# An Algebraic Formulation of Hypergraph Colorings

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**Abstract**

A hypergraph is properly vertex-colored if no edge contains vertices which are assigned the same color. We provide an algebraic formulation of the $k$-colorability of uniform and non-uniform hypergraphs. This formulation provides an algebraic algorithm, via Gröbner bases, which can determine whether a given hypergraph is $k$-colorable or not. We further study new families of $k$-colorings with additional restrictions on permissible colorings. These new families of colorings generalize several recently studied variations of $k$-colorings.

**Mathematics Subject Classifications:** 05C15

## 1 Introduction

A hypergraph is a pair $(V, E)$ where $V$ is a nonempty finite set of *vertices* and $E$ is a nonempty set of nonempty subsets of $V$ called *edges*. A hypergraph is called *$r$-uniform*, or simply uniform, if all elements of $E$ have the same cardinality $r$. For an integer $k \geqslant 2$, we call a hypergraph properly $k$-colorable if there is a map from the set of vertices to a set of colors $C$, $|C| = k$, such that no edge is colored by a single color. We will call such mappings proper colorings and call a hypergraph $k$-colorable for short in this paper if it admits a proper $k$-coloring. Our notation for hypergraphs follows Diestel [10].

In this paper we examine several algebraic formulations of the vertex $k$-colorability of hypergraphs. Our primary goal is to establish the existence of an algorithm which will determine if a given hypergraph is properly $k$-colorable, and find proper $k$-colorings should they exist. This is accomplished by computing Gröbner bases for polynomial ideals. In addition, we introduce $k$-coloring *schemes* for hypergraphs which classify proper colorings based on the distribution of colors to the vertices of each edge. As with $k$-colorability, we will provide an algorithm which will determine if such a coloring exists and finds the colorings if they exist. The concept of coloring schemes generalizes recently studied families of colorings: stably bounded hypergraphs and (unoriented) pattern hypergraphs ([6], [15], [23], [24]).

The current paper was inspired by algebraic characterizations of graph colorability presented by Hillar and Windfeldt in [18]. Our first goal is to prove a partial generalization of an algebraic characterization of graph colorability proved in [18]. The full algebraic characterization presented in [18] builds on results from many authors including Alon, Bayer, de Loera, Lovász, and Tarsi, ([1], [2], [4], [11], [12], [13], [21]). As a main algebraic tool, we introduce an ideal corresponding to a zero-dimensional variety which represents all of the proper colorings of a given hypergraph. A similar algebraic characterization of 2-colorability for uniform hypergraphs was provided in [20]. The method presented there does not seem to immediately generalize to the case of $k$-colorability with $k > 2$.

The second goal of this paper is to introduce a similar algebraic ideal corresponding to proper colorings which satisfy additional restrictions on colors used on the edges. We introduce *coloring schemes* for edges to formalize additional restrictions on proper colorings. The concept of coloring schemes generalizes recently studied families of mixed hypergraphs, stably bounded hypergraphs, and pattern hypergraphs, ([5], [6], [15], [24]). We explicitly describe the connections in section 3.2.

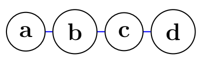
We will restrict our discussion to simple hypergraphs; that is, hypergraphs without singleton edges, and with no edge being contained in another. This is analogous to removing loops and multiple edges from a graph. All colorings considered in this paper are *vertex* colorings, though we will often refer to the vertices (once colored) which compose an individual edge as a 'colored edge'. Additionally, we will first formulate our results for *uniform* hypergraphs for clarity. We extend our results to the non-uniform case in a later section.

### 1.1 Color Patterns and Coloring Schemes

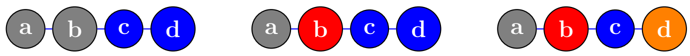
In Section 3, we study $k$-colorings with additional restrictions on colors assigned to vertices of each edge. An example of such a coloring, and our motivation for this aspect of the paper, is the conflict-free coloring which requires each edge to have a non-repeating color, [16]. We would like to provide an informal description of the idea of color patterns and coloring schemes in this subsection. Formal definitions are presented in Section 3.

Coloring schemes differentiate proper colorings of hypergraphs in a sense that is not possible for colorings of graphs. Each edge in a graph, which may be considered as a 2-uniform hypergraph, may only be (vertex) colored in two ways: either proper, with the vertices having different colors assigned, or improper, with the vertices both assigned the same color. For an $r$-uniform hypergraph, with $r > 2$, each edge may also be either properly or improperly colored. However, we will further distinguish proper colorings by the *color patterns* on each edge. These color patterns depend on the size of the edge, $r$, and the number of colors, $k$, used.

Consider the following edge of a 4-uniform hypergraph containing vertices $\{a, b, c, d\}$



with the following colorings:




Each coloring is a proper coloring of the edge, yet each is associated with a different color pattern. We will describe color patterns using partitions of $r$. In the case above $r = 4$ and we say that the first edge has a color pattern $\{2, 2\}$ meaning that two vertices of the edge are colored by one color and the remaining two vertices are colored by another color. Note that $\{2, 2\}$ is a partition of 4. We say the second edge has a color pattern $\{2, 1, 1\}$ and the third edge a color pattern $\{1, 1, 1, 1\}$. All of these are partitions of 4.

For the sake of an example, let us say we want to study colorings of 4-uniform hypergraphs with edges colored by two colors used on a pair of vertices or edges with all vertices colored by a distinct color. To describe such colorings we introduce the notion of coloring schemes. This example corresponds to the coloring scheme $\{\{2, 2\}, \{1, 1, 1, 1\}\}$. That is, a coloring scheme consists of one or more color patterns by which an edge may be colored in a vertex coloring.

A formal definition of color patterns and coloring schemes, along with their connections to pattern and stably bounded hypergraphs is provided in Section 3.

### 1.2 Outline

The paper is organized in the following way. Proper $k$-colorings of uniform hypergraphs and the necessary definitions are presented in Section 2. Section 3 addresses the characterization of proper colorings by specific color patterns which define coloring schemes. We present the proofs of the results from Sections 2 and 3 in Sections 5 and 6. Section 4 contains a short overview of the required tools from commutative algebra and algebraic geometry. Section 7 presents an example of a hypergraph which admits specific coloring schemes, and does not allow others. In Section 8 we discuss the implementation of the algorithms implied by the results in Sections 2 and 3. We present some special hypergraph colorings and show how to present some recently studied hypergraph coloring classes in Section 8.2. The $k$-colorability of non-uniform hypergraphs and the extensions of coloring schemes are addressed in Section 9.

## 2 Uniform $k$-Colorability

Let us start by reviewing an algebraic characterization of the $k$-colorability of graphs. Let $k, n$ be positive integers and $K_n$ denote the complete graph on $n$ vertices. For a graph $G$ on $n$ vertices $\{1, \dots, n\}$, the graph polynomial, $P_G$, of $G$ is defined by

$$P_G(x_1, \dots, x_n) = \prod_{\{i,j\} \in E(G)} (x_i - x_j) \ .$$

Next, let the ideals $T_{n,k}$, $R_{n,k}$, and $R_{G,k}$ of $\mathbb{C}[x_1, \ldots, x_n]$ be defined as in [18]

$$\begin{aligned}
T_{n,k} &= \langle P_G : G \text{ consists of a copy of } K_{k+1} \text{ and a set of } n-(k+1) \text{ isolated vertices.}\rangle, \\
R_{n,k} &= \langle x_i^k - 1 : i \in [n]\rangle, \\
R_{G,k} &= R_{n,k} + \left\langle x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_i x_j^{k-2} + x_j^{k-1} : \{i,j\} \in E(G) \right\rangle.
\end{aligned}$$

The following theorem from [18] gives a complete characterization of $k$-colorability of graphs. Let $\Bbbk$ be an algebraically closed field with characteristic not dividing $k$, and set $R = \Bbbk[x_1, \ldots, x_n]$.

**Theorem** 2.1 (Theorem 1.1, [18]). *Let $k < n$ be positive integers and $G$ be a graph on $n$ vertices. The following statements are equivalent:*

(1) *The graph $G$ is not $k$-colorable.*

(2) $\dim_{\mathbb{C}} R/R_{G,k} = 0$ *as a vector space.*

(3) *The constant polynomial 1 belongs to the ideal $R_{G,k}$.*

(4) *The graph polynomial $P_G$ belongs to the ideal $R_{n,k}$.*

(5) *The graph polynomial $P_G$ belongs to the ideal $T_{n,k}$.*

Let $r \geqslant 2$ be an integer and $\mathcal{H}$ be an $r$-uniform hypergraph on $n$ vertices with $m$ edges. We will assume that the vertices of $\mathcal{H}$ are labeled by the first $n$ positive integers, i.e., $V(\mathcal{H}) = [n]$. For an integer $k \geqslant 2$, a *$k$-coloring* of the vertex set of $\mathcal{H}$ is defined to be a map, $\mathscr{C} : V(\mathcal{H}) \to C$, where $C$ is a set of $k$ distinct *colors*. We will often refer to a coloring explicitly as an $n$-tuple: $\mathscr{C} = (c_1, \ldots, c_n)$, where $c_i \in C$. A *proper* $k$-coloring of $\mathcal{H}$ is a $k$-coloring where no edge $e \in E(\mathcal{H})$ is monochromatic, i.e., no edge is colored by a single color. The $k^{th}$ roots of unity are frequently used to represent colors in the literature, ([4], [18], [20]). However, using these colors does not seem to immediately generalize to colorings of (uniform) hypergraphs by more than two colors. Instead, for the general $k$-colorability of hypergraphs we will utilize prime numbers as our colors.

For an integer $k \geqslant 2$, let $\mathcal{P}_k$ be the set of the first $k$ primes. We will equivalently define a $k$-coloring of the uniform hypergraph $\mathcal{H}$ as a map

$$\mathscr{C} : V(\mathcal{H}) \to \mathcal{P}_k.$$

Often, we will represent a particular $k$-coloring as an $n$-tuple $\mathbf{c} = (c_1, \ldots, c_n)$ where $c_i = \mathscr{C}(i)$, $i \in [n]$. Additionally, for each edge $e \in E(\mathcal{H})$, we let

$$e = \{e_1, e_2, \ldots, e_r\},$$

where $e_j \in V(\mathcal{H}) = [n]$ denote the vertices of the edge. Finally, in an abuse of notation, we allow the edge $e$ colored by $\mathscr{C}$ to be represented as:

$$\mathscr{C}(e) = (\mathscr{C}(e_1), \ldots, \mathscr{C}(e_r)).$$

Let $R$ be the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. A standard association of the vertices of a hypergraph to the corresponding variables in the ring $R$ is given by

$$i \to x_i \ , \quad i \in [n] \ .$$

With this association a coloring, $\mathscr{C} = (c_1, \ldots, c_n)$, of the hypergraph corresponds to the evaluation of each variable $x_i$ to the color $c_i = \mathscr{C}(i)$.

To describe the $k$-colorings of $\mathcal{H}$, we introduce several ideals in $R = \mathbb{C}[x_1, \ldots, x_n]$. First, we define the *color ideal*, $C_k$, by

$$C_k = \left\langle \prod_{p \in \mathcal{P}_k} (x_i - p) : i \in V(\mathcal{H}) \right\rangle .$$

The ideal $C_k$ is an equivalent form of $R_{n,k}$ from Theorem 2.1 using colors from $\mathcal{P}_k$. Thus $C_k$ captures, via its *variety*, every $k$-coloring of $\mathcal{H}$, including the improper colorings. We therefore need some additional conditions to ensure that only the proper colorings are described.

We introduce polynomials $f_e(x_{e_1}, \ldots, x_{e_r})$, $e = \{e_1, \ldots, e_r\} \in E(\mathcal{H})$, such that $f_e = 0$ if and only if the edge $e$ is properly colored. To achieve that, let $\mathcal{S}$ be the set of all submultisets of $\mathcal{P}_k$ of size $r$, i.e., repeating elements are allowed, and let $\mathcal{R} \subset \mathcal{S}$ be the set of all submultisets of $\mathcal{P}_k$ which represent a proper coloring of an edge $e \in E(\mathcal{H})$. Let $N\mathcal{R} = \mathcal{S} \setminus \mathcal{R}$.

Define a polynomial $h$ with $r$ variables $y_1, \ldots, y_r$ by

$$h(y_1, \ldots, y_r) = \sum_{1 \leqslant i < j \leqslant r} (y_i - y_j)^2$$

and note that $h$ vanishes only on $r$-tuples corresponding to improper colorings in $N\mathcal{R}$, (Lemma 5.1). Now, we introduce polynomials $f_e$, $e \in E(\mathcal{H})$, as follows

$$f_e = \prod_{u \in \mathcal{R}} \left( h(x_{e_1}, \ldots, x_{e_r}) - h(u) \right) \ .$$

Combining these polynomials with the color ideal $C_k$ we can introduce an ideal that will capture only the proper $k$-colorings of $\mathcal{H}$. Define the *$k$-colorability ideal* for the hypergraph $\mathcal{H}$, $I_{\mathcal{H},r}(k)$, by

$$I_{\mathcal{H},r}(k) = C_k + \langle f_e : e \in E(\mathcal{H}) \rangle \ .$$

The ideal $I_{\mathcal{H},r}(k)$ is a hypergraph generalization of $R_{G,k}$ from Theorem 2.1. We present the generators of $I_{\mathcal{H},r}(k)$ for the Fano Plane below in Example 2.1. In Section 5 we describe properties of the $k$-colorability ideal and its role in determining the colorability of a given hypergraph. To generalize Theorem 2.1, we define the *hypergraph polynomial for $k$-colorability*, $P_{\mathcal{H},r,k}(x_1, \ldots, x_n)$, by

$$P_{\mathcal{H},r,k}(x_1, \ldots, x_n) = \prod_{e \in E(\mathcal{H})} h(x_{e_1}, \ldots, x_{e_r}) \ .$$

In Section 5 we prove that $P_{\mathcal{H},r,k}$ characterizes all improper colorings of $\mathcal{H}$. Combining the above, we are now ready to state our generalization of Theorem 2.1 by Hillar and Windfeldt [18].

**Theorem** 2.2. *Let $r$, $k \geqslant 2$ be positive integers and let $\mathcal{H}$ be an $r$-uniform hypergraph on $n$ vertices. Let $R = \mathbb{C}[x_1, \ldots, x_n]$. Let $I_{\mathcal{H},r}(k)$ be the $k$-colorability ideal for $\mathcal{H}$ and $P_{\mathcal{H},r,k}$ be the hypergraph polynomial for $k$-colorability for $\mathcal{H}$. Then the following are equivalent:*

1. *The hypergraph $\mathcal{H}$ is not $k$-colorable.*
2. *$\dim_{\mathbb{C}} R/I_{\mathcal{H},r}(k) = 0$ as a vector space.*
3. *The constant polynomial 1 belongs to the ideal $I_{\mathcal{H},r}(k)$.*
4. *The hypergraph polynomial $P_{\mathcal{H},r,k}$ belongs to the ideal $C_k$.*

We prove the theorem in Section 5.

**Remark** 2.1. In a survey, Francisco et. al. [17], Problem 3.13, asks for an algebraic algorithm to compute the chromatic number of a hypergraph using properties of its edge ideal. Using Gröbner bases, Theorem 2.2 implies the existence of an algebraic algorithm to decide the $k$-colorability of $\mathcal{H}$, based however, on a different kind of an ideal, the $k$-colorability ideal.

In Section 9 we extend the results of this section to the non-uniform case. We close with an example to illustrate the $k$-colorability ideal.

**Example** 2.1. In this example, we will illustrate the set of generators of $I_{\mathcal{H},r}(k)$ for the Fano Plane and $k = 2, 3$ colors. The Fano Plane, $FP$, is a 3-uniform hypergraph on 7 vertices $V(FP) = \{1, 2, 3, 4, 5, 6, 7\}$ with 7 edges

$$E(FP) = \{\{1, 2, 5\}, \{1, 3, 7\}, \{1, 4, 6\}, \{2, 3, 6\}, \{3, 4, 5\}, \{2, 4, 7\}, \{5, 6, 7\}\} .$$

First, let us consider the case $k = 2$ colors. Then our colors are $\mathcal{P}_2 = \{2, 3\}$ and $\mathcal{R} = \{\{2, 2, 3\}, \{2, 3, 3\}\}$, $N\mathcal{R} = \{\{2, 2, 2\}, \{3, 3, 3\}\}$. Hence, the generators of $I_{FP,3}(2)$ are

$$\begin{aligned}
I_{FP,3}(2) = &\langle (x_i - 2)(x_i - 3) : i \in [7]\rangle + \\
&\langle ((x_1 - x_2)^2 + (x_1 - x_5)^2 + (x_2 - x_5)^2 - 2)((x_1 - x_2)^2 + (x_1 - x_5)^2 + (x_2 - x_5)^2 - 2), \\
&((x_1 - x_3)^2 + (x_1 - x_7)^2 + (x_3 - x_7)^2 - 2)((x_1 - x_3)^2 + (x_1 - x_7)^2 + (x_3 - x_7)^2 - 2), \\
&((x_1 - x_4)^2 + (x_1 - x_6)^2 + (x_4 - x_6)^2 - 2)((x_1 - x_4)^2 + (x_1 - x_6)^2 + (x_4 - x_6)^2 - 2), \\
&((x_2 - x_3)^2 + (x_2 - x_6)^2 + (x_3 - x_6)^2 - 2)((x_2 - x_3)^2 + (x_2 - x_6)^2 + (x_3 - x_6)^2 - 2), \\
&((x_3 - x_4)^2 + (x_3 - x_5)^2 + (x_4 - x_5)^2 - 2)((x_3 - x_4)^2 + (x_3 - x_5)^2 + (x_4 - x_5)^2 - 2), \\
&((x_2 - x_4)^2 + (x_2 - x_7)^2 + (x_4 - x_7)^2 - 2)((x_2 - x_4)^2 + (x_2 - x_7)^2 + (x_4 - x_7)^2 - 2), \\
&((x_5 - x_6)^2 + (x_5 - x_7)^2 + (x_6 - x_7)^2 - 2)((x_5 - x_6)^2 + (x_5 - x_7)^2 + (x_6 - x_7)^2 - 2)\rangle .
\end{aligned}$$

Next, consider the case of $k = 3$ colors. Now, our colors are $\mathcal{P}_3 = \{2,3,5\}$ and $\mathcal{R} = \{\{2,3,5\},\{2,2,3\},\{2,2,5\},\{2,3,3\},\{2,5,5\},\{3,3,5\},\{3,5,5\}\}$, $N\mathcal{R} = \{\{2,2,2\},\{3,3,3\},\{5,5,5\}\}$. Hence, the generators of $I_{FP,3}(3)$ are

$$
\begin{aligned}
&I_{FP,3}(3) = \langle (x_i-2)(x_i-3)(x_i-5) : i \in [7]\rangle + \\
&\langle\ ((x_1-x_2)^2+(x_1-x_5)^2+(x_2-x_5)^2-14)((x_1-x_2)^2+(x_1-x_5)^2+(x_2-x_5)^2-2) \\
&\quad ((x_1-x_2)^2+(x_1-x_5)^2+(x_2-x_5)^2-18)((x_1-x_2)^2+(x_1-x_5)^2+(x_2-x_5)^2-2) \\
&\quad ((x_1-x_2)^2+(x_1-x_5)^2+(x_2-x_5)^2-18)((x_1-x_2)^2+(x_1-x_5)^2+(x_2-x_5)^2-8) \\
&\quad ((x_1-x_2)^2+(x_1-x_5)^2+(x_2-x_5)^2-8), \\
&((x_1-x_3)^2+(x_1-x_7)^2+(x_3-x_7)^2-14)((x_1-x_3)^2+(x_1-x_7)^2+(x_3-x_7)^2-2) \\
&\quad ((x_1-x_3)^2+(x_1-x_7)^2+(x_3-x_7)^2-18)((x_1-x_3)^2+(x_1-x_7)^2+(x_3-x_7)^2-2) \\
&\quad ((x_1-x_3)^2+(x_1-x_7)^2+(x_3-x_7)^2-18)((x_1-x_3)^2+(x_1-x_7)^2+(x_3-x_7)^2-8) \\
&\quad ((x_1-x_3)^2+(x_1-x_7)^2+(x_3-x_7)^2-8), \\
&((x_1-x_4)^2+(x_1-x_6)^2+(x_4-x_6)^2-14)((x_1-x_4)^2+(x_1-x_6)^2+(x_4-x_6)^2-2) \\
&\quad ((x_1-x_4)^2+(x_1-x_6)^2+(x_4-x_6)^2-18)((x_1-x_4)^2+(x_1-x_6)^2+(x_4-x_6)^2-2) \\
&\quad ((x_1-x_4)^2+(x_1-x_6)^2+(x_4-x_6)^2-18)((x_1-x_4)^2+(x_1-x_6)^2+(x_4-x_6)^2-8) \\
&\quad ((x_1-x_4)^2+(x_1-x_6)^2+(x_4-x_6)^2-8), \\
&((x_2-x_3)^2+(x_2-x_6)^2+(x_3-x_6)^2-14)((x_2-x_3)^2+(x_2-x_6)^2+(x_3-x_6)^2-2) \\
&\quad ((x_2-x_3)^2+(x_2-x_6)^2+(x_3-x_6)^2-18)((x_2-x_3)^2+(x_2-x_6)^2+(x_3-x_6)^2-2) \\
&\quad ((x_2-x_3)^2+(x_2-x_6)^2+(x_3-x_6)^2-18)((x_2-x_3)^2+(x_2-x_6)^2+(x_3-x_6)^2-8) \\
&\quad ((x_2-x_3)^2+(x_2-x_6)^2+(x_3-x_6)^2-8), \\
&((x_3-x_4)^2+(x_3-x_5)^2+(x_4-x_5)^2-14)((x_3-x_4)^2+(x_3-x_5)^2+(x_4-x_5)^2-2) \\
&\quad ((x_3-x_4)^2+(x_3-x_5)^2+(x_4-x_5)^2-18)((x_3-x_4)^2+(x_3-x_5)^2+(x_4-x_5)^2-2) \\
&\quad ((x_3-x_4)^2+(x_3-x_5)^2+(x_4-x_5)^2-18)((x_3-x_4)^2+(x_3-x_5)^2+(x_4-x_5)^2-8) \\
&\quad ((x_3-x_4)^2+(x_3-x_5)^2+(x_4-x_5)^2-8), \\
&((x_2-x_4)^2+(x_2-x_7)^2+(x_4-x_7)^2-14)((x_2-x_4)^2+(x_2-x_7)^2+(x_4-x_7)^2-2) \\
&\quad ((x_2-x_4)^2+(x_2-x_7)^2+(x_4-x_7)^2-18)((x_2-x_4)^2+(x_2-x_7)^2+(x_4-x_7)^2-2) \\
&\quad ((x_2-x_4)^2+(x_2-x_7)^2+(x_4-x_7)^2-18)((x_2-x_4)^2+(x_2-x_7)^2+(x_4-x_7)^2-8) \\
&\quad ((x_2-x_4)^2+(x_2-x_7)^2+(x_4-x_7)^2-8), \\
&((x_5-x_6)^2+(x_5-x_7)^2+(x_6-x_7)^2-14)((x_5-x_6)^2+(x_5-x_7)^2+(x_6-x_7)^2-2) \\
&\quad ((x_5-x_6)^2+(x_5-x_7)^2+(x_6-x_7)^2-18)((x_5-x_6)^2+(x_5-x_7)^2+(x_6-x_7)^2-2) \\
&\quad ((x_5-x_6)^2+(x_5-x_7)^2+(x_6-x_7)^2-18)((x_5-x_6)^2+(x_5-x_7)^2+(x_6-x_7)^2-8) \\
&\quad ((x_5-x_6)^2+(x_5-x_7)^2+(x_6-x_7)^2-8)\ \rangle\ .
\end{aligned}
$$

Note that we did not simplify the generators of $I_{FP,3}(2)$ and $I_{FP,3}(3)$ above. Duplicate factors may arise in certain cases, for instance above when $k = 2$:

$$h(\{2,2,3\}) = (2-2)^2 + (2-3)^2 + (2-3)^2 = 0 + 1 + 1 = 2,$$

$$h(\{2,3,3\}) = (2-3)^2 + (2-3)^2 + (3-3)^2 = 1 + 1 + 0 = 2.$$

When used in actual computation, we can, of course, delete any repeating factors in the individual generating polynomials.

# 3 Coloring Schemes

Theorem 2.2 allows us to determine if a given uniform hypergraph admits any proper coloring. In this section, we will further distinguish proper colorings based on the *color schemes* used to color edges. We provide formal definitions for color patterns, coloring schemes, and the objects which determine them below. We aim to describe the ideals which will encode coloring schemes and provide a method for determining whether a given hypergraph admits a proper coloring with a given coloring scheme.

## 3.1 Coloring Schemes for Uniform Hypergraphs

Let $R$ be the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Let $\mathcal{H}$ be an $r$-uniform hypergraph on $n$ vertices. Fix an enumeration of $\mathcal{P}_k = \{p_1, \ldots, p_k\}$ the first $k$ prime numbers used as colors. We continue to use all notation introduced in Section 2.

We set

$$\Gamma[e] = \prod_{i=1}^{r} x_{e_i}$$

to be the product of the variables assigned to the vertices in edge $e = \{e_1, \ldots, e_r\} \in E(\mathcal{H})$. The *edge product* of an edge $e \in E(\mathcal{H})$ colored by a $k$-coloring $\mathscr{C} = (c_1, \ldots, c_n)$ is defined to be the product of the colors $\mathscr{C}$ assigns to the vertices in $e$

$$\Gamma[\mathscr{C}(e)] = \prod_{i=1}^{r} \mathscr{C}(e_i) = \prod_{i=1}^{r} c_{e_i}\ .$$

Each edge product in a $k$-coloring may be represented by an exponent vector of non-negative integers, $(\alpha_1, \ldots, \alpha_k)$, indexed in correspondence with the colors in $\mathcal{P}_k$. That is, $\alpha_i$ corresponds to the frequency of the color $p_i$ in the product. Note that the frequency of a particular color may be zero, as not all colors are required to be present on each edge. We wish to focus on the integer partition of $r$ embedded in $(\alpha_1, \ldots, \alpha_k)$ as the collection of non-zero $\alpha_i$ correspond to the colors used on the edge in the given $k$-coloring. We say such a partition is *extracted* from $(\alpha_1, \ldots, \alpha_k)$.

For a given edge product, only a single partition may be extracted. However, the correspondence between edge products and extracted partitions is not one-to-one, the same partition may be extracted from several distinct edge products. For example, consider $k = 5$ and $\mathcal{P}_5 = \{2, 3, 5, 7, 11\}$. The edge products $5 \cdot 5 \cdot 7 \cdot 3 \cdot 5$ and $3 \cdot 11 \cdot 2 \cdot 11 \cdot 11$ produce the exponent vectors $(0, 1, 3, 1, 0)$ and $(1, 1, 0, 0, 3)$ respectively. However, the same partition of 5, $\{3, 1, 1\}$, is extracted from each. It is this partition which we associate with a color pattern.

A partition $\lambda$ of $r$ is called a *color pattern*. Let $\mathscr{C}$ be a $k$-coloring of the $r$-uniform hypergraph $\mathcal{H}$. For an edge $e \in E(\mathcal{H})$, we say that $\mathscr{C}$ colors $e$ by the color pattern $\lambda$ if $\lambda$ is the partition of $r$ extracted from the exponent vector $(\alpha_1, \ldots, \alpha_k)$ representing $\Gamma[\mathscr{C}(e)]$.

Let $\Lambda_r$ be the set of all partitions of $r$, and $\Lambda_{\hat{r}}$ be the set of all partitions of $r$ which contain no part equal to $r$; we will refer to such partitions as *proper partitions* of $r$. For any $\lambda \in \Lambda_{\hat{r}}$ let the *expansion* of $\lambda$, $Ex(\lambda)$, denote the set of all $k$-tuples containing each part of $\lambda$ exactly once, and zeros in all other locations. In other words, $Ex(\lambda)$ is the collection of all exponent vectors of edge products from which $\lambda$ can be extracted.

**Example** 3.1. Let $r = 4$ and $k = 3$. Then

$$\Lambda_{\hat{r}} = \{\{3,1\},\{2,2\},\{2,1,1\},\{1,1,1,1\}\},$$

and the partition $\{4\}$ is omitted. Choosing $\lambda = \{3,1\}$, we have

$$Ex(\lambda) = \{(3,1,0),(3,0,1),(0,3,1),(0,1,3),(1,0,3),(1,3,0)\}.$$

Note, for $\tau = \{1,1,1,1\}$, $Ex(\tau) = \{\}$ as there are more parts in $\tau$ than there are positions in a 3-tuple.

Further, we define

$$EP(\lambda) = \left\{ \prod_{t=1}^{k} p_t^{\alpha_t} : (\alpha_1, \ldots, \alpha_k) \in Ex(\lambda) \right\}$$

to be the collection of all edge products associated with the partition $\lambda$. Finally, note that a proper coloring will correspond to partitions in which no part has size $r$. Thus we may collect all proper edge products, via $\Lambda_{\hat{r}}$, in the following set,

$$A = \{EP(\lambda) : \lambda \in \Lambda_{\hat{r}}\} \ .$$

**Example** 3.2. Consider $r = 4$ and $k = 5$. Then $\Lambda_{\hat{r}} = \{\{3,1\},\{2,2\},\{2,1,1\},\{1,1,1,1\}\}$, as above. However, the expansion of $\lambda = \{3,1\}$ is larger:

$$\begin{array}{rllll} Ex(\lambda) = & \{(3,1,0,0), & (1,3,0,0), & (1,0,3,0), & (1,0,0,3), \\ & (3,0,1,0), & (0,3,1,0), & (0,1,3,0), & (0,1,0,3), \\ & (3,0,0,1), & (0,3,0,1), & (0,0,3,1), & (0,0,1,3)\} \end{array}$$

When we combine these expansions with those of the other proper partitions in $\Lambda_{\hat{4}}$, we have a total of 31 exponent vectors viable for a proper coloring. Further, when we consider the $k = 5$ possible colors which may be applied to each vertex, we have a total of 155 ways to properly 5-color the vertices of a single edge in a 4-uniform hypergraph.

As with proper $k$-colorings, we may describe proper colorings of uniform hypergraphs which admit specified color patterns via polynomial ideals.

**Definition** 3.1. Let $k, r \geqslant 2$ be integers. A non-empty subset $M = \{\lambda_1, \ldots, \lambda_t\}$ of $\Lambda_{\hat{r}}$ is called a *$k$-coloring scheme* for an $r$-uniform hypergraph.

Note that a coloring scheme may consist of a single partition, we will refer to these coloring schemes as *homogeneous*, or may include all possible (proper) partitions of $r$. To translate coloring schemes to polynomial ideals, we provide the following.

**Definition** 3.2. Let $k, r \geqslant 2$ be integers and $M = \{\lambda_1, \ldots, \lambda_t\} \subseteq \Lambda_{\hat{r}}$ a $k$-coloring scheme. We define the *$k$-coloring scheme ideal* $S_{\mathcal{H},r}(M,k)$ by

$$S_{\mathcal{H},r}(M,k) = C_k + \left\langle \prod_{\lambda \in M} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{i=1}^{r} x_{e_i} - a \right) \right] : e \in E(\mathcal{H}) \right\rangle .$$

The ideals $I_{\mathcal{H},r}(k)$ and $S_{\mathcal{H},r}(M,k)$ and their associated (algebraic) varieties $\mathcal{V}(I_{\mathcal{H},r}(k))$ and $\mathcal{V}(S_{\mathcal{H},r}(M,k))$ are related in the following way. The *variety* of an ideal is defined in Section 4, Definition 4.2.

**Theorem** 3.1. Let $r$, $k \geqslant 2$ be integers and let $\mathcal{H}$ be an $r$-uniform hypergraph. Then

$$\mathcal{V}(I_{\mathcal{H},r}(k)) = \bigcup_{\substack{M \subseteq \Lambda_{\hat{r}} \\ M \neq \emptyset}} \mathcal{V}(S_{\mathcal{H},r}(M,k))$$

and

$$I_{\mathcal{H},r}(k) = \bigcap_{\substack{M \subseteq \Lambda_{\hat{r}} \\ M \neq \emptyset}} S_{\mathcal{H},r}(M,k) \; .$$

The proof of Theorem 3.1 is postponed to Section 6.

### 3.2 Coloring Classes of Hypergraphs

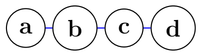
The color patterns and coloring schemes presented here are similar to the *edge types* used to define *pattern hypergraphs* introduced by Dvořák et. al. in [15], and *stably bounded hypergraphs* introduced by Bujtás and Tuza in [6]. Both pattern and stably bounded hypergraphs generalize several hypergraph vertex-coloring concepts including mixed hypergraphs, color-bounded hypergraphs, B-hypergraphs, and S-hypergraphs, [23]. We will review the basics of stably bounded and pattern hypergraphs in order to discuss their relationships to coloring schemes.

A *stably bounded hypergraph*, $\mathcal{H}$, is a six-tuple: $(V(\mathcal{H}), E(\mathcal{H}), s, t, a, b)$ where $V(\mathcal{H})$ and $E(\mathcal{H})$ are the vertex and edge sets respectively, and $s, t, a$, and $b$ are integer valued functions from $E(\mathcal{H})$ to $\mathbb{N}$, [6]. For each edge, the functions $s$ and $t$ are the lower and upper bounds on the number of distinct colors permissible on the edge. While $a$ and $b$ establish the minimum number of vertices required to share a color, and the maximum number of vertices which may share a color within the edge.

A *pattern* hypergraph is equipped with an ordering on the vertex set and a collection of *edge types* assigned to each edge [15]. An edge type is a list of specific color patterns, with respect to the ordering of the vertices, which constitute a proper vertex coloring. If an edge type contains all permutations of its color patterns, then the edge type is said to be *unoriented*. As the color patterns and coloring schemes introduced in this section do not rely on an ordering of the vertices, they correspond directly with unoriented pattern hypergraphs (see Example 3.3).

At the individual edge level, the coloring schemes developed in this paper constitute a coloring class which may be viewed as a refinement of stably bounded hypergraphs. Conversely, due to the possible ordering imposed on the vertices, pattern hypergraphs may be viewed as a further refinement of coloring schemes. By refinement, we mean the level of control each class permits. Each class restricts the coloring of vertices in an edge to certain patterns, with some limits. For example, let us return to the 4-uniform edge from Section 1.1.

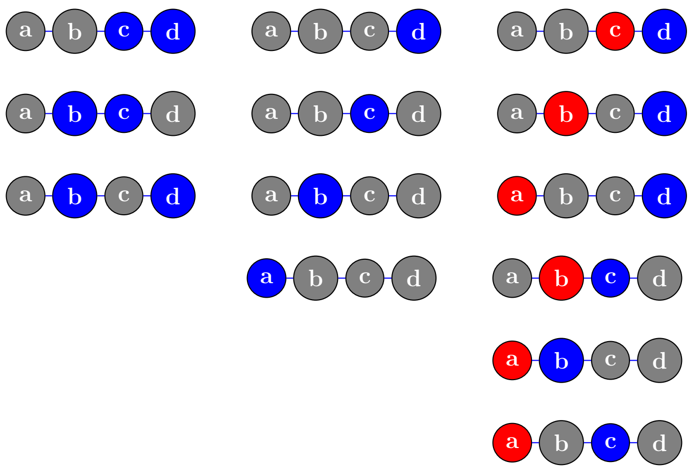
**Example** 3.3. Consider the four-vertex edge:



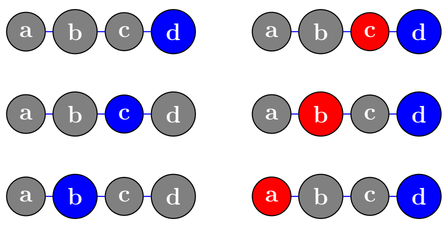

Setting the parameters for this edge in a stably bounded hypergraph to:

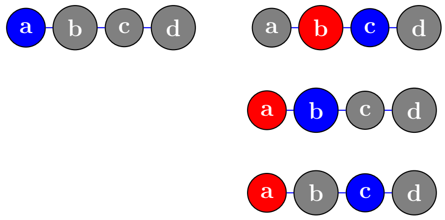
$$s = 2, \quad t = 4, \quad a = 2, \quad b = 3,$$

we have the following permissible edge patterns with three colors (up to a permutation of the colors):



With a coloring scheme comprised of the color patterns $\{3, 1\}$ and $\{2, 1, 1\}$, we can further restrict the permissible edge patters to the second and third columns above only. This coloring scheme corresponds to a *conflict-free* 3-coloring of this edge (see Section 8.2).






Both coloring schemes and stably bounded hypergraphs consider each column above as a single color pattern as both classes do not impose an ordering on the vertices of a hypergraph. Thus the two classes will not distinguish between the rows in each column. A pattern hypergraph (with an ordering on the vertices) may further restrict the permissible edge patterns down to *any* single pattern (column *and* row) above, if desired.

In general, both stably bounded and pattern hypergraphs may assign patterns to each edge individually. We will refer to classes with edge-level specifications as *local* coloring classes. Coloring schemes are presented in this paper first as a *global* coloring class for clarity in part, but also as they were inspired by conflict-free colorings introduced in [16]. In Section 9, we extend coloring schemes to admit color patterns for individual edges, and to non-uniform hypergraphs. We also illustrate how coloring schemes may be used to test a given hypergraph under various local coloring classes in Section 9.4.

## 4 Algebraic Background

In this section we introduce some of the algebraic tools required to prove the results in Sections 2 and 3. We collect several well known results from commutative algebra and algebraic geometry and hope the reader will find this section helpful. The use of radical, square-free generated ideals greatly simplifies the arguments involving the determination of the variety of the ideal in question. Since our main goal is to determine which ideals give rise to desired varieties, for example the variety that contains all proper k-colorings of a hypergraph, we review these below. The majority of the results in this section come from [7] and [8].

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and $I$ and $J$ be ideals in $R$.

**Definition** 4.1. The *radical* of $I$, denoted $\sqrt{I}$, is the set

$$\sqrt{I} = \{f \in R : f^m \in I, \text{ for some } m \in \mathbb{Z}^+\} .$$

Moreover, the ideal $I$ is *radical* if $I = \sqrt{I}$.

Working with the radical of an ideal or a radical ideal greatly simplifies computation and the geometric structure associated with the ideal.

**Definition** 4.2. The subset of $\mathbb{C}^n$ consisting of all of the solutions common to the polynomials in $I$ is the *variety* of $I$, denoted $\mathcal{V}(I)$. Conversely, given a subset $V \subseteq \mathbb{C}^n$, the *vanishing ideal* is the set of all polynomials in $R$ that vanish at every point in $V$ and is denoted $\mathcal{I}(V)$.

Varieties and vanishing ideals are related in the following way:

$$\mathcal{V}(\mathcal{I}(V)) = V, \quad \text{and} \quad \mathcal{I}(\mathcal{V}(I)) = \sqrt{I}\ .$$

Since $\mathbb{C}$ is algebraically closed, the second equality follows from Hilbert's Strong Nullstellensatz. Moreover, the relationships are inclusion-reversing, a fact which we will utilize in the following way: if $J \subseteq I$ are ideals, then $\mathcal{V}(J) \supseteq \mathcal{V}(I)$ (see Chapter 4, Section 2, Theorem 7 of [7]).

Some operations for radical ideals correspond to operations on their varieties as summarized in the following theorem.

**Theorem** 4.1 (Chapter 4 in [7]). Let $I$ and $J$ be ideals of $R$. If $I$ and $J$ are radical, then we have:

$$\begin{aligned} \mathcal{V}(IJ) &= \mathcal{V}(I) \cup \mathcal{V}(J)\ , \\ \mathcal{V}(I \cap J) &= \mathcal{V}(I) \cup \mathcal{V}(J)\ , \\ \mathcal{V}(I + J) &= \mathcal{V}(I) \cap \mathcal{V}(J). \end{aligned}$$

Moreover, $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$ .

Hilbert's Weak Nullstellensatz gives an equivalent condition for the variety of an ideal to be nonempty.

**Theorem** 4.2 (Weak Nullstellensatz). Let $I$ be a polynomial ideal. Then the polynomials in $I$ have no common solution if and only if the constant polynomial 1 is an element of $I$.

The next lemma gives a characterization of a radical ideal. First, we need to introduce a monomial ordering. A *monomial ordering* on the monomials of the polynomial ring $R$ is a multiplicative well-ordering $<$ defined on the set of monomials such that the constant polynomial 1 is the least element in this ordering. Given a monomial ordering, we can define the leading term of any polynomial $f \in R$. The leading term of a polynomial $f \in R$, $LT(f)$, is the monomial in $f$ that is largest with respect to the monomial ordering $<$. Any monomial which is not a leading term of a polynomial in an ideal $I$ is called a *standard monomial* and the set of all such monomials is denoted $\mathcal{B}_<(I)$.

An ideal $I$ in $R$ is called *zero-dimensional*, as an ideal, if its variety $\mathcal{V}(I)$ contains only a finite number of points. In [18], Hillar and Windfeldt collect several equivalent properties for a zero-dimensional ideal to be radical. We state their lemma here for completeness.

**Lemma** 4.1 (Lemma 2.1, [18]). Let $I$ be a zero-dimensional ideal in $R$ and fix a monomial ordering $<$. Then,

$$\dim_{\mathbb{C}} R/I = |\mathcal{B}_<(I)| \geqslant |\mathcal{V}(I)| \quad \text{(as a vector space)}.$$

Moreover, the following are equivalent:

1. $I$ is a radical ideal.

2. $I$ contains a univariate square-free polynomial in each indeterminate.

3. $|\mathcal{B}_<(I)| = |\mathcal{V}(I)|$.

Finally, we can use the monomial ordering of an ideal to help find common solutions to the polynomials contained in the ideal. What follows is a description of the primary tool for establishing the existence of an algorithm which can be used to determine if a hypergraph may be properly $k$-colored, or may be colored by a given coloring scheme.

Given an ideal $I$ in $R$, the ideal of leading terms is defined as $LT(I) = \langle LT(f) : f \in I \rangle$. A *Gröbner basis* for an ideal $I$ is a finite set of generators $\{g_1, \ldots, g_m\}$ for $I$ whose leading terms generate the ideal of all leading terms in $I$, i.e.,

$$I = \langle g_1, \ldots, g_m \rangle \quad \text{and} \quad LT(I) = \langle LT(g_1), \ldots, LT(g_m) \rangle \ .$$

A *reduced* Gröbner basis $\mathcal{G}$, is a Gröbner basis whose elements are all monic, and are such that no leading term in $\mathcal{G}$ divides any other term in any polynomial in $\mathcal{G}$. Gröbner bases are very useful in the study of polynomial ideals, they are used to determine ideal membership, compute intersections, and establish equality between ideals. We will utilize the following as our primary tool for determining which $k$-colorings a given hypergraph admits.

**Theorem** 4.3. The polynomials in an ideal $I = \langle f_1, \ldots, f_m \rangle$ have a common solution if and only if any Gröbner basis for $I$ is non-trivial.

## 5 Uniform $k$-Colorability Proofs

We give proofs of properties of the ideals defined in Section 2 culminating with a proof of Theorem 2.2.

Let $\mathcal{H}$ be an $r$-uniform hypergraph on $n$ vertices and let $\mathcal{P}_k$ be the set of the first $k$ prime numbers. Recall that in a multiset repeated elements are permitted. As in Section 2, let $\mathcal{S}$ be the set of all submultisets of $\mathcal{P}_k$ of size $r$, and let $\mathcal{R} \subset \mathcal{S}$ be the set of all submultisets of $\mathcal{P}_k$ which properly color an edge $e \in E(\mathcal{H})$. Let $N\mathcal{R} = \mathcal{S} \setminus \mathcal{R}$, and note that $N\mathcal{R}$ is the set of all submultisets in which all elements are the same, corresponding to monochromatic colorings of an edge. We begin with a series of propositions utilized in the proof of Theorem 2.2.

**Proposition** 5.1. The variety $\mathcal{V}(C_k)$ is the set of all $k$-colorings of $\mathcal{H}$.

*Proof.* For any $i \in V(\mathcal{H})$ the associated polynomial in $C_k$,

$$\prod_{p \in \mathcal{P}_k} (x_i - p) \ ,$$

vanishes if and only if $x_i = p$ for some $p \in \mathcal{P}_k$. That is, if and only if the vertex $i$ is colored by a color in $\mathcal{P}_k$.

Since the polynomials in $C_k$ have a common solution, $\mathbf{c}$, if and only if $\mathbf{c}$ is a $k$-coloring of $\mathcal{H}$, the variety $\mathcal{V}(C_k)$ is the set of all $k$-colorings of $\mathcal{H}$. □

As $C_k$ contains the information on all $k$-colorings for the hypergraph $\mathcal{H}$, including improper colorings, we need to be able to eliminate those colorings and identify the proper $k$-colorings only. This is achieved through the $k$-colorability ideal defined in Section 2. First we include a simple but crucial property of the polynomial $h$.

**Lemma** 5.1. Let $\bar{\mathscr{C}}(e) = (c_{e_1}, \ldots, c_{e_r})$ be a $k$-coloring of the vertices in an edge of $\mathcal{H}$. Then $h(\bar{\mathscr{C}}(e)) = 0$ if and only if the corresponding multiset for $\bar{\mathscr{C}}(e)$ is an element of $N\mathcal{R}$.

*Proof.* Let $e \in E(\mathcal{H})$, and $\bar{\mathscr{C}}(e) = (c_{e_1}, \ldots, c_{e_r})$ be the coloring of $e$ under $\bar{\mathscr{C}}$. Since all colors in $\mathcal{P}_k$ are positive integers,

$$h(\bar{\mathscr{C}}(e)) = (c_{e_1} - c_{e_2})^2 + (c_{e_1} - c_{e_3})^2 + \cdots + (c_{e_2} - c_{e_3})^2 + (c_{e_2} - c_{e_4})^2 + \cdots + (c_{e_{r-1}} - c_{e_r})^2 = 0$$

if and only if, $c_{e_i} = c_{e_j}$ for all $1 \leqslant i < j \leqslant r$. Thus $h(\bar{\mathscr{C}}(e)) = 0$ if and only if the corresponding multiset for $\bar{\mathscr{C}}(e)$ is an element of $N\mathcal{R}$. □

**Proposition** 5.2. The polynomials in the ideal

$$I_{\mathcal{H},r}(k) = C_k + \langle f_e : e \in E(\mathcal{H}) \rangle$$

have a common solution if and only if $\mathcal{H}$ is properly $k$-colorable.

*Proof.* ($\Rightarrow$) Let $\mathbf{c} = (c_1, \ldots, c_n)$ be a common solution to the polynomials in $I_{\mathcal{H},r}(k)$. Since $\mathbf{c}$ is common solution to the polynomials in $C_k$, $\mathbf{c} \in \mathcal{V}(C_k)$. By Proposition 5.1 $\mathbf{c} = (c_1, \ldots, c_n)$ is a $k$-coloring of $\mathcal{H}$ and thus each $c_i$ takes on a value in $\mathcal{P}_k$. Moreover, as $\mathbf{c}$ is a common solution for the polynomials in $\langle f_e : e \in E(\mathcal{H}) \rangle$, for every edge $e = \{e_1, \ldots, e_r\} \in E(\mathcal{H})$, $f_e(c_{e_1}, \ldots c_{e_r}) = 0$, thus there is some $u = \{u_1, \ldots, u_r\} \in \mathcal{R}$ such that

$$h(c_{e_1}, \ldots, c_{e_r}) - h(u_1, \ldots, u_r) = 0 \ .$$

By Lemma 5.1, $h(u_1, \ldots, u_r) \neq 0$, thus $h(c_{e_1}, \ldots, c_{e_r}) \neq 0$ as well, and the edge $e$ is properly colored. Hence, $\mathbf{c}$ is a proper coloring of $\mathcal{H}$.

($\Leftarrow$) Let $\tilde{\mathscr{C}} = (c_1, \ldots, c_n)$ be a proper $k$-coloring of $\mathcal{H}$. That is, $c_i \in \mathcal{P}_k$ and $\tilde{\mathscr{C}}$ colors no edge monochromatically. Since each $c_i \in \mathcal{P}_k$, $(c_1, \ldots, c_n)$ is a common solution of the polynomials in $C_k$ by Proposition 5.1.

Further, by Lemma 5.1, since no edge is colored by a single color, $h(c_{e_1}, \ldots c_{e_r}) \neq 0$ for every edge $e = \{e_1, \ldots, e_r\} \in E(\mathcal{H})$. Thus, for every edge $e \in E(\mathcal{H})$, there exists $u \in \mathcal{R}$ such that $h(u) = h(c_{e_1}, \ldots c_{e_r})$. We can choose $u$ as the multiset $\{c_{e_1}, \ldots, c_{e_r}\} \in \mathcal{R}$, since $\tilde{\mathscr{C}}$ is a proper coloring. Hence, $f_e(c_{e_1}, \ldots c_{e_r}) = 0$ for every $e \in E(\mathcal{H})$ and, thus, $\tilde{\mathscr{C}}$ is a common solution to the polynomials in $I_{\mathcal{H},r}(k)$. □

**Proposition** 5.3. $\mathcal{V}(I_{\mathcal{H},r}(k))$ is the set of all proper $k$-colorings of $\mathcal{H}$.

*Proof.* This follows from Proposition 5.2 and the fact that the variety of a polynomial ideal is the collection of all common solutions to the polynomials contained in that ideal. □

At this point, we are able to apply Theorem 4.3 to the $k$-colorability ideal and determine if a given uniform hypergraph is properly $k$-colorable or not.

The $k$-colorability ideal allows us to describe the proper $k$-colorings of $\mathcal{H}$. On the other hand, the hypergraph polynomial for $k$-colorability identifies the improper colorings of $\mathcal{H}$ as we show below. Both formulations of colorability are useful in characterizing the colorability of a uniform hypergraph.

**Proposition** 5.4. *Let $\mathscr{C} = (c_1, \dots, c_n)$ be a $k$-coloring of $\mathcal{H}$. Then $\mathscr{C}$ is not a proper $k$-coloring if and only if $(c_1, \dots, c_n)$ is a common solution to the polynomials in $C_k + \langle P_{\mathcal{H},r,k}\rangle$.*

*Proof.* ($\Rightarrow$) Let $\mathscr{C} = (c_1, \dots, c_n)$ be a $k$-coloring of the $r$-uniform hypergraph $\mathcal{H}$. Assume $\mathscr{C}$ is not a proper coloring. Then there is some edge in $\mathcal{H}$, $\hat{e}$, which is monochromatically colored. Thus, the corresponding color multiset for $\hat{e}$ colored by $\mathscr{C}$ is an element of $N\mathcal{R}$. By Lemma 5.1, $h(c_{e_1}, \dots, c_{e_r}) = 0$ for the edge $\hat{e}$, and hence $P_{\mathcal{H},r,k}(c_1, \dots, c_n) = 0$. Further, since $\mathscr{C}$ is a $k$-coloring, $(c_1, \dots, c_n)$ is a common solution to the polynomials in $\langle C_k\rangle$ and thus also for $C_k + \langle P_{\mathcal{H},r,k}\rangle$.

($\Leftarrow$) Assume $\mathbf{c} = (c_1, \dots, c_n)$ is a common solution to the polynomials in $C_k + \langle P_{\mathcal{H},r,k}\rangle$. By Proposition 5.1, since $\mathbf{c}$ is a common solution to the polynomials in $C_k$, $\mathbf{c}$ is a $k$-coloring of $\mathcal{H}$; set $\mathscr{C} = (c_1, \dots, c_n)$ as the $k$-coloring which corresponds to the $n$-tuple $\mathbf{c}$.

Since $P_{\mathcal{H},r,k}(c_1, \dots, c_n) = 0$, there is some factor, $h_{\bar{e}}$, corresponding to the edge, $\bar{e} \in E(\mathcal{H})$, which evaluates to zero at $\mathbf{c}$. By Lemma 5.1, since $h(c_{\bar{e}_1}, \dots, c_{\bar{e}_n}) = 0$, the color multiset for $\bar{e}$ is an element of $N\mathcal{R}$. Hence the edge $\bar{e}$ is monochromatically colored by $(c_1, \dots, c_n)$, and $\mathscr{C}$ is an improper coloring of $\mathcal{H}$. $\square$

**Proposition** 5.5. *$\mathcal{V}(C_k + \langle P_{\mathcal{H},r,k}\rangle)$ is the set of all improper $k$-colorings of $\mathcal{H}$.*

*Proof.* This follows from Proposition 5.4 and the fact that the variety of a polynomial ideal is the collection of all common solutions to the polynomials contained in that ideal. $\square$

We are now ready to prove our main result for arbitrary $k$-colorings of uniform hypergraphs, which we re-state for convenience.

**Theorem** 2.2. *Let $r$, $k \geqslant 2$ be positive integers and let $\mathcal{H}$ be an $r$-uniform hypergraph on $n$ vertices. Let $R = \mathbb{C}[x_1, \dots, x_n]$. Let $I_{\mathcal{H},r}(k)$ be the $k$-colorability ideal for $\mathcal{H}$ and $P_{\mathcal{H},r,k}$ be the hypergraph polynomial for $k$-colorability for $\mathcal{H}$. Then the following are equivalent:*

1. *The hypergraph $\mathcal{H}$ is not $k$-colorable.*

2. *$\dim_{\mathbb{C}} R/I_{\mathcal{H},r}(k) = 0$ as a vector space.*

3. *The constant polynomial 1 belongs to the ideal $I_{\mathcal{H},r}(k)$.*

4. *The hypergraph polynomial $P_{\mathcal{H},r,k}$ belongs to the ideal $C_k$.*

*Proof.* The equivalence of (1), (2), and (3) is given by Lemma 4.1, Proposition 5.2, and the Weak Nullstellensatz.

It remains to show (1) is equivalent to (4).

(1) $\Rightarrow$ (4): Assume $\mathcal{H}$ is not $k$-colorable. Let $\mathscr{C} = (c_1, \ldots, c_n)$ be a $k$-coloring of $\mathcal{H}$; thus

$$(c_1, \ldots, c_n) \in \mathcal{V}(C_k).$$

Since $\mathcal{H}$ is not $k$-colorable, $\mathscr{C}$ is not a proper coloring, so by Proposition 5.5,

$$(c_1, \ldots, c_n) \in \mathcal{V}\left(C_k + \langle P_{\mathcal{H},r,k}\rangle\right).$$

Thus,

$$\mathcal{V}\left(C_k + \langle P_{\mathcal{H},r,k}\rangle\right) \supseteq \mathcal{V}\left(C_k\right) \ ,$$

and so,

$$C_k + \langle P_{\mathcal{H},r,k}\rangle \subseteq C_k \ .$$

Hence, $P_{\mathcal{H},r,k} \in C_k$.
(4) $\Rightarrow$ (1): Assume $P_{\mathcal{H},r,k} \in C_k$. Then

$$C_k + \langle P_{\mathcal{H},r,k}\rangle \subseteq C_k$$

and

$$\mathcal{V}\left(C_k + \langle P_{\mathcal{H},r,k}\rangle\right) \supseteq \mathcal{V}\left(C_k\right).$$

Let $\mathscr{C} = (c_1, \ldots, c_n)$ be a $k$-coloring of $\mathcal{H}$. Then since $(c_1, \ldots, c_n)$ is a point in $\mathcal{V}(C_k)$, by Proposition 5.5, $(c_1, \ldots, c_n) \in \mathcal{V}\left(C_k + \langle P_{\mathcal{H},r,k}\rangle\right)$ and $\mathscr{C}$ is not proper. So $\mathcal{H}$ is not $k$-colorable.

This completes the proof of Theorem 2.2. $\square$

**Remark** 5.1. We would like to make a remark regarding the choice of the polynomial $h$ in Section 2. We can choose $h$ to be any symmetric polynomial in $r$ variables such that $h(u) \neq 0$ for all $u \in \mathcal{R}$ and $h(v) = 0$ for all $v \in N\mathcal{R}$. Taking into account computational considerations and for the sake of concreteness, we chose $h$ to be a second degree symmetric polynomial.

## 6 Uniform Coloring Scheme Proofs

In this section, we give a proof of Theorem 3.1. Further, we describe properties of the $k$-coloring scheme ideals $S_{\mathcal{H},r}(M, k)$.

Let $\mathcal{H}$ be an $r$-uniform hypergraph on $n$ vertices. We let $\mathcal{P}_k$ be the set of the first $k$ prime numbers and $\Lambda_{\hat{r}}$ be the set of all proper partitions of $r$. We begin by formalizing the importance of $\Lambda_{\hat{r}}$. Recall that $\Lambda_{\hat{r}}$ contains the proper partitions of $r$, and excludes what we refer to as the improper partition containing the single part, $r$.

**Lemma** 6.1. Any improper $k$-coloring of $\mathcal{H}$ produces at least one edge product which corresponds to the improper partition of $r$, $\tau = \{r\}$. Moreover, this is the only partition of $r$ which produces a monochromatically colored edge.

*Proof.* Let $\mathscr{C}$ be an improper coloring of $\mathcal{H}$. Since $\mathscr{C}$ is improper, there exists an edge, $e = \{e_1, e_2, \ldots, e_r\} \in E(\mathcal{H})$, for which

$$\mathscr{C}(e_i) = p$$

for all $i = 1, 2, \ldots, r$ and some $p \in \mathcal{P}_k$. Therefore, the edge product of $e$ is

$$\Gamma[\mathscr{C}(e)] = \prod_{i=1}^{r} \mathscr{C}(e_i) = \prod_{i=1}^{r} p = p^r.$$

We assert that this improperly colored edge product is only produced by the improper partition of $r$. Let $\tau = \{r\}$ be the improper partition of $r$. Since the expansion of $\tau$:

$$Ex(\tau) = \{(r, 0, 0, \ldots, 0), (0, r, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, r)\},$$

produces the edge products:

$$EP(\tau) = \left\{ \prod_{l=1}^{k} p_l^{\alpha_l} : (\alpha_1, \ldots, \alpha_k) \in Ex(\tau) \right\} = \{p_1^r, p_2^r, \ldots, p_k^r\}.$$

We have that $\tau$ produces an edge product which improperly colors an edge. As $\tau$ is the only partition of $r$ which admits a part of size $r$, it is the only partition which may produce such improper edge products. $\square$

**Corollary** 6.1. *The partitions in $\Lambda_{\hat{r}}$ produce edge products which properly color the vertices in an edge.*

*Proof.* Since $\tau = \{r\}$ is not a partition contained in $\Lambda_{\hat{r}}$, the corollary follows from Lemma 6.1. $\square$

The next lemma establish the connection between color patterns in a coloring scheme, edge products, and zeros of the polynomials which generate coloring scheme ideals. As this lemma will be useful in classifying coloring schemes in non-uniform hypergraphs, the cardinality of each edge, $r(e)$, is kept arbitrary (see Section 9.1).

**Lemma** 6.2. *Let $\mathcal{H}$ be a simple hypergraph and $e = \{e_1, e_2, \ldots, e_{r(e)}\}$ an edge in $\mathcal{H}$. Let $CP(e)$ be a collection of color patterns for $e$. Then the polynomial*

$$g(e, CP(e)) = \prod_{\lambda \in CP(e)} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right]$$

*has zeros which correspond to coloring $e$ with color patterns from $CP(e)$.*

*Proof.* Let $\mathscr{C} = (c_1, \ldots, c_n)$ be a $k$-coloring of $\mathcal{H}$ for which the coordinates $\{c_{e_i}\}_{e_i \in e}$ zero $g(e, CP(e))$. Then there exists a partition $\tilde{\lambda} \in CP(e)$ for which

$$\prod_{a \in EP(\tilde{\lambda})} \left( \prod_{e_i \in e} x_{e_i} - a \right)$$

must vanish. Further, the product

$$\prod_{a \in EP(\tilde{\lambda})} \left( \prod_{e_i \in e} x_{e_i} - a \right)$$

vanishes if and only if

$$\prod_{e_i \in e} x_{e_i} - a = 0, \text{ or, } \prod_{e_i \in e} x_{e_i} = a$$

for some $a \in EP(\tilde{\lambda})$. So, for some expansion of $\tilde{\lambda}$, under the coloring $\mathscr{C}$, the edge $e$ has the edge product

$$\prod_{e_i \in e} \mathscr{C}(e_i) = \prod_{e_i \in e} c_{e_i} = a$$

where $a \in EP(\tilde{\lambda})$. Since $\tilde{\lambda} \in CP(e)$, $\mathscr{C}$ colors $e$ with a coloring pattern from $CP(e)$. $\square$

We are now ready to provide the connection between the $k$-coloring scheme ideal and proper colorings of a hypergraph with a coloring scheme.

**Theorem** 6.1. Let $k, r \geqslant 2$ be integers and $\mathcal{H}$ be an $r$-uniform hypergraph. Let $M = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$ be a non-empty subset of $\Lambda_{\hat{r}}$. The polynomials in the ideal $S_{\mathcal{H},r}(M, k)$ have a common solution if and only if the hypergraph $\mathcal{H}$ may be properly colored by the coloring scheme $M$.

*Proof.* ($\Rightarrow$) Assume $\mathbf{c} = (c_1, \ldots, c_n)$ is a common solution of the polynomials in

$$S_{\mathcal{H},r}(M, k) = C_k + \left\langle \prod_{\lambda \in M} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{i=1}^{r} x_{e_i} - a \right) \right] : e \in E(\mathcal{H}) \right\rangle .$$

By Proposition 5.1, $(c_1, \ldots, c_n)$ is a $k$-coloring of $\mathcal{H}$. Let $\mathscr{C} = (c_1, \ldots, c_n)$ be the $k$-coloring of $\mathcal{H}$ associated with $\mathbf{c}$. It remains to show that the generators of the ideal

$$\left\langle \prod_{\lambda \in M} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{i=1}^{r} x_{e_i} - a \right) \right] : e \in E(\mathcal{H}) \right\rangle$$

force $\mathscr{C}$ to be a proper coloring with the coloring scheme $M = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$.

Let $e = \{e_1, e_2, \ldots, e_r\} \in E(\mathcal{H})$ be an arbitrary edge in $\mathcal{H}$. Since $\mathscr{C} = (c_1, \ldots, c_n)$ zeros

$$\prod_{\lambda \in M} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{i=1}^{r} x_{e_i} - a \right) \right],$$

Lemma 6.2 implies $\mathscr{C}$ colors $e$ with a color pattern in $M$. Since $e$ was chosen arbitrarily, $\mathscr{C}$ colors $\mathcal{H}$ with the coloring scheme $M$. Moreover, $\mathscr{C}$ is a proper coloring of $\mathcal{H}$ by Corollary 6.1 since $M \subseteq \Lambda_{\hat{r}}$.

($\Leftarrow$) Assume $\mathcal{H}$ is properly colorable with the coloring scheme $M = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$. Let $\mathscr{C} = (c_1, \ldots, c_n)$ be such a $k$-coloring. By Proposition 5.1, $(c_1, \ldots, c_n) \in \mathcal{V}(C_k)$ and is a common solution to the polynomials of $C_k$. Let $\mathbf{c} = (c_1, \ldots, c_n)$ represent the point in $\mathcal{V}(C_k)$. Additionally, $(c_1, \ldots, c_n)$ assigns a color pattern from $M$ to each edge. Let $e \in E(\mathcal{H})$ and $\prod_{i=1}^{r} c_{e_i}$ be the edge product of $e$ colored by $\mathscr{C}$. Since $\mathscr{C}$ colors $e$ with a color pattern from $M$, there exists some $\lambda \in M$ such that

$$\prod_{i=1}^{r} c_{e_i} = a \in EP(\lambda).$$

Hence, the polynomial

$$\prod_{\lambda \in M} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{i=1}^{r} x_{e_i} - a \right) \right]$$

vanishes at $(c_1, \ldots, c_n)$, and $\mathbf{c} = (c_1, \ldots, c_n)$ is a common solution to the polynomials in

$$\left\langle \prod_{\lambda \in M} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{i=1}^{r} x_{e_i} - a \right) \right] : e \in E(\mathcal{H}) \right\rangle .$$

□

A description of the variety $\mathcal{V}(S_{\mathcal{H},r}(M,k))$ follows immediately.

**Corollary** 6.2. Let $M$ be a nonempty subset of $\Lambda_{\hat{r}}$. Then $\mathcal{V}(S_{\mathcal{H},r}(M,k))$ is the set of all proper $k$-colorings of $\mathcal{H}$ with the coloring scheme $M$.

*Proof.* This follows from Theorem 6.1 and the fact that the variety of a polynomial ideal is the collection of all common solutions to the polynomials contained in that ideal. □

At this point we are able to apply Theorem 4.3 to the $k$-coloring scheme ideal to determine if a hypergraph admits a proper coloring with a given coloring scheme.

We now prove Theorem 3.1 relating the ideals $I_{\mathcal{H},r}(k)$ and $S_{\mathcal{H},r}(M,k)$.

*Proof. (Theorem 3.1)* Note that since $C_k \subset I_{\mathcal{H},r}(k)$, $C_k \subset S_{\mathcal{H},r}(M,k)$, and $C_k$ contains univariate square-free polynomials in each indeterminate, $C_k$, $I_{\mathcal{H},r}(k)$, and $S_{\mathcal{H},r}(M,k)$, are all radical by Theorem 4.1. Since $\Lambda_{\hat{r}}$ is the collection of all proper partitions of $r$, we have by Corollary 5.3 and Theorem 6.1, that

$$\mathcal{V}(I_{\mathcal{H},r}(k)) = \bigcup_{\substack{M \subseteq \Lambda_{\hat{r}} \\ M \neq \emptyset}} \mathcal{V}(S_{\mathcal{H},r}(M,k)) .$$

Moreover, since $I_{\mathcal{H},r}(k)$ and $S_{\mathcal{H},r}(M,k)$ are radical, we have that

$$I_{\mathcal{H},r}(k) = \bigcap_{\substack{M \subseteq \Lambda_{\hat{r}} \\ M \neq \emptyset}} S_{\mathcal{H},r}(M,k)$$

by the one-to-one correspondence of varieties and radical ideals. $\square$

## 7 A Homogeneous Coloring Scheme Example

In this section we consider a hypergraph introduced in [20] that permits certain coloring schemes, while forbidding others. This illustrates the type of questions we can analyze and answer using coloring schemes. We will collect the results in this section and list the Gröbner bases for the appropriate coloring schemes in Appendix A.

Let $\mathcal{H}$ be the 4-uniform hypergraph on 7 vertices with the following edge set:

$$\begin{array}{cccccccc} \{\{1, & 2, & 3, & 4\}, & \{1, & 5, & 6, & 7\}, \\ \{2, & 5, & 6, & 7\}, & \{3, & 5, & 6, & 7\}, \\ \{3, & 4, & 5, & 6\}, & \{3, & 4, & 6, & 7\}, \\ \{3, & 4, & 5, & 7\}, & \{1, & 2, & 3, & 5\}, \\ \{1, & 2, & 3, & 6\}, & \{1, & 2, & 3, & 7\}\}\,. \end{array}$$

It was shown in [20] that $\mathcal{H}$ is 2-colorable with both the $\{3,1\}$ and the $\{\{3,1\},\{2,2\}\}$ coloring schemes, with the latter corresponding to the general 2-colorability of the hypergraph. Here we will only consider coloring schemes on 3 or more colors for $\mathcal{H}$; moreover we will consider only *homogeneous* color schemes, consisting of a single color pattern. Homogeneous coloring schemes will be represented by their individual partition, $\lambda$, in place of $\{\lambda\}$ for convenience.

First, we show that the only way to color $\mathcal{H}$ with a $\{3,1\}$ coloring scheme is with 2 colors. When $k=3$, the color pattern $\lambda = \{3,1\}$ produces the expansion

$$Ex(\lambda) = \{(3,1,0),(1,3,0),(3,0,1),(1,0,3),(0,3,1),(0,1,3)\}\,.$$

This forces each edge to only admit two distinct colors, though not necessarily the same two colors for every edge.

By arbitrarily choosing $2,3$, and $5$ as the color for $x_1$ to expedite computation (see Section 8), we receive the following Gröbner bases for $S_{\mathcal{H},4}(\{3,1\},3)$:

$$\begin{array}{lll} x_1 = 2 & \to & \{x_6 - x_7,\, x_5 - x_7,\, x_4 - x_7,\, x_3 - 2,\, x_2 - 2,\, 15 - 8x_7 + x_7^2\} \\ x_1 = 3 & \to & \{x_6 - x_7,\, x_5 - x_7,\, x_4 - x_7,\, x_3 - 3,\, x_2 - 3,\, 10 - 7x_7 + x_7^2\} \\ x_1 = 5 & \to & \{x_6 - x_7,\, x_5 - x_7,\, x_4 - x_7,\, x_3 - 5,\, x_2 - 5,\, 6 - 5x_7 + x_7^2\} \end{array}$$

The three bases above show that at most two colors are used in the $\lambda = \{3,1\}$ scheme. This can be seen by noting that the vertices associated with the variables $x_2$ and $x_3$ only admit a single color, which always matches $x_1$. Moreover, by factoring the final polynomial in each basis, we see that $x_7$, and therefore $x_4$, $x_5$, and $x_6$, must all be assigned one of the remaining colors.

For the remaining individual color schemes, $\lambda = \{2, 1, 1\}$ and $\lambda = \{2, 2\}$, we have trivial Gröbner bases, indicating that $\mathcal{H}$ is not colorable by those coloring schemes. Further, we have that the $\{2, 2\}$ coloring scheme is impossible on this hypergraph as $k = 3$ is the maximum number of colors possible on $n = 7$ vertices with this scheme. In order to use four *distinct* colors with the $\lambda = \{2, 2\}$ coloring scheme at least eight vertices would be required. Four colors distributed in pairs (i.e., parts of size two) is impossible on seven vertices.

We find that the only other homogeneous coloring scheme possible on $\mathcal{H}$ is $\lambda = \{2, 1, 1\}$, with $Ex(\lambda)$ equal to all permutations of the exponent vector $(0, 1, 1, 2)$. Despite coloring each edge with exactly three colors, this coloring scheme is possible with a minimum of $k = 4$ colors. Here, both $x_1, x_2$ were assigned the color 2 arbitrarily to expedite computation; the resulting Gröbner basis has ten generating polynomials, listed in Appendix A below.

Clearly, since the $\lambda = \{2, 1, 1\}$ scheme works for $k = 4$, we can always color $\mathcal{H}$ with $\lambda$ when $k > 4$ by simply leaving some colors out. When $k = 5$ and $k = 6$ we are interested to see that colorings of $\mathcal{H}$ using $\lambda = \{2, 1, 1\}$ will indeed allow *every* color to be used. This can be seen by solving the polynomials for $S_{\mathcal{H},4}(\lambda, k)$ listed in Appendix A by elimination. Finally, it is interesting to note that if $k = 4$, 5, or 6 it is impossible to use the homogeneous coloring scheme $\lambda = \{1, 1, 1, 1\}$. This shows that the only such coloring of this hypergraph is the trivial 7-coloring.

# 8 Implementation

The primary goal of Sections 2, 3, and 9 is to determine if a given hypergraph, $\mathcal{H}$, is properly colorable. This is accomplished by computing a Gröbner basis for the appropriate ideal and applying Theorem 4.3. If the Gröbner basis is non-trivial, then each point in the associated variety corresponds to a proper coloring of $\mathcal{H}$; this is the second goal of this paper. Throughout this section we will refer to any coloring that satisfies the corresponding color class as a *proper coloring.*

## 8.1 Computation of Gröbner Bases

Gröbner basis computation can be time consuming, even with specialized computer algebra systems such as *SINGULAR*. To help accelerate these computations, we arbitrarily assign a color to one or more vertices of the hypergraph. This corresponds to working in an *elimination ideal* of the appropriate coloring ideal and applying the *Extension Theorem* (see, for example, Chapter 3 in [7]). We refer to this process as *partially coloring* a hypergraph $\mathcal{H}$.

This technique allows us to quickly determine that a hypergraph is properly colorable without knowing the Gröbner basis of the entire colorability ideal. If a single partial coloring does not extend to a proper coloring, we cannot conclude that the hypergraph is not colorable, as a different partial coloring may extend to a proper coloring. Extending a partial coloring can be used to test for non-colorability, as long as all possible initial colorings are tested on the chosen vertices. However, if a partial coloring does produce a non-trivial Gröbner basis, then we can conclude the hypergraph is properly colorable.

In Section 7 we saw an example of a 4-uniform hypergraph with 10 edges on 7 vertices. To test for colorability with a $\{3, 1\}$ coloring scheme, we used a partial coloring by assigning the vertex 1 a particular color. Not only does this dramatically speed up the computation of a Gröbner basis, but in this case it makes extracting explicit colorings trivial.

**Example** 8.1. The explicit proper colorings from Section 7 using a $\{3, 1\}$ coloring scheme:

$$(2, 2, 2, 3, 3, 3, 3),\ (3, 3, 3, 2, 2, 2, 2),\ (5, 5, 5, 2, 2, 2, 2),$$
$$(2, 2, 2, 5, 5, 5, 5),\ (3, 3, 3, 5, 5, 5, 5),\ (5, 5, 5, 3, 3, 3, 3).$$

In particular, other than choice of color, we see that this coloring is unique,

Computing the Gröbner basis directly results in a slightly more complicated set of polynomials, which required more than 96 hours of computing time. On the other hand, each partially colored Gröbner basis in Section 7 was computed in under *three seconds*. Moreover, these bases were computed in Mathematica, which is not as optimized for this type of algebraic computation as SINGULAR.

**Example** 8.2. The Gröbner basis for the hypergraph in Section 7, without a partial coloring.

$$\begin{aligned}
\{&x_6 - x_7,\ x_5 - x_7,\ x_4 - x_7,\ x_2 - x_3,\ x_1 - x_3,\\
&x_3^2 + x_3x_7 + x_7^2 - 10x_3 - 10x_7 + 31,\\
&x_7^3 - 10x_7^2 + 31x_7 - 30\}
\end{aligned}$$

Note the tuples listed in Example 8.1 are precisely the solutions to this system of equations.

### 8.2 Some Global Coloring Schemes

Colorings schemes are most directly applicable to global coloring classes. In fact, our inspiration for classifying proper hypergraph $k$-colorings are *conflict-free* colorings, described below. Additionally we will describe coloring schemes for two confict-free variants and the most natural extension of a proper *graph* coloring.

The natural extension of a proper *graph* coloring, where each vertex in an edge is assigned a unique color, to a hypergraph is called a *strong* coloring. Let $\mathcal{H}$ be an $r$-uniform hypergraph on $n$ vertices and let $k \geqslant r$. A *strong* $k$-coloring of the hypergraph $\mathcal{H}$ is a coloring, $\mathscr{C}$, where $\mathscr{C}(e_i) \neq \mathscr{C}(e_j)$ for all $e_i, e_j \in e$. As each color may be used at most once on each edge, the strong $k$-coloring scheme contains only the partition of $r$ with all parts equal to 1. The definition remains the same for non-uniform hypergraphs, so long as there are at least as many colors used as vertices contained in the largest edge.

Let $\Lambda_{\hat{r},\mathrm{CF}}$ be the subset of $\Lambda_{\hat{r}}$ containing the proper partitions of $r$ with at least one part of size 1. The variety of the $k$-coloring scheme ideal $S_{\mathcal{H},r}(\Lambda_{\hat{r},\mathrm{CF}}, k)$ consists of the proper $k$-colorings of $\mathcal{H}$ in which every edge has at least one non-repeating color. Such $k$-colorings are called *conflict-free colorings* and were introduced by Even et al. in [16]. Example 3.3 illustrates a conflict-free coloring on a 4-uniform edge.

Recently, Cui and Hu [9] have presented some bounds on variants of conflict-free colorings introduced by Smorodinsky [22]. We list the coloring schemes corresponding to these variants for uniform hypergraphs. Let $k \leqslant t$, and let $\mathscr{C}$ be a $k$-coloring of $\mathcal{H}$. Then $\mathscr{C}$ is called a *$t$-conflict-free coloring* of $\mathcal{H}$ if:

$$\forall\, e \in E(\mathcal{H}),\ \exists\, j \in [k] \text{ s.t. } 1 \leqslant |\{v \in e : c(v) = p_j\}| \leqslant t.$$

That is, $\mathscr{C}$ is a coloring where each edge has a color that is assigned to no more than $t$ vertices in the edge. Here $t$ is the 'upper bound' on the number of vertices an edge may be colored by some color. The $t$-conflict-free coloring scheme is the subset of $\Lambda_{\hat{r}}$ which contains partitions with at least one part within the range $[1, t]$.

Let $r, t, k \in \mathbb{Z}^+, r \leqslant t \leqslant k$. Let $\mathscr{C}$ be a $k$-coloring of the $r$-uniform hypergraph $\mathcal{H}$. Then $\mathscr{C}$ is a *$t$-strong-conflict-free-coloring* of $\mathcal{H}$ if for every edge $e \in E(\mathcal{H})$ there are at least $t$ colors which appear only once in $e$. Note this definition is a modification of the one given in [3] and [9], which considers non-uniform hypergraphs. The $t$-strong-conflict-free coloring scheme is the subset of $\Lambda_{\hat{r}}$ which contains partitions with at least $t$ parts equal to 1.

**Global Coloring Scheme Summary**

| Title | Edge Restrictions | Coloring Scheme |
|---|---|---|
| strong | No color may be used twice. | $M = M_r = \mathcal{M} = \{\underbrace{\{1, 1, \ldots, 1\}}_{r \text{ copies}}\}$ |
| conflict-free | At least one color is used only once. | $1 \in \lambda, \forall\, \lambda \in M, M_r, \mathcal{M}$ |
| $t$-conflict-free | At least one color is used no more than $t$ times. | $\forall\, \lambda \in M, M_r, \mathcal{M}$, $\exists\, \lambda_i \in \lambda\, s.t.\, \lambda_i \leqslant t$ |
| $t$-strong-conflict-free | At least $t$ colors are used exactly once. | $\forall\, \lambda \in M, M_r, \mathcal{M}$, $\{\underbrace{1, 1, \ldots, 1}_{t \text{ copies}}\} \subseteq \lambda$ |

## 9 Non-Uniform Hypergraphs and Extended Coloring Schemes

The results from Sections 2 and 3 may easily be extended in several ways. The $k$-colorability of non-uniform hypergraphs may be determined by an ideal similar to the $k$-colorability ideal by simply introducing the function $r : E(\mathcal{H}) \to \mathbb{N}$ which maps each edge to its cardinality. Moreover, the coloring scheme ideals may be further decomposed so that each edge is given a particular set of color patterns which are permissible. The latter extension will permit coloring scheme ideals to completely refine stably bounded hypergraphs. In addition, this extension is equivalent to unoriented pattern hypergraphs. It should be noted that all results in this section are more technical extensions of the uniform case.

Let $\mathcal{H}$ be a hypergraph on $n$ vertices with $m$ edges. The definitions of $k$-colorings and proper $k$-colorings remain the same as for uniform hypergraphs.

### 9.1 General $k$-Colorability

Let $r : E(\mathcal{H}) \to \mathbb{N}$ be the function that maps each edge to its cardinality and let $[\mathcal{H}] = \{r(e) : e \in E(\mathcal{H})\}$ be the set of edge cardinalities of $\mathcal{H}$. For any edge $e \in E(\mathcal{H})$, let $e = \{e_1, \ldots, e_{r(e)}\}$.

As with uniform hypergraphs we must categorize proper and improper edge products. For $r \in [\mathcal{H}]$, let $\mathcal{S}(r)$ be the set of all submultisets of $\mathcal{P}_k$ of size $r$, and let $\mathcal{R}(r) \subset \mathcal{S}(r)$ be the set of all submultisets of $\mathcal{P}_k$ which represent a proper $k$-coloring of edges with cardinality $r$. Let $N\mathcal{R}(r) = \mathcal{S}(r) \backslash \mathcal{R}(r)$, and note that $N\mathcal{R}(r)$ is the set of all submultisets in which all elements are the same.

The ideal capturing all colorings remains unchanged when moving to non-uniform hypergraphs:

$$C_k = \left\langle \prod_{p \in \mathcal{P}_k} (x_i - p) : i \in V(\mathcal{H}) \right\rangle$$

is again the ideal of all $k$-colorings of $\mathcal{H}$.

For non-uniform hypergraphs, we extend the definition of $h(y_1, \ldots, y_r)$ from Section 2 to vary with $r(e)$. We define a polynomial in $r(e)$ variables, $h_{r(e)}$, by

$$h_{r(e)}(y_1, \ldots, y_{r(e)}) = \sum_{1 \leqslant i < j \leqslant r(e)} (y_i - y_j)^2$$

and note that $h_{r(e)}$ vanishes exactly on multisets in $N\mathcal{R}(r(e))$. We state this as a generalization of Lemma 5.1.

**Lemma** 9.1. Let $\tilde{c} = (c_1, \ldots, c_{r(e)})$ be a $k$-coloring of the vertices in an edge, $e$, of $\mathcal{H}$. Then $h_{r(e)}(\tilde{c}) = 0$ if and only if the corresponding multiset for $\tilde{c}$ is an element of $N\mathcal{R}(r(e))$.

*Proof.* Since all colors in $\mathcal{P}_k$ are positive integers,

$$h_{r(e)}(\tilde{c}) = (c_1 - c_2)^2 + (c_1 - c_3)^2 + \cdots + (c_2 - c_3)^2 + (c_2 - c_4)^2 + \cdots + (c_{r(e)-1} - c_{r(e)})^2 = 0$$

if and only if, $c_i = c_j$ for all $1 \leqslant i < j \leqslant r(e)$. Thus $h_{r(e)}(\tilde{c}) = 0$ if and only if the corresponding multiset for $\tilde{c}$ is an element of $N\mathcal{R}(r(e))$. ☐

As with the uniform case, we will classify proper colorings via $h_{r(e)}$ polynomials. We define the polynomial $Gf_e$ as follows:

$$Gf_e(x_{e_1}, \ldots, x_{e_{r(e)}}) = \prod_{u \in \mathcal{R}(r(e))} \left( h_{r(e)}(x_{e_1}, \ldots, x_{e_{r(e)}}) - h_{r(e)}(u) \right) .$$

The polynomials $Gf_e$ define the *general $k$-colorability ideal*, $I_{\mathcal{H}}(k)$

$$I_{\mathcal{H}}(k) = C_k + \langle Gf_e : e \in E(\mathcal{H}) \rangle .$$

**Proposition** 9.1. *The polynomials in the ideal $I_{\mathcal{H}}(k)$ have a common solution if and only if $\mathcal{H}$ is properly $k$-colorable.*

*Proof.* ($\Rightarrow$) Let $\mathbf{c} = (c_1, \ldots, c_n)$ be a common solution to the polynomials in $I_{\mathcal{H}}(k)$. Since $\mathbf{c}$ is common solution to the polynomials in $C_k$, $\mathbf{c} \in \mathcal{V}(C_k)$. By Proposition 5.1 $\mathbf{c} = (c_1, \ldots, c_n)$ is a $k$-coloring of $\mathcal{H}$ and thus each $c_i$ takes on a value in $\mathcal{P}_k$. Moreover, as $\mathbf{c}$ is a common solution for the polynomials in $\langle Gf_e : e \in E(\mathcal{H})\rangle$, for every edge $e = \{e_1, \ldots, e_{r(e)}\} \in E(\mathcal{H})$, $Gf_e(c_{e_1}, \ldots c_{e_{r(e)}}) = 0$, thus there is some $u = \{u_1, \ldots, u_{r(e)}\} \in \mathcal{R}(r(e))$ such that

$$h_{r(e)}(c_{e_1}, \ldots, c_{e_{r(e)}}) - h_{r(e)}(u_1, \ldots, u_{r(e)}) = 0 .$$

By Lemma 9.1, $h_{r(e)}(u_1, \ldots, u_{r(e)}) \neq 0$, thus $h_{r(e)}(c_{e_1}, \ldots, c_{e_{r(e)}}) \neq 0$ as well, and the edge $e$ is properly colored. Hence, $\mathbf{c}$ is a proper coloring of $\mathcal{H}$.

($\Leftarrow$) Let $\tilde{\mathbf{c}} = (c_1, \ldots, c_n)$ be a proper $k$-coloring of $\mathcal{H}$. That is, $c_i \in \mathcal{P}_k$ and $\tilde{\mathbf{c}}$ colors no edge monochromatically. Since each $c_i \in \mathcal{P}_k$, $\tilde{\mathbf{c}}$ is a common solution of the polynomials in $C_k$ by Proposition 5.1.

Further, by Lemma 9.1, since no edge is colored by a single color, $h_{r(e)}(c_{e_1}, \ldots c_{e_{r(e)}}) \neq 0$ for every edge $e = \{e_1, \ldots, e_{r(e)}\} \in E(\mathcal{H})$. Thus, for every edge $e \in E(\mathcal{H})$, there exists $u \in \mathcal{R}(r(e))$ such that $h_{r(e)}(u) = h_{r(e)}(c_{e_1}, \ldots c_{e_{r(e)}})$. We can choose $u$ as the multiset $\{c_{e_1}, \ldots, c_{e_{r(e)}}\} \in \mathcal{R}(r(e))$, since $\tilde{\mathbf{c}}$ is a proper coloring. Hence, $Gf_e(c_{e_1}, \ldots c_{e_{r(e)}}) = 0$ for every $e \in E(\mathcal{H})$ and, thus, $\tilde{\mathbf{c}}$ is a common solution to the polynomials in $I_{\mathcal{H}}(k)$. ☐

**Corollary** 9.1. *$\mathcal{V}(I_{\mathcal{H}}(k))$ is the set of all proper $k$-colorings of $\mathcal{H}$.*

*Proof.* This follows from Proposition 9.1 and the fact that the variety of a polynomial ideal is the collection of all common solutions to the polynomials contained in that ideal. ☐

To describe the improper $k$-colorings of $\mathcal{H}$, we define the *general hypergraph polynomial for $k$-colorability $GP_{\mathcal{H},k}$* by

$$GP_{\mathcal{H},k}(x_1, \ldots, x_n) = \prod_{e \in E(\mathcal{H})} h_{r(e)}(x_{e_1}, \ldots, x_{e_{r(e)}}) .$$

**Proposition** 9.2. *$\mathcal{H}$ is not properly $k$-colorable if and only if the polynomials in $C_k + \langle GP_{\mathcal{H},k}\rangle$ have a common solution.*

*Proof.* ($\Rightarrow$) Let $\mathbf{c}$ be a $k$-coloring of the hypergraph $\mathcal{H}$. Assume $\mathbf{c}$ is not a proper coloring. Then there exists an edge, $\hat{e} \in E(\mathcal{H})$, which is monochromatically colored. Thus the corresponding color multiset for $\hat{e}$ colored by $\mathbf{c}$ is an element of $N\mathcal{R}(r(\hat{e}))$. By Lemma 9.1, $h_{r(\hat{e})}(\mathbf{c}) = 0$ and $GP_{\mathcal{H},r,k}(\mathbf{c}) = 0$. Further, since $\mathbf{c}$ is a $k$-coloring, $\mathbf{c}$ is a common solution to the polynomials in $\langle C_k\rangle$ by Proposition 5.1 and thus a common solution to $C_k + \langle GP_{\mathcal{H},k}\rangle$.

($\Leftarrow$) Assume $\mathbf{c}$ is a common solution to the polynomials in $C_k + \langle GP_{\mathcal{H},k}\rangle$. By Proposition 5.1, since $\mathbf{c}$ is a common solution to the polynomials in $C_k$, $\mathbf{c}$ is a $k$-coloring of $\mathcal{H}$. Since $GP_{\mathcal{H},k}(\mathbf{c}) = 0$, there is some factor, $h_{r(\bar{e})}$, corresponding to an edge, $\bar{e} \in E(\mathcal{H})$, which evaluates to zero under $\mathbf{c}$. By Lemma 9.1, since $h_{r(\bar{e})}(\mathbf{c}) = 0$, the color multiset for $\bar{e}$ is an element of $N\mathcal{R}(r(\bar{e}))$. Hence the edge $\bar{e}$ is monochromatically colored by $\mathbf{c}$, and $\mathbf{c}$ is an improper coloring of $\mathcal{H}$. ☐

**Corollary** 9.2. $\mathcal{V}(C_k + \langle GP_{\mathcal{H},k}\rangle)$ is the set of all improper $k$-colorings of $\mathcal{H}$.

*Proof.* This follows from Proposition 9.1 and the fact that the variety of a polynomial ideal is the collection of all common solutions to the polynomials contained in that ideal. ☐

As with uniform hypergraphs, we can characterize the $k$-colorability of a non-uniform hypergraph with the following partial generalization of Theorem 1.1 by Hillar and Windfeldt [18].

**Theorem** 9.1. Let $k \geqslant 2$ be a positive integer and let $\mathcal{H}$ be a simple hypergraph on $n$ vertices. Let $R = \mathbb{C}[x_1, \ldots, x_n]$. Let $I_{\mathcal{H}}(k)$ be the general $k$-colorability ideal for $\mathcal{H}$ and let $GP_{\mathcal{H},k}$ be the general hypergraph polynomial for $k$-colorability for $\mathcal{H}$. Then the following are equivalent:

(1) The hypergraph $\mathcal{H}$ is not $k$-colorable.

(2) The constant 1 is an element of the ideal $I_{\mathcal{H}}(k)$.

(3) $\dim_{\mathbb{C}} R/I_{\mathcal{H}}(k) = 0$ as a vector space.

(4) The hypergraph polynomial $GP_{\mathcal{H},k}$ belongs to the ideal $C_k$.

*Proof.* The equivalence of (1), (2), and (3) is given by Lemma 4.1, Proposition 9.1, and the Weak Nullstellensatz.

It remains to show (1) is equivalent to (4).

(1) $\Rightarrow$ (4): Assume $\mathcal{H}$ is not $k$-colorable. Let $\mathscr{C} = (c_1, \ldots, c_n)$ be a $k$-coloring of $\mathcal{H}$; thus

$$(c_1, \ldots, c_n) \in \mathcal{V}(C_k).$$

Since $\mathcal{H}$ is not $k$-colorable, $\mathscr{C}$ is not a proper coloring, so by Proposition 5.5,

$$(c_1, \ldots, c_n) \in \mathcal{V}\left(C_k + \langle GP_{\mathcal{H},k}\rangle\right).$$

Thus,

$$\mathcal{V}\left(C_k + \langle GP_{\mathcal{H},k}\rangle\right) \supseteq \mathcal{V}\left(C_k\right) \ ,$$

and so,

$$C_k + \langle GP_{\mathcal{H},k}\rangle \subseteq C_k \ .$$

Hence, $GP_{\mathcal{H},k} \in C_k$.

(4) $\Rightarrow$ (1): Assume $GP_{\mathcal{H},k} \in C_k$. Then

$$C_k + \langle GP_{\mathcal{H},k}\rangle \subseteq C_k$$

and

$$\mathcal{V}\left(C_k + \langle GP_{\mathcal{H},k}\rangle\right) \supseteq \mathcal{V}\left(C_k\right).$$

Let $\mathscr{C} = (c_1, \ldots, c_n)$ be a $k$-coloring of $\mathcal{H}$. Then since $(c_1, \ldots, c_n)$ is a point in $\mathcal{V}(C_k)$, by Proposition 5.5, $(c_1, \ldots, c_n) \in \mathcal{V}(C_k + \langle GP_{\mathcal{H},k} \rangle)$ and $\mathscr{C}$ is not proper. So $\mathcal{H}$ is not $k$-colorable.

This completes the proof of Theorem 9.1. ☐

### 9.2 General Coloring Schemes

To address coloring schemes for non-uniform hypergraphs we must select color patterns for each edge. In this section, we consider the case where all edges of a given size are permitted the same color patterns. We will address colorings schemes with non-global (with respect to edge cardinality) color patterns subsequently.

We define the *general edge product* of an edge $e \in E(\mathcal{H})$ to be

$$\prod_{i=1}^{r(e)} x_{e_i} \ .$$

If the edge $e$ is colored by a $k$-coloring $\mathscr{C} = (c_1, \ldots c_n)$, then the general edge product is the value $\mathscr{C}$ assigns to the corresponding general edge product:

$$\prod_{i=1}^{r(e)} \mathscr{C}(x_{e_i}) = \prod_{i=1}^{r(e)} c_{e_i} \ .$$

For convenience, when working with an arbitrary general edge product, we will write

$$\prod_{e_i \in e} x_{e_i}, \quad \prod_{e_i \in e} \mathscr{C}(e_i) \quad \text{and,} \quad \prod_{e_i \in e} c_{e_i}.$$

As with the uniform case, we create a coloring scheme by collecting the partitions of $r(e)$ that have expansions which produce the desired general edge products for the $k$-coloring we wish to consider. Whereas we were able to list a single set of proper partitions of $r$, $\Lambda_{\hat{r}}$, we must now consider a set of partitions for each edge size present in $\mathcal{H}$. Moreover, as we see in Section 8.2, we may be interested in *improper* edge products, i.e., those corresponding to partitions of size one. Therefore, for any $e \in E(\mathcal{H})$ we will utilize $\Lambda_{r(e)}$: the collection of *all* partitions of $r(e)$ when defining general coloring schemes.

**Definition** 9.1. Let $\mathcal{H}$ be a simple hypergraph and let $[\mathcal{H}] = \{r(e) : e \in E(\mathcal{H})\}$ be its set of cardinalities. A non-empty subset $M_r = \{\lambda_1(r), \ldots, \lambda_t(r)\}$ of $\Lambda_r$ is called a *$k$-coloring $r$-scheme*. A collection of $k$-coloring $r$-schemes, $\{M_r : r \in [\mathcal{H}]\}$ is called a *general $k$-coloring scheme* for $\mathcal{H}$. When the cardinality is obvious from $M_r$, we will write $\lambda(r) = \lambda$ for convenience.

We may then define an ideal which will encode a given general $k$-coloring scheme for $\mathcal{H}$.

**Definition** 9.2. Let $\mathcal{H}$ be a simple hypergraph and let $[\mathcal{H}] = \{r(e) : e \in E(\mathcal{H})\}$ be its set of cardinalities. Given a general $k$-coloring scheme, $\{M_r : r \in [\mathcal{H}]\}$, we define the *general $k$-coloring scheme ideal* by

$$S_{\mathcal{H}}(\{M_r : r \in [\mathcal{H}]\}, k) = C_k + \sum_{r \in [\mathcal{H}]} \left\langle \prod_{\lambda \in M_r} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right] : e \in E(\mathcal{H}), |e| = r \right\rangle$$

The general $k$-coloring scheme ideal allows us to determine if a given hypergraph is colorable with a general $k$-coloring scheme.

**Theorem** 9.2. Let $\mathcal{H}$ be a simple hypergraph and let $[\mathcal{H}] = \{r(e) : e \in E(\mathcal{H})\}$ be its set of cardinalities. Let $\{M_r : r \in [\mathcal{H}]\}$ be a general $k$-coloring scheme for $\mathcal{H}$. The polynomials in the ideal $S_{\mathcal{H}}(\{M_r : r \in [\mathcal{H}]\}, k)$ have a common solution if and only if the hypergraph $\mathcal{H}$ may be properly colored by the general coloring scheme $\{M_r : r \in [\mathcal{H}]\}$.

*Proof.* ($\Rightarrow$) Assume $\mathbf{c} = (c_1, \ldots, c_n)$ is a common solution to the polynomials in

$$S_{\mathcal{H}}(\{M_r : r \in [\mathcal{H}]\}, k) = C_k + \sum_{r \in [\mathcal{H}]} \left\langle \prod_{\lambda \in M_r} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right] : e \in E(\mathcal{H}), |e| = r \right\rangle.$$

By Proposition 5.1, $(c_1, \ldots, c_n)$ is a $k$-coloring of $\mathcal{H}$. Let $\mathscr{C} = (c_1, \ldots, c_n)$ be the $k$-coloring of $\mathcal{H}$ associated with $\mathbf{c}$. It remains to show that the generators of the ideal

$$\sum_{r \in [\mathcal{H}]} \left\langle \prod_{\lambda \in M_r} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right] : e \in E(\mathcal{H}), |e| = r \right\rangle$$

force $\mathscr{C}$ to be a coloring of $\mathcal{H}$ by the general $k$-coloring scheme $\{M_r : r \in [\mathcal{H}]\}$.

Let $e = \{e_1, e_2, \ldots, e_{r(e)}\} \in E(\mathcal{H})$ be an arbitrary edge in $\mathcal{H}$. Since $\mathscr{C} = (c_1, \ldots, c_n)$ zeros

$$\prod_{\lambda \in M_r} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right],$$

Lemma 6.2 implies $\mathscr{C}$ colors $e$ with a color pattern in $M_r$. Since $e$ was chosen arbitrarily, $\mathscr{C}$ colors $\mathcal{H}$ with the general $k$-coloring scheme $\{M_r : r \in [\mathcal{H}]\}$.

($\Leftarrow$) Assume $\mathcal{H}$ is properly colorable with the general $k$-coloring scheme $\{M_r : r \in [\mathcal{H}]\}$. Let $\mathscr{C} = (c_1, \ldots, c_n)$ be such a $k$-coloring. By Proposition 5.1, $(c_1, \ldots, c_n) \in \mathcal{V}(C_k)$ and is a common solution to the polynomials of $C_k$. Let $\mathbf{c} = (c_1, \ldots, c_n)$ represent the point in $\mathcal{V}(C_k)$. Additionally, $(c_1, \ldots, c_n)$ assigns a color pattern from $M_r$ to each edge of cardinality $r$. Let $r \in [\mathcal{H}]$ and $e = (e_1, \ldots, e_r) \in E(\mathcal{H})$ have cardinality $r$. Let $\prod_{e \in e_i} c_{e_i}$ be the edge product of $e$ colored by $\mathscr{C}$. Since $\mathscr{C}$ colors $e$ with a color pattern from $M_r$, there exists some $\lambda \in M_r$ such that

$$\prod_{e_i \in e} c_{e_i} = a \in EP(\lambda).$$

Hence, the polynomial

$$\prod_{\lambda\in M_r}\left[\prod_{a\in EP(\lambda)}\left(\prod_{e_i\in e}x_{e_i}-a\right)\right]$$

vanishes at $(c_1,\ldots,c_n)$. Since $r$ was chosen arbitrarily, $\mathbf{c}=(c_1,\ldots,c_n)$ is a common solution to the polynomials in

$$\sum_{r\in[\mathcal{H}]}\left\langle\prod_{\lambda\in M_r}\left[\prod_{a\in EP(\lambda)}\left(\prod_{e_i\in e}x_{e_i}-a\right)\right]:e\in E(\mathcal{H}),|e|=r\right\rangle.$$

□

A description of the variety $\mathcal{V}(S_{\mathcal{H}}(\{M_r:r\in[\mathcal{H}]\},k))$ follows immediately.

**Corollary** 9.3. Let $\{M_r:r\in[\mathcal{H}]\}$ be a general $k$-coloring scheme. Then

$$\mathcal{V}(S_{\mathcal{H}}(\{M_r:r\in[\mathcal{H}]\},k))$$

is the set of all $k$-colorings of $\mathcal{H}$ with the general $k$-coloring scheme $\{M_r:r\in[\mathcal{H}]\}$.

*Proof.* This follows from Theorem 9.2 and the fact that the variety of a polynomial ideal is the collection of all common solutions to the polynomials contained in that ideal. □

Let $\mathcal{H}$ be a simple hypergraph. In order to state a decomposition theorem for the variety $\mathcal{V}(I_{\mathcal{H}}(k))$ analogous to Theorem 3.1, we must consider only general $k$-coloring schemes which are *proper*. Let $\mathbb{M}$ be the collection of all proper general $k$-coloring schemes for $\mathcal{H}$. That is, no edge of any cardinality in $\mathcal{H}$ is colored monochromatically by any general $k$-coloring scheme in $\mathbb{M}$.

**Corollary** 9.4. Let $\mathcal{H}$ be a simple hypergraph. If $\{M_r:r\in[\mathcal{H}]\}$ contains no color patterns consisting of a single part, and $\mathcal{H}$ is colorable by the general $k$-color scheme $\{M_r:r\in[\mathcal{H}]\}$, then $\mathcal{H}$ is properly $k$-colorable.

*Proof.* Since $\{M_r:r\in[\mathcal{H}]\}$ contains no color patterns consisting of a single part, any coloring of $\mathcal{H}$ consistent with $\{M_r:r\in[\mathcal{H}]\}$ will be proper by Lemma 6.1. □

**Theorem** 9.3. Let $k\geqslant 2$ be an integer and let $\mathcal{H}$ be a simple hypergraph. Let $\mathbb{M}$ be the collection of all proper general $k$-coloring schemes for $\mathcal{H}$. Then

$$\mathcal{V}(I_{\mathcal{H}}(k))=\bigcup_{\{M_r:r\in[\mathcal{H}]\}\in\mathbb{M}}\mathcal{V}(S_{\mathcal{H}}(\{M_r:r\in[\mathcal{H}]\},k))$$

and

$$I_{\mathcal{H}}(k)=\bigcap_{\{M_r:r\in[\mathcal{H}]\}\in\mathbb{M}}S_{\mathcal{H}}(\{M_r:r\in[\mathcal{H}]\},k)\ .$$

*Proof.* Note that since $C_k \subset I_{\mathcal{H}}(k)$, $C_k \subset S_{\mathcal{H}}(\{M_r : r \in [\mathcal{H}]\}, k)$, and $C_k$ contains univariate square-free polynomials in each indeterminate, $C_k$, $I_{\mathcal{H}}(k)$, and $S_{\mathcal{H}}(\{M_r : r \in [\mathcal{H}]\}, k)$, are all radical by Theorem 4.1.

Since $\mathbb{M}$ is the collection of all proper general $k$-coloring schemes of $\mathcal{H}$, we have by Corollaries 9.1 and 9.4, that

$$\mathcal{V}(I_{\mathcal{H}}(k)) = \bigcup_{\{M_r : r \in [\mathcal{H}]\} \in \mathbb{M}} \mathcal{V}(S_{\mathcal{H}}(\{M_r : r \in [\mathcal{H}]\}, k))$$

Moreover, since $I_{\mathcal{H}}(k)$ and $S_{\mathcal{H}}(\{M_r : r \in [\mathcal{H}]\}, k)$ are radical, we have that

$$I_{\mathcal{H}}(k) = \bigcap_{\{M_r : r \in [\mathcal{H}]\} \in \mathbb{M}} S_{\mathcal{H}}(\{M_r : r \in [\mathcal{H}]\}, k) \ .$$

by the one-to-one correspondence of varieties and radical ideals. $\square$

### 9.3 Extended Coloring Schemes

Finally, we may construct a coloring scheme which mimics completely both stably bounded hypergraphs and unoriented pattern hypergraphs discussed in Section 3.2.

The modifications required are to first choose a collection of color patterns for each edge, $e \in \mathcal{H}$.

**Definition** 9.3. Let $\mathcal{H}$ be a simple hypergraph. For each edge, $e \in E(\mathcal{H})$, the non-empty subsets

$$M(e) = \{\lambda_1(e), \ldots, \lambda_{t(e)}(e)\} \subset \Lambda_{r(e)},$$

are called *extended $k$-coloring schemes* of $\mathcal{H}$.

Secondly, for convenience we may re-group the generators of the general $k$-coloring scheme ideal, $S_{\mathcal{H}}(M, k)$, based on the coloring schemes assigned to each edge.

**Definition** 9.4. Let $\mathcal{H}$ be a simple hypergraph. We define the *extended $k$-coloring scheme ideal* for $\mathcal{H}$ by

$$ES_{\mathcal{H}}(\mathcal{M}, k) = C_k + \sum_{e \in E(\mathcal{H})} \left\langle \prod_{\lambda \in M(e)} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right] \right\rangle$$

where $\mathcal{M} = \{M(e) : e \in E(\mathcal{H})\}$ is a collection of extended $k$-coloring schemes for the edges of $\mathcal{H}$.

**Theorem** 9.4. Let $k \geqslant 2$ be an integer and $\mathcal{H}$ be a simple hypergraph. Let $\mathcal{M} = \{M(e) : e \in E(\mathcal{H})\}$ be a collection of extended $k$-coloring schemes for $\mathcal{H}$. Then $\mathcal{H}$ is colorable by the extended $k$-coloring schemes $\mathcal{M}$ if and only if polynomials in the extended $k$-coloring scheme ideal

$$ES_{\mathcal{H}}(\mathcal{M}, k) = C_k + \sum_{e \in E(\mathcal{H})} \left\langle \prod_{\lambda \in M(e)} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right] \right\rangle$$

have a common solution.

*Proof.* ($\Rightarrow$) Assume $\mathbf{c} = (c_1, \ldots, c_n)$ is a common solution to the polynomials in

$$ES_{\mathcal{H}}(\mathcal{M}, k) = C_k + \sum_{e \in E(\mathcal{H})} \left\langle \prod_{\lambda \in M(e)} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right] \right\rangle$$

By Proposition 5.1, $(c_1, \ldots, c_n)$ is a $k$-coloring of $\mathcal{H}$. Let $\mathscr{C} = (c_1, \ldots, c_n)$ be the $k$-coloring of $\mathcal{H}$ associated with $\mathbf{c}$. It remains to show that the generators of the ideal

$$\sum_{e \in E(\mathcal{H})} \left\langle \prod_{\lambda \in M(e)} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right] \right\rangle$$

force $\mathscr{C}$ to be a coloring of $\mathcal{H}$ by the collection of extended $k$-coloring schemes, $\mathcal{M}$.

Let $e = \{e_1, e_2, \ldots, e_{r(e)}\} \in E(\mathcal{H})$ be an arbitrary edge in $\mathcal{H}$. Since $\mathscr{C} = (c_1, \ldots, c_n)$ zeros

$$\prod_{\lambda \in M(e)} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right],$$

Lemma 6.2 implies $\mathscr{C}$ colors $e$ with a color pattern in $M(e)$. Since $e$ was chosen arbitrarily, $\mathscr{C}$ colors $\mathcal{H}$ with the general $k$-coloring scheme $\mathcal{M}$.

($\Leftarrow$) Assume $\mathcal{H}$ is properly colorable by the extended $k$-coloring schemes $\mathcal{M}$. Let $\mathscr{C} = (c_1, \ldots, c_n)$ be such a $k$-coloring. By Proposition 5.1, $(c_1, \ldots, c_n) \in \mathcal{V}(C_k)$ and is a common solution to the polynomials of $C_k$. Let $\mathbf{c} = (c_1, \ldots, c_n)$ represent the point in $\mathcal{V}(C_k)$. Additionally, $(c_1, \ldots, c_n)$ assigns a color pattern from $\mathcal{M}$ to each edge. Let $e = (e_1, \ldots, e_{r(e)}) \in E(\mathcal{H})$. Let $\prod_{e \in e_i} c_{e_i}$ be the edge product of $e$ colored by $\mathscr{C}$. Since $\mathscr{C}$ colors $e$ with a color pattern from $\mathcal{M}$, there exists some $\lambda \in M(e)$ such that

$$\prod_{e_i \in e} c_{e_i} = a \in EP(\lambda).$$

Hence, the polynomial

$$\prod_{\lambda \in M(e)} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right]$$

vanishes at $(c_1, \ldots, c_n)$. Since $e$ was chosen arbitrarily, $\mathbf{c} = (c_1, \ldots, c_n)$ is a common solution to the polynomials in

$$\sum_{e \in E(\mathcal{H})} \left\langle \prod_{\lambda \in M(e)} \left[ \prod_{a \in EP(\lambda)} \left( \prod_{e_i \in e} x_{e_i} - a \right) \right] \right\rangle$$

$\square$

**Corollary** 9.5. Let $k \geqslant 2$ be an integer and $\mathcal{H}$ be a simple hypergraph. Let $\mathcal{M}$ be a collection of extended $k$-coloring schemes. Then

$$\mathcal{V}(ES_{\mathcal{H}}(\mathcal{M}, k))$$

is the set of all $k$-colorings of $\mathcal{H}$ with the extended $k$-coloring schemes $\mathcal{M}$.

*Proof.* This follows from Theorem 9.4 and the fact that the variety of a polynomial ideal is the collection of all common solutions to the polynomials contained in that ideal. □

### 9.4 Coloring Schemes and Local Coloring Classes

Utilizing Theorems 9.2 and 9.4 we may encode stably-bounded and unoriented pattern hypergraphs as coloring schemes. The primary drawback to encoding local coloring classes with coloring schemes is specifying particular color patterns for each edge. We begin with *mixed hypergraphs*: a coloring class which has inspired a wide range of research.

Mixed hypergraphs were introduced by Voloshin in 1993, for a full treatment see the monograph [24]. A mixed hypergraph, $\mathcal{H}$, is a triple $(V(\mathcal{H}), \mathcal{C}, \mathcal{D})$ consisting of the vertices, and collections of subsets of vertices: the $\mathcal{C}$-edges, and the $\mathcal{D}$-edges, respectively. The primary difference, from a vertex-coloring perspective, between a traditional hypergraph and a mixed hypergraph is that $\mathcal{C}$-edges are permitted to be monochromatically colored in a proper vertex coloring. In such a coloring, all $\mathcal{C}$-edges are required to have at least two vertices assigned the same color. Further, all $\mathcal{D}$-edges forbid monochromatic coloring by requiring the use of at least two different colors in a proper coloring.

In the context of coloring schemes, a mixed hypergraph may be encoded by an extended $k$-coloring scheme ideal with the following extended $k$-coloring schemes: edges in $\mathcal{D}$, $e_D$, have extended coloring schemes, $M(e_D)$, consisting of partitions of $r(e_D)$ with

$$\lambda_i \in \{1, 2, \ldots, r(e_d) - 1\}, \text{ for all } \lambda = \{\lambda_1, \ldots, \lambda_s\} \in M(e_D).$$

While edges in $\mathcal{C}$, $e_C$, have extended coloring schemes, $M(e_C)$, consisting of partitions of $r(e_C)$ with

$$\lambda_i \in \{2, 3, \ldots, r(e_C)\}, \text{ for all } \lambda = \{\lambda_1, \ldots, \lambda_s\} \in M(e_C).$$

Moreover, any edge $e \in \mathcal{C} \cap \mathcal{D}$ is considered a *bi-edge* and may be assigned an extended coloring scheme consisting of *any* partition of $r(e)$, so long as a part of size at least two is included.

The next class provides a complete generalization of traditional and mixed hypergraph colorings. Stably bounded hypergraphs were introduced by Bujtás and Tuza in a series of papers including [5] and [6]. As introduced in Section 3.2, this class assigns four parameters to each edge which control how many, and with what frequency, colors are assigned to the vertices of an edge.

Let $\mathcal{H}$ be a simple hypergraph and $e \in E(\mathcal{H})$. The parameters $s$ and $t$ are the minimum and maximum number of colors permitted on an edge. In terms of coloring schemes, these correspond to the minimum and maximum number of *parts* allowed in a partition of $r(e)$ used to create a permitted color pattern for the edge $e$. The $b$ parameter is the maximum size of a monochromatically colored subset of $e$, which corresponds to the largest permitted part in a partition of $r(e)$. The final parameter, $a$, is more complicated to describe in terms of coloring schemes. For $e$, $a(e)$ is the minimum number of vertices which *must* share a color in a proper coloring. Thus, the corresponding coloring scheme must be composed of partitions of $r(e)$ which contain a part of size at least $a(e)$. Note this does not force *all* parts to be at least $a(e)$, the partition must contain at least one part which is at least $a(e)$.

In the next example we use a 3-critical hypergraph, i.e., a hypergraph which is not 2-colorable unless any edge is removed.

**Example** 9.1. In [20], a non-2-colorable 4-uniform hypergraphs on 11 vertices with 24 edges was introduced. This is a stably bounded 3-colorable hypergraph with:

$$s = 2, t = 4, \boxed{a = 2}, b = 4,$$

but not 5-colorable with:

$$s = 2, t = 4, \boxed{a = 3}, b = 4.$$

The corresponding coloring schemes are:

$$k = 3: \quad \{\{2,1,1\},\{2,2\},\{3,1\}\}, \quad \text{and} \quad k = 5: \quad \{\{3,1\}\}$$

respectively.

### 9.5 A Non-Uniform Hypergraph Example

In our final example, we would like to illustrate the definitions and theory developed for non-uniform hypergrpahs. Let $\mathcal{H}$ be the following modification of the 4-uniform hypergraph from Section 7. This non-uniform simple hypergraph contains 8 vertices with 14 edges; there are 3 edges of size 3, 10 edges of size 4, and 1 edge of size 5.

$$\begin{aligned}\mathcal{H} = \{&\{1,2,5,6,8\},\{3,4,8\},\{4,6,8\},\{1,7,8\},\\&\{1,2,3,4\},\{1,5,6,7\},\{2,5,6,7\},\{3,5,6,7\},\{3,4,5,6\},\\&\{3,4,6,7\},\{3,4,5,7\},\{1,2,3,5\},\{1,2,3,6\},\{1,2,3,7\}\}.\end{aligned}$$

The hypergraph is arbitrarily 2-colorable, see Appendix B for the 24-polynomial Gröbner basis, however, when restricted to the extended coloring scheme listed below, 4 colors are required to properly color the hypergraph.

| Edge(s) | Coloring Scheme |
|---|---|
| $\{1,2,5,6,8\}$ | $\{2,1,1,1\}$ |
| All 3-edges. | $\{1,1,1\}$ |
| $\{1,2,3,4\},\{1,5,6,7\}$ | $\{2,2\}$ |
| All other 4-edges | $\{2,2\}$, $\{2,1,1\}$, $\{1,1,1,1\}$ |

Computation of the Gröbner basis for the extended coloring scheme above was assisted by partially coloring two variables, $x_1$ and $x_2$. Assigning these two variables the same color leads to a trivial Gröbner basis. Thus, the two variables must be assigned different colors ($x_1 = 2, x_2 = 3$), which produces this Gröbner basis containing six polynomials:

$$\{x_7 + x_8 - 12,\ x_6 + x_8 - 12,\ x_5 - 2,\ x_4 - 2,\ x_3 - 3,\ x_8^2 - 12x_8 + 35\}.$$

This yields two possible colorings:

$$(2, 3, 3, 2, 2, 7, 7, 5) \quad \text{and} \quad (2, 3, 3, 2, 2, 5, 5, 7),$$

which are isomorphic, so we conclude that this coloring scheme provides a unique 4-coloring of $\mathcal{H}$.

## Conclusions

We note that Propositions 5.2 and 9.1 imply the existence of algebraic algorithms to determine if a given hypergraph is $k$-colorable. In addition Theorems 6.1, 9.2, and 9.4 imply the existence of similar algorithms to determine the colorability of a given hypergraph by a coloring described using a coloring scheme.

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## Appendix A

All computations were performed using *SINGULAR* in conjunction with *Mathematica* via the *Singular.m* Mathematica package written by Manuel Kauers which can be found at [19].

Let $\lambda = \{2, 1, 1\}$ and $\mathcal{H}$ be the 4-uniform hypergraph on 7 vertices from Section 7. In this appendix we collect the Gröbner bases for the coloring scheme ideals $\mathcal{S}_{\mathcal{H},4}(\lambda, k)$.

Let $k = 4$. With $x_1 = x_2 = 2$, the reduced Gröbner basis for $S_{\mathcal{H},4}(\lambda, 4)$ contains ten polynomials:

$$
\begin{aligned}
p_1 =& x_3^2 + x_3x_4 + x_3x_5 + x_3x_6 + x_3x_7 + x_4^2 + x_4x_5 + x_4x_6 + x_4x_7 + x_5^2 + x_5x_6 + \\
& x_5x_7 + x_6^2 + x_6x_7 + x_7^2 - 30x_3 - 30x_4 - 30x_5 - 30x_6 - 30x_7 + 367 \\
p_2 =& x_7^3 - 15x_7^2 + 71x_7 - 105 \\
p_3 =& x_6^3 - 15x_6^2 + 71x_6 - 105 \\
p_4 =& x_5^3 - 15x_5^2 + 71x_5 - 105 \\
p_5 =& x_4^2x_5 + x_4^2x_6 + x_4^2x_7 + x_4x_5^2 + x_4x_5x_6 + x_4x_5x_7 + x_4x_6^2 + x_4x_6x_7 + x_4x_7^2 + \\
& x_5^2x_6 + x_5^2x_7 + x_5x_6^2 + x_5x_6x_7 + x_5x_7^2 + x_6^2x_7 + x_6x_7^2 - 15x_4^2 - \\
& 30x_4x_5 - 30x_4x_6 - 30x_4x_7 - 15x_5^2 - 30x_5x_6 - 30x_5x_7 - 15x_6^2 - 30x_6x_7 - 15x_7^2 + \\
& 296x_4 + 296x_5 + 296x_6 + 296x_7 - 1920 \\
p_6 =& x_4^3 - 15x_4^2 + 71x_4 - 105 \\
p_7 =& x_3x_4x_5 + x_3x_4x_6 + x_3x_4x_7 + x_3x_5x_6 + x_3x_5x_7 + x_3x_6x_7 + x_4x_5x_6 + x_4x_5x_7 + \\
& x_4x_6x_7 + x_5x_6x_7 - 15x_3x_4 - 15x_3x_5 - 15x_3x_6 - 15x_3x_7 - 15x_4x_5 - 15x_4x_6 - \\
& 15x_4x_7 - 15x_5x_6 - 15x_5x_7 - 15x_6x_7 + 154x_3 + 154x_4 + 154x_5 + \\
& 154x_6 + 154x_7 - 1350 \\
p_8 =& x_5^2x_6^2 + x_5^2x_6x_7 + x_5^2x_7^2 + x_5x_6^2x_7 + x_5x_6x_7^2 + x_6^2x_7^2 - 15x_5^2x_6 - 15x_5^2x_7 - \\
& 15x_5x_6^2 - 30x_5x_6x_7 - 15x_5x_7^2 - 15x_6^2x_7 - 5x_6x_7^2 + 71x_5^2 + 225x_5x_6 + \\
& 225x_5x_7 + 71x_6^2 + 225x_6x_7 + 71x_7^2 - 960x_5 - 960x_6 - 960x_7 + 3466 \\
p_9 =& x_4^2x_6^2 + x_4^2x_6x_7 + x_4^2x_7^2 + x_4x_6^2x_7 + x_4x_6x_7^2 + x_6^2x_7^2 - 15x_4^2x_6 - \\
& 15x_4^2x_7 - 15x_4x_6^2 - 30x_4x_6x_7 - 15x_4x_7^2 - 15x_6^2x_7 - 15x_6x_7^2 + 71x_4^2 + \\
& 225x_4x_6 + 225x_4x_7 + 71x_6^2 + 225x_6x_7 + 71x_7^2 - 960x_4 - 960x_6 - 960x_7 + 3466 \\
p_{10} =& x_3x_4x_6^2 + x_3x_4x_6x_7 + x_3x_4x_7^2 + x_3x_5x_6^2 + x_3x_5x_6x_7 + x_3x_5x_7^2 + x_3x_6^2x_7 + \\
& x_3x_6x_7^2 + x_4x_5x_6^2 + x_4x_5x_6x_7 + x_4x_5x_7^2 + x_4x_6^2x_7 + x_4x_6x_7^2 + x_5x_6^2x_7 + \\
& x_5x_6x_7^2 - 15x_3x_4x_6 - 15x_3x_4x_7 - 15x_3x_5x_6 - 15x_3x_5x_7 - 15x_3x_6^2 - 30x_3x_6x_7 - \\
& 15x_3x_7^2 - 15x_4x_5x_6 - 15x_4x_5x_7 - 15x_4x_6^2 - 30x_4x_6x_7 - 15x_4x_7^2 - 15x_5x_6^2 - \\
& 30x_5x_6x_7 - 15x_5x_7^2 - 15x_6^2x_7 - 15x_6x_7^2 + 71x_3x_4 + 71x_3x_5 + 225x_3x_6 + 225x_3x_7 + \\
& 71x_4x_5 + 225x_4x_6 + 225x_4x_7 + 225x_5x_6 + 225x_5x_7 + 154x_6^2 + 379x_6x_7 + 154x_7^2 - \\
& 960x_3 - 960x_4 - 960x_5 - 2310x_6 - 2310x_7 + 9359
\end{aligned}
$$

Let $k = 5$. With $x_1 = x_2 = 2, x_3 = 3$, and $x_4 = 5$, the reduced Gröbner basis for $S_{\mathcal{H},4}(\lambda, 5)$ has seven polynomials:

$$\begin{aligned}
p_1 =& x_7^4 - 26x_7^3 + 236x_7^2 - 886x_7 + 1155 \\
p_2 =& x_6^4 - 26x_6^3 + 236x_6^2 - 886x_6 + 1155 \\
p_3 =& x_5^3x_6 + x_5^3x_7 + x_5^2x_6^2 + x_5^2x_6x_7 + x_5^2x_7^2 + x_5x_6^3 + x_5x_6^2x_7 + x_5x_6x_7^2 + x_5x_7^3 + x_6^3x_7 + \\
& x_6^2x_7^2 + x_6x_7^3 \\
& - 18x_5^3 - 44x_5^2x_6 - 44x_5^2x_7 - 44x_5x_6^2 - 44x_5x_6x_7 - 44x_5x_7^2 - 18x_6^3 - 44x_6^2x_7 - \\
& 44x_6x_7^2 - 18x_7^3 + 545x_5^2 + 781x_5x_6 + 781x_5x_7 + 545x_6^2 + 781x_6x_7 + 545x_7^2 - \\
& 6250x_5 - 6250x_6 - 6250x_7 + 31810 \\
p_4 =& x_5^4 - 26x_5^3 + 236x_5^2 - 886x_5 + 1155 \\
p_5 =& x_6^3x_7^2 + x_6^2x_7^3 - 18x_6^3x_7 - 44x_6^2x_7^2 - 18x_6x_7^3 + 77x_6^3 + 545x_6^2x_7 + 545x_6x_7^2 \\
& + 77x_7^3 - 2002x_6^2 - 5364x_6x_7 - 2002x_7^2 + 17017x_6 + 17017x_7 - 47432 \\
p_6 =& x_5^3x_7^2 + x_5^2x_7^3 - 18x_5^3x_7 - 44x_5^2x_7^2 - 18x_5x_7^3 + 77x_5^3 + 545x_5^2x_7 + \\
& 545x_5x_7^2 + 77x_7^3 - 2002x_5^2 - 5364x_5x_7 - 2002x_7^2 + 17017x_5 + 17017x_7 - 47432 \\
p_7 =& x_5^2x_6^2x_7^2 - 18x_5^2x_6^2x_7 - 18x_5^2x_6x_7^2 - 18x_5x_6^2x_7^2 + 77x_5^2x_6^2 + 324x_5^2x_6x_7 + \\
& 77x_5^2x_7^2 + 324x_5x_6^2x_7 + 324x_5x_6x_7^2 + 77x_6^2x_7^2 - 1386x_5^2x_6 - 1386x_5^2x_7 - \\
& 1386x_5x_6^2 - 5832x_5x_6x_7 - 1386x_5x_7^2 - 1386x_6^2x_7 - 1386x_6x_7^2 + 5929x_5^2 + \\
& 24948x_5x_6 + 24948x_5x_7 + 5929x_6^2 + 24948x_6x_7 + 5929x_7^2 - 106722x_5 - \\
& 106722x_6 - 106722x_7 + 456533
\end{aligned}$$

Let $k = 6$. With $x_1 = x_2 = 2, x_3 = 3$, and $x_4 = 5$, the reduced Gröbner basis for $S_{\mathcal{H},4}(\lambda, 6)$ also has seven polynomials;

$$\begin{aligned}
p_1 =& x_7^5 - 39x_7^4 + 574x_7^3 - 3954x_7^2 + 12673x_7 - 15015\\
p_2 =& x_6^5 - 39x_6^4 + 574x_6^3 - 3954x_6^2 + 12673x_6 - 15015\\
p_3 =& x_5^5 - 39x_5^4 + 574x_5^3 - 3954x_5^2 + 12673x_5 - 15015\\
p_4 =& x_5^4x_6^2 + x_5^4x_6x_7 + x_5^4x_7^2 + x_5^3x_6^3 + x_5^3x_6^2x_7 + x_5^3x_6x_7^2 + x_5^3x_7^3 + x_5^2x_6^4 + x_5^2x_6^3x_7+\\
& x_5^2x_6^2x_7^2 + x_5^2x_6x_7^3 + x_5^2x_7^4 + x_5x_6^4x_7 + x_5x_6^3x_7^2 + x_5x_6^2x_7^3 + x_5x_6x_7^4 + x_6^4x_7^2+\\
& x_6^3x_7^3 + x_6^2x_7^4 - 31x_5^4x_6 - 31x_5^4x_7 - 70x_5^3x_6^2 - 70x_5^3x_6x_7 - 70x_5^3x_7^2 - 70x_5^2x_6^3-\\
& 70x_5^2x_6^2x_7 - 70x_5^2x_6x_7^2 - 70x_5^2x_7^3 - 31x_5x_6^4 - 70x_5x_6^3x_7 - 70x_5x_6^2x_7^2 - 70x_5x_6x_7^3-\\
& 31x_5x_7^4 - 31x_6^4x_7 - 70x_6^3x_7^2 - 70x_6^2x_7^3 - 31x_6x_7^4 + 311x_5^4 + 1520x_5^3x_6 + 1520x_5^3x_7+\\
& 2094x_5^2x_6^2 + 2094x_5^2x_6x_7 + 2094x_5^2x_7^2 + 1520x_5x_6^3 + 2094x_5x_6^2x_7 + 2094x_5x_6x_7^2+\\
& 1520x_5x_7^3 + 311x_6^4 + 1520x_6^3x_7 + 2094x_6^2x_7^2 + 1520x_6x_7^3 + 311x_7^4 - 13130x_5^3-\\
& 30924x_5^2x_6 - 30924x_5^2x_7 - 30924x_5x_6^2 - 34878x_5x_6x_7 - 30924x_5x_7^2 - 13130x_6^3-\\
& 30924x_6^2x_7 - 30924x_6x_7^2 - 13130x_7^3 + 217553x_5^2 + 327454x_5x_6 + 327454x_5x_7+\\
& 217553x_6^2 + 327454x_6x_7 + 217553x_7^2 - 1774238x_5 - 1774238x_6-\\
& 1774238x_7 + 6968327\\
p_5 =& x_6^4x_7^3 + x_6^3x_7^4 - 31x_6^4x_7^2 - 70x_6^3x_7^3 - 31x_6^2x_7^4 + 311x_6^4x_7 + 1520x_6^3x_7^2 + 1520x_6^2x_7^3+\\
& 311x_6x_7^4 - 1001x_6^4 - 13130x_6^3x_7 - 26970x_6^2x_7^2 - 13130x_6x_7^3 - 1001x_7^4 + 39039x_6^3+\\
& 204880x_6^2x_7 + 204880x_6x_7^2 + 39039x_7^3 - 559559x_6^2 - 1396390x_6x_7 - 559559x_7^2+\\
& 3492489x_6 + 3492489x_7 - 8016008\\
p_6 =& x_5^4x_7^3 + x_5^3x_7^4 - 31x_5^4x_7^2 - 70x_5^3x_7^3 - 31x_5^2x_7^4 + 311x_5^4x_7 + 1520x_5^3x_7^2 + 1520x_5^2x_7^3+\\
& 311x_5x_7^4 - 1001x_5^4 - 13130x_5^3x_7 - 26970x_5^2x_7^2 - 13130x_5x_7^3 - 1001x_7^4 + 39039x_5^3+\\
& 204880x_5^2x_7 + 204880x_5x_7^2 + 39039x_7^3 - 559559x_5^2 - 1396390x_5x_7 - 559559x_7^2+\\
& 3492489x_5 + 3492489x_7 - 8016008\\
p_7 =& x_5^3x_6^3x_7^3 - 31x_5^3x_6^3x_7^2 - 31x_5^3x_6^2x_7^3 - 31x_5^2x_6^3x_7^3 + 311x_5^3x_6^3x_7 + 961x_5^3x_6^2x_7^2 + 311x_5^3x_6x_7^3+\\
& 961x_5^2x_6^3x_7^2 + 961x_5^2x_6^2x_7^3 + 311x_5x_6^3x_7^3 - 1001x_5^3x_6^3 - 9641x_5^3x_6^2x_7 - 9641x_5^3x_6x_7^2-\\
& 1001x_5^3x_7^3 - 9641x_5^2x_6^3x_7 - 29791x_5^2x_6^2x_7^2 - 9641x_5^2x_6x_7^3 - 9641x_5x_6^3x_7^2 - 9641x_5x_6^2x_7^3-\\
& 1001x_6^3x_7^3 + 31031x_5^3x_6^2 + 96721x_5^3x_6x_7 + 31031x_5^3x_7^2 + 31031x_5^2x_6^3 + 298871x_5^2x_6^2x_7+\\
& 298871x_5^2x_6x_7^2 + 31031x_5^2x_7^3 + 96721x_5x_6^3x_7 + 298871x_5x_6^2x_7^2 + 96721x_5x_6x_7^3+\\
& 31031x_6^3x_7^2 + 31031x_6^2x_7^3 - 311311x_5^3x_6 - 311311x_5^3x_7 - 961961x_5^2x_6^2-\\
& 2998351x_5^2x_6x_7 - 961961x_5^2x_7^2 - 311311x_5x_6^3 - 2998351x_5x_6^2x_7 - 2998351x_5x_6x_7^2-\\
& 311311x_5x_7^3 - 311311x_6^3x_7 - 961961x_6^2x_7^2 - 311311x_6x_7^3 + 1002001x_5^3+\\
& 9650641x_5^2x_6 + 9650641x_5^2x_7 + 9650641x_5x_6^2 + 30080231x_5x_6x_7+\\
& 9650641x_5x_7^2 + 1002001x_6^3 + 9650641x_6^2x_7 + 9650641x_6x_7^2 + 1002001x_7^3-\\
& 31062031x_5^2 - 96817721x_5x_6 - 96817721x_5x_7 - 31062031x_6^2 - 96817721x_6x_7-\\
& 31062031x_7^2 + 311622311x_5 + 311622311x_6 + 311622311x_7 - 1003003001
\end{aligned}$$

## Appendix B

The Gröbner basis for $I_{\mathcal{H}}(2)$, where $\mathcal{H}$ is the non-uniform example from Section 9.5, computed in Mathematica.

$$
\begin{aligned}
p_1 =& x_8^2 - 5x_8 + 6,\\
p_2 =& x_7^2 - 5x_7 + 6,\\
p_3 =& x_7x_1 + x_8x_1 - 5x_1 - 5x_7 + x_7x_8 - 5x_8 + 19,\\
p_4 =& x_6^2 - 5x_6 + 6,\\
p_5 =& x_6x_4 + x_8x_4 - 5x_4 - 5x_6 + x_6x_8 - 5x_8 + 19,\\
p_6 =& x_5^2 - 5x_5 + 6,\\
p_7 =& x_5x_3 + x_6x_3 + x_7x_3 - x_8x_3 - 5x_3 + x_4x_5 - 5x_5 + x_4x_7 - 5x_7 - 2x_4x_8\\
&- x_6x_8 + 10x_8 + 6,\\
p_8 =& x_4^2 - 5x_4 + 6,\\
p_9 =& x_4x_3 + x_8x_3 - 5x_3 - 5x_4 + x_4x_8 - 5x_8 + 19,\\
p_{10} =& x_3^2 - 5x_3 + 6,\\
p_{11} =& x_3x_1 + x_4x_1 - 5x_1 - 5x_2 + x_2x_3 - 5x_3 + x_2x_4 - 5x_4 + 25,\\
p_{12} =& x_2^2 - 5x_2 + 6,\\
p_{13} =& x_1x_2 - x_4x_2 - x_1x_4 + 5x_4 + 5x_5 - x_5x_6 + 5x_6 - x_5x_7 - x_6x_7 + 5x_7 - 25,\\
p_{14} =& x_1^2 - 5x_1 + 6,\\
p_{15} =& -5x_6x_5 + x_6x_7x_5 - 5x_7x_5 + x_6x_8x_5 + x_7x_8x_5 - 5x_8x_5 + 19x_5 + 19x_6 - 5x_6x_7+\\
&19x_7 - 5x_6x_8 + x_6x_7x_8 - 5x_7x_8 + 19x_8 - 65,\\
p_{16} =& x_6x_7x_3 - x_6x_8x_3 - x_7x_8x_3 + 5x_8x_3 - 6x_3 - 12x_4 - 6x_6 + 5x_4x_7 - 19x_7 + 5x_4x_8+\\
&5x_6x_8 - 2x_4x_7x_8 - x_6x_7x_8 + 10x_7x_8 - 31x_8 + 60,\\
p_{17} =& x_5x_7x_4 - x_5x_8x_4 - x_7x_8x_4 + 5x_8x_4 - 6x_4 + 6x_5 - 5x_5x_7 + 6x_7 + x_5x_7x_8 - 6x_8,\\
p_{18} =& -5x_4x_2 + x_4x_7x_2 - 5x_7x_2 + x_4x_8x_2 + x_7x_8x_2 - 5x_8x_2 + 19x_2 + 6x_4 + 6x_7-\\
&x_4x_7x_8 + 6x_8 - 30,\\
p_{19} =& -5x_3x_2 + x_3x_7x_2 + x_3x_8x_2 - x_7x_8x_2 + 6x_2 + 6x_3 - 6x_7 - x_3x_7x_8 + 5x_7x_8 - 6x_8,\\
p_{20} =& -5x_5x_2 + x_5x_6x_2 - 5x_6x_2 + x_5x_7x_2 + x_6x_7x_2 - 5x_7x_2 + 19x_2 + 5x_5x_8-\\
&x_5x_6x_8 + 5x_6x_8 - x_5x_7x_8 - x_6x_7x_8 + 5x_7x_8 - 19x_8,\\
p_{21} =& x_5x_6x_1 - x_5x_8x_1 - x_6x_8x_1 + 5x_8x_1 - 6x_1 - 19x_5 - 19x_6 + 5x_5x_7 + 5x_6x_7 - 25x_7+\\
&10x_5x_8 - x_5x_6x_8 + 10x_6x_8 - 2x_5x_7x_8 - 2x_6x_7x_8 + 10x_7x_8 - 44x_8 + 95,\\
p_{22} =& 5x_4x_2 + x_3x_6x_2 - x_3x_8x_2 - 2x_4x_8x_2 - x_6x_8x_2 + 10x_8x_2 - 19x_2 - 6x_3-\\
&12x_4 - 6x_6 + 5x_3x_8 + 5x_4x_8 - x_3x_6x_8 + 5x_6x_8 - 31x_8 + 60,\\
p_{23} =& x_4x_5x_2 - 5x_5x_2 - x_4x_8x_2 + x_5x_8x_2 + 6x_2 - 6x_4 + 6x_5 + 5x_4x_8 - x_4x_5x_8 - 6x_8,\\
p_{24} =& x_4x_5x_1 - 5x_5x_1 - x_4x_8x_1 + x_5x_8x_1 + 6x_1 + 6x_4 - 5x_4x_5 + 19x_5 + x_4x_5x_8-\\
&5x_5x_8 + 6x_8 - 30
\end{aligned}
$$