ and we may assume that u_i has a neighbor in G_i for $i = 1, 2$. Now, by Lemma 11, for distinct $x, y, z \in S$, cycles of the forms $u_1, u_2, G_2, z, G_1, u_1$ and x, G_3, y, G_4, x avoid each other, a contradiction. This establishes (21.3). $\diamondsuit$

Let $k \in \{1, 2\}$. By (21.1) and (21.3), u_k has a neighbor $v_k \in S$ as $d_G(u_k) \geqslant 4$. Let $x \in S - \{v_1, v_2\}$. If x has two neighbors in G_1, then $G_1 + x$ has a cycle avoiding a cycle of the form $u_1, u_2, v_2, G_2, v_1, u_1$ by Lemma 11, a contradiction. Hence, By Lemma 11, x has only one neighbor in G_1 and, analogously, only one neighbor in G_2. By Lemma 12, x has a neighbor $y \in S$. If $y \notin \{v_1, v_2\}$, by Lemma 11, a cycle of the form $u_1, u_2, v_2, G_1, v_1, u_1$ avoids a cycle of the form x, y, G_2, x, a contradiction. So, we may assume that $y = v_1$. As $d_G(u_1) \geqslant 4$, there is $w \in N_G(u_1) - \{u_2, v_1\}$. By Lemma 11 and (21.3), for some $i \in \{1, 2\}$, either $w \in G_i$ or w has a neighbor in G_i, and, therefore $G_i + w$ is connected. Now, by Lemma 11, a cycle of the form x, v_1, G_{3-i}, x avoids a cycle of the form $u_1, u_2, v_2, (G_i + w), u_1$, a contradiction. □

Lemma 22. *Let G be an e-Dirac graph with $w(e, G) = 4$ and let S^+ be a minimum-sized vertex-cut containing $V(e)$. Suppose that $G - S^+$ has a connected component J with neighbors of both endvertices of e. Then $G - S^+$ has a unique connected component K other than J and, for $S := S^+ - V(e)$, $G[V(K) \cup S]$ is a path with endvertices in S. Moreover, each vertex of K has a neighbor in $V(e)$.*

Proof. Define $\{x, y\} := S$. Note that $J + V(e)$ has a cycle C containing e. If $G - S^+$ has two different connected components K and K' differing from J, by Lemma 11, C avoids a cycle of the form x, K, y, K', x, a contradiction. This establishes the uniqueness of K.

Now we prove that $P := G[V(K) \cup S]$ is an (x, y)-path. By Lemma 11, P is connected. As C avoids P and G is e-Dirac, P is a tree. By Lemma 11, $P - x$ is connected, and, therefore, x is a leaf of P. Analogously, y is a leaf of P.

Suppose for a contradiction that l is a leaf of P differing from x and y. As $d_G(l) \geqslant 3$, $V(e) \subseteq N_G(l)$. As l is a leaf of P, $P - l$ is connected. By Lemma 11 x and y have neighbors in J. Now the cycle u_1, u_2, l, u_1 avoids a cycle of the form x, P, y, J, x, a contradiction. Hence x and y are the only leaves of P and P is an (x, y)-path and the first part of the lemma holds.

For the second part, simply observe that, as each inner vertex of P has degree two in P, it must have a neighbor in $V(e)$ in order to have degree at least three in G. □

The next lemma gives a full characterization of the e-Dirac graphs when $w(e, G) = 3$.

Lemma 23. *Let G be a 3-connected graph. Suppose that $e := uv \in E(G)$ is an edge of G and $w \in V(G)$ is such that $\{u, v, w\}$ is a 3-vertex cut of G. Then the following assertions are equivalent:*

(a) G is e-Dirac.

(b) Each connected component K of $G - \{u, v, w\}$ is a tree with $|N_G(w) \cap V(K)| = 1$.

Proof. Suppose (a). As G is 3-connected each vertex of $\{u, v, w\}$ has at least one neighbor in each connected component of $G - \{u, v, w\}$. Let K and L be different connected components of this graph. Suppose for a contradiction that either K is not a tree or K has more than one neighbor of w. In both cases $K + w$ has a cycle, which avoids a cycle of the form u, L, v, u, a contradiction to (a). This implies (b).

Conversely, if (b) holds, it is easy to check that $G - \{u, v\}$ is a tree and G is e-Dirac. □

Let G be an u_1u_2-Dirac graph with $w(u_1u_2, G) = 4$. Let X be a 4-vertex-cut of G containing $\{u_1, u_2\}$. We say that X is **two-sided** if, for each component K of $G - X$, there is $\{i, j\} = \{1, 2\}$ such that there is an edge from u_i to K but no edge from u_j to K; note that X is two sided if and only if $X - \{u_1, u_2\}$ is a vertex cut of $G\backslash u_1u_2$. We say that X is **one-sided** if, for each component K of $G - X$, there are edges from both u_1 and u_2 to K. We say that X is **no-sided** if X is neither one-sided nor two-sided.

Lemma 24. *Let G be an e-Dirac with $w(e, G) \geqslant 4$ and write $u_1u_2 := e$. Let S be pair of vertices in $V(G) - V(e)$. The following conditions are equivalent.*

(a) S is a vertex-cut of $G\backslash e$ and

(b) either

(b1) for some $\{i, j\} = \{1, 2\}$, $N_G(u_i) = S \cup u_j$; or

(b2) $S \cup V(e)$ is a two-sided vertex-cut of G.

Proof. Suppose that (a) holds and (b1) fails. As $G - S$ is connected, $G\backslash e - S = (G - S)\backslash e$ has exactly two connected components K_1 and K_2 with $u_i \in V(K_i)$ for $i = 1, 2$. As (b1) fails, u_i has a neighbor out of $S \cup u_{3-i}$, which must be in K_i, implying that $K_i - u_i$ is nonempty. Now each connected component of $G - (V(e) \cup S) = G\backslash e - (V(e) \cup S)$ is, for some $i \in \{1, 2\}$, a connected component of $K_i - u_i$ and, therefore, meets $N_G(u_i)$ but not $N_G(u_{3-1})$. Now (b2) holds and (a) implies (b).

If (b1) holds, as $w(e, G) \geqslant 4$, $|G| \geqslant 6$; so there is $v \in V(G) - N_G(u_i)$. Observe now that S separates u_i from v in $G\backslash e$ because $N_{G\backslash e}(u_i) = S$ and (a) holds. So (b1) implies (a).

It is left to prove that (b2) implies (a). Suppose (b2). Let, for $i = 1, 2$, U_i be the union of the vertices in components of $G - (V(e) \cup S)$ with a neighbor of u_i. As $S \cup V(e)$ is two-sided, $U_1 \cap U_2$ is empty and $U_2 \cup U_2 = V(G) - (V(e) \cup S)$. Now, for $i = 1, 2$, $G[U_i \cup u_i]$ is the connected component of $G\backslash e - S$ containing u_i and (a) holds. □

Lemma 25. *Suppose that G is an e-Dirac graph on more than five vertices, $w(e, G) = 4$, the endvertices of e has degree at least four, and all 4-vertex-cuts of G containing $V(e)$ are no-sided. Then $G - V(e)$ is a cycle.*

Proof. Suppose that the lemma fails. Denote $\{u_1, u_2\} := V(e)$. It follows from Lemma 22 that, for each 4-vertex-cut X containing $V(e)$, $G - X$ has exactly two connected components.

Now, for each 2-subset S of $V(G) - V(e)$ such that $S \cup V(e)$ is a 4-vertex cut of G, $S \cup V(e)$ is no-sided by hypothesis. The connected component of $G - (S \cup V(e))$ containing neighbors of both u_1 and u_2 will be called the S-*double* component and denoted by $D(S)$, while the component containing neighbors of only one element of $V(e)$ will be called the S-*single* component and denoted by $D^*(S)$. Choose S maximizing $|D^*(S)|$. Say that u_1 has a neighbor in $D^*(S)$ and u_2 does not.

Write $S = \{v_1, v_2\}$. By Lemma 22, $G[D^*(S) \cup S]$ is an (v_1, v_2)-path with all vertices in the neighborhood of u_1. If $|D(S)| = 1$, the lemma follows from Lemma 11. So, $|D(S)| \geqslant 2$.

(25.1). *For $i = 1, 2$, v_i has at least two neighbors in $D(S)$. Moreover, $v_1u_2, v_2u_2, v_1v_2 \notin E(G)$ and u_2 has at least three neighbors in $D(S)$.*

For the first part, first suppose for a contradiction that v_1 has at most one neighbor in $D(S)$. By Lemma 11, v_1 has a unique neighbor x in $D(S)$. For $A := \{x, v_2\}$, $A \cup V(e)$ separates $D^*(S) + v_1$ from $D(S) - x$, which is nonempty as we already verified that $|D(S)| \geqslant 2$. By Lemma 11, $D^*(S) + v_1$ is connected. By Hypothesis, $A \cup V(e)$ is no-sided and, by Lemma 22, $G - (A \cup V(e))$ has only two connected components. Hence $D(S) - x$ is connected. If $u_2v_1 \notin E(G)$, then, as u_2 has no neighbor in $D^*(S)$, there are no edges from u_2 to $D^*(S) + v_1$ and $D^*(A) = D^*(S) + v_1$; this contradicts the maximality of $|D^*(S)|$. Thus $u_2v_1 \in E(G)$. So, both u_1 and u_2 have neighbors in $D^*(S) + v_1$. This implies that $D(A) = D^*(S) + v_1$ and $D^*(A) = D(S) - x$. By Lemma 22, $G[D^*(S) \cup S]$ is a (v_1, v_2)-path. By Lemma 22 again, but now with A playing the role of S, $G[D^*(A) \cup A] = G[D(S) \cup v_2]$ is a (v_2, x)-path. Recall that x is the unique neighbor of v_1 in $D(S)$. Hence $G - V(e)$ is a cycle and the lemma holds in this case, a contradiction. Therefore, v_1 has at least two neighbors in $D(S)$ and, analogously, so does v_2. This proves the first part of (25.1).

Now, suppose for a contradiction that $v_2u_2 \in E(G)$. Since there are two neighbors of v_1 in $D(S)$, $D(S) + v_1$ has a cycle, which, by Lemma 11, must avoid a cycle of the form $u_1, u_2, v_2, D^*(S), u_1$, a contradiction. So, $v_2u_2 \notin E(G)$. Similarly $v_1u_2 \notin E(G)$. As $d_G(u_2) \geqslant 4$, u_2 has at least 3 neighbors in $D(S)$. We already saw that $v_1v_2 \notin E(G)$, since $G[D^*(S) \cup S]$ is a (v_1, v_2)-path. So (25.1) holds. $\Diamond$

(25.2). *$D(S)$ is a tree.*

Suppose for a contradiction that C is a cycle of $D(S)$. By Lemma 24, $G \backslash e$ is 3-connected. By Menger's Theorem, there are three pairwise disjoint $(\{v_1, v_2, u_1\}, V(C))$-paths α_1, α_2 and α_3 in $G \backslash e$. As $\{u_1, v_1, v_2\}$ separates $D^*(S)$ from C in $G \backslash e$, none of these paths meet $D^*(S)$.

Say that $\{v_1, a_1\}$, $\{v_2, a_2\}$ and $\{u_1, a_3\}$ are the respective pairs of endvertices of α_1, α_2 and α_3. Consider also $(u_2, V(C))$-paths β_1 and β_2 of the 2-connected graph $G - u_1$ intersecting only in u_2 and let b_1 and b_2 be the respective endvertices of β_1 and β_2 in C.

If for some $j \in \{1, 2\}$ and $i \in \{1, 2, 3\}$, β_j meets α_i out of C, then G has an $(u_2, \{v_1, v_2, u_1\})$-path γ avoiding C and, as a consequence, by Lemma 11, C avoids a cycle of the form $u_1, u_2, \gamma, D^*(S) + \{v_1, v_2, u_1\}$ containing e, a contradiction. So, α_i and β_j do not meet out of C. In particular, $u_2 \notin V(\alpha_i)$ for $i = 1, 2, 3$.

Let δ and ε be the (a_1, a_2)-paths of C meeting and avoiding a_3 respectively. If β_j has an endvertex in $\mathrm{Int}(\delta)$, then, the cycle $u_1, u_2, \beta_j, b_j, \delta, a_3, \alpha_3, u_1$ avoids a cycle of the form $v_1, \alpha_1, a_1, \varepsilon, a_2, v_2, D^*(S), v_1$, a contradiction. So, b_1 and b_2 are in ε and we may assume that b_1 is closer to a_1 than b_2 in ε. See an illustration in Figure 7.

We define H as the subgraph of G in Figure 7, more precisely, for edges e_1, e_2, and e_3 linking v_1, v_2 and u_1 to $D^*(S)$ (which exist by Lemma 11):

$$H := (D^*(S) \cup C \cup \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \beta_1 \cup \beta_2) + \{e_1, e_2, e_3\}.$$

By (25.1), v_1 has two different neighbors in $D(S)$ and there is a $(v_1, V(H))$-path φ of $D(S) + v_1$ beginning with an edge out of H. Let x be the other endvertex of φ. Next we

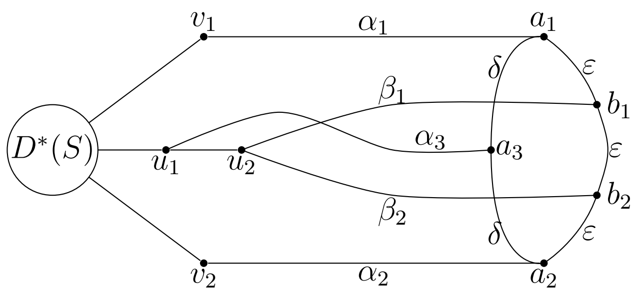

Figure 7: An illustration of the proof of (25.2).

consider, for $i = 1, 2$, the following cycle:

$$D_i := u_1, u_2, \beta_i, b_i, \varepsilon, a_i, \delta, a_3, \alpha_3, u_1.$$

Note that $H - V(D_i)$ is connected for $i = 1, 2$. By Lemma 13, $x \in V(D_i)$ for $i = 1, 2$. Hence, $x \in V(D_1) \cap V(D_2) \cap V(D(S)) = V(\alpha_3) - u_1$. Now, $v_1, \alpha_1, a_1, \delta, a_3, \alpha_3, x, \varphi, v_1$ avoids a cycle of the form $u_1, u_2, \beta_2, \varepsilon, a_2, \alpha_2, v_2, D^*(S), u_1$, a contradiction. ♢

(25.3). *u_1 and u_2 have no common neighbor as a leaf of $D(S)$.*

Suppose that l is a leaf contradicting (25.3). By (25.1), v_1 and v_2 have neighbors in $D(S) - l$, which is connected as l is a leaf. Then, by Lemma 11, a cycle of the form $v_1, D(S) - l, v_2, D^*(S), v_1$ avoids the cycle u_1, u_2, l, u_1, a contradiction. ♢

(25.4). *Let l be a leaf of $D(S)$. Then, $u_2 l \notin E(G)$, $u_1 l \in E(G)$ and there is a unique index $i \in \{1, 2\}$ such that $l v_i \in E(G)$.*

First let us prove that there is no leaf l of $D(S)$ such that $lv_1, lv_2 \in E(G)$. Suppose that l is such a leaf. If u_1 has a neighbor in $V(D(S)) - l$, then by (25.1), so does u_2 and, by Lemma 11, cycles of the form $u_1, u_2, (D(S) - l), u_1$ and $v_1, l, v_2, D^*(S), v_1$ avoid each other, a contradiction. Hence l is the unique neighbor of u_1 in $D(S)$. As we can repeat the same argument to any other leaf l' of $D(S)$ in the neighborhood of both v_1 and v_2 to conclude that $u_1 l' \in E(G)$ and l' is the unique neighbor of u_1 in $D(S)$, we conclude that:

$$l \text{ is the unique leaf of } D(S) \text{ incident to both } v_1 \text{ and } v_2. \tag{1}$$

By (25.1), $|D(S)| \geqslant 2$, so $D(S)$ has a leaf l' differing from l. As $N_G(u_1) \cap D(S) = \{l\}$, we have $N_G(l') - D(S) \subseteq \{u_2, v_1, v_2\}$ and, as $d_G(l') \geqslant 3$, we may assume that $l' v_1 \in E(G)$. By (1), it follows that $l' v_2 \notin E(G)$ and, therefore, $l' u_2 \in E(G)$. By (25.1), v_2 has a neighbor in $D(S) - l$. Now, since $v_1 l' \in E(G)$, (1) implies that v_2 has a neighbor in $D(S) - \{l, l'\}$. So, $(D(S) - l') + v_2$ has a cycle, which, by Lemma 11, avoids a cycle of the form $u_1, u_2, l', v_1, D^*(S), u_1$, a contradiction. So, no leaf of $D(S)$ is a common neighbor to v_1 and v_2.

By (25.3), this implies that each leaf of $D(S)$ is adjacent to a unique element of $\{u_1, u_2\}$ and a unique element of $\{v_1, v_2\}$. Let us prove that no leaf is adjacent to u_2. Suppose that some leaf l'' is adjacent to u_2. As l'' is adjacent to a unique vertex of $\{v_1, v_2\}$ we may assume that $l''v_1 \in E(G)$ and $l''v_2 \notin E(G)$. By (25.1), v_2 has at least two neighbors in $D(S) - l''$ and, by Lemma 11, cycles of the form $v_2, D(S) - l'', v_2$ and $u_1, u_2, l'', v_1, D^*(S), u_1$ avoid each other, a contradiction. This implies (25.4). $\diamondsuit$

(25.5). *$D(S)$ has exactly two leaves.*

Suppose the contrary. Then, $D(S)$ has a vertex w with $d_{D(S)}(w) \geqslant 3$. Furthermore, $D(S) - w$ has different connected components K_1, K_2 and K_3, each one containing a leaf of $D(S)$. Let $t \in \{1, 2, 3\}$. By (25.4), K_t has a neighbor of u_1 and a neighbor of some $w_t \in \{v_1, v_2\}$. By (25.1), u_2 has a neighbor in $D(S) - w$, so, we may assume that u_2 has a neighbor in K_1 and, therefore, $K_1 + \{u_1, u_2\}$ has a cycle C containing e. By Lemma 11, $D^*(S) + \{v_1, v_2\}$ is connected, and as each one of K_2 and K_3 has an edge to $\{v_1, v_2\}$, there are two edges linking $D^*(S) + \{v_1, v_2\}$ to $(K_2 \cup K_3) + w$, which is also connected. This implies that there is a cycle in $(D^*(S) \cup K_2 \cup K_3) + \{w, v_1, v_2\}$ avoiding C, a contradiction. $\diamondsuit$

Now, we may write $D(S)$ as a path $w_1, \ldots, w_n$. By (25.4), $u_1w_1, u_1w_n \in E(G)$. By (25.1) and (25.4), there are $1 < a < b < n$ such that $w_a, w_b \in N_G(u_2)$. Also by (25.1) and (25.4), there are edges from $H := D^*(S) + \{v_1, v_2\}$ to w_1, w_n and a vertex w_d with $1 < d < n$. If $d < b$, then a cycle of the form $w_1, \ldots, w_d, H, w_1$ avoids $u_1, u_2, w_b, \ldots, w_n, u_1$, a contradiction. Thus $d \geqslant b$. But, symmetrically, we have $d \leqslant a$, a contradiction. $\square$

Proof of Theorem 8: Let us first prove that if (a), (b) or (c) holds, then G is e-Dirac. This is clear if (b) holds. If (a) holds, this follows from Lemma 23. So, assume that (c) holds. Suppose for a contradiction that C and D are vertex-disjoint cycles of G with $e \in E(D)$. Note that $D \backslash e$ is an (R_1, R_2)-path and, therefore, $D \backslash e$ meets $\{x, y\}$. We may assume that $x \in D$. Note that $\{y\}$ is a vertex-cut of $G - V(D)$ separating $R_1 - V(D)$ from $R_2 - V(D)$; hence C is entirely contained in one of these graphs. Say that C is cycle of $R_1 - V(D)$. Note that $\delta := D \cap V(R_1)$ is an (u_1, x)-path avoiding C, which is a cycle of $R_1 - \{u_1, x\}$. Since G is 3-connected, $G \backslash e$ is 2-connected and, by Menger's Theorem has internally disjoint (u_1, u_2)-paths ζ_x and ζ_y. As $\{x, y\}$ separates u_1 from u_2 in $G \backslash e$, we may assume that $z \in V(\zeta_z)$ for each $z \in \{x, y\}$; define $\rho_z := \zeta_z \cap R_1$. As G is 3-connected, each vertex of $R_1 - \{x, y, u_1\}$ has degree at least 3 in G, and as $\{x, y, u_1\}$ separates $R_1 - \{x, y, u_1\}$ from the other vertices of G, the vertices of $R_1 - \{x, y, u_1\}$ also have degree at least 3 in R_1. Now δ and C contradict Lemma 17. This proves one of the implications of the theorem.

For the other implication, suppose that G is e-Dirac and let us prove that one of the assertions (a), (b) or (c) holds.

First suppose that G has no vertex cut containing $V(e)$. By Lemma 10, $G \cong K_{3,3}^{1,1}$ and the endvertices of e have degree three. For $\{x, y\} = N_G(u_1) - u_2$ we have item (c). So we may assume that G has a vertex-cut containing $V(e)$.

If $w(e, G) = 3$, (a) follows from Lemma 23. So, we may assume that $w(e, G) \geqslant 4$. Now we split the proof into two cases.

Case 1. $d_G(u_1), d_G(u_2) \geqslant 4$ and G has no two-sided 4-vertex cuts.

As $d_G(u_1), d_G(u_2) \geqslant 4$, Lemma 21 implies that $w(e, G) = 4$. If all vertex-cuts containing $V(e)$ are no-sided, the result follows from Lemma 25. So we may assume that there is a one-sided 4-vertex-cut $S^+ := \{u_1, u_2, x, y\}$. By Lemma 22, $G - S^+$ has exactly two connected components K_1 and K_2 and $K_1 + \{x, y\}$ and $K_2 + \{x, y\}$ are (x, y)-paths. This implies that $G - \{u_1, u_2\}$ is a cycle. Finally, by Lemma 24, $G \backslash e$ is 3-connected. This establishes (b).

Case 2. Either some endvertex of e has degree 3 or G has a two-sided vertex cut S^+ containing $V(e)$.

By Lemma 24, $G \backslash e$ has a 2-vertex-cut $\{x, y\}$ and, as $w(e, G) = 4$, $G - \{u_1, u_2\}$ is 2-connected. Now we have to prove that (c.1) and (c.2) hold.

Define $\{u_1, u_2, x, y\} := S^+$. Let K_i be the union of the connected components of $G - S^+$ intersecting $N_G(u_i)$. In Case 2 each connected component of $G - S^+$ intersects only one of the sets $N_G(u_1)$ or $N_G(u_2)$, hence $V(K_1)$ and $V(K_2)$ are disjoint. Define $R_i := G[V(K_i) \cup \{u_i, x, y\}]$. We have to prove that R_i is an (u_i, x, y)-rope bridge for $i = 1, 2$ to establish (c.1).

Note that $\{u_2, x, y\}$ separates u_1 from $V(R_2)$ in G. This implies that $\{x, y\}$ separates u_1 from $V(R_2)$ in $G - u_2$. By Menger's Theorem, there is a pair of $(u_1, V(R_2))$-paths in $G - u_2$. As $x, y \in V(R_2)$, it follows that x and y are end-vertices of these paths and these paths lay in R_1. Now R_1 satisfies the hypothesis for Lemma 17. Suppose for a contradiction that R_1 is not an (u_i, x, y)-rope bridge. By Lemma 17, there is $z \in \{x, y\}$, a cycle C of $R_1 - \{u_1, z\}$ and an (u_1, z)-path γ of R_1 avoiding C. By Lemma 11, C avoids a cycle of the form $u_1, \gamma, z, R_2, u_2, u_1$, a contradiction. Hence, R_1 is an (u_1, x, y)-rope bridge. Analogously, R_2 is an (u_2, x, y)-rope bridge and (c.1) holds.

To prove (c.2) we may assume that there are two different connected components H_1 and H_2 of $G - S^+$ meeting $N_G(u_1)$.

Let us prove that $H_1 + x$ is a tree. Suppose for a contradiction that $H_1 + x$ is not a tree. By Lemma 11, each connected component of $G - S^+$ meet both $N_G(x)$ and $N_G(y)$. Hence $H_1 + x$ is connected and one of its cycles avoids a cycle of the form $u_1, u_2, R_2, y, H_1, u_1$. This implies that G is not e-Dirac, a contradiction. So $H_1 + x$ is a tree. Analogously, $H_1 + y$, $H_2 + x$, and $H_2 + y$ are trees.

As H_1 is connected and $H_1 + x$ is a tree, this implies that H_1 has a unique neighbor of x and, analogously, a unique neighbor of y. For each leaf l of H_1, $N_G(l) - V(H) \subseteq \{x, y, u_1\}$, and therefore l is adjacent to x or y. By the uniqueness of the neighbors of x and y in H_1, this implies that $H_1 + \{x, y\}$ is an (x, y)-path. As each one of its inner vertices has degree at least three, this implies that the inner vertices are adjacent to u_1. this proves (c.2) and concludes the proof of the theorem. □

Proof of Theorem 6: If items (a) or (b) of Theorem 8 hold, then items (b) or (a) of Theorem 6 hold respectively. So Assume that item (c) of Theorem 8 holds; for $i = 1, 2$ consider the (u_i, x, y)-rope bridge R_i as in that theorem. By Lemma 19, R_i is a minor of a graph R'_i as in Figure 5 or 6 replacing the label of u by u_i. We may pick R'_i with $V(R'_1) \cap V(R'_2) = \{x, y\}$. Now item (c) of Theorem 6 holds. Indeed, the graph in Figure 2 is obtained by concatenating two mirrored copies of the graph in Figure 5 and adding the edge e. Idem for Figures 3 and 6. Figure 4 is obtained by the concatenation of a mirrored version of Figure 6 on the left side with a copy of Figure 5 on the right side and the addition of the edge e. □

Proof of Theorem 7: Suppose that the result fails. As G is 3-connected G does not satisfy items (a) nor (c) of Theorem 8; hence item (b) holds. So $G - \{u, v\}$ is a cycle. As G is 4-connected, all vertices in $G - \{u, v\}$ have degree at least 4 and, therefore are adjacent to u and v. This establishes the theorem. □

5 Proofs of Theorems 4 and 5

In the end of this section we prove Theorems 4 and 5. Next we establish some lemmas and terminologies. If G contains no edge-disjoint cycles using e, then G is said to be **strongly e-Dirac**. It is clear that all e-Dirac graphs are strongly e-Dirac, by the main results the converse is not true. Although the class of e-Dirac graphs is closed under minors that use e, the same is not true for the class of strongly e-Dirac graphs; this class is closed under deletions but not under contractions of other edges than e. For example, when G is the prism, and e and f are edges not belonging to triangles, then G is strongly e-Dirac, but G/f is not.

Lemma 26. *If G is an e-Dirac graph with $|G| \geqslant 6$ satisfying item (a) of Theorem 8, then G is not strongly e-Dirac.*

Proof. Suppose that G is a graph contradicting the lemma and let $e = u_1u_2$. Consider a 3-vertex-cut $S := \{u_1, u_2, w\}$ of G as in item (a) of Theorem 8. If $G - S$ has three distinct connected components K_1, K_2 and K_3, then G has edge-disjoint cycles of the form u_1, u_2, K_1, u_1 and u_2, K_2, w, K_3, u_2, a contradiction. Thus $G - S$ has exactly two connected components, K and K'. As $|G| \geqslant 6$, we may assume that $|K| \geqslant 2$. So, there are two leaves l_1 and l_2 in K. According to item (a) of Theorem 8, we may suppose that $wl_1 \notin E(G)$. As $d_G(l_1) \geqslant 3$, $l_1u_1, l_1u_2 \in E(G)$. As $d_G(l_2) \geqslant 3$, l_2 has a neighbor in $\{u_1, u_2\}$, say $l_2u_1 \in E(G)$. Consider the (l_1, l_2)-path γ of K. Now, G has the cycle $u_1, l_1, \gamma, l_2, u_1$ edge-disjoint from a cycle of the form u_1, K', u_2, u_1, a contradiction. □

Lemma 27. *If G is an e-Dirac graph with $|G| \geqslant 6$, satisfying item (b) of Theorem 8, then G is not strongly e-Dirac.*

Proof. Suppose that G contradicts the lemma. Let $e = u_1u_2$, and $x_1, \ldots, x_n$ be a cyclic ordering of the cycle $G - \{u_1, u_2\}$ with $u_1x_1 \in E(G)$. By item (ii) of Theorem 8, $d_G(u_1) \geqslant 4$ and there are indices $1 < a < b \leqslant n$ for which $u_1x_a, u_1x_b \in E(G)$. Let c be an index for which $u_2x_c \in E(G)$. If $c \leqslant a$, then the cycle $u_1, x_1, x_2, \ldots, x_c, u_2, u_1$ avoids the edges of $u_1, x_a, x_{a+1}, \ldots, x_b, u_1$, a contradiction. So $c > a$. Let C be the cycle $u_1, x_1, x_2, \ldots, x_a, u_1$ and let α be the (x_b, x_c)-path of $G - V(C)$. Now C is edge-disjoint from the cycle $u_1, x_b, \alpha, x_c, u_2, u_1$. □

We say that an (u, x, y)-rope bridge is **strong** if all its steps have length one, no pair of steps have a common endvertex and u is incident only to the edges that start the ropes.

Lemma 28. *A graph R with a 3-subset $\{u, x, y\} \subseteq V(G)$ is a strong (u, x, y)-rope bridge with ropes ρ_x and ρ_y if and only if R has internally disjoint paths $\rho_x := v_0, \dots, v_n$ and $\rho_y := w_0, \dots, w_n$, with $v_n = x$, $w_n = y$ and $v_0 = w_0 = u$ and there is a family $\mathcal{P}$ of pairwise disjoint pairs of consecutive elements of $\{1, \dots, n\}$ such that*

(a) $V(R) = V(\rho_x) \cup V(\rho_y)$ *and*

(b) For $F = \{v_a w_b, v_b w_a : \{a, b\} \in \mathcal{P}\} \cup \{v_c w_c : c \text{ is in no member of } \mathcal{P}\} \cup E(\rho_x) \cup E(\rho_y)$, *either* $E(R) = F$ *of* $E(R) = F - \{xy\}$.

Proof. It is clear that a graph satisfying the given conditions is a strong (u, x, y)-rope bridge with ropes ρ_x and ρ_y. Let us prove the converse. Let $\rho_x := v_0, \dots, v_m$ and $\rho_y := w_0, \dots, w_n$ be the ropes with $v_m = x$, $w_n = y$ and $v_0 = w_0 = u$. By the definition of strong rope bridge, no pair of steps shares the same endvertices and u is adjacent only to v_1 and w_1; as no pair of steps may have a common endvertex in a strong rope bridge, it follows that $m = n$. We call σ_i the step with extremity in v_i. Let $\mathcal{P}$ be the family of pairs $\{i, j\}$ such that σ_i crosses σ_j. By (RB3), the members of $\mathcal{P}$ are pairwise disjoint. It also follows from (RB3) that each pair in $\mathcal{P}$ contains consecutive indices. Analogously, for a pair $\{i, j\} \in \mathcal{P}$, σ_i and σ_j also have endvertices that are neighbors in ρ_y. Now it is a simple induction on k to verify that the edge of σ_k is in F and this implies the lemma. □

Lemma 29. *Let R be a connected graph with vertices u, x, y and paths ρ_x and ρ_y from u to x and y respectively, satisfying $V(\rho_y) \cap V(\rho_x) = \{u\}$. Suppose that $d_G(v) \geqslant 3$ for all $v \in V(R) - \{u, x, y\}$. Then the following assertions are equivalent:*

(a) R is a strong (u, x, y)-rope bridge with ropes ρ_x and ρ_y.

(b) If C is a cycle of R, then R has no (u, z)-path edge-disjoint from C for each $z \in \{x, y\}$,.

Proof. It follows from Lemma 28 that (a) implies (b). Suppose (b). This implies item (b) of Lemma 17. So, R is a (u, x, y)-rope bridge with ropes ρ_x and ρ_y. Let us prove the R is strong as a rope bridge.

First we check that no step has an inner vertex. If σ is a step with an inner vertex z, then ρ_x is edge-disjoint from the cycle $u, z, \sigma, y_\sigma, \rho_y, u$, a contradiction to item (b).

Next we check that at most one step arrive at each vertex. If α and β are steps with a common end-vertex, say $x_\alpha = x_\beta$, then ρ_x is edge-disjoint from the cycle

$$x_\beta, \beta, y_\beta, \rho_y, y_\alpha, \alpha, x_\alpha,$$

a contradiction again to item (b).

Finally let us check that u is only adjacent to the edges in the end-vertices of ρ_x and ρ_y. Suppose that u is incident to an edge uz not in ρ_x or ρ_y. As we already proved that no steps have inner vertices, it follows from (RB4) that $z \in V(\rho_x) \cup V(\rho_y)$. Say $z \in V(\rho_x)$, now ρ_y is edge-disjoint from the cycle u, ρ_x, z, u, a contradiction to (b) again. This establishes (a) and finishes the proof of the lemma. □

Proof of Theorem 4: First note that all graphs described in the theorem are strongly e-Dirac. Let us prove the converse. Let G be a strongly e-Dirac graph. By Lemmas 27 and 26, G satisfies item (c) of Theorem 8, so consider distinct vertices $x, y \in V(G) - \{u_1, u_2\}$ and, for each $i \in \{1, 2\}$, an (u_i, x, y)-rope bridge R_i such that $V(R_1) \cap V(R_2) = \{x, y\}$ and $G = (R_1 \cup R_2) + e$ and R_1 and R_2 are induced subgraphs of G.

Let us prove that R_1 is a strong rope bridge. Suppose the contrary. By Lemma 29, there is a cycle C of R_1 and an (u_1, z)-path α of R_1 edge-disjoint from C for some $z \in \{x, y\}$. We may assume $z = x$. As R_2 is an (u_2, x, y)-rope bridge it has a rope ρ with an extreme in x, which avoids y. This implies that $xy \notin E(\rho)$. As xy is the unique edge that possibly is in $E(R_1) \cap E(R_2)$, it follows that ρ is edge-disjoint from R_2 and, consequently, from C. Now C is edge-disjoint from the cycle $u_1, u_2, \rho, x, \alpha, u_1$, a contradiction. So R_1 is a strong rope bridge. Analogously, R_2 is also a strong rope bridge.

If $xy \in E(R_i)$ for some $i \in \{1, 2\}$, then xy is in $E(G)$ and as R_i is an induced subgraph of G, $xy \in E(R_1) \cap E(R_2)$. Now the theorem follows from Lemma 28. So assume that $xy \notin E(G)$.

For each $\in \{x, y\}$ and $i \in \{1, 2\}$ let ρ_z^i be the rope of R_i with extreme in z. If both R_1 and R_2 have no steps arriving in x, by Lemma 28 on R_1 and R_2, it follows that $d_G(x) = 2$, a contradiction. So we may assume that R_1 has a step α arriving at x. As $xy \notin E(G)$, by Lemma 28, α is of the form x, y' with $y'y \in E(\rho_y^1)$ and x, y' crosses a step of the form x', y with $x'x \in E(\rho_x^1)$. If R_2 has no steps arriving in x, Lemma 28 implies the theorem. So assume that R_2 has a step arriving at x. As argued for R_1, R_2 has vertices x'' and y'' such that $x''x \in E(\rho_x^2)$, $y''y \in E(\rho_x^2)$ and x, y'' and y, x'' are steps crossing each other. Now the cycle x, y', y, y'', x is edge-disjoint from the cycle $u_1, u_2, \rho_x^2, x'', y, x', \rho_x^1, u_1$, a contradiction. □

Proof of Theorem 5: Consider a graph G as described in the theorem. By items (a)-(d), all cycles not contained in one of the G_i's are the cycles containing U, which are exactly the cycles containing e. As each $G_i + u_{i-1}u_i$ is $u_{i-1}u_i$-Dirac, it follows that G is e-Dirac. For the converse, suppose for a contradiction that G is a 2-connected strongly e-Dirac graph not fitting into the description of the theorem minimizing $|G|$.

The 3-connected strongly e-Dirac graphs, described in Theorem 4, fit trivially in the description in this theorem. So G is not 3-connected. As the theorem also holds trivially if $|G| \leqslant 3$, then G has a 2-vertex-cut $\{x, y\}$. This implies that we may write G as the union of two graphs H and K such that $|H|, |K| \geqslant 3$, $V(H) \cap V(K) = \{x, y\}$, $H + xy$ and $K + xy$ are 2-connected, $e \in V(H)$ and $xy \notin E(K)$.

Let us check that $H + xy$ is a strongly e-Dirac graph. Suppose for a contradiction that $H + xy$ has a pair of edge-disjoint cycles D_1 and D_2 with $e \in E(D_1)$. Then for some $i \in \{1, 2\}$, D_i is not a cycle of G. So, $xy \in E(D_i) - E(H)$. Let D be a cycle of $K + xy$ containing xy. Now $(D_i \cup D) \backslash xy$ and D_{3-i} are disjoint cycles of G whose union contains e, contradicting the fact that G is strongly e-Dirac. So $H + xy$ is a 2-connected strongly e-Dirac graph. As $|H| < |G|$, the theorem holds for $H + xy$. Consider, for $H + xy$, graphs $G'_1, \dots, G'_{n'}$ and vertices $u'_0, \dots, u'_{n'}$ as in the theorem, with $e = u'_0u'_{n'}$.

As $K + xy$ is 2-connected and $xy \notin E(K)$, then either K has a cycle or K is an xy-path. In the later case, G is isomorphic to a subdivision of H and as, the theorem holds for H, it is straighforward to verify that it also holds for G, so K has a cycle C_K.

If H has a cycle C_H containing e, then C_H and C_K are edge-disjoint cycles of G, a contradiction. So e is in no cycle of H. As $H + xy$ is 2-connected, this implies that $xy \notin E(H)$ and $\{e, xy\}$ is an edge-cut of $H + xy$. By the description of $H + xy$ as in the theorem, $\{x, y\} = \{u'_{i-1}, u'_i\}$ for some index $i \in \{1, \dots, n'\}$ such that $V(G'_i) = \{u'_{i-1}, u'_i\}$. We may assume without loss of generality that $(x, y) = (u'_{i-1}, u'_i)$.

Let us check that $K + xy$ is strongly xy-Dirac. Suppose for a contradiction that $K + xy$ has a pair of edge-disjoint cycles C_1 and C_2 with $xy \in E(C_1)$. Then for a cycle C of $H + xy$ with $e, xy \in E(C)$, $((C \cup C_1)\backslash xy, C_2)$ is a pair of edge-disjoint cycles of G whose union contains e, a contradiction. So $K + xy$ is a 2-connected strongly xy-Dirac graph. As $|K| < |G|$, we may apply the theorem on $K + xy$ in respect to the edge xy. Consider, for $K + xy$, graphs $G''_1, \dots, G''_{n''}$ and vertices $u''_0, \dots, u''_{n''}$ as in the theorem with $(x, y) = (u''_0, u''_{n''})$.

Now the graphs

$$(G_1, \dots, G_n) := (G'_1, \dots, G'_{i-1}, G''_1, \dots, G''_{n''}, G'_{i+1}, \dots, G'_{n'})$$

and vertices

$$(u_0, \dots, u_n) := (u'_0, \dots, u'_{i-1}, u''_1, \dots, u''_{n''-1}, u'_i, \dots, u'_{n'})$$

give a description of G according to the theorem. □

6 Prism-Minors

In this section we prove Theorem 3. The theorem follows straightforwardly from Lemmas 31, 32 and 34.

If G is a graph with a subgraph H' isomorphic to the subdivision of a graph H, we say that an H-minor of H' is an ***H*-topological minor** of G. If G has an H-topological minor using an edge e, then it is clear that G has an H-minor using e. The converse does not hold in general, but it is easy to verify that it is true provided G and H are 3-connected and H is cubic, which is the case in our concern: when H is the prism and G is 3-connected. We will use this fact with no mentions.

Let G be a 3-connected graph with an edge e. By Menger's Theorem, G is not e-Dirac if and only if G has a prism-minor H using e as an edge in a triangle of H. So, our problem lies within the class of e-Dirac graphs. As the prism minor are topological minor in our case, the following lemma is valid.

Lemma 30. *A 3-connected e-Dirac graph has a prism-minor using e if and only if it has vertex-disjoint cycles C and D and three vertex-disjoint $(V(C), V(D))$-paths α_1, α_2, and α_3 such that $e \in E(\alpha_3)$.*

The next two lemmas proves Theorem 3 for e-Dirac graphs satisfying items (a) and (b) of Theorem 8.

Lemma 31. *If G is a 3-connected e-Dirac graph with $|G| \geqslant 6$ satisfying item (b) of Theorem 8, then G has a prism-minor using e.*

Proof. Let $e = uv$. First suppose that $|V(G)-\{u,v\}| \geqslant 5$. Denote for each $\{x,y\} = \{u,v\}$, $A_x := N_G(x) - y$. Assume without losing generality that $|A_v| \geqslant |A_v|$. Choose distinct vertices $a, b \in A_u$ with the property that $|A_v - \{a,b\}|$ is maximum. As $|A_u \cup A_v| \geqslant 5$ and $|A_u|, |A_v| \geqslant 3$, this maximality implies that $|A_v - \{a,b\}| \geqslant 3$. Hence v has neighbors c and d with the property that a, b, c and d appear in this order in some cycle ordering of the cycle $G - \{u,v\}$. This implies that G has a subdivision of the prism containing e and this implies the lemma if $|V(G) - \{u,v\}| \geqslant 5$.

As $|G| \geqslant 6$, we may assume now that $|V(G) - \{u,v\}| = 4$. As $d_G(v) \geqslant 4$, there is a cycle ordering a, b, c, d of $G - \{u,v\}$ with the property that b, c, and d are neighbors of v and d is a neighbor of u. As $d_G(u) \geqslant 4$, either b or d is a neighbor of u, we may assume $ub \in E(G)$ as swapping the labels of b and d just inverts the cycle ordering. Now $\{u,a,b\}$ and $\{v,c,d\}$ induce triangles in G. But ad, bc and $e = uv$ are edges of G. So e is in a prism-minor of G. This finishes the proof. □

Lemma 32. *If G is a 3-connected e-Dirac graph satisfying item (a) of Theorem 8, then G has no prism-minor using e.*

Proof. For $e = uv$ and some vertex w, $A := \{u,v,w\}$ is a vertex-cut of G and each connected component of $G - A$ is a tree with a unique neighbor of w. Suppose for a contradiction that there is a prism-minor of G using e. By lemma 30, G has vertex disjoint cycles C and D and vertex-disjoint $(V(C), V(D))$-paths α_1, α_2 and α_3 with $e \in E(\alpha_3)$.

As the components of $G - A$ are trees, both C and D meet A. Since w has a unique neighbor in each connected component of $G-A$, then it is not possible that $V(C) \cap A = \{w\}$ or $V(D) \cap A = \{w\}$. So, we may assume that $u \in V(C)$ and $v \in V(D)$.

For $i = 1, 2$, let c_i and d_i be the endvertices of α_i in C and D respectively. Consider the cycle:

$$C' := c_1, C - u, c_2, \alpha_2, d_2, D - v, d_1, \alpha_1, c_1.$$

As the connected components of $G - A$ are trees, C' meets A. But $u, v \notin V(C')$. Hence $V(C') \cap A = \{w\}$. So $V(C') - w$ is entirely contained in a connected component of $G - A$, which, therefore, contains the two neighbors of w in C', a contradiction. □

Lemma 33. *Suppose that G is a 3-connected graph with an edge uv with the property that $G - \{u,v\}$ is 2-connected. Suppose that C and D are vertex-disjoint cycles of G such that $\{u,v\} \subseteq V(C) \cup V(D)$. Then G has a prism-minor using uv.*

Proof. If both u and v are in one of these cycles, say C, then we may choose C in such a way that $uv \in E(C)$; by applying Menger's Theorem on G to obtain three vertex-disjoint $(V(C), V(D))$-paths, we get a subdivision of the prism in G using e. So, assume that $u \in C$ and $v \in D$. As $G-\{u,v\}$ is 2-connected, we apply Menger's Theorem on $G-\{u,v\}$ to obtain two $(V(C) - u, V(D) - v)$ vertex disjoint-paths that, together with u, v, are three vertex-disjoint $(V(C), V(D))$-paths and G has subdivision of the prism containing uv in all cases. Therefore, G has a prism-minor and the lemma holds. □

Lemma 34. *If G is an e-Dirac graph with $|G| \geqslant 6$ satisfying item (c) of Theorem 8, then either G has a prism-minor using e or $G \cong W_n$, $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$ or $K'''_{3,n}$ for some $n \geqslant 3$.*

Proof. Suppose that the lemma fails. By Theorem 8 (c), G has vertices x and y and induced subgraphs R_1 and R_2 such that $V(R_1) \cap V(R_2) = \{x, y\}$ and, for $i = 1, 2$, R_i is an (u_i, x, y)-rope bridge with ropes ρ^i_x and ρ^i_y. Denote, for $i \in \{1, 2\}$ and $z \in \{x, y\}$, $\rho^i_z = u_i, z^i_1, \dots, z^i_{m(z,i)}$ with $z^i_{m(z,i)} = z$. Next we prove:

(34.1). *For some $i \in \{1, 2\}$, $d_G(u_i) = 3$.*

Suppose the contrary. This implies that R_1 and R_2 have steps. As $d_G(u_i) \geqslant 4$, u_i has a neighbor v_i in R_i other than x^i_1 and y^i_1.

First assume that R_1 has a step α with an inner vertex v, which is incident to u_1 by (RB5). Either v_2 is in a rope of R_2, say ρ^2_x if it is the case, or v_2 is in an inner step of R_2 by (RB4). In both cases there is an $(u_2, V(\rho^2_x) - u_2)$ path β internally vertex-disjoint from ρ^2_x and avoiding ρ^2_y. This implies that R_2 has a cycle C avoiding $\rho^2_y - u_2$ and containing u_2. Now C is vertex disjoint from $u_1, v, \alpha, \alpha_y, \rho^1_y, u_1$ and, by Lemma 33, G has a prism-minor using e, a contradiction. So R_1 has no step with an inner vertex. Analogously R_2 also has no step with an inner vertex.

This implies that, for $i = \{1, 2\}$, v_i is in a rope $\rho^i_{z_i}$. So we have the cycle $C_i := u_i, \rho^i_{z_i}, v_i, u_i$. If $v_1 \neq v_2$, C_1 and C_2 are vertex-disjoint and, by Lemma 33, G has a prism-minor using e, a contradiction. So $v_1 = v_2$. As $V(R_1) \cap V(R_2) = \{x, y\}$, we may assume that $v_1 = x = v_2$. As $v_1 \neq x^1_1$, and $d_{R_1}(x^1_1) \geqslant 3$, R_1 has a step γ with extreme in x^1_1. Now C_2 is vertex disjoint from the cycle $u_1, x^1_1, \gamma, y_\gamma, \rho^1_y, u_1$ and, again we have a contradiction to Lemma 33. ♢

Now we may assume that $d_G(u_2) = 3$. In particular this allow us to pick $\{x, y\} = N_G(u_2) - u_1$. As R_2 is a trivial rope-bridge, now it makes sense simplifying our notation by $\rho^1_z := \rho_z$, $m(z) := m(z, 1)$ and $x_k := x^1_k$ for each $z \in \{x, y\}$ and $i = 1, \dots, m(z)$. Also, when we talk about ropes and steps, is about the structures of R_1. In particular, note that the unique possible step of R_2 is x, y, but it is also a step of R_1 in this case, so we do not need to consider the steps of R_2 any longer.

(34.2). *If a step has an inner vertex, then its end-vertices are x and y.*

Suppose that α is a step with an inner vertex z and extreme other than x or y. By (RB4), $u_1 z \in E(G)$. Say that $\alpha_x \neq x$. As $d_G(x) \geqslant 3$, there is a step β with an extreme in x. Now $u_2, x, \beta, y_\beta, \rho_y, y, u_2$ is a cycle vertex-disjoint from u_1, z, α, α_x. By Lemma 33, there is a prism-minor using e, a contradiction. ♢

(34.3). *Each step has at most one inner vertex.*

Suppose that the claim fails. So R_1 has a step α with two adjacent inner vertices a and b.

Suppose first that R_1 has a step β other than α. Now the cycle u_1, a, b, u_1 is vertex-disjoint from the cycle

$$u_2, x, \rho_x, x_\beta, \beta, y_\beta, \rho_y, y, u_2.$$

Now, by Lemma 33, G has a prism-minor using e, a contradiction. So α is the unique step of R_1.

Now $\rho_x = u_1, x$ and $\rho_y = u_1, y$ because the existence of any other vertex in the ropes would imply the existence of some step arriving at this vertex. Now note that each vertex of G is a neighbor of u_1 and $G - u_1$ is the cycle u_2, x, α, y, x. This implies that G is isomorphic to a wheel and the lemma holds, a contradiction. ◊

(34.4). *There is no step with an inner vertex.*

Suppose that the claim fails. We will prove that the lemma is valid proving that $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$. For this purpose it suffices to prove that $N_G(v) = \{u_1, x, y\}$ for each $v \in V(G) - \{u_1, x, y\}$. This is true for $v = u_2$ by our assumption after proving (34.1).

First suppose that v is an inner vertex of a step, by (34.2) and (34.3), $x, y \in N_G(v)$. As $d_G(v) \geqslant 3$, v is incident to at least one edge out of the step that v is in. By (RB5), u_1v is the only such an edge. So, we proved that $N_G(v) = \{u_1, x, y\}$ for each inner vertex v of each step.

Now it is left to prove that the vertices in the ropes either are in $\{u_1, x, y\}$ or have this set as neighborhood. Suppose that v is a vertex in $V(\rho_x) - \{u_1, x\}$. As $d_G(v) \geqslant 3$, v is the extreme of a step β. As the claim fails, there is a step α with an inner vertex. By (34.2) x and y are the extremes of α. By (RB6) applied on α, β is the unique step with an extreme in $\mathrm{Int}(\rho_x)$. This implies that $\rho_x = u_1, v, x$. Moreover, by (RB5), this uniqueness of β implies that the unique neighbor of v other than u_1 and x is its neighbor in β, call it w. We have to prove that $w = y$. Since β does not have x and y as extremes, by (34.2), $\beta = v, w$ and $w \in V(\rho_y)$. If $w \neq y$, it follows that the cycles u_1, v, w, ρ_y, u_1 and u_2, x, α, y, u_2 are vertex-disjoint, contradicting Lemma 33. Hence $\beta = v, y$. So $N_G(v) = \{u_1, x, y\}$. Analogously this also holds if $v \in \mathrm{Int}(\rho_y)$ and the claim holds. ◊

(34.5). *There is a pair of crossing steps.*

Suppose that the claim fails. By (34.4) each step has only two vertices. If there are steps x_a, y_b and x_c, y_d with $a < c$ and $b < d$, then we have the vertex-disjoint cycles $u_1, \rho_x, x_a, y_b, \rho_y, u_1$ and $u_2, x, \rho_x, x_c, y_d, \rho_y, y, u_2$, contradicting Lemma 33. Hence there is no such a pair of steps. This fact together with the absence of crossing steps implies that either all steps arrive in the same vertex of ρ_x or all steps arrive in the same vertex of ρ_y, we may assume the later case. In particular, this implies that $\rho_y = u_1, y$. Note that $G - y$ is the cycle u_1, ρ_x, x, u_2, u_1. This implies that G is a wheel and the lemma holds, a contradiction. ◊

(34.6). *$\rho_z = u_1, z_1, z$ for each $z \in \{x, y\}$*

By (34.5), there are steps x_a, y_b and x_c, y_d crossing each other. Say that $a < c$ and $d < b$. By (34.4), all steps are of the form x_i, y_j for some indices i and j.

Let x_i be a vertex of ρ_x, let us prove that $i \in \{a, b\}$. Assume the contrary.

First suppose that $i < a$. As $d_G(x_i) \geqslant 3$, there is a step of the form x_i, y_j. By (RB3), x_c, y_d do not cross x_i, y_j, so $j \leqslant d$. Now we have the vertex-disjoint cycles $u_1, \rho_x, x_i, y_i, \rho_y, u_1$ and $u_2, x, \rho_x, x_a, y_b, \rho_y, y, u_2$ contradicting Lemma 33. So, $i > a$.

Next suppose that $i > c$. By (RB3), x_a, y_b do not cross x_i, y_j, so $j \geqslant b$. Now we have the vertex-disjoint cycles $u_1, \rho_x, x_c, y_d, \rho_y, u_1$ and $u_2, x, \rho_x, x_i, y_j, \rho_y, y, u_2$ contradicting Lemma 33. So, $a < i < b$.

As $b > d$, either $j > d$ and x_i, y_j crosses x_c, y_d or $j < b$ and x_i, y_j crosses x_a, y_b. In both cases we have a contradiction to (RB3). Hence $i \in \{a, b\}$. This implies that ρ_x has only two vertices other than u_1 and $\rho_x = u_1, x_1, x$. Analogously $\rho_y = u_1, y_1, y$ and the claim holds. ♢

Now (34.6), (34.4) and (RB4) imply that $V(G) = \{u_1, u_2, x_1, y_1, x, y\}$.

Note that (34.4) and (34.5) and (34.6) imply that x_1, y and x, y_1 are steps. By our assumption after proving (34.1), $N_G(u_2) = \{u_1, x, y\}$. Hence $u_1x_1, u_1y_1, x_1x, y_1y \in E(G)$. This implies that G has a subgraph isomorphic to $K_{3,3}$ with stable sets $A := \{u_1, x, y\}$ and $B := \{u_2, x_1, y_1\}$. If $G[A]$ or $G[B]$ has no edges, then $G \cong K_{3,3}$, $K'_{3,3}$, $K''_{3,3}$ or $K'''_{3,3}$ and the lemma holds. So both $G[A]$ and $G[B]$ have edges. As $N_G(u_2) = \{u_1, x, y\}$, x_1y_1 is the unique edge of $G[B]$. So two steps arrive in x_1 and, by (RB6), $u_1x \notin E(G)$. Analogously, $u_1y \notin E(G)$. Hence xy is the unique edge of $G[A]$. Now we have the vertex-disjoint cycles u_1, x_1, y_1, u_1 and u_2, x, y, u_2, contradicting Lemma 33. □

Theorem 3 now follows from Lemmas 31, 32 and 34.

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