Perfect Colorings of the Infinite Square Grid: Coverings and Twin Colors

Denis S. Krotov*

Sobolev Institute of Mathematics
Novosibirsk, Russia

krotov@math.nsc.ru

Submitted: Nov 8, 2020; Accepted: Mar 20, 2023; Published: Apr 7, 2023

Abstract

A perfect coloring (equivalent concepts are equitable partition and partition design) of a graph G is a function f from the set of vertices onto some finite set (of colors) such that every node of color i has exactly $S(i,j)$ neighbors of color j, where $S(i,j)$ are constants, forming the matrix S called quotient. If S is an adjacency matrix of some simple graph T on the set of colors, then f is called a covering of the target graph T by the cover graph G. We characterize all coverings by the infinite square grid, proving that every such coloring is either orbit (that is, corresponds to the orbit partition under the action of some group of graph automorphisms) or has twin colors (that is, two colors such that unifying them keeps the coloring perfect). The case of twin colors is separately classified.

Mathematics Subject Classifications: 05C15, 05B99

1 Introduction

This work is focused on the problem of classification of perfect colorings of the infinite square grid $G(\mathbb{Z}^2)$. We classify an important subcase of perfect colorings when there are no two nodes of the same color at distance one or two from each other. Separately, we characterize bipartite perfect colorings in the case when two colors can be merged keeping the perfectness of the coloring.

Perfect colorings (equivalent concepts are equitable partitions, regular partitions, partition designs) of graphs are usually considered as objects of algebraic combinatorics, see, e.g., [15, 9.3], because of their connection with the eigenspaces of graphs, which

*The study was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (Project FWNF-2022-0017) and supported by the Russian Foundation for Basic Research (Grant 20-51-53023).

determine many of their properties. While perfect colorings can be considered for any graph or even multigraph, the most natural classes of graphs to study from this point of view are distance-regular graphs [7] and vertex-transitive graphs. Perfect colorings of vertex-transitive infinite grids can be considered as combinatorial analogs of crystals.

One of motivations of the study of perfect colorings is that they naturally generalize more special classes of objects with similar regular properties. The list of concepts that can be defined as perfect colorings with special parameters includes perfect codes, MDS codes (maximum distance separable codes [22, Ch. 11], not necessarily linear) and MRD (maximum rank distance) codes [14] with distance 2 or 3, latin squares [12] and latin hypercubes [23], Steiner triple and quadruple systems [9], some other types of combinatorial designs and their subspace analogues [6], Cameron–Liebler line classes [8], intriguing sets [11] (which are essentially perfect 2-colorings). For some subclasses of classical objects, their equivalence to perfect colorings with corresponding parameters are less straightforward and the study of such connections can be considered as a notable direction in discrete mathematics. For example, orthogonal arrays [17] and error-correcting codes [22] whose parameters attain certain bounds are equivalent to perfect colorings with certain parameters (see e.g. [13], [25], [20, Th.1], [19], [5]), Boolean bent functions are equivalent to perfect 4-colorings of Grassmann graphs [26, Th. 2], some optimal edge cuts in a regular graphs correspond to perfect 2-colorings [16, Th. 2.4].

Perfect colorings of $G(\mathbb{Z}^2)$ were studied in the following preceding papers. The most important result was proved by Puzynina, who showed in [27] that every such coloring is either periodic (in two independent directions), or has so-called binary diagonals and can be made periodic by shifting some of them. In [4, 28, 18], perfect colorings of $G(\mathbb{Z}^2)$ up to, respectively, 2, 3, and 9 colors are listed (the last result is computational). In [3], there is a characterization of *completely regular codes* in $G(\mathbb{Z}^2)$, i.e., sets C of vertices such that the distance coloring with respect to C is perfect.

Similar results on perfect colorings of other notable infinite grids are worth mentioning. A partial analogs of the result of [27] was obtained in [29], where it was proved that for every perfect coloring of the hexagonal or triangular grid (of degree 3 and 6, respectively), there is a periodic (in two directions) perfect coloring with the same quotient matrix. Completely regular codes in the hexagonal or triangular grids were described in [1] and [31], [32] respectively. In [24], perfect colorings of the infinite hexagonal grid with three colors were described. Some results on completely regular codes in n-dimensional square grid were obtained in [2]. A systematic study of perfect colorings of infinite one-dimensional periodic graphs (finite-degree periodic graphs on $\mathbb{Z}$: infinite circulant graphs, infinite multipath graphs, etc.) is carried out by Avgustinovich, Lisitsyna, and Parshina, see one of the latest works [21] and references there. That study is also related to perfect colorings of infinite grids because some of those graphs are quotient graphs of one of the grids mentioned above.

Consider the set $\mathbb{Z}^2$ of pairs $[x, y]$ of integers x, y. For convenience, the elements of $\mathbb{Z}^2$ will be called *nodes* and pictured as squares. Pictures that show a fragment of $\mathbb{Z}^2$ with colored nodes are very frequent in this paper and formally can be considered as mathematical expressions, which can be a part of a sentence. For example, the phrase "a coloring $\mathcal{F}$ contains the fragment  " means "there are integers x and y such that $\mathcal{F}([x,y+1]) = \mathcal{F}([x+3,y]) = \boxed{1}$ and $\mathcal{F}([x+1,y]) = \mathcal{F}([x+4,y+1]) = \boxed{2}$". In such pictures the first coordinate x grows in the right (sometimes, right-down) direction, while the second, y, grows in the upward (sometimes, up-right) direction. If a picture is implied to show a coloring of the whole grid (or of some of the nodes, for example only the even nodes), see, e.g., Fig. A, B, ..., then the whole coloring is reconstructed from the shown fragment by translations with periods that are obvious from the picture.

A node $[x,y]$ is *even* (*odd*) iff the number $x+y$ is even (odd). We define the *distance* between two nodes $\bar{v} = [x,y]$ and $\bar{v}' = [x',y']$ as $d(\bar{v},\bar{v}') \triangleq |x'-x| + |y'-y|$. Two nodes at distance one from each other are *adjacent* or *neighbors*. In such a manner, an infinite graph $G(\mathbb{Z}^2)$ is defined, called the *infinite square grid* (sometimes, infinite rectangular grid). The *neighborhood* of a node is the set of all (four) its neighbors. We say that two nodes $[x,y]$ and $[x',y']$ are placed *diagonally* from each other iff $|x'-x| = |y'-y| = 1$. For a node $[x,y]$ the set $\{[x+i,y+i] \mid i \in \mathbb{Z}\}$ is called the *R-diagonal* through $[x,y]$; the set $\{[x+i,y-i] \mid i \in \mathbb{Z}\}$ is called the *L-diagonal* through $[x,y]$. By definition, a *diagonal* is an R-diagonal (R is for "right") or an L-diagonal (L is for "left"). Two R-diagonals (L-diagonals) through neighbor nodes are also called *neighbor*.

Let $G = (V(G), E(G))$ be a simple graph; let $\Sigma = (\boxed{1}, \boxed{2}, \ldots, \boxed{n})$ be some fixed collection of distinct elements, which will be referred to as *colors*; and let $S = (S_{ij})_{ij}$ be an $n \times n$ nonnegative integer matrix. A surjective function $\mathcal{F} : V(G) \to \Sigma$ is called a *perfect coloring* (of G) with (quotient) matrix S iff for every i and j from 1 to n every node $v \in V(G)$ such that $\mathcal{F}(v) = \boxed{i}$ has exactly S_{ij} neighbors colored with $\boxed{j}$.

The convenient term *semicoloring*, suggested in [21], refers to any function from a part of a bipartite graph to a finite set of colors.

Two colorings $\mathcal{F}, \mathcal{G} : \mathbb{Z}^2 \to \Sigma$ (or semicolorings) are called *equivalent*, $\mathcal{F} \sim \mathcal{G}$, iff $\mathcal{F}(\cdot) \equiv \pi\mathcal{G}(\tau(\cdot))$ where π is a permutation of the colors, τ is an adjacency-preserving transformation of $\mathbb{Z}^2$ (an *automorphism* of $G(\mathbb{Z}^2)$), i.e., translation, rotation, reflection, or a sliding symmetry (the composition of a reflection and a translation).

Next, we define three important classes of perfect colorings: bipartite perfect colorings, coverings, and perfect colorings with twin colors.

A perfect coloring is called *bipartite* iff the set of nodes of each color consists of nodes of the same parity (even or odd). Every perfect coloring $\mathcal{F}$ (of $G(\mathbb{Z}^2)$ or any other bipartite graph) can be treated as a bipartite perfect coloring: if $\mathcal{F}$ is not bipartite itself, then each color can be split into the even subcolor and the odd subcolor, resulting in a bipartite perfect coloring $\overline{\mathcal{F}}$. Note that $\overline{\mathcal{F}} \sim \overline{\mathcal{G}}$ neither means that $\mathcal{F}$ and $\mathcal{G}$ are equivalent, nor that they have the same quotient matrix (up to permutation of colors). On the other hand, some bipartite perfect colorings do not correspond to any non-bipartite ones.

If the quotient matrix of a perfect coloring $\mathcal{F}$ of G is a $\{0,1\}$-matrix with zero diagonal, then it is the adjacency matrix of some graph T on the set of colors. In this case, $\mathcal{F}$ is known as a *covering* of the *target* graph T by the *cover* graph G.

Assume that we have a perfect coloring with an n-by-n quotient matrix S. Two different colors $\boxed{a}$ and $\boxed{b}$ are called *twin* (twin colors or just *twins*) iff identifying them results in a perfect coloring $\mathcal{F}'$ with $n-1$ colors. Equivalently, two different colors $\boxed{a}$ and $\boxed{b}$ are twin if and only if $S_{aj} = S_{bj}$ for all $j \neq a, b$ (however, S_{ja} and S_{jb} might be distinct, and the "densities" of two twin colors in a perfect coloring can be different).

Remark. In preceding papers, twin colors were called "equivalent". However, equivalence is a general mathematical concept: any reflexive, symmetric, and transitive relation is an equivalence. In this paper, a specific term "twin" is suggested for that reason.

In this paper, we (1) characterize coverings by $G(\mathbb{Z}^2)$, and (2) classify all bipartite perfect colorings of $G(\mathbb{Z}^2)$ with twin colors.

In the next section, some additional concepts and two main theorems are introduced. Sections 3 and 4 contain proofs of the theorems. In Section 5, we briefly discuss the existence of non-orbit perfect colorings. Appendix A completes Theorem 1 by describing the subgroups of the automorphism group of the infinite square grid whose orbit coloring is a covering of a simple graph. Appendix B contains a catalogue of perfect colorings that, together with Theorem 2, form the classification of perfect colorings of the infinite square grid with twin colors.

2 Main results

Assume that some perfect coloring $\mathcal{F}$ with $n \times n$ matrix $S = (S_{ij})_{ij}$ is fixed. Let $\boxed{a}$ and $\boxed{b}$ be some colors, C some set of colors. We say that $\bar{v} \in \mathbb{Z}^2$ is an $\boxed{a}$-*node* (or $\bar{v}$ is colored with $\boxed{a}$, or $\bar{v}$ has color $\boxed{a}$) iff $\mathcal{F}(\bar{v}) = \boxed{a}$. We say that $\bar{v} \in \mathbb{Z}^2$ is a C-*node* iff $\mathcal{F}(\bar{v}) \in C$.

A diagonal (R-diagonal, L-diaginal) is called an $\boxed{a}\boxed{b}$-diagonal ($\boxed{a}\boxed{b}$-R-diagonal or $\boxed{a}\boxed{b}$-L-diaginal, depending on its direction) iff it consists of $\{\boxed{a}, \boxed{b}\}$-nodes and, moreover, the colors alternate on the diagonal, i.e., all its nodes of type $[2i, \cdot]$ ($i \in \mathbb{Z}$) are $\boxed{a}$-nodes and all its nodes of type $[2i+1, \cdot]$ ($i \in \mathbb{Z}$) are $\boxed{b}$-nodes, or vice versa. An $\boxed{a}\boxed{b}$-diagonal is called a *binary diagonal* iff $\boxed{a} \neq \boxed{b}$. The following simple fact is crucial for the rest of the article, and it will be used without explicit references.

Claim ([27, Proposition 6])**.** *If there is an $\boxed{a}\boxed{b}$-diagonal with $\boxed{a} \neq \boxed{b}$, then $\boxed{a}$ and $\boxed{b}$ are twins.*

Proof. Assume without loss of generality that

$$\mathcal{F}([0,0]) = \mathcal{F}([2,2]) = \mathcal{F}([4,4]) = \boxed{a} \quad \text{and} \quad \mathcal{F}([1,1]) = \mathcal{F}([3,3]) = \mathcal{F}([5,5]) = \boxed{b}.$$

Denote by N_a and N_b the union of the neighborhoods of $[0,0]$, $[2,2]$, $[4,4]$ and $[1,1]$, $[3,3]$, $[5,5]$, respectively. For a color $\boxed{i}$, the number of $\boxed{i}$-nodes in N_a is $3S_{ai}$, and the number of $\boxed{i}$-nodes in N_b is $3S_{bi}$. Since $|N_a \backslash N_b| = |N_b \backslash N_a| = 2$, the difference between $3S_{ai}$ and $3S_{bi}$ cannot be larger than 2. Hence, $S_{ai} = S_{bi}$. □

By *shifting* a binary $\boxed{a}\boxed{b}$-diagonal we mean swapping the colors $\boxed{a}$ and $\boxed{b}$ on that diagonal. Such operation does not change the perfectness of the coloring of the whole grid. Two colorings or semicolorings are *shifting equivalent* iff one of them is equivalent to a coloring (respectively, semicoloring) obtained from the other by shifting binary diagonals.

A coloring of a graph G is called *orbit*, if there is a subgroup Φ of the automorphism group $\mathrm{Aut}(G)$ of G such that

- any two nodes $\bar{x}$ and $\bar{y}$ have the same color if and only if $\bar{x} = \varphi(\bar{y})$ for some φ in Φ (i.e., $\bar{x}$ and $\bar{y}$ belong to the same orbit under the action of Φ).

It is straightforward and well known that every orbit coloring is a perfect coloring.

We are now ready to state the main results of the paper.

Theorem 1. *Every perfect coloring of $G(\mathbb{Z}^2)$ with quotient $\{0,1\}$-matrix with zero diagonal (that is, the coloring is a covering of a simple graph) is either orbit or has a binary diagonal (and, hence, twin colors).*

Two corollaries (5 and 6) of the next theorem, of independent interest, complete Theorem 1 by characterizing coverings with binary diagonals.

Theorem 2. *Every bipartite perfect coloring of $G(\mathbb{Z}^2)$ with at least one pair of twin colors is either shifting equivalent to one of the colorings listed in Fig. A–W with the corresponding matrices (see Appendix B) or can be obtained from such a coloring by merging groups of (two, three, or four) mutually twin colors.*

A special case in the claim of Theorem 2 should be highlighted because of the risk to be forgotten. The only case that satisfies the hypothesis of the theorem with twin colors of different parity is the *chessboard coloring*, obtained from the coloring in Fig. I by merging two groups (even and odd) from four mutually twin colors.

Remark 3. Each of the infinite families of colorings shown in Fig. K–W has one or two small-parameter cases that are included in other, finite, families. In each of such cases, the coloring has binary diagonals of both directions, in contrast to the other colorings of the corresponding infinite family. For example the coloring in Fig. M($n = 6$) is equivalent to a coloring in Fig. I with shifted L-diagonals and some merged colors. That coloring has an additional pair ($\boxed{3}$, $\boxed{4}$) of twin colors and binary $\boxed{3}\boxed{4}$-L-diagonals, which are not illustrated in the general diagram (Fig. M). By this reason, one can find natural to exclude such special cases from the infinite series (which keeps Theorem 2 complete). To make this easy, such special cases are explicitly indicated by the reference to a finite family in the parenthesis after the corresponding number of the colors, see the description to each of Fig. K–W.

Remark 4. The coloring shown in Fig. A has intersecting binary diagonals, which cannot be shifted independently. Actually, any shifting R-diagonals, except shifting *all* R-diagonals (which has the same effect as renaming colors $\boxed{4} \leftrightarrow \boxed{5}$) results in a coloring without binary L-diagonals. Since the original coloring has an automorphism (horizontal reflection plus swapping colors $\boxed{5} \leftrightarrow \boxed{6}$) that changes the roles of R-diagonals and L-diagonals, it is safe to say that we can use only R-diagonal shifts and ordinary equivalence transformations (automorphisms of the grid and renaming colors) to exhaust the shifting equivalence class of the coloring shown in Fig. A.

A similar situation is with Fig. I. However, in that case there are L-diagonals and R-diagonals of both parities. Shifting some (but not all) $\boxed{1}\boxed{2}$-diagonals or some (but not all) $\boxed{3}\boxed{4}$-diagonals breaks all L-diagonals of the same parity, but does not affect the diagonals of the opposite parity. Similar "nontrivial" shifting $\boxed{1}\boxed{3}$- and $\boxed{2}\boxed{4}$-diagonals breaks all R-diagonals of the same parity. Similarly, for $\boxed{5}\boxed{7}$- and $\boxed{6}\boxed{8}$-diagonals and for $\boxed{5}\boxed{6}$- and $\boxed{7}\boxed{8}$-diagonals. It is not difficult to conclude that the shifting equivalence class is the union of two families. One family is obtained by shifting only diagonals of the same direction (say, R-diagonals); the second family is obtained by shifting only even diagonal of the same direction (say, R-diagonals) and odd diagonals of the opposite direction (respectively, L-diagonals). The two families intersect in the colorings equivalent to the one shown in Fig. I.

If Fig. J, both binary R-diagonals and binary L-diagonals can be found, but they do not intersect and can be shifted independently.

Corollary 5. *Every bipartite perfect colorings of $G(\mathbb{Z}^2)$ with quotient $\{0,1\}$-matrix and twin colors is shifting equivalent to a coloring shown in Fig. I, Fig. K, or Fig. T. All such colorings have binary diagonals.*

Any non-bipartite perfect coloring $\mathcal{F}$ of $G(\mathbb{Z}^2)$ with equal rows in the quotient matrix corresponds to a bipartite perfect coloring $\overline{\mathcal{F}}$ (with the number of colors twice larger than in $\mathcal{F}$) with twin colors. Using this fact and Theorem 2, we can derive the following.

Corollary 6. *Every non-bipartite perfect colorings of $G(\mathbb{Z}^2)$ whose quotient matrix is a $\{0,1\}$-matrix with zero diagonal and two equal rows is shown in Fig. K, where $n = 6, 10, 14, \ldots$ in the non-bipartite case. All such colorings have binary diagonals.*

Corollary 7. *There are two quotient matrices admitting perfect colorings of $G(\mathbb{Z}^2)$ both with binary diagonals and without binary diagonals. These matrices are shown in Fig. C (Fig. J) and Fig. E (Fig. W, $n = 10$).*

3 Proof of Theorem 2: cases and subcases

Let us consider a bipartite perfect coloring $\mathcal{F}$ with twin colors $\boxed{1}$ and $\boxed{2}$. We start the proof with a very simple observation.

Claim 8. *Either $\mathcal{F}$ is the chessboard coloring (i.e., obtained from the coloring in Fig. I by merging the groups $\{\boxed{1},\boxed{2},\boxed{3},\boxed{4}\}$ and $\{\boxed{5},\boxed{6},\boxed{7},\boxed{8}\}$ of twin colors) or all the $\{\boxed{1},\boxed{2}\}$-nodes are of the same parity.*

Proof. Since $\mathcal{F}$ is bipartite, there are no two neighbor nodes of the same color. Therefore, either $\boxed{1}$-nodes have only $\boxed{2}$-neighbors and $\boxed{2}$-nodes have only $\boxed{1}$-neighbors (hence the coloring is chessboard), or both $\boxed{1}$-nodes and $\boxed{2}$-nodes have neighbors of some third color, say $\boxed{3}$. If the $\boxed{3}$-nodes are even then the $\{\boxed{1},\boxed{2}\}$-nodes are odd and vice versa. □

Since the chessboard coloring satisfies the conclusion of Theorem 2, we further assume w.l.o.g. that the $\{\boxed{1},\boxed{2}\}$-nodes are even. We will say that a node is of type $k{:}l$ iff it is adjacent to exactly k $\boxed{1}$-nodes and l $\boxed{2}$-nodes. Clearly, nodes of the same color are of the same type. Moreover, nodes of the same color are adjacent to the same number of nodes of some fixed type.

From the definition of twin colors we directly get the following fact. A node is adjacent to a $\boxed{1}$-node if and only if it is adjacent to a $\boxed{2}$-node. To put it differently, there are no nodes of type $0{:}k$ or $k{:}0$, where $k > 0$. On the other hand, the number of $\boxed{1}$-nodes and the number of $\boxed{2}$-nodes that are adjacent to a given node can be different.

We divide the situation into the following cases.

1. There exists a node of type 3:1 or 1:3. W.l.o.g., we can consider only the first case.
2. There exists a node of type 2:1 or 1:2. W.l.o.g., we can consider only the first case.
3. All the neighbors of a $\{\boxed{1},\boxed{2}\}$-node are of type 2:2.
4. Every $\{\boxed{1},\boxed{2}\}$-node has two neighbors of type 2:2 and two neighbors of type 1:1.
5. Every $\{\boxed{1},\boxed{2}\}$-node has one neighbor of type 2:2 and three neighbors of type 1:1.
6. All the neighbors of a $\{\boxed{1},\boxed{2}\}$-node are of type 1:1.

It is easy to see that if a node has three neighbors of type 2:2 then the fourth neighbor can not be of type 1:1. So, the completeness of the case list is obvious. Before starting to prove Theorem 2 for each of the cases, we introduce one convenient tool related with the number of paths of special color types in a perfect coloring.

Assume that $\mathcal{F}$ is a perfect coloring of $G(\mathbb{Z}^2)$ (or any finite-degree graph G) with quotient $n \times n$ matrix $S = (S_{ij})_{i,j\in\{1,\dots,n\}}$. For a node $\bar{v}_0$ of color $\boxed{b}$, a positive integer k, and colors $\boxed{b_1},\dots,\boxed{b_k}$ denote by $d_{\bar{v}_0}(\boxed{b},\boxed{b_1},\dots,\boxed{b_k})$ the number of k-tuples $(\bar{v}_1,\dots,\bar{v}_k)$ such that $\bar{v}_{i-1}$ and $\bar{v}_i$ are adjacent in G and $\mathcal{F}(\bar{v}_i) = \boxed{b_i}$ for every $i \in \{1,\dots,k\}$; denote by $d^k_{\bar{v}_0}(\boxed{b},\boxed{b_k})$ the number of k-tuples $(\bar{v}_1,\dots,\bar{v}_k)$ such that $\bar{v}_{i-1}$ and $\bar{v}_i$ are adjacent in G for every $i \in \{1,\dots,k\}$ and $\mathcal{F}(\bar{v}_k) = \boxed{b_k}$. The following straightforward statement is well known.

Lemma 9. *The value $d^k_{\bar{v}}(\boxed{b},\boxed{b_k})$ does not depend on the choice of the $\boxed{b}$-node $\bar{v}$.*

Proof. It is clear that

$$d^k_{\bar{v}}(\boxed{b},\boxed{b_k}) = \sum_{b_1=1}^{n} \cdots \sum_{b_{k-1}=1}^{n} d_{\bar{v}}(\boxed{b},\boxed{b_1},\dots,\boxed{b_k}).$$

By induction, $d_{\bar{v}}(\boxed{b},\boxed{b_1},\dots,\boxed{b_k})$ equals $S_{b,b_1}S_{b_1,b_2}\cdots S_{b_{k-1},b_k}$ and does not depend on the choice of the $\boxed{b}$-node $\bar{v}$. (The induction base and step are straightforward from the definition of perfect coloring.) $\square$

As $d^k_{\bar{v}}(\boxed{b}, \boxed{b_k})$ does not depend on $\bar{v}$, we will use the notation $d^k(\boxed{b}, \boxed{b_k})$ instead. We will also need the following fact.

Lemma 10 ([28, Claim 9])**.** *There are positive rational numbers $P_1, \ldots, P_n$ (the "densities" of the corresponding colors) such that $\sum_{i=1}^{n} P_i = 1$ and $S_{ij}P_i = S_{ji}P_j$ for all $i, j \in \{1, \ldots, n\}$.*

Remark 11. The density P_i equals the limit of the portion of the $\boxed{i}$-nodes in a square of growing size.

I There exists a node of type 3:1

We assume that a node of type 3:1 has color $\boxed{3}$.

Claim 12. *Every node is of type* 0:0 *or of type* 3:1.

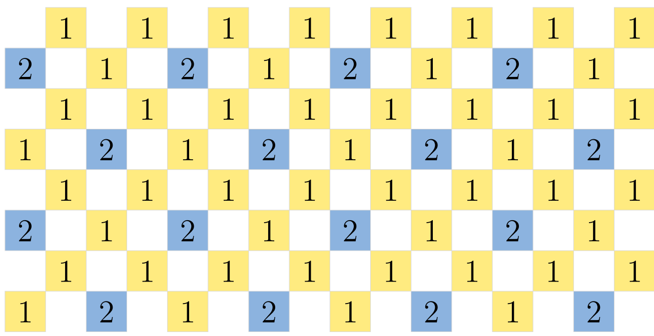
Proof. By Lemma 10, for any color $\boxed{i}$ we have

$$S_{1,3}S_{3,2}S_{2,i}S_{i,1} = S_{3,1}S_{2,3}S_{i,2}S_{1,i}.$$

Since $S_{1,3} = S_{2,3}$, $S_{2,i} = S_{1,i}$, $S_{3,1} = 3$, and $S_{3,2} = 1$, we get $S_{i,1} = 3S_{i,2}$. Hence, either $S_{i,1} = S_{i,2} = 0$, or $S_{i,1} = 3$ and $S_{i,2} = 1$. □

It is easy to see that all even nodes are $\{\boxed{1}, \boxed{2}\}$-nodes. Note that there are only $\boxed{1}$-nodes at the distance 2 from any $\boxed{2}$-node. Consider two possibilities.

I.a) $\boxed{2}$-nodes are placed periodically with periods $[2, 2]$ and $[2, -2]$.

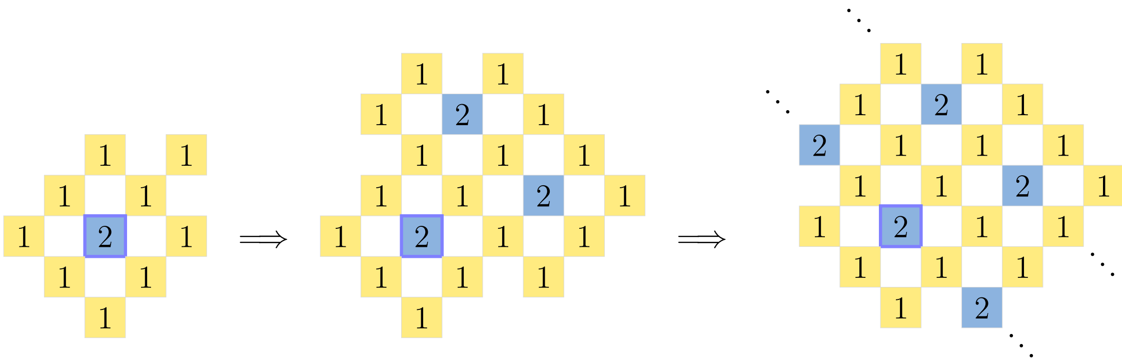


I.b) In some $\boxed{2}$-node $\bar{v}$ one of the periods is broken, i.e., $\mathcal{F}(\bar{v} + \bar{d}) = \boxed{1}$, where

I.b') $\bar{d} = [2, 2]$, or $\bar{d} = [-2, -2]$, or

I.b") $\bar{d} = [2, -2]$, or $\bar{d} = [-2, 2]$.

Without loss of generality, we assume $\bar{v} = [0, 0]$ and $\bar{d} = [2, 2]$.



From Claim 12 we get that $[1,3]$ and $[3,1]$ are 2-nodes; then, $[-2,2]$ and $[2,-2]$ are $\boxed{2}$-nodes, and, by induction, $k[-2,2]$ and $k[-2,2]+[3,1]$ are $\boxed{2}$-nodes for any integer k. Further, if $[4,4]$ is a $\boxed{2}$-node, then in a similar manner we have that $k[-2,2]+[4,4]$ is a $\boxed{2}$-node for any k; otherwise the nodes $k[-2,2]+[4,4]+[-1,1]$ are $\boxed{2}$-nodes. In both cases we have one more $\boxed{1}\boxed{2}$-L-diagonal. In a similar manner, by induction, we get that for each integer n either the nodes $k[-2,2]+n[2,2]$ or the nodes $k[-2,2]+[-1,1]+n[2,2]$ are $\boxed{2}$-nodes. So, we can conclude the following.

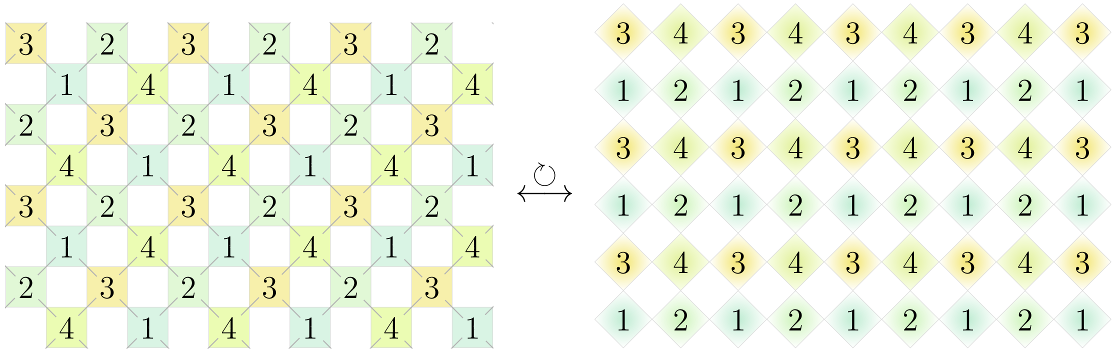
Proposition 13. *A coloring corresponding to subcase* I.b *can be obtained from a coloring corresponding to subcase* I.a *by shifting* $\boxed{1}\boxed{2}$*-L-diagonals or by shifting* $\boxed{1}\boxed{2}$*-R-diagonals (but not both).*

So, the colors of the even nodes are described. Every odd node is adjacent to a $\boxed{1}$-node, and hence, the quotient matrix S is completely determined by the colors in the neighborhood of a $\boxed{1}$-node. Up to renaming the colors, S is one of the following:

$$\begin{pmatrix} 0&0&4\\ 0&0&4\\ 3&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&3&1\\ 0&0&3&1\\ 3&1&0&0\\ 3&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&2&2\\ 0&0&2&2\\ 3&1&0&0\\ 3&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&2&1&1\\ 0&0&2&1&1\\ 3&1&0&0&0\\ 3&1&0&0&0\\ 3&1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1&1&1\\ 0&0&1&1&1&1\\ 3&1&0&0&0&0\\ 3&1&0&0&0&0\\ 3&1&0&0&0&0\\ 3&1&0&0&0&0 \end{pmatrix}.$$

If S is the first matrix, then the odd nodes are colored with one color. If the second one, then we can change the roles of the even and odd nodes to see that the colors of the odd nodes are also described by Proposition 13 (with colors $\boxed{3}$ and $\boxed{4}$ instead of $\boxed{1}$ and $\boxed{2}$). For the last three matrices, we also see that every two odd colors are twin, but after changing the roles of the even and odd nodes we get one of cases that will be considered later, namely, in cases II, III, and VI. To help structuring the proof, we will state the following lemma as the summary of these cases. The corollary of the lemma finalizes the proof of Theorem 2 for cases I, II, III, and VI.a.

Lemma 14. *Assume that every two nodes of the same parity are colored with the same or twin colors. Then the coloring of the even (similarly, odd) nodes is equivalent to a semicoloring obtained from the following semicoloring $\mathcal{H}$ by shifting R-diagonals and/or* *merging colors (two, three, or all four colors, or two groups of two colors):*



Proof. If there is only one even color, then the claim is trivial. If there are at least two even colors, $\boxed{1}$ and $\boxed{2}$, then they are twin and all odd nodes are of the same type by the hypothesis of the lemma. Depending on the type, 3:1, 2:1, 2:2, or 1:1, the situation falls into the scope of cases I, II, III, VI, respectively.

The claim of the lemma is proved in Proposition 13 for case I, Proposition 19(a) for case II, Proposition 20 for case III, Propositions 26(ii,iii) and 32(a) for subcases VI.a and VI.d of case VI, while in the other subcases, the claim of Theorem 2 is proved directly, which also means that either the claim of the lemma is true (Fig. I) or its hypothesis is not satisfied (Fig. A–H, J–W). □

Applying Lemma 14 to both even and odd nodes, we get

Corollary 15. *Under the hypothesis of Lemma 14, the conclusion of Theorem 2 holds with the corresponding Fig. I.*

II There exists a node of type 2:1

Similarly to Claim 12, we have

Claim 16. *Every node has one of the types* 0:0*,* 2:1*.*

It is not difficult to see that the type 0:0 nodes are the even nodes and all the odd nodes have the type 2:1 (indeed, if, seeking a contradiction, we assume that an odd node $\bar{u}$ is of type 0:0, then $\bar{u} + [\pm 1, \pm 1]$ are also type 0:0; by induction, all odd nodes are type 0:0). Consider a new coloring $\mathcal{G}$ with

- $\mathcal{G}(\bar{v}) = \mathcal{F}(\bar{v})$ if $\mathcal{F}(\bar{v}) \in \{\boxed{1}, \boxed{2}\}$,
- $\mathcal{G}(\bar{v}) = \boxed{4}$ if $\bar{v}$ is odd,
- $\mathcal{G}(\bar{v}) = \boxed{3}$ otherwise.

By the definition, $\mathcal{G}$ is a bipartite perfect coloring with the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 \\ 2 & 1 & 1 & 0 \end{pmatrix}.$$

Joining together twin colors $\boxed{1}$ and $\boxed{3}$ (or $\boxed{1}$ and $\boxed{2}$), we get the situation of case I. Hence, the location of the $\boxed{2}$-nodes (respectively, $\boxed{3}$-nodes) is described by subcases I.a and I.b.

Claim 17. *The coloring $\mathcal{G}$ has a period $(2,2)$ or $(2,-2)$.*

Proof. If the set of $\boxed{2}$-nodes corresponds to subcases I.a, then it has both periods $(2,2)$ and $(2,-2)$; if it corresponds to subcases I.b', then it has period $(2,-2)$; if it corresponds to subcases I.b", then it has period $(2,2)$. The same can be stated for the set of $\boxed{3}$-nodes. It remains to show that subcases I.b' and I.b" cannot happen simultaneously for the $\boxed{2}$-nodes and the $\boxed{3}$-nodes (or, similarly, for the $\boxed{3}$-nodes and the $\boxed{2}$-nodes) respectively. Seeking a contradiction, assume that the $\boxed{2}$-nodes correspond to subcases I.b' and the $\boxed{3}$-nodes correspond to subcases I.b". This means that for some even $\bar{v} = [x,y]$,

$$\mathcal{G}([x+2i, y-2i]) = \mathcal{G}([x+3+2i, y+1-2i]) = \boxed{2}, \quad \text{for all } i \in \mathbb{Z}, \tag{1}$$

and, similarly, for some even $[x', y']$,

$$\mathcal{G}([x'+2j, y'+2j]) = \mathcal{G}([x'+3+2j, y'-1+2j]) = \boxed{3}, \quad \text{for all } j \in \mathbb{Z}. \tag{2}$$

Now, at least one of the four pairs $\left(\frac{y-x-y'+x'}{4}, \frac{y+x-y'-x'}{4}\right)$, $\left(\frac{y-x-y'+x'-2}{4}, \frac{y+x-y'-x'+4}{4}\right)$, $\left(\frac{y-x-y'+x'+4}{4}, \frac{y+x-y'-x'-2}{4}\right)$, $\left(\frac{y-x-y'+x'+2}{4}, \frac{y+x-y'-x'+2}{4}\right)$ is integer; denote it (i_0, j_0). Substituting $i = i_0$ in (1) and $j = j_0$ in (2), we find that the same node has color $\boxed{2}$ and $\boxed{3}$ (for example, if (i,j) is equal to the first pair, then $[x+2i, y-2i] = [x'+2j, y'+2j]$). The contradiction obtained proves the claim. □

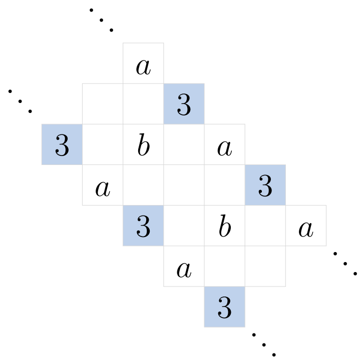
See Proposition 13 for the placement of nodes with colors $\boxed{2}$ and $\boxed{3}$. Moreover, if we assume that $\mathcal{G}$ has period $[2,-2]$, then we find that up to equivalence, the coloring $\mathcal{G}$ can be obtained by shifting L-diagonals from one of the following two colorings:

3	4	2	4	3	4	2	4	3	4	2	4
4	1	4	1	4	1	4	1	4	1	4	1
2	4	3	4	2	4	3	4	2	4	3	4
4	1	4	1	4	1	4	1	4	1	4	1
3	4	2	4	3	4	2	4	3	4	2	4
4	1	4	1	4	1	4	1	4	1	4	1
2	4	3	4	2	4	3	4	2	4	3	4
4	1	4	1	4	1	4	1	4	1	4	1

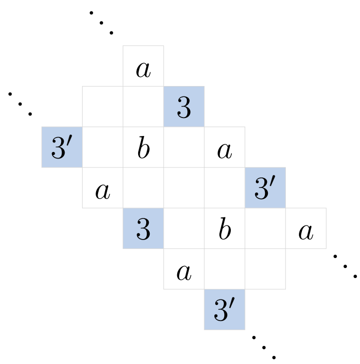
1	4	2	4	1	4	2	4	1	4	2	4
4	1	4	3	4	1	4	3	4	1	4	3
2	4	1	4	2	4	1	4	2	4	1	4
4	3	4	1	4	3	4	1	4	3	4	1
1	4	2	4	1	4	2	4	1	4	2	4
4	1	4	3	4	1	4	3	4	1	4	3
2	4	1	4	2	4	1	4	2	4	1	4
4	3	4	1	4	3	4	1	4	3	4	1

Claim 18. *One of the following assertions is true.*
a) *The color* $\boxed{3}$ *of the coloring* $\mathcal{G}$ *corresponds to one color of the coloring* $\mathcal{F}$.
b) *The color* $\boxed{3}$ *of the coloring* $\mathcal{G}$ *corresponds to two colors of the coloring* $\mathcal{F}$ *and the corresponding nodes are placed periodically with periods* $[0,4]$ *and* $[4,0]$.

Proof. 1. Assume the placement of $\boxed{3}$-nodes corresponds to case I.b. Then the coloring $\mathcal{G}$ contains (up to reflection) the fragment



1.1. Consider the values $d^2(\boxed{a},\boxed{x})$ for the coloring $\mathcal{F}$ and each color $\boxed{x}$ that corresponds to the color $\boxed{3}$ of $\mathcal{G}$. Using Lemma 9 we get that there are not more than two such colors (say, $\boxed{3}$ and $\boxed{3'}$), moreover, their nodes alternate with period $[4,-4]$.

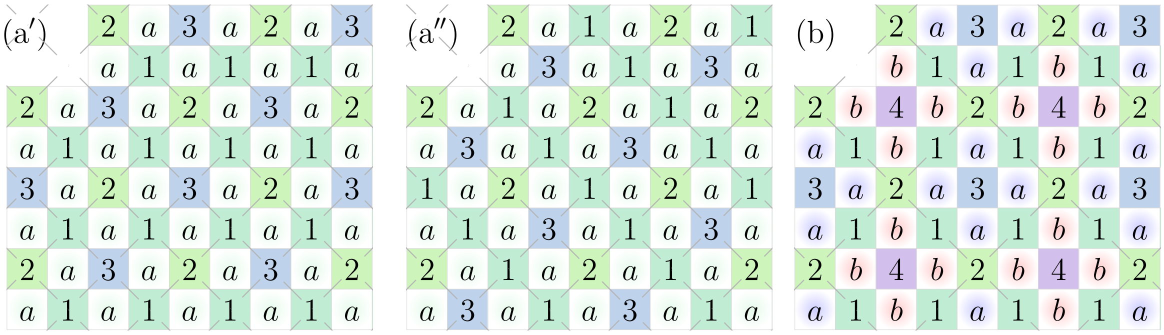


1.2. Considering the values $d^2(\boxed{b},\boxed{3'})$ for two $\boxed{b}$-nodes we get that $\boxed{3}$ and $\boxed{3'}$ are the same color.

2. Assume the placement of $\boxed{3}$-nodes corresponds to the case I.a. Then by arguments similar to p. 1.1 we get that one of the statements a), b) holds. ☐

Now we can conclude (with a similar argument than the one given in the proof of Claim 18) that the following proposition is true.

Proposition 19. *In case* II, $\mathcal{F}$ *is equivalent to a coloring obtained by shifting L-diagonals from a colorings of the following three types, where the colors of any two a-nodes (b-nodes)* *are either twin or the same:*



Proposition 19(a',a") confirms the claim of Lemma 14 with the following correspondence of colors:

(a′) :	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$
$\mathcal{H}$:	$\boxed{1}\cup\boxed{4}$	$\boxed{2}$	$\boxed{3}$

,

(a″) :	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$
$\mathcal{H}$:	$\boxed{1}\cup\boxed{2}$	$\boxed{3}$	$\boxed{4}$

.

Corollary 15 finalizes the proof of Theorem 2, case II in subcases (a') and (a") (according to the conclusion of Proposition 19). In subcase (b), the claim of Theorem 2 holds corresponding to Fig. A in accordance to the following table:

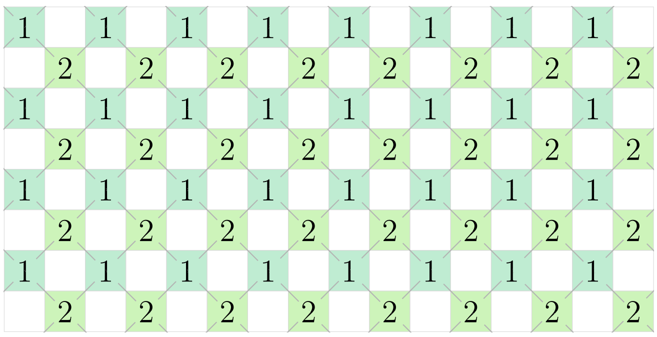
(b) :	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	a	b
Fig. A :	$\boxed{5}\cup\boxed{6}$	$\boxed{4}$	$\boxed{1}$	$\boxed{9}$	$\boxed{2}\cup\boxed{3}$	$\boxed{7}\cup\boxed{8}$

.

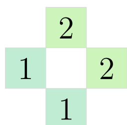
III All the neighbors of a $\{\boxed{1},\boxed{2}\}$-node are of type 2:2

Consider two subcases.

III.a) The coloring $\mathcal{F}$ is of the following type, where the odd nodes are colored with not more than four pairwise twin colors:

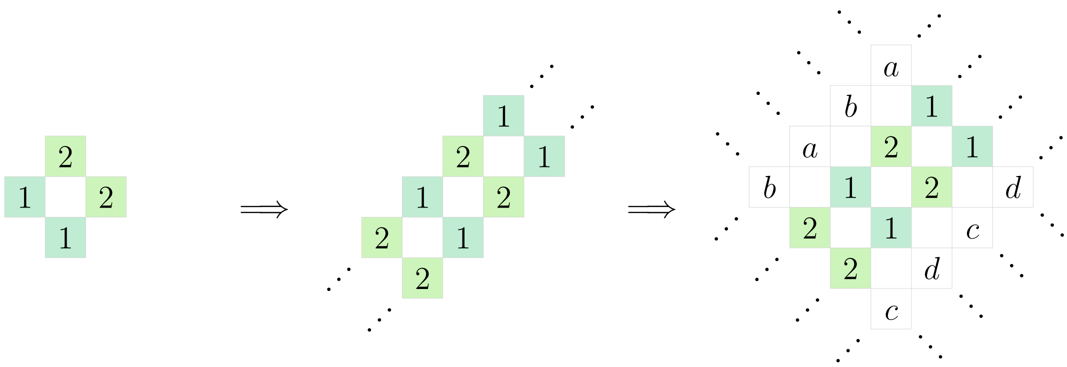


III.b) The coloring $\mathcal{F}$ contains up to reflection the following fragment:



In the last subcase, by the hypothesis of case III we get

- the fragment is uniquely extended to two $\boxed{1}\boxed{2}$-R-diagonals,
- the set of all even nodes consists of $\boxed{1}\boxed{2}$-R-diagonals.



$$\{a, b\} = \{c, d\} = \{1, 2\}$$

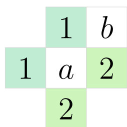
So, we have the following fact.

Proposition 20. *A coloring corresponding to subcase* III.b *can be obtained from a coloring corresponding to subcase* III.a *by shifting* $\boxed{1}\boxed{2}$*-R-diagonals or by shifting* $\boxed{1}\boxed{2}$*-L-diagonals (but not both).*

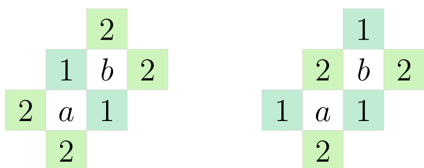
Again, we see that the claim of Lemma 14 is confirmed by Proposition 20 in the considered case, where the colors $\boxed{1}$ and $\boxed{2}$ in Proposition 20 correspond to the merged $\boxed{1}$ and $\boxed{3}$ and merged $\boxed{2}$ and $\boxed{4}$ in Lemma 14. Corollary 15 finalizes the proof of Theorem 2 in this case.

IV Every $\{\boxed{1}, \boxed{2}\}$-node has two neighbors of type 2:2 and two neighbors of type 1:1

Consider any $\boxed{1}$-node. Let its neighbors of type 2:2 have colors $\boxed{a}$ and $\boxed{b}$ (it is possible that $\boxed{a} = \boxed{b}$). Note that these neighbors must be placed diagonally from each other, otherwise the other two neighbors cannot be of type 1:1. By the same reason, a fragment like



is impossible. Hence we have two possibilities up to rotation and reflection.



In the both cases the fragment is uniquely extended to two $\boxed{1}\boxed{2}$-diagonals and $\boxed{a}\boxed{b}$-diagonal.

The rest of the proof of the conclusion of Theorem 2 in Case IV is contained in the following proposition, which reflects a more general situation and will be repeatedly used in Section VI.

Proposition 21. *If a bipartite perfect coloring $\mathcal{F}$ of $G(\mathbb{Z}^2)$ has neighbor $\boxed{a}\boxed{b}$ and $\boxed{e}\boxed{f}$ diagonals, where $\boxed{a}$, $\boxed{b}$, $\boxed{e}$, $\boxed{f}$ are some colors, not necessarily distinct, then $\mathcal{F}$ is equivalent to a coloring obtained from a coloring in Fig I, Fig K, or Fig L by shifting R-diagonals and merging some groups of twin colors.*

Proof. Without loss of generality, we assume that $\mathcal{F}$ is a bipartite perfect coloring with two neighbor binary or one-color R-diagonals.

Claim 22. *Every R-diagonal is either binary or one-color.*

Indeed, consider the nodes $[1, 0]$ and $[0, -1]$. Their neighbors have colors $\boxed{a}$, $\boxed{b}$, $\mathcal{F}([1, -1])$, $\mathcal{F}([2, 0])$ and $\boxed{a}$, $\boxed{b}$, $\mathcal{F}([1, -1])$, $\mathcal{F}([0, -2])$, respectively. Since $[1, 0]$ and $[0, -1]$ are of the same or twin colors, we conclude $\mathcal{F}([2, 0]) = \mathcal{F}([0, -2])$. Similarly, by induction on $|i|$, we get $\mathcal{F}([i, j]) = \mathcal{F}([i + 2, j + 2])$ for every $i, j \in \mathbb{Z}$, which proves Claim 22.

We now consider two subcases.

(i) For every two R-diagonals, the sets of their colors coincide or do not intersect. In this case, for an R-diagonal, its set of colors uniquely determines the sets of colors of the two neighbor diagonals. In other words, we have a perfect coloring of the infinite path graph $G(\mathbb{Z})$, where the node $[j]$ is colored with the set of colors of the diagonal through $[j, 0]$ in the coloring $\mathcal{F}$. It is easy to find (see, e.g., [21, Lemma 4]) that there are only two classes of bipartite perfect colorings of $G(\mathbb{Z})$:

(a) *cyclic* colorings $\cdots\boxed{c_1}\boxed{c_2}\cdots\boxed{c_s}\boxed{c_1}\boxed{c_2}\cdots\boxed{c_s}\boxed{c_1}\boxed{c_2}\cdots$ (the number s of colors is even because the coloring is bipartite);

(b) *mirror* colorings $\cdots\boxed{c_2}\boxed{c_1}\boxed{c_2}\cdots\boxed{c_{s-1}}\boxed{c_s}\boxed{c_{s-1}}\cdots\boxed{c_2}\boxed{c_1}\boxed{c_2}\cdots$ (any number s of colors).

Once we fix the coloring of $G(\mathbb{Z})$, the colors of the one-color R-diagonals of $\mathcal{F}$ are determined, and the colors of the binary R-diagonals of $\mathcal{F}$ are determined up to shifting. If there are only binary R-diagonals, then $\mathcal{F}$ is obtained by shifting from the coloring in Fig. K (the case $n = 8$ is shown separately in Fig. I, and the case $n = 4$ is obtained from Fig. I by merging twins $\boxed{1}$ and $\boxed{3}$, $\boxed{2}$ and $\boxed{4}$, $\boxed{5}$ and $\boxed{6}$, $\boxed{7}$ and $\boxed{8}$) or in Fig. L (again the case $n = 4$ is obtained from Fig. I by merging twin colors as above, and the case $n = 6$ is obtained from Fig. I by merging $\boxed{5}$ and $\boxed{6}$, $\boxed{7}$ and $\boxed{8}$), for subcases (a) and (b) respectively.

(ii) There are two R-diagonals, say $\boxed{a}\boxed{b}$- and $\boxed{a}\boxed{c}$-diagonals, whose sets of colors are different but with non-zero intersection, i.e., $\boxed{b} \neq \boxed{c}$. Clearly, the colors from $\{\boxed{a}, \boxed{b}, \boxed{c}\}$ are pairwise twin. We assume w.l.o.g. that $\boxed{c} \neq \boxed{a}$, $\mathcal{F}([0, 0]) = \boxed{a}$, the R-diagonal through $[0, 0]$ is an $\boxed{a}\boxed{b}$-diagonal, and the R-diagonal through $[1, 0]$ is an $\boxed{e}\boxed{f}$-diagonal. Denote $\boxed{g} = \mathcal{F}([-1, 0])$, $\boxed{h} = \mathcal{F}([0, 1])$, (hence, the whole diagonal through the last two nodes is a $\boxed{g}\boxed{h}$-diagonal). Since $\boxed{a}$ and $\boxed{c}$ are twins (this is the only place where we use the existence of an $\boxed{a}\boxed{c}$-diagonal), every $\boxed{e}$-node has a $\boxed{c}$-neighbor. Therefore, the diagonal through $[2,0]$ is a $\boxed{c}\boxed{d}$-diagonal for some $\boxed{d}$. Similarly, the diagonal through $[-2,0]$ is a $\boxed{c}\boxed{d'}$-diagonal for some $\boxed{d'}$. The colors from $\{\boxed{a},\boxed{b},\boxed{c},\boxed{d},\boxed{d'}\}$ are mutually twin, and we see that the diagonal through $[0,3]$ is a $\boxed{g}\boxed{h}$-diagonal and the diagonal through $[0,-3]$ is an $\boxed{e}\boxed{f}$-diagonal. Similarly (by induction on $|i|$), for every $i \in \mathbb{Z}$ the colors of the diagonal through $[2i,0]$ are equal or twin to $\boxed{a}$ and the diagonal through $[2i+1,0]$ is an $\boxed{e}\boxed{f}$- or $\boxed{g}\boxed{h}$-diagonal, depending on the parity of i. In particular, every odd node is an $\{\boxed{e},\boxed{f},\boxed{g},\boxed{h}\}$-node, where $\boxed{e}$ and $\boxed{f}$ are equal or twin, $\boxed{g}$ and $\boxed{h}$ are equal or twin. By shifting some of R-diagonals, we can make a $\boxed{e}\boxed{g}$-L-diagonal. Therefore, any two different colors from $\{\boxed{e},\boxed{f},\boxed{g},\boxed{h}\}$ are twins. It follows that $\boxed{d}=\boxed{d'}$ and for every even (odd) i the R-diagonal through $[2i,0]$ is an $\boxed{a}\boxed{b}$-diagonal (a $\boxed{c}\boxed{d}$-diagonal, respectively). We now see that if all colors $\boxed{a}$, $\boxed{b}$, $\boxed{c}$, $\boxed{d}$, $\boxed{e}$, $\boxed{f}$, $\boxed{g}$, $\boxed{h}$ are pairwise different, then $\mathcal{F}$ is shifting equivalent to the coloring in Fig. I. If there are some equalities between them, then $\mathcal{F}$ is shifting equivalent to a coloring obtained from the one in Fig. I by merging twin colors. □

V Each $\{\boxed{1},\boxed{2}\}$-node has one neighbor of type 2:2 and three neighbors of type 1:1

Denote by $\boxed{3}$ the color of the type 2:2 nodes. Up to rotation the coloring contains one of the following fragments.

	2	
1	3	2
	1	

	1	
2	3	2
	1	

It is easy to see that the first fragment contradicts the condition of case V. Directly from this condition we get that the second fragment uniquely determines the placement of the $\{\boxed{1},\boxed{2},\boxed{3}\}$-nodes:

	·				·				·				·		
			1				2				1				2
1		2	3	2		1	3	1		2	3	2		1	3
			1				2				1				2
	·				·				·				·		
			2				1				2				1
2		1	3	1		2	3	2		1	3	1		2	3
			2				1				2				1
	·				·				·				·		
			1				2				1				2

Denote by E the set of the colors of the even nodes excluding $\boxed{1}$ and $\boxed{2}$.

Claim 23. $|E| \leqslant 2$. *If* $|E| = 2$, *then* $d^3(\boxed{3},\boxed{e}) = 12$ *for each* $\boxed{e}$ *in* E.

Proof. First we note that

$$\sum_{e\in E} d^2(\boxed{1},\boxed{e}) = 6, \qquad \sum_{e\in E} d^2(\boxed{2},\boxed{e}) = 6,$$

and

$$d^2(\boxed{1},\boxed{e}) \geqslant 2, \qquad d^2(\boxed{2},\boxed{e}) \geqslant 2 \qquad \text{for all } \boxed{e}\in E \tag{3}$$

(the last follows from the fact that every E-node is at $[\pm1,\pm1]$ from some $\boxed{1}$-node). Then,

$$d^3(\boxed{3},\boxed{e}) = 2d^2(\boxed{1},\boxed{e}) + 2d^2(\boxed{2},\boxed{e}) \equiv 0 \bmod 3 \tag{4}$$

(indeed, there are exactly 3 paths of length 3 from every $\boxed{3}$-node to every nearest E-node, each path containing exactly one $\{\boxed{1},\boxed{2}\}$-node); hence, $d^3(\boxed{3},\boxed{e})$ is even, not less than 8, and divisible by 3. Since

$$\sum_{e\in E} d^3(\boxed{3},\boxed{e}) = 24. \tag{5}$$

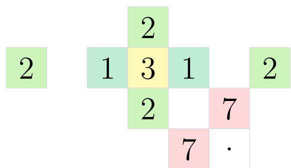
the only possibilities for $d^3(\boxed{3},\boxed{e})$ are 12 (with $|E|=2$) and 24 (with $|E|=1$). □

Proposition 24. *In case* V*, the coloring* $\mathcal{F}$ *either*

(a) *is equivalent to the coloring shown in Fig. B, or*

(b) $\mathcal{F}$ *can be obtained from the coloring shown in Fig. A by shifting* $\boxed{4}\boxed{5}$*- or* $\boxed{4}\boxed{6}$*-diagonals (but not both), unifying the colors* $\boxed{7}$ *and* $\boxed{8}$*, unifying two or three colors from* $\{\boxed{4},\boxed{5},\boxed{6}\}$*, and/or renaming colors.*

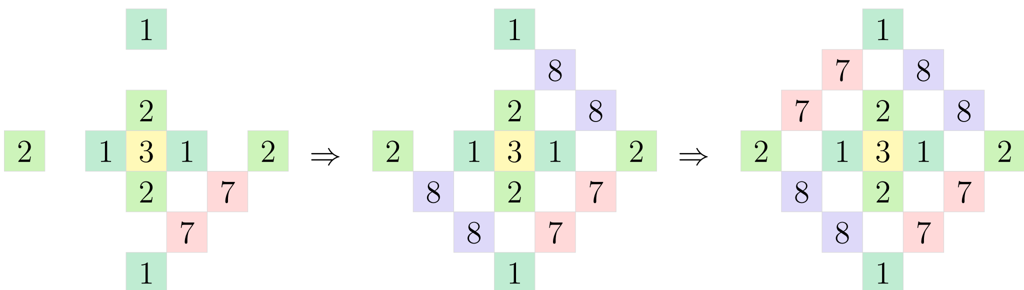
Proof. (a) We first consider the subcase when $|E|=2$, say $E=\{\boxed{7},\boxed{8}\}$ (in agree with Fig. B), and an odd node of type 0:0 has two non-opposite neighbors of the same color.



We see that $d^2(\boxed{1},\boxed{7}) \geqslant 3$ and $d^2(\boxed{2},\boxed{7}) \geqslant 3$. From this, $d^3(\boxed{3},\boxed{7}) = 12$ (Claim 23), and $d^3(\boxed{3},\boxed{7}) = 4\cdot d^2(\boxed{1},\boxed{7})$, we find

$$d^2(\boxed{1},\boxed{7}) = d^2(\boxed{2},\boxed{7}) = 3, \quad \text{and} \quad d^2(\boxed{1},\boxed{8}) = d^2(\boxed{2},\boxed{8}) = 3. \tag{6}$$

With the last equations, the colors of all nodes at distance 3 from the $\boxed{3}$-node are uniquely reconstructed.



Next, we see that the cells at distance 2 from the $\boxed{3}$-cell are uniquely colored with three new colors. From this point, the fragment considered is continued in a straightforward manner, resulting in the coloring in Fig. B.

(b') Next, we consider the subcase when $|E| = 2$, say $E = \{\boxed{7}, \boxed{8}\}$ and any two non-opposite neighbors of each odd node of type 0:0 have different colors. We have a fragment like the following:

	2		$\bar{y}$	
1	3	1		2
	2		8	
$\bar{x}$		7	9	7
	1		8	

We claim that $\mathcal{F}(\bar{x}) = \boxed{8}$ and similarly $\mathcal{F}(\bar{y}) = \boxed{7}$. Indeed, if $\mathcal{F}(\bar{x}) = \boxed{7}$, then $d^2(\boxed{1}, \boxed{7}) \geqslant 4$ and $d^2(\boxed{2}, \boxed{7}) \geqslant 4$, which means $d^3(\boxed{3}, \boxed{7}) \geqslant 16$ and contradicts Claim 23.

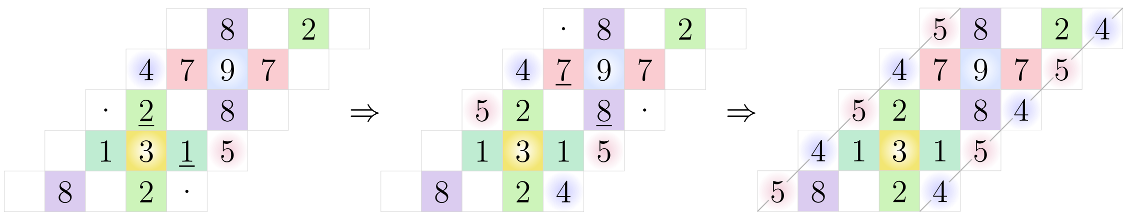
In a similar manner, we can now determine the colors of all nodes except those at distance 2 from the $\boxed{3}$-nodes. In the picture below, these nodes are marked by the labels a, b, and c.

1	a	2	3	2	a	1	3	1	a	2	3	2	a	1	3
c	7	b	1	c	8	b	2	c	7	b	1	c	8	b	2
8	9	8	a	7	9	7	a	8	9	8	a	7	9	7	a
b	7	c	2	b	8	c	1	b	7	c	2	b	8	c	1
2	a	1	3	1	a	2	3	2	a	1	3	1	a	2	3
c	8	b	2	c	7	b	1	c	8	b	2	c	7	b	1
7	9	7	a	8	9	8	a	7	9	7	a	8	9	8	a
b	8	c	1	b	7	c	2	b	8	c	1	b	7	c	2

Clearly, the remaining nodes are colored with one, two, or three mutually twin colors. Moreover, it is easy to see that $S_{1,i} = S_{2,i} = S_{7,i} = S_{8,i}$ for every color $\boxed{i}$ of a node marked by a, b, or c. Hence, $\boxed{7}$ and $\boxed{8}$ are twins.

If the nodes marked by a are of the same color, then it is not difficult to find that the same is true for the nodes marked by b and for the nodes marked by c. Then p. (b) of the proposition takes place.

Otherwise, there are two a-nodes $\bar{z}$ and $\bar{z}'$ of different colors and with difference $\bar{z}' - \bar{z} = [2, \pm 2]$. W.l.o.g., assume $\bar{z} = [0, 0]$, $\bar{z} = [2, -2]$, $\mathcal{F}(\bar{z}) = \boxed{4}$, $\mathcal{F}(\bar{z}) = \boxed{5}$.



Since $\boxed{1}$ and $\boxed{2}$ are twins, the neighbors of each of the nodes $[0, -1]$, $[1, -2]$ (underlined in the picture above) have colors $\boxed{3}$, $\boxed{4}$, $\boxed{5}$, and $\mathcal{F}([1, -1])$, independently on the value

$\mathcal{F}([1,-1])$ (it can be $\boxed{4}$, $\boxed{5}$, or a new color, say $\boxed{6}$). We see that $\mathcal{F}([-1,-1]) = \boxed{5}$ and $\mathcal{F}([1,-3]) = \boxed{4}$. Similarly, considering the neighborhoods of the $\{\boxed{7}, \boxed{8}\}$-nodes $[1,0]$ and $[2,-1]$, we see $\mathcal{F}([1,1]) = \boxed{5}$ and $\mathcal{F}([3,-1]) = \boxed{4}$. Processing in a similar way, we find that the both R-diagonals through $[0,0]$ and $[2,-2]$ are $\boxed{4}\boxed{5}$-diagonals.

The same argument show that the R-diagonal through $[4,-4]$ is a $\boxed{4}\boxed{5}$-diagonal, independently on the value $\mathcal{F}([4,-4])$, which can be $\boxed{4}$ or $\boxed{5}$. Similarly, by induction, for every i the R-diagonal through $[2i,-2i]$ is a $\boxed{4}\boxed{5}$-diagonal. It remains to note that all nodes marked by b, i.e., of form $[2+2i+2j, 2i-2j]$ are colored with the same color, $\boxed{4}$, $\boxed{5}$, or a new color $\boxed{6}$.

(b") The subcase $|E| = 1$ is considered similarly to (b'). □

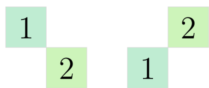
VI All the neighbors of a $\{\boxed{1}, \boxed{2}\}$-node are of type 1:1

We divide the case into the following four subcases.

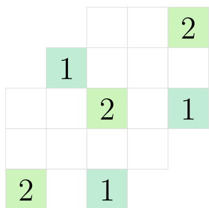
Claim 25. *One of the following four assertions takes place.*

VI.a. *The $\{1,2\}$-nodes are all the nodes with even coordinates or all the nodes with odd coordinates (without loss of generality, we consider the even subcase).*

VI.b. *The coloring $\mathcal{F}$ contains the following fragment, up to rotation and reflection:*



VI.c. *The coloring $\mathcal{F}$ contains the following fragment, up to rotation and renaming the colors $\boxed{1}$ and $\boxed{2}$:*



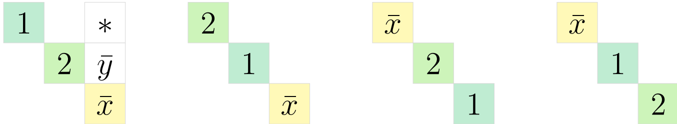
VI.d. *The coloring $\mathcal{F}$ contains a $\boxed{1}\boxed{2}$-diagonal.*

Proof. At first, if every type 1:1 node has opposite $\boxed{1}$- and $\boxed{2}$-neighbors, then we obviously have subcase VI.a.

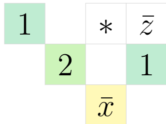
Thus, we can assume that there are two $\{\boxed{1}, \boxed{2}\}$-neighbors of a type 1:1 node that are placed diagonally from each other. W.l.o.g., assume $\mathcal{F}([0,0]) = 2$ and $\mathcal{F}([-1,1]) = 1$. Let us consider the diagonal through these two nodes.

If it is a $\boxed{1}\boxed{2}$-diagonal, then we have case VI.d.

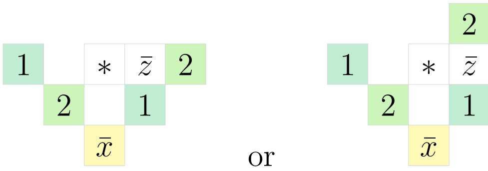
Otherwise, there is an integer i with the minimum absolute value such that the node $\bar{x} = [i,-i]$ is not a $\{\boxed{1}, \boxed{2}\}$-node. Depending on the sign and the parity of i, we have one of the four situations:



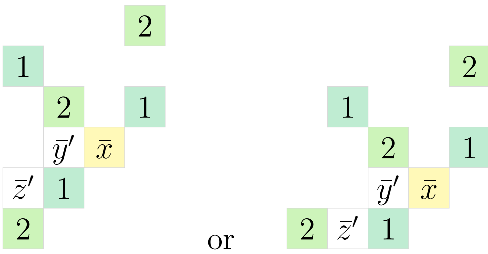
We consider only the first one, as the other three are similar. We observe that the node $[i, -i+2]$, marked by $*$ in the picture, is not a $\{\boxed{1}, \boxed{2}\}$-node, by the condition of case VI. The node $\bar{y} = [i, -i+1]$ is of type 1:1, and the only place for its $\boxed{1}$-neighbor is $[i+1, -i+1]$:



The node $\bar{z} = [i+1, -i+2]$ is also of type 1:1, and there are two possibilities for its $\boxed{2}$-neighbor:



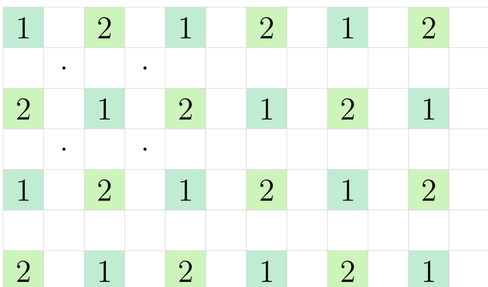
The first possibility leads to subcase VI.b. In the remaining case, considering similarly the type 1:1 nodes $\bar{y}' = [i-1, -i]$ and $\bar{z}' = [i-2, -i-1]$, we find that subcase VI.b or subcase VI.c takes place:



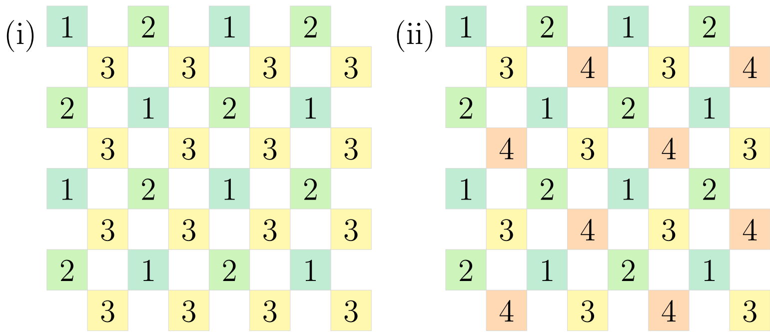
$\square$

In the following four subsections, we show that the claim of Theorem 2 holds in each of the four subcases.

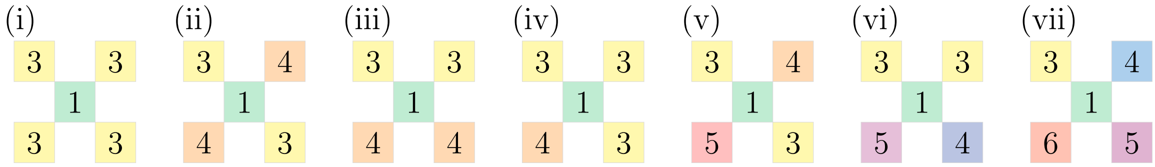
VI.a All the $\{\boxed{1}, \boxed{2}\}$-nodes are all the nodes with even coordinates



Proposition 26. *In case* VI.a*, either the coloring $\mathcal{F}$ is equivalent to one of the colorings in Fig. A, C, D, E where some twin colors can be merged, or the coloring of the even vertices is equivalent to one of the following two semicolorings:*



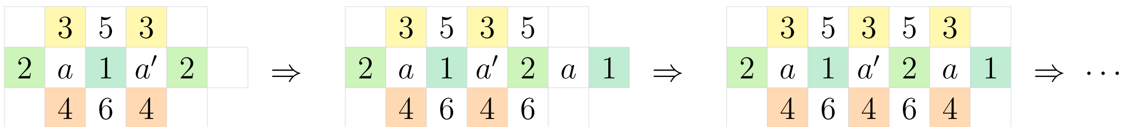
Proof. Up to equivalence, there are seven ways to color the four nodes placed diagonally from a $\boxed{1}$-node:



In case (i), all non-$\{\boxed{1},\boxed{2}\}$ even nodes have color $\boxed{3}$, corresponding to p. (i) of the claim.

In case (ii) each neighbor of a $\boxed{1}$-node has odd node has one $\boxed{1}$-, one $\boxed{2}$-, one $\boxed{3}$-, and one $\boxed{4}$-neighbor. And this is true for every odd node. It is not difficult to see that the coloring of the even nodes is uniquely determined. Since every odd node is a neighbor of a $\boxed{1}$-node,

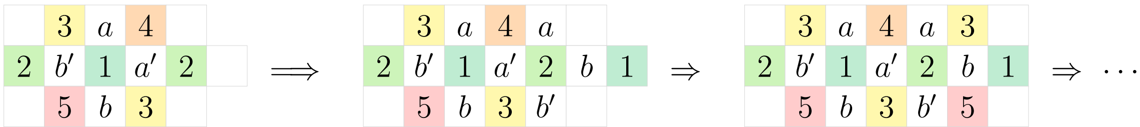
Now consider case (iii). We denote the colors of the 1-node neighbors at the picture as follows, where each of $\boxed{a}$, $\boxed{a'}$ is different from $\boxed{5}$ and $\boxed{6}$ but it is possible that $\boxed{a} = \boxed{a'}$:



Since $\boxed{1}$ and $\boxed{2}$ are twins, the rightest $\boxed{2}$-node in the picture has neighbors of colors $\boxed{5}$, $\boxed{6}$, $\boxed{a}$, and $\boxed{a'}$. The only places for the $\boxed{5}$- and $\boxed{6}$-neighbors are above and below the considered $\boxed{2}$-node, respectively. Now, this last $\boxed{5}$-node must have a second $\boxed{3}$-neighbor, and the only place is at the right (because of the $\boxed{1}$- and $\boxed{2}$-neighbors in the vertical direction). Similarly, all nodes in the three horizontal rows are uniquely colored. The whole coloring is now uniquely determined, whether $\boxed{a} = \boxed{a'}$ or $\boxed{a} \neq \boxed{a'}$, see Fig. C.

Case (iv) leads to a contradiction when trying to reconstruct the coloring, which is straightforward but not too short. We will use more intuitive arguments. It follows from Lemma 9 that there are exactly three $\boxed{3}$-nodes at distance 2 from every $\boxed{1}$-node and exactly two $\boxed{1}$-nodes at distance 2 from every $\boxed{3}$-node. Considering a sufficiently large square, we find (Remark 11) that the densities P_1 and P_3 of the colors $\boxed{1}$ and $\boxed{3}$ are related as $P_1 : P_3 = 2 : 3$. On the other hand, it is easy to see that there are no nodes that have more than 2 neighbors of the same color. It follows from Lemma 10 that the quotient of the densities of any two colors is a power of two, a contradiction.

Now consider case (v). We denote the colors of the 1-node neighbors at the picture as follows, where $\{\boxed{a}, \boxed{a'}\}$ and $\{\boxed{b}, \boxed{b'}\}$ are disjoint but it is possible that $\boxed{a} = \boxed{a'}$ or $\boxed{b} = \boxed{b'}$:



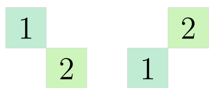
Let us determine the colors of the neighbors of the rightest $\boxed{2}$-node in the picture. The neighbor above has color $\boxed{a}$ because $\{\boxed{b}, \boxed{b'}\}$-nodes have no $\boxed{4}$-neighbors. The neighbor below has color $\boxed{b'}$ because by Lemma 10 $S_{1b} = S_{1b'}$ implies $S_{3b} = S_{3b'}$. So, arguing as in case (iii), we reconstruct the colors in the three horizontal rows and then in the whole grid, obtaining the coloring corresponding to Fig A as follows:

$\mathcal{F}$:	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{a}$	$\boxed{a'}$	$\boxed{b}$	$\boxed{b'}$
Fig. A:	$\boxed{6}$	$\boxed{5}$	$\boxed{4}$	$\boxed{9}$	$\boxed{1}$	$\boxed{7}$	$\boxed{8}$	$\boxed{2}$	$\boxed{3}$

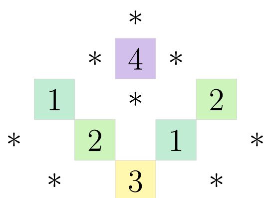
In each of cases (vi), (vii), the $\boxed{1}$-node has neighbors of four distinct colors, and with the same strategy as in cases (iii) and (v) the coloring is uniquely reconstructed, corresponding to Fig. D or Fig. E, respectively. □

We see that in subcases (i) and (ii) in Proposition 26 the even colors are pairwise twin (we remember that twin colors might have different densities like $\boxed{1}$ and $\boxed{3}$ in subcase (i)). The same is true for the odd colors (if there are more than one such colors) because the neighborhoods of all odd nodes are colored with the same colors. Proposition 26 confirms the claim of Lemma 14 and Corollary 15 finalizes the proof of Theorem 2 in the considered case VI.a.

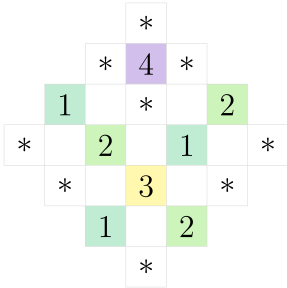
VI.b The coloring $\mathcal{F}$ contains the fragment



Without loss of generality, we assume that the $\boxed{1}$-nodes of the fragment are $[-2, 2]$ and $[1, 1]$ and the $\boxed{2}$-nodes are $[-1, 1]$ and $[2, 2]$. By the condition of the case VI the colors of the nodes $[0, 0]$, $[-2, 0]$, $[2, 0]$, $[3, 1]$, $[0, 2]$, $[-1, 3]$, $[1, 3]$, $[0, 4]$, $[-3, 1]$ differ from $\boxed{1}$ and $\boxed{2}$. Denote by $\boxed{3}$ and $\boxed{4}$ the colors of $[0, 0]$ and $[0, 3]$ respectively. At the pictures, an even node marked by $*$ is not a $\{\boxed{1}, \boxed{2}\}$-node.



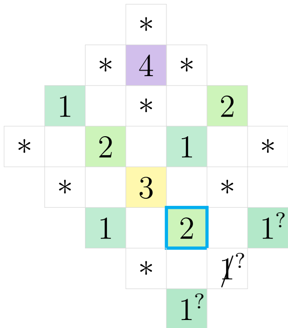
Using the condition of case VI, we see that $[-1,-1]$ is a $\boxed{1}$-node, $[1,-1]$ is a $\boxed{2}$-node, $[0,-2]$ is not a $\{\boxed{1},\boxed{2}\}$-node.



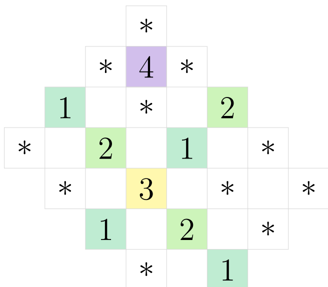
From the picture above, we see

$$d^3(\boxed{2},\boxed{4}) > 0. \tag{7}$$

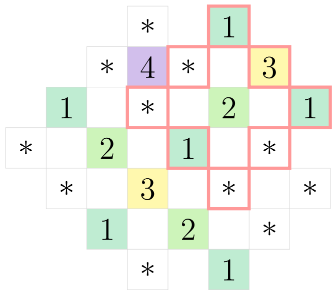
If $[2,-2]$ is not a $\boxed{1}$-node, then $[3,-1]$ and $[1,-3]$ are $\boxed{1}$-nodes, by the condition of case VI.



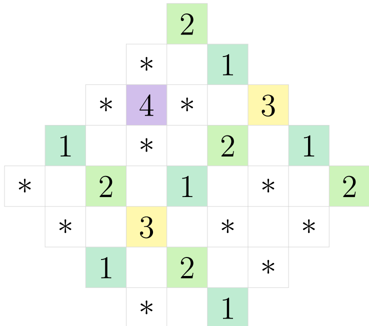
But then we have a contradiction with (7), because there is no room to place a $\boxed{4}$-node at distance 3 from the $\boxed{2}$-node $[1,-1]$ (taking into account that the $\boxed{4}$-nodes are of type 0:0). Hence, the color of $[2,-2]$ is $\boxed{1}$.



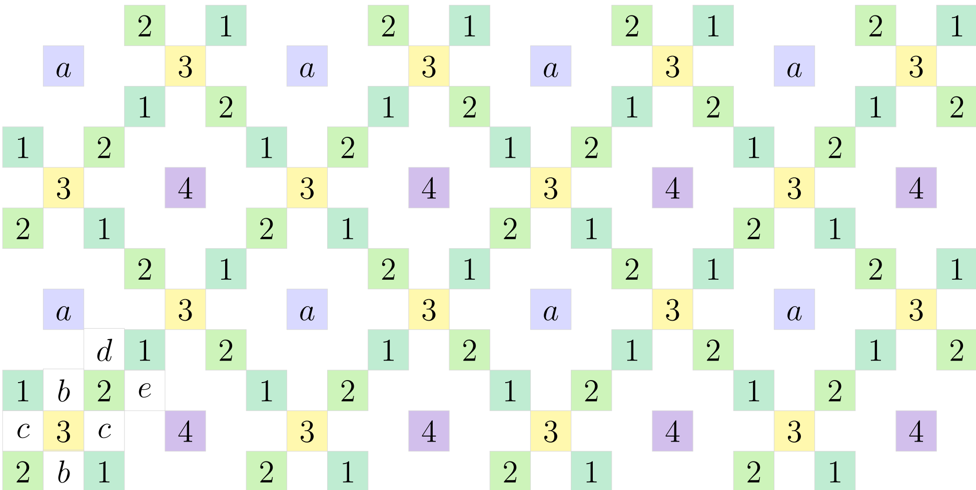
We see that $d^2(\boxed{1},\boxed{3}) = 2$, $d^2(\boxed{2},\boxed{3}) = 2$, and $d^2(\boxed{2},\boxed{1}) = 4$. Applying the last two equalities to $[2,2]$ and noting that a $\boxed{3}$-node cannot be neighbor to a type 0:0 node, we find that there is only one way to color the nodes $[2,4]$, $[3,3]$, and $[4,2]$:



Directly from the condition of case VI, we find two more $\boxed{2}$-nodes:



Now, by analogy, we can recover the placement of all the $\{\boxed{1}, \boxed{2}, \boxed{3}\}$-nodes.



Moreover, since $d^3(\boxed{1}, \boxed{4}) = 3$ or $d^3(\boxed{1}, \boxed{4}) = 6$, we see that there are at most two colors of odd type 0:0 nodes and these nodes are colored periodically with periods $[6, 0]$ and $[0, 6]$. Applying the definition of a perfect coloring to the neighborhoods of $\boxed{1}$- and $\boxed{2}$-nodes, we conclude that the opposite neighbors of a $\boxed{3}$-node must have the same color. We separate five subcases, according to the equalities between the colors of the marked nodes in the last picture:

(i) $\boxed{a} = \boxed{4}$, $\boxed{b} = \boxed{c}$, $\boxed{d} = \boxed{e}$;

(ii) $\boxed{a} = \boxed{4}$, $\boxed{b} = \boxed{c}$, $\boxed{d} \neq \boxed{e}$;

(iii) $\boxed{a} = \boxed{4}$, $\boxed{b} \neq \boxed{c}$, $\boxed{d} = \boxed{e}$;

(iv) $\boxed{a} = \boxed{4}$, $\boxed{b} \neq \boxed{c}$, $\boxed{d} \neq \boxed{e}$;

(v) $\boxed{a} \neq \boxed{4}$ (it follow that $\boxed{b} \neq \boxed{c}$ and $\boxed{d} \neq \boxed{e}$).

Leaving the further arguments as an exercise, we formulate the result.

Proposition 27. *In subcase VI.b, the coloring $\mathcal{F}$ is equivalent to one of the following five colorings:*

(i) *the coloring shown at Fig. F, with merged colors* $\boxed{6}$ *and* $\boxed{7}$*;*

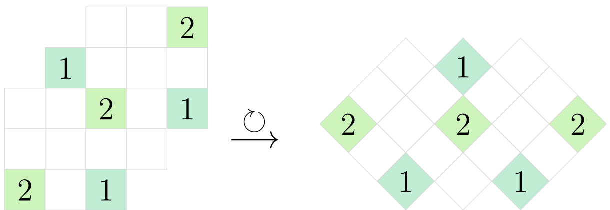
(ii) *the coloring shown at Fig. F;*

(iii) *the coloring shown at Fig. G, with merged colors* $\boxed{7}$ *and* $\boxed{8}$*;*

(iv) *the coloring shown at Fig. G;*

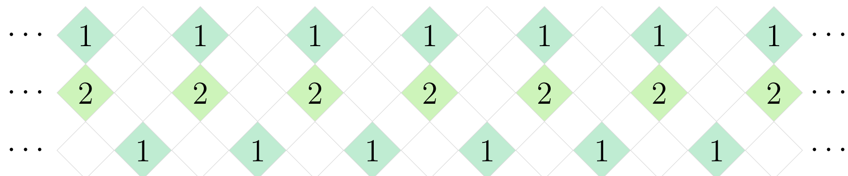
(v) *the coloring shown at Fig. H.*

VI.c The coloring $\mathcal{F}$ contains the following fragment:



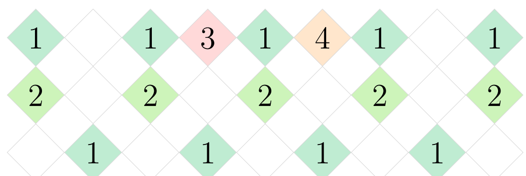
In this subsection, we will draw the grid 45° rotated and give indexed names to R-diagonals as illustrated in the figure above.

By the hypothesis of case VI, all neighbors of a $\{\boxed{1}, \boxed{2}\}$-node are of type 1:1. Subsequently applying this condition to the nodes of the diagonals D_1 and D_3, we uniquely reconstruct all $\{\boxed{1}, \boxed{2}\}$-nodes in the diagonals D_0, D_2, and D_4:

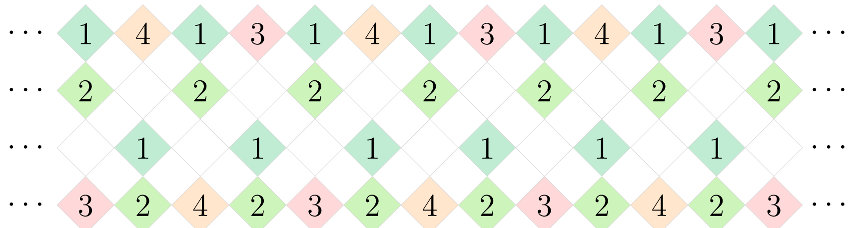


Claim 28. *The diagonal D_0 is binary.*

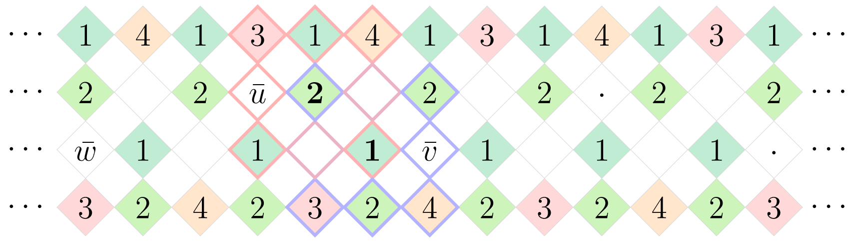
Proof. Assume the contrary. Then D_0 contains two nodes of different colors, say $\boxed{3}$ and $\boxed{4}$, at distance 4 from each other.



Considering the central $\boxed{2}$-node of the diagonal D_2 in the diagram above, we see that $d^2(\boxed{2},\boxed{i})$ is odd for $\boxed{i} = \boxed{3},\boxed{4}$ and even for any other $\boxed{i}$. Since $\boxed{1}$ and $\boxed{2}$ are twins, the same holds for $d^2(\boxed{1},\boxed{i})$. Considering the other $\boxed{2}$-nodes of D_2, we see that the colors of the diagonal D_0 alternate like $\boxed{1}$, $\boxed{3}$, $\boxed{1}$, $\boxed{4}$, $\boxed{1}$, $\boxed{3}$, $\boxed{1}$, $\boxed{4}$, ... (we will talk about a $\boxed{1}\boxed{3}\boxed{1}\boxed{4}$-diagonal in this case). Considering the $\boxed{1}$-nodes of the diagonal D_4 (recall that $d^2(\boxed{1},\boxed{3})$ and $d^2(\boxed{1},\boxed{3})$ are odd), we see that the diagonal D_6 is a $\boxed{2}\boxed{3}\boxed{2}\boxed{4}$-diagonal, and the situation is, up to the reflection, as in the following figure:



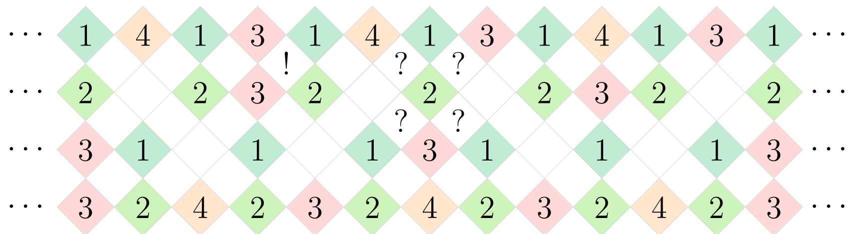
As we see from the diagonal D_0, $d^2(\boxed{1},\boxed{3})$ and $d^2(\boxed{1},\boxed{4})$ are not less than 2. Since they are known to be odd, at least one of these two values is 3. Let, w.l.o.g., $d^2(\boxed{1},\boxed{3}) = 3$. Then $d^2(\boxed{2},\boxed{3}) = 3$ too. Considering these values for the bold $\boxed{1}$-node $\bar{u} + [3,1]$ and $\boxed{2}$-node $\bar{u} + [1,1]$ from the diagonals D_2 and D_4, see the diagram



we find the following:

Claim 29. *A node $\bar{u}$ from D_2 has color $\boxed{3}$ if and only if $\bar{v}$ is a $\boxed{3}$-node too, where $\bar{v} = \bar{u} + [4,2]$ (similarly, for $\bar{w} = \bar{u} - [4,2]$).*

Starting from an arbitrary $\boxed{3}$-node in D_2 or D_4 and applying Claim 29, we inductively derive that the $\boxed{3}$-nodes of the diagonal D_2, as well as of D_4, are placed with period $[6,6]$:



Now we see an obvious contradiction with the definition of a perfect coloring: there are nodes with the neighbors of colors $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{3}$; but there is a $\boxed{2}$-node that is not adjacent with such a node. The contradiction proves the statement. □

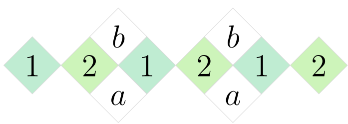
Thus, there is a binary diagonal. Well, it is not a $\boxed{1}\boxed{2}$-diagonal, but nevertheless, its colors are twin and hence the situation is described in some other considered case, I, II, III, or VI.d.

VI.d The coloring $\mathcal{F}$ contains a $\boxed{1}\boxed{2}$-diagonal

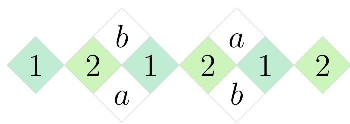
Without loss of generality, we assume that $\mathcal{F}$ contains a $\boxed{1}\boxed{2}$-R-diagonal through $[0, 0]$.

Proposition 30. *Either the coloring is periodic with period* $[2, 2]$ *or* $[4, 4]$*, or the two diagonals at distance* 4 *from the given "main"* $\boxed{1}\boxed{2}$*-diagonal are* $\boxed{1}\boxed{2}$*-diagonals too.*

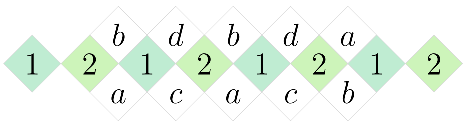
Proof. We first note that if some nodes $[x, x+1]$ and $[x+1, x]$ have colors $\boxed{a}$ and $\boxed{b}$, then $[x+2, x+3]$ and $[x+3, x+2]$ have, respectively, the colors $\boxed{a}$ and $\boxed{b}$ or $\boxed{b}$ and $\boxed{a}$; the same is true for $[x-2, x-1]$ and $[x-1, x-2]$:



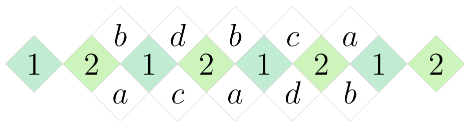
or



If both diagonals neighbor to the considered diagonal are periodic with period $[4, 4]$, then the coloring is periodic too. Otherwise there is one of the following two fragments:



or



where $\boxed{a} \neq \boxed{b}$ but other equalities are allowed. We see that in both cases $d^2(\boxed{c}, \boxed{a}) \geqslant 4$ and $d^2(\boxed{c}, \boxed{b}) \geqslant 3$; hence, there are $\{\boxed{a}, \boxed{b}\}$-nodes at distance 3 from the main diagonal. It follows that there are $\{\boxed{1}, \boxed{2}\}$-nodes at distance 4 from the main diagonal. By the hypothesis of case VI, such a node must belong to a $\boxed{1}\boxed{2}$-diagonal. Similarly, there is an "opposite", with respect to the main diagonal, $\boxed{1}\boxed{2}$-diagonal. $\square$

We separate Case VI.d into two subcases.

VI.d.1 The coloring $\mathcal{F}$ is not periodic with period $[4, 4]$ or $[2, 2]$

By Proposition 30, the $\boxed{1}\boxed{2}$-diagonals in $\mathcal{F}$ occur with period $[2, -2]$. The other even nodes also constitute R-diagonals that occur with period $[2, -2]$.

Claim 31. *Every even R-diagonal is colored in such a way that the color of every of its nodes* $\bar{x}$ *uniquely determines the unordered pair of colors of nearest diagonal nodes* $\bar{x} - [1, 1]$ *and* $\bar{x} + [1, 1]$ *(in other words, we have a perfect coloring of the diagonal considered as a chain graph).*

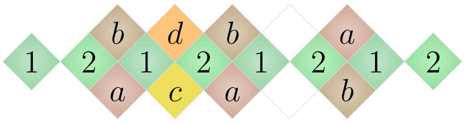
Proof. For every two colors $\boxed{i}$ and $\boxed{j}$ occuring on the considered diagonal, the value $d^2(\boxed{i}, \boxed{j})$ does not depend on the choice of the initial $\boxed{i}$-node $\bar{x}$. Moreover, it coincides to the number of $\boxed{j}$-nodes among $\bar{x} - [1,1]$ and $\bar{x} + [1,1]$ (indeed, the two R-diagonals at distance 2 from $\bar{x}$ have no $\boxed{j}$-nodes). $\square$

Proposition 32. *In subcase VI.d, one of the following takes place.*

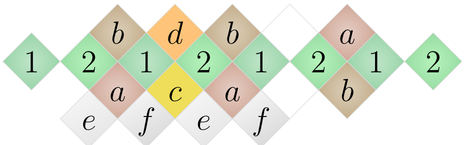
(i) *The hypothesis and the conclusion of Lemma 14 (and hence, of Corollary 15) are satisfied both for even and odd nodes.*

(ii) *The coloring is equivalent, up to shifting* $\boxed{1}\boxed{2}$- *and* $\boxed{3}\boxed{4}$- *diagonals, to the coloring in Fig. J.*

Proof. We already know that $\boxed{1}\boxed{2}$-R-diagonals occur at distance 4 from each other. Let us focus on an R-diagonal at distance 2 from a $\boxed{1}\boxed{2}$-diagonal. As shown above (see the proof of Proposition 30), there is a fragment like the following:

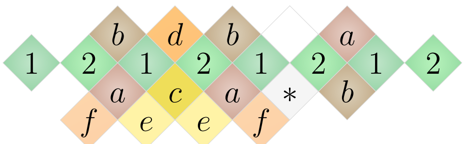


Now consider the neighbors of the $\boxed{a}$-nodes. If the fragment expands like

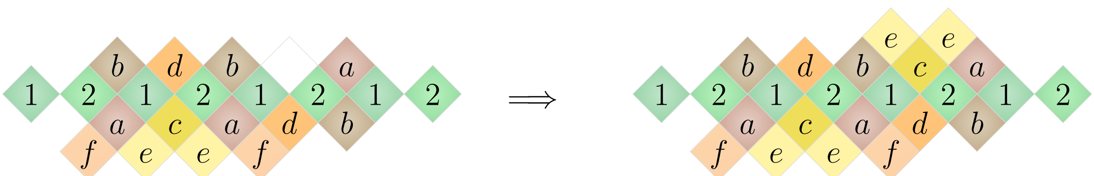


for some (maybe equal) $\boxed{e}$ and $\boxed{f}$, then by Claim 31 there is an $\boxed{e}\boxed{f}$-diagonal. It follows immediately that all even R-diagonals are either $\boxed{1}\boxed{2}$ or $\boxed{e}\boxed{f}$. Then all colors of nodes of the same parity are mutually twin, and p. (i) of the proposition statement takes place.

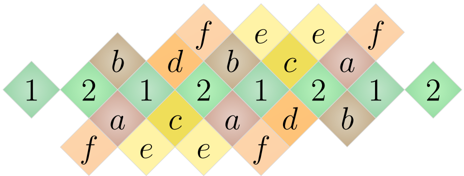
It remains to consider the following case, where $\boxed{f} \neq \boxed{e}$:



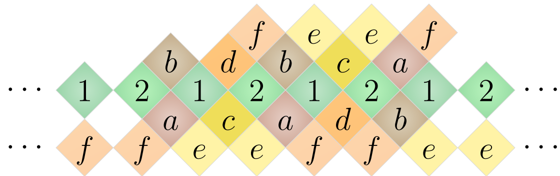
Since the $\boxed{c}$-nodes have no $\boxed{f}$-neighbors, the node marked by $*$ at the figure above is a $\boxed{d}$-node:



Since $d^2(\boxed{e},\boxed{f}) = 2$, we get



Now we see that a $\boxed{b}$-node has neighbors of colors $\boxed{e}$ and $\boxed{f}$. By Claim 31, a $\boxed{e}\boxed{e}\boxed{f}\boxed{f}$-diagonal is uniquely reconstructed:



It is easy to see now that the whole coloring, up to equivalence and shifting binary diagonals, is as in p. (ii) of the proposition statement. $\square$

VI.d.2 The coloring $\mathcal{F}$ is periodic with period $[4, 4]$

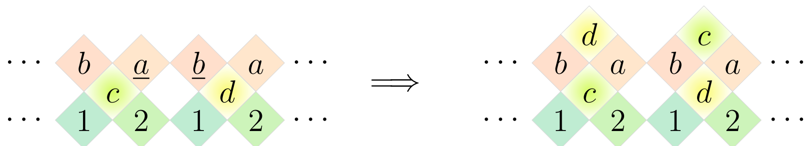
In this subcase, $\mathcal{F}$ has a $\boxed{1}\boxed{2}$-R-diagonal and is periodic with period $[4, 4]$. We further divide this subcase depending on the colors of nodes at distance 1 and 2 from the $\boxed{1}\boxed{2}$-diagonal.

VI.d.2.i There is an $\boxed{a}\boxed{b}$-diagonal neighbor to a $\boxed{1}\boxed{2}$-diagonal

In this subcase, the conclusion of Theorem 2 holds by Proposition 21.

VI.d.2.ii There is an $\boxed{a}\boxed{b}$-diagonal at distance 2 from a $\boxed{1}\boxed{2}$-diagonal

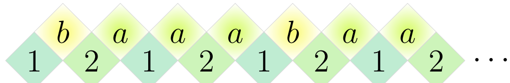
Let us consider the diagonal between the $\boxed{a}\boxed{b}$-diagonal and the $\boxed{1}\boxed{2}$-diagonal. If it is binary or one-color, then subcase VI.d takes place. Otherwise, it contains two nodes at distance 4 colored with different colors, say $\boxed{c}$ and $\boxed{d}$:



Since $\boxed{a}$ and $\boxed{b}$ are either the same or twins, we find that there are $\{\boxed{c},\boxed{d}\}$-nodes at distance 3 from the $\boxed{1}\boxed{2}$-diagonal. Hence, there are $\{\boxed{1},\boxed{2}\}$-nodes at distance 4 from the $\boxed{1}\boxed{2}$-diagonal; of course, they belong to a $\boxed{1}\boxed{2}$-diagonal because neighbors of $\{\boxed{1},\boxed{2}\}$ nodes are of type 1:1. Similarly, there is an $\boxed{a}\boxed{b}$-diagonal at distance 4 from the considered $\boxed{a}\boxed{b}$-diagonal. Now we can see that all odd colors are mutually twin and we get the situation described in Proposition 32(i), with the only nonessential difference that now there is the period $[4, 4]$.

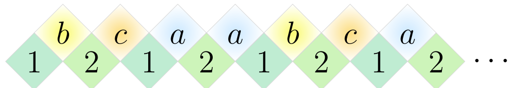
In the following subcases we automatically assume that the R-diagonals at distance 2 from every $\boxed{1}\boxed{2}$-diagonal are not one-color or binary.

VI.d.2.iii ***There is an $\boxed{a}\boxed{a}\boxed{a}\boxed{b}$-diagonal neighbor to a $\boxed{1}\boxed{2}$-diagonal, $\boxed{a} \neq \boxed{b}$***

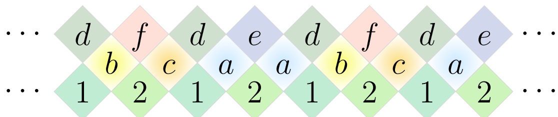


It is easy to find that the diagonal next to the $\boxed{a}\boxed{a}\boxed{a}\boxed{b}$-diagonal (at distance 2 from the $\boxed{1}\boxed{2}$-diagonal) is one-color or binary.

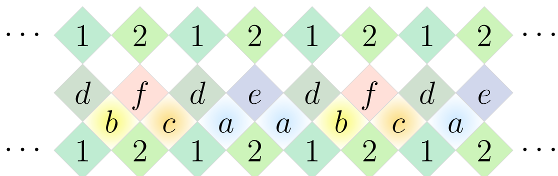
VI.d.2.iv ***There is an $\boxed{a}\boxed{a}\boxed{b}\boxed{c}$-diagonal neighbor to a $\boxed{1}\boxed{2}$-diagonal, $\boxed{a} \neq \boxed{b} \neq \boxed{c} \neq \boxed{a}$***



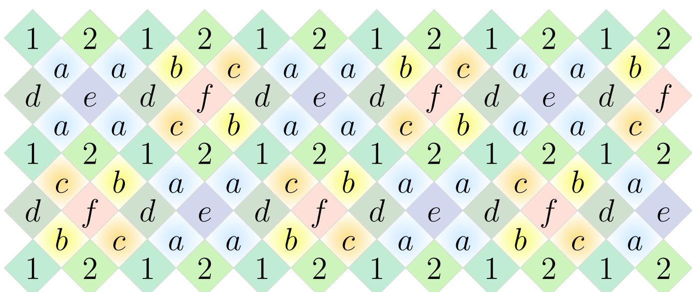
We have:



where $\boxed{f} \neq \boxed{e}$ (otherwise, we obtain a $\boxed{d}\boxed{e}$-diagonal and return to subcase VI.d). We see that a $\boxed{d}$-node has neighbors of all three colors $\boxed{a}$, $\boxed{b}$, and $\boxed{c}$. This means that there are two $\{\boxed{1},\boxed{2}\}$-diagonals at distance 2 from a $\boxed{d}$-node:



Since $d^2(\boxed{d},\boxed{f}) = 2 \neq d^2(\boxed{e},\boxed{f})$, we see that $\boxed{d}$ is different from $\boxed{e}$. Similarly, since $d^2(\boxed{d},\boxed{e}) = 2 \neq d^2(\boxed{f},\boxed{e})$, we have $\boxed{d} \neq \boxed{f}$. Since $\{\boxed{b},\boxed{c}\}$-nodes do not have $\boxed{e}$-neighbors, every $\boxed{e}$-node has four $\boxed{a}$-neighbors (looking to the neighborhoods of $\{\boxed{1},\boxed{2}\}$-nodes, we find that there are only three odd colors). Then the whole coloring can be reconstructed up to shifting $\boxed{1}\boxed{2}$-diagonals:

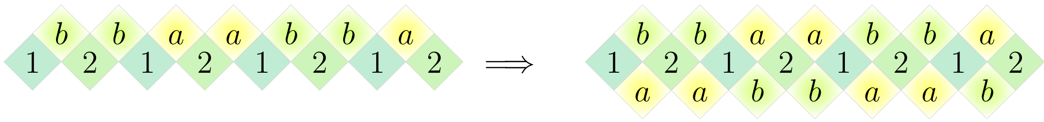


We get the coloring shown in Fig. A with the following color correspondence:

$\mathcal{F}$:	$\boxed{1}$	$\boxed{2}$	$\boxed{a}$	$\boxed{b}$	$\boxed{c}$	$\boxed{d}$	$\boxed{e}$	$\boxed{f}$
Fig. A :	$\boxed{4}$	$\boxed{5}$	$\boxed{7}\cup\boxed{8}$	$\boxed{2}$	$\boxed{3}$	$\boxed{6}$	$\boxed{9}$	$\boxed{1}$

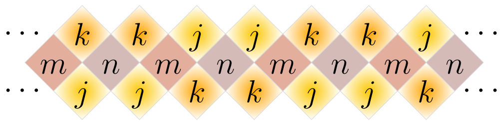
Remark 33. The colors $\boxed{b}$ and $\boxed{c}$ are twin, and we considered colorings with such two twin colors in Case V, which also leads to the coloring in Fig. A in one of the subcases.

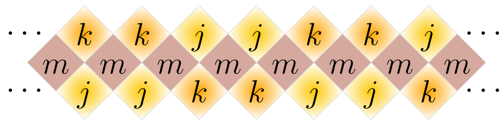
VI.d.2.v There is an $\boxed{a}\boxed{a}\boxed{b}\boxed{b}$-diagonal neighbor to a $\boxed{1}\boxed{2}$-diagonal, $\boxed{a} \neq \boxed{b}$

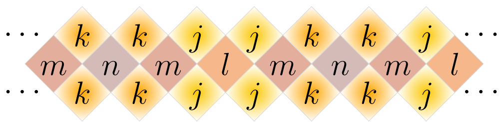


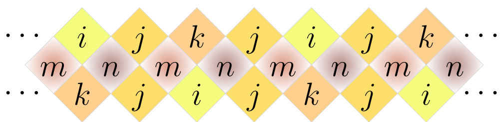
It is immediately follows that the $\boxed{1}\boxed{2}$-diagonal is surrounded by two $\boxed{a}\boxed{a}\boxed{b}\boxed{b}$-diagonals, as in the figure. Next, a diagonal, which is not binary or one-color, at distance 2 from the $\boxed{1}\boxed{2}$-diagonal must be a a $\boxed{c}\boxed{d}\boxed{e}\boxed{d}$-diagonal for some distinct $\boxed{c}$, $\boxed{d}$, and $\boxed{e}$.

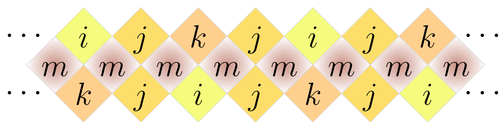
Proposition 34. *In subcase VI.d, the coloring has the following structure, up to renaming colors. Every $\boxed{1}\boxed{2}$-diagonal is surrounded by two $\boxed{3}\boxed{3}\boxed{4}\boxed{4}$-diagonals. Each of the $\boxed{3}\boxed{3}\boxed{4}\boxed{4}$-diagonals is followed by a $\boxed{5}\boxed{6}\boxed{7}\boxed{6}$-diagonal. Then, a $\boxed{8}\boxed{8}\boxed{9}\boxed{9}$-diagonal can follow; then a $\boxed{10}\boxed{11}\boxed{12}\boxed{11}$-diagonal, and so on, until we meet one of the following fragments:*

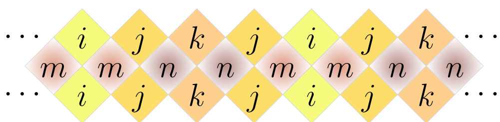
(a) 

or 

(b) 

(c) 

or 

(d) 

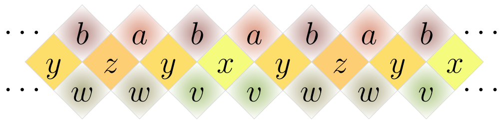
After that, the same diagonals follow in the reverse order, until a $\boxed{1}\boxed{2}$-diagonal; then the situation repeats.

The corresponding colorings are shown in Fig. M–P.

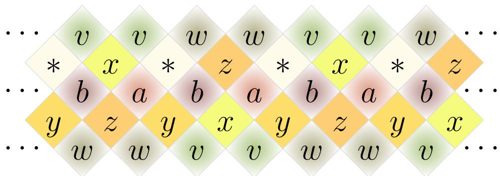
Proof. We use induction on i, the distance to the $\boxed{1}\boxed{2}$-diagonal. We assume that every $\boxed{1}\boxed{2}$-diagonal is surrounded by two $\boxed{3}\boxed{3}\boxed{4}\boxed{4}$-diagonals. Each of the $\boxed{3}\boxed{3}\boxed{4}\boxed{4}$-diagonals is followed by a $\boxed{5}\boxed{6}\boxed{7}\boxed{6}$-diagonal. Then, a $\boxed{8}\boxed{8}\boxed{9}\boxed{9}$-diagonal and a $\boxed{10}\boxed{11}\boxed{12}\boxed{11}$-diagonal, and so on, until we have colored a diagonal at distance i away from the initial $\boxed{1}\boxed{2}$-diagonal. We also assume that the colors have not been exhausted, that is, we have not yet crossed the diagonal where the pattern folds and the colors start to repeat (those diagonals correspond, as will see, to cases (a)–(d) in the statement of the proposition). In what follows we will show what the possibilities are for the diagonal at distance $i+1$. The arguments above the proposition provide the induction base, and now we can assume $i \geqslant 2$. Consider two cases according to the parity of i.

(A) i is even. We consider two neighbor $\boxed{v}\boxed{v}\boxed{w}\boxed{w}$- and $\boxed{x}\boxed{y}\boxed{z}\boxed{y}$-diagonals at distance $i-1$ and i from a $\boxed{1}\boxed{2}$-diagonal, respectively. Considering the neighbors of $\boxed{y}$-nodes, we see that the diagonal next to the $\boxed{x}\boxed{y}\boxed{z}\boxed{y}$-diagonal must be an $\boxed{a}\boxed{b}$- or $\boxed{a}\boxed{a}\boxed{b}\boxed{b}$-diagonal for some (maybe equal) $\boxed{a}$, $\boxed{b}$.

At first, assume that it is an $\boxed{a}\boxed{b}$-diagonal.

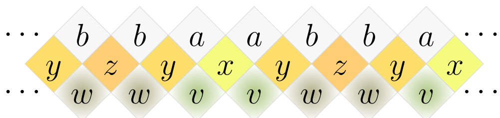


The colors $\boxed{a}$ and $\boxed{b}$ are the same or twins, and considering the neighborhoods of the corresponding nodes, we find positions of some $\{\boxed{x}, \boxed{z}\}$-nodes; then we uniquely color the neighborhoods of these nodes:



We note that $\boxed{a}, \boxed{b} \notin \{\boxed{v}, \boxed{w}\}$ (indeed, each $\{\boxed{v}, \boxed{w}\}$-node has two non-$\{\boxed{x}, \boxed{y}, \boxed{z}\}$ neighbors). Now we see that the nodes marked by $*$ in the figure must be colored with $\boxed{y}$ (as we see from the four previous diagonals, $\boxed{y}$ has no twins). So, we obtain fragment (c).

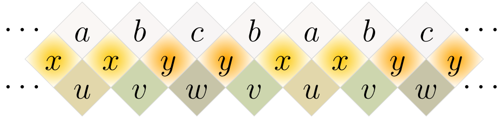
Now consider the second variant:



where $\boxed{a} \neq \boxed{b}$. It is not difficult to see that either $\boxed{a} = \boxed{v}$ and $\boxed{b} = \boxed{w}$ or $\boxed{a}$ and $\boxed{b}$ are "new" colors (that is, they do not occur between the $\boxed{x}\boxed{y}\boxed{z}\boxed{y}$-diagonal and the nearest $\boxed{1}\boxed{2}$-diagonal). Indeed, if $\boxed{a}$ is an old color, then $\boxed{a} = \boxed{v}$ because all other old colors have no $\boxed{x}$-neighbors. Similarly, if $\boxed{b}$ is old, then $\boxed{b} = \boxed{w}$. If $\boxed{a}$ is new, then we see $d^2(\boxed{w}, \boxed{a}) = 1$ while $d^2(\boxed{b}, \boxed{a}) \geqslant 2$; so, $\boxed{b} \neq \boxed{w}$ and hence $\boxed{b}$ is new too. Similarly, if $\boxed{b}$ is new, then $\boxed{a}$ is new.

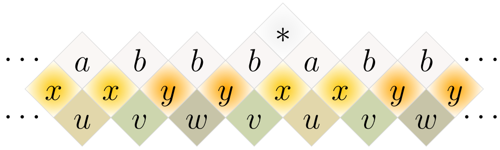
In the first case, we have fragment (b); in the second, we move to the next step and consider the $\boxed{x}\boxed{y}\boxed{z}\boxed{y}$- and $\boxed{a}\boxed{a}\boxed{b}\boxed{b}$-diagonals, see p. (B) of the proof.

(B) i is odd. We consider two neighbor $\boxed{u}\boxed{v}\boxed{w}\boxed{v}$- and $\boxed{x}\boxed{x}\boxed{y}\boxed{y}$-diagonals at distance $i-1$ and i from a $\boxed{1}\boxed{2}$-diagonal, respectively. Considering the neighbors of $\boxed{y}$-nodes, we see that the diagonal next to the $\boxed{x}\boxed{x}\boxed{y}\boxed{y}$-diagonal must be an $\boxed{a}\boxed{b}\boxed{c}\boxed{b}$-diagonal for some $\boxed{a}$, $\boxed{b}$, $\boxed{c}$.



At first, assume $\boxed{a} = \boxed{c}$, i.e., we have an $\boxed{a}\boxed{b}$-diagonal, where $\boxed{a}$ and $\boxed{b}$ are the same color or twins. As every $\{\boxed{a}, \boxed{b}\}$-node has two $\boxed{x}$- and two $\boxed{y}$-neighbors, the next diagonal is uniquely colored. We get fragment (a).

Then, assume $\boxed{c} = \boxed{b}$:

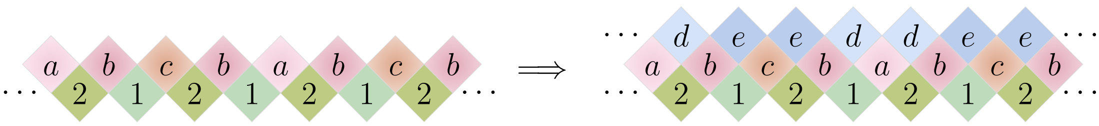


Considering the neighborhoods of $\boxed{b}$-nodes, we find that the node marked by $*$ has the color $\boxed{y}$. Since the $\boxed{y}$-nodes have neighbors of colors $\boxed{v}$, $\boxed{w}$, $\boxed{b}$ only, we conclude that $\boxed{a} = \boxed{b}$; hence, $\boxed{a} = \boxed{c}$, which case was considered above.

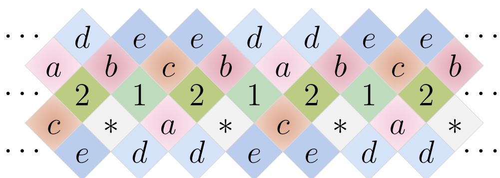
The last possibility is $\boxed{a} \neq \boxed{b} \neq \boxed{c} \neq \boxed{a}$. If $\{\boxed{a}, \boxed{b}, \boxed{c}\} \in \{\boxed{u}, \boxed{v}, \boxed{w}\}$, then the color of a $\{\boxed{a}, \boxed{b}, \boxed{c}\}$-node is uniquely determined by the occurence of $\boxed{x}$- or $\boxed{y}$-nodes in its neighborhood. Namely, $\boxed{a} = \boxed{u}$, $\boxed{b} = \boxed{v}$, $\boxed{c} = \boxed{w}$. We get fragment (d). Otherwise, $\boxed{a}$, $\boxed{b}$, and $\boxed{c}$ are new colors, and we continue considering the $\boxed{x}\boxed{x}\boxed{y}\boxed{y}$ and $\boxed{a}\boxed{b}\boxed{c}\boxed{b}$–diagonals with p. (A) of the current proof.

As the number of colors is finite, we have to meet one of the fragments (a)–(d) at some step. □

VI.d.2.vi There is an $\boxed{a}\boxed{b}\boxed{c}\boxed{b}$-diagonal neighbor to a $\boxed{1}\boxed{2}$-diagonal, $\boxed{a} \neq \boxed{b} \neq \boxed{c} \neq \boxed{a}$



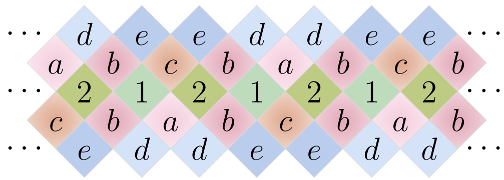
Considering the neighborhoods of the $\boxed{b}$-nodes, we find that the next diagonal is a $\boxed{d}\boxed{d}\boxed{e}\boxed{e}$-diagonal for some different $\boxed{d}$ and $\boxed{e}$ (recall that we avoid the possibility for this diagonal to be binary or one-color, considered in subcase VI.d). We see that every $\{\boxed{1}, \boxed{2}\}$-node has neighbors of colors $\boxed{a}$, $\boxed{b}$, and $\boxed{c}$. This allows us to reconstruct the colors of the following nodes:



If the nodes marked by $*$ have a color different from $\boxed{b}$, then there are $\boxed{b}$-nodes at distance 3 from the $\boxed{1}\boxed{2}$-diagonal, and, hence, there are $\{\boxed{1}, \boxed{2}\}$-nodes at distance 4 from the

$\boxed{1}\boxed{2}$-diagonal. We finally obtain the situation of Proposition 32(ii), i.e. the coloring is equivalent, up to shifting of $\boxed{1}\boxed{2}$-diagonals, to the coloring of Fig. J.

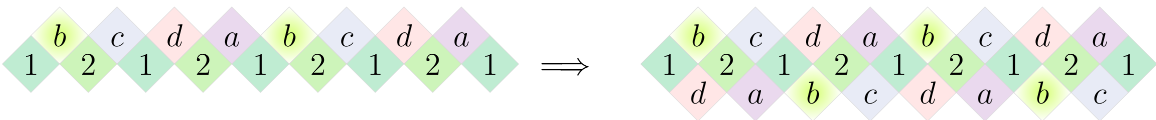
Otherwise, we have the situation as follows, which can be solved similarly to subcase VI.d:



Proposition 35. *In the subcase VI.d, the coloring either is described in Proposition 32 or has the following structure, up to renaming colors: Every $\boxed{1}\boxed{2}$-diagonal is surrounded by two $\boxed{3}\boxed{4}\boxed{5}\boxed{4}$-diagonals. Each of the $\boxed{3}\boxed{4}\boxed{5}\boxed{4}$-diagonals is followed by a $\boxed{6}\boxed{6}\boxed{7}\boxed{7}$-diagonal. Then, a $\boxed{8}\boxed{9}\boxed{10}\boxed{9}$-diagonal can follow; then a $\boxed{11}\boxed{11}\boxed{12}\boxed{12}$-diagonal, and so on, until we meet one of the fragments* (a)–(d) *of Proposition 34. After that, the same diagonals follow in the reverse order, until a $\boxed{1}\boxed{2}$-diagonal; then the situation repeats.*

The corresponding colorings are shown in Fig. O, S, Q, and R (the special case $n = 7$ corresponds to in Fig. J with merged colors $\boxed{3}$ and $\boxed{4}$).

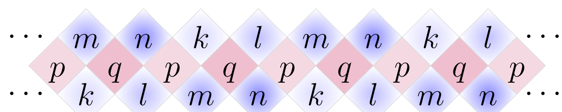
VI.d.2.vii There is an $\boxed{a}\boxed{b}\boxed{c}\boxed{d}$-diagonal neighbor to a $\boxed{1}\boxed{2}$-diagonal; $\boxed{a}$–$\boxed{d}$ are pairwise distinct



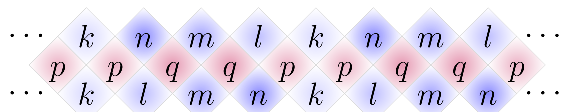
This subcase is considered similarly to the previous two subcases, resulting in:

Proposition 36. *In the subcase VI.d, the coloring has the following structure, up to renaming colors. Every $\boxed{1}\boxed{2}$-diagonal is surrounded by two $\boxed{3}\boxed{4}\boxed{5}\boxed{6}$-diagonals. Then, every $\boxed{a}\boxed{b}\boxed{c}\boxed{d}$-diagonal is followed by an $\boxed{e}\boxed{f}\boxed{g}\boxed{h}$-diagonal, where* $h = g + 1 = f + 2 = e + 3 = d + 4 = c + 5 = b + 6 = a + 7 \equiv 2 \bmod 4$, *until we meet one of the following fragments:*

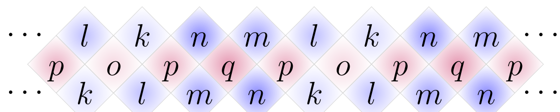
(a)



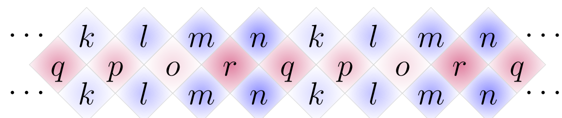
(b)



(c)



(d)



(where all symbols denote different colors except maybe $\boxed{p}$ *and* $\boxed{q}$ *in* (a)*). After that, the same diagonals follow in the reverse order, until a* $\boxed{1}\boxed{2}$*-diagonal; then the situation repeats.*

The corresponding colorings are shown in Fig. T, U, V, and W.

4 Proof of Theorem 1

We start with an obvious claim, then prove two lemmas, considering two special cases, then give a proof of Theorem 1, which consists of crucial Proposition 41 and a short concluding part.

Claim 37. *For a perfect coloring of* $G(\mathbb{Z}^2)$ *with quotient* $\{0,1\}$*-matrix with zero diagonal and without equal rows, any two nodes with difference in* $\{\,[0,\pm1],\ [\pm1,0],\ [\pm1,\pm1],\ [0,\pm2],\ [\pm2,0],\ [\pm2,\pm2]\,\}$ *are colored with different colors.*

Proof. The only non-trivial case is with the difference of form $[\pm2,\pm2]$. If, w.l.o.g., $\mathcal{F}([0,0]) = \mathcal{F}([2,2]) = \boxed{1}$ and $\mathcal{F}([1,1]) = \boxed{2}$, then each of $[0,1]$, $[0,1]$, $[1,2]$, $[2,1]$ is adjacent to a $\boxed{1}$-node and to a $\boxed{2}$-node. Since the colors of $[0,1]$, $[0,1]$, $[1,2]$, $[2,1]$ are pairwise different, we see that the 1st and 2nd rows of the quotient matrix coincide, contradicting the hypothesis of the claim. □

Lemma 38. *If a bipartite perfect coloring* $\mathcal{F}$ *of* $G(\mathbb{Z}^2)$ *with quotient* $\{0,1\}$*-matrix contains the fragment*

		5		
	4	B	1	
6	A	0	C	6
	3	D	2	
		5		

then $\mathcal{F}$*, up to equivalence, is one of the following two colorings, with periods, respectively,* $[4,0]$ *and* $[0,4]$, $[4,0]$ *and* $[0,8]$.

7	G	5	H	7	G	5	H	7
E	4	B	1	E	4	B	1	E
6	A	0	C	6	A	0	C	6
F	3	D	2	F	3	D	2	F
7	G	5	H	7	G	5	H	7
E	4	B	1	E	4	B	1	E
6	A	0	C	6	A	0	C	6
F	3	D	2	F	3	D	2	F
7	G	5	H	7	G	5	H	7

7	H	5	G	7	H	5	G	7
E	4	B	1	E	4	B	1	E
6	A	0	C	6	A	0	C	6
F	3	D	2	F	3	D	2	F
7	G	5	H	7	G	5	H	7
E	1	B	4	E	1	B	4	E
6	C	0	A	6	C	0	A	6
F	2	D	3	F	2	D	3	F
7	H	5	G	7	H	5	G	7

Proof. The only $\boxed{1}$-node in the fragment cannot be adjacent to an $\boxed{A}$- or $\boxed{D}$-node; so, its right neighbor has a color, say $\boxed{E}$, different from $\boxed{A}$, $\boxed{B}$, $\boxed{C}$, $\boxed{D}$. The same can be said about the right neighbor of the $\boxed{2}$-node, denote its color by $\boxed{F}$.

		5		
	4	B	1	E
6	A	0	C	6
	3	D	2	F
		5		

The fourth neighbor of the right $\boxed{6}$-node has color $\boxed{A}$. In its turn, this $\boxed{A}$-node has a $\boxed{0}$-neighbor, and it can be only the right neighbor.

		5				
	4	B	1	E		
6	A	0	C	6	A	0
	3	D	2	F		
		5				

There are only two possibilities for the neighborhood of the right $\boxed{A}$-node. In each case, the neighborhood of the right $\boxed{0}$-node is uniquely colored:

(a)

		5				5		
	4	B	1	E	4	B	1	
6	A	0	C	6	A	0	C	6
	3	D	2	F	3	D	2	
		5				5		

or (b)

		5				5		
	4	B	1	E	3	D	2	
6	A	0	C	6	A	0	C	6
	3	D	2	F	4	B	1	
		5				5		

The same situation takes place in the vertical direction

(c)

		5		
	4	B	1	
6	A	0	C	6
	3	D	2	
	G	5	H	
	4	B	1	
6	A	0	C	6
	3	D	2	
		5		

or (d)

		5		
	4	B	1	
6	A	0	C	6
	3	D	2	
	G	5	H	
	1	B	4	
6	C	0	A	6
	2	D	3	
		5		

Each of (a), (b) can be combined with each of (c), (d), see e.g. (a)+(c) and (a)+(d) in the first two pictures below, except (b)+(d), see the contradiction at the third picture (the color of the node marked with "?" must be $\boxed{F}$ because of the $\boxed{4}$-node at the left; on the other hand, it cannot be $\boxed{F}$ because the $\boxed{7}$-node above already has an $\boxed{F}$-neighbor).

		5				5		
	4	B	1	E	4	B	1	
6	A	0	C	6	A	0	C	6
	3	D	2	F	3	D	2	
	G	5	H	7		5		
	4	B	1					
6	A	0	C	6				
	3	D	2					
		5						

(a)+(c)

		5				5		
	4	B	1	E	4	B	1	
6	A	0	C	6	A	0	C	6
	3	D	2	F	3	D	2	
	G	5	H	7		5		
	1	B	4					
6	C	0	A	6				
	2	D	3					
		5						

(a)+(d)

		5				5		
	4	B	1	E	3	D	2	
6	A	0	C	6	A	0	C	6
	3	D	2	F	4	B	1	
	G	5	H	7		5		
	1	B	4	?				
6	C	0	A	6				
	2	D	3					
		5						

(b)+(d)

In cases (a)+(c) and (a)+(d), it is easy to see that the quotient matrix and the whole coloring are uniquely reconstructed. Cases (a)+(d) and (b)+(c) result in equivalent colorings. $\square$

Next, we consider the case where two nodes with difference in $\{[\pm 3, \pm 1], [\pm 1, \pm 3]\}$ (w.l.o.g., $[3, 1]$) have the same color.

Lemma 39. *Let $\mathcal{F}$ be a perfect coloring with quotient $\{0, 1\}$-matrix with zero diagonal and without equal rows. If $\mathcal{F}(\bar{x}) = \mathcal{F}(\bar{x} + [3, 1])$ for some $\bar{x} \in \mathbb{Z}^2$, then either*

- (a) *$\mathcal{F}$ has 3β colors, $\beta \in \{3, 4, 5, \ldots\}$, periods $[6, 0]$ and $[0, \beta]$, and satisfies $\mathcal{F}(i, j) \equiv \mathcal{F}(i + 3, -j + 1)$, or*
- (b) *$\mathcal{F}$ has α colors, $\alpha \in \{5, 7, 8, 9, \ldots\}$, and periods $[\alpha, 0]$ and $[3, 1]$.*

In both cases, $\mathcal{F}$ is an orbit coloring.

Proof. Without loss of generality, assume $\mathcal{F}([0, 0]) = \mathcal{F}([3, 1]) = \boxed{1}$.

			·	
		·	1	2
1	2		·	

Denote by $\boxed{2}$ the color of $[1, 0]$. By Claim 37, the $\boxed{2}$-neighbor of $[3, 1]$ is not $[2, 1]$, $[3, 0]$, or $[3, 2]$. Hence, it is $[4, 1]$. Similarly, the colors of $[-1, 0]$ and $[2, 1]$ coincide. By induction, we have

$$\mathcal{F}([i, 0]) = \mathcal{F}([i + 3, 1]) \qquad \text{for all } i \in \mathbb{Z}. \tag{8}$$

Denote by α the first positive integer such that $\mathcal{F}([i, 0]) = \mathcal{F}([i + \alpha, 0])$ for some i. From $\mathcal{F}([i, 0]) = \mathcal{F}([i + 3, 1])$ and Claim 37, we see $\alpha \geqslant 5$. We consider two cases.

(a) $\alpha = 6$. Without loss of generality, $\mathcal{F}([0, 0]) = \mathcal{F}([6, 0])$.

			1	A	2	B	3	C
1	A	2	B	3	C	1		

Similarly to (8), we have

$$\mathcal{F}([i,1]) = \mathcal{F}([i+3,0]) \qquad \text{for all } i \in \mathbb{Z},$$

and we know the coloring of the two rows

$\cdots$	2	B	3	C	1	A	2	B	3	C	1	A	$\cdots$
$\cdots$	C	1	A	2	B	3	C	1	A	2	B	3	$\cdots$

Claim 40. *There are no* $\{\boxed{1},\boxed{2},\boxed{3},\boxed{A},\boxed{B},\boxed{C}\}$*-nodes in the row* $[i,2]$*,* $i \in \mathbb{Z}$*.*

W.l.o.g., we will show this for $\boxed{1}$-nodes. If i is even or $i \equiv 3 \bmod 6$, then the claim follows from Claim 37. Seeking a contradiction in the remaining case, we assume without loss of generality that $\mathcal{F}([1,2]) = \boxed{1}$.

		1		3	$\bar{x}$	?	
2	B	3	C	1	A	2	B
C	1	A	2	B	3	C	1

Then $[3,2]$ is a $\boxed{3}$-node, $\bar{x} = [4,2]$ is an $\{\boxed{A},\boxed{B},\boxed{C}\}$-node, and $[5,2]$ is an $\{\boxed{1},\boxed{2},\boxed{3}\}$-node. However, $\mathcal{F}([5,2]) \in \{\boxed{2},\boxed{3}\}$ contradicts Claim 37, while $\mathcal{F}([5,2]) = \boxed{1}$ contradicts the fact that the $\boxed{1}$-node $[3,1]$ is not adjacent to any $\boxed{2}$-node. The contradiction obtained proves Claim 40.

It immediately follows that the row $[i,2]$, $i \in \mathbb{Z}$, is colored periodically with period $[6,0]$ with new colors. We divide this situation into two subcases, depending on the number of different colors in this row.

(a') The row $[i,2]$ is colored with less than 6 colors. Then it contains two nodes of the same color at distance 3 from each other. W.l.o.g., $\mathcal{F}([0,2]) = \mathcal{F}([3,2]) = \boxed{X}$; denote $\boxed{Y} = \mathcal{F}([2,2])$, $\boxed{Z} = \mathcal{F}([1,2])$.

	X	Z	Y	X	
2	B	3	C	1	A
C	1	A	2	B	3

$\Rightarrow$

	1			B	
	X	Z	Y	X	
2	B	3	C	1	A
C	1	A	2	B	3

$\Rightarrow$

	1	A	2	B	
	X	Z	Y	X	
2	B	3	C	1	A
C	1	A	2	B	3

The right $\boxed{X}$-node at the picture must have a $\boxed{B}$-neighbor, which has three $\{\boxed{1},\boxed{2},\boxed{3}\}$-neighbors. The node $[2,3]$ abobe the $\boxed{Y}$-node can only be a $\boxed{2}$-node. Similarly, $[1,3]$ is an $\boxed{A}$-node. Now (see the last picture above) the quotient matrix is completely determined, and it is easy to see that the coloring is uniquely reconstructed and meets assertion (a) of the lemma with $\beta = 3$.

(a") The row $[i,2]$ is colored with exactly 6 colors.

$\cdots$	D	6	E	4	F	5	D	6	E	4	F	5	$\cdots$
$\cdots$	2	B	3	C	1	A	2	B	3	C	1	A	$\cdots$
$\cdots$	C	1	A	2	B	3	C	1	A	2	B	3	$\cdots$

It is easy to see that the row $[i,3]$, $i \in \mathbb{Z}$, has no $\{\boxed{1},\boxed{2},\boxed{3},\boxed{A},\boxed{B},\boxed{C}\}$-nodes (for example, a $\boxed{1}$-node cannot have a $\{\boxed{4},\boxed{5},\boxed{6},\boxed{D},\boxed{E}\}$-neighbor, and an $\boxed{F}$-node cannot have two $\boxed{1}$-neighbors). So, there are two possibilities for that row.

1. It can have a $\{\boxed{4},\boxed{5},\boxed{6},\boxed{D},\boxed{E},\boxed{F}\}$-node, w.l.o.g., a $\boxed{4}$-node. According to Claim 37, that $\boxed{4}$-node can only be above a $\boxed{D}$-node, and by arguments similar as above we have $\mathcal{F}([i,3]) = \mathcal{F}([i-3,2])$ for all $i \in \mathbb{Z}$.

⋯	4	F	5	D	6	E	4	F	5	D	6	E	⋯
⋯	D	6	E	4	F	5	D	6	E	4	F	5	⋯
⋯	2	B	3	C	1	A	2	B	3	C	1	A	⋯
⋯	C	1	A	2	B	3	C	1	A	2	B	3	⋯

From these four rows, we know the quotient matrix, and the coloring is uniquely reconstructed and satisfies the statement of the lemma with $\beta = 4$.

2. In the other subcase, the row $[i,3]$, $i \in \mathbb{Z}$, is colored with new colors so that $\mathcal{F}([i,3]) = \mathcal{F}([i+6,3])$. If the number of those new colors is less than 6, the arguments similar to (a') show that this number is 3 and that assertion (a) of the lemma holds with $\beta = 5$. Otherwise, we have six new colors.

In its turn, the next row $[i,4]$, $i \in \mathbb{Z}$, either satisfies $\mathcal{F}([i,4]) = \mathcal{F}([i-3,3])$, $i \in \mathbb{Z}$, or colored with new three of six colors.

In the last case, we apply the same argument to the row $[i,4]$, then $[i,5]$, and so on, until the colors are exhausted after some row $[i,\gamma]$ in the sense that the number of remaining colors is less than 6. If at least one new color remains, then similarly to (a') the next row $[i,\gamma+1]$ is colored with period $[3,0]$ and assertion (a) of the lemma holds with $\beta = 2\gamma+1$. Otherwise, the only possibility is $\mathcal{F}([i,\gamma+1]) = \mathcal{F}([i-3,\gamma])$, $i \in \mathbb{Z}$. From the $\gamma+2$ rows $[i,0]$, …, $[i,\gamma+1]$, the quotient matrix and the coloring is uniquely reconstructed and satisfies assertion (a) of the lemma with $\beta = 2\gamma$.

(b) $\alpha = 5$ *or* $\alpha \geqslant 7$.

The colors of $[0,0]$, $[1,0]$, …, $[\alpha-1,0]$ are pairwise distinct, by the definition of α. The node $[\alpha,0]$ (the right $\boxed{1}$-node in the picture below) is colored with $\boxed{1}$.

			1	2	3	4	5	⋯				l	
1	2	3	4	5				⋯	l	m	n	1	$\star$

We see that the neighborhood of a $\boxed{1}$-node has colors $\boxed{2}$, $\boxed{4}$, $\boxed{l}$, $\boxed{n}$. The color of $[\alpha+1,0]$ (marked by $\star$) cannot be $\boxed{l}$ or $\boxed{n}$. And it cannot be $\boxed{4}$, because $\mathcal{F}([3,0]) = \mathcal{F}([\alpha+1,0])$ contradicts the definition of α. So, it is $\boxed{2}$. Analogously, the color of $[2,1]$, and hence, of $[-1,0]$, is $\boxed{n}$. Using similar arguments, we can prove by induction that $\mathcal{F}([i,0]) = \mathcal{F}([i+\alpha,0])$ for any $i \in \mathbb{Z}$, i.e., the row $[i,0]$ (and hence, $[i,1]$) is colored with α colors in a cyclic way.

Next, we see that a $\boxed{4}$-node has a $\boxed{1}$-neighbor. Also we know that $\mathcal{F}([6,0]) \neq \boxed{1}$ because

$\alpha \neq 6$. Considering the $\boxed{4}$-node $[6,1]$, we find that the color of $[6,2]$ is $\boxed{1}$:

			1	2	3	4	5
1	2	3	4	5		$\not{1}$	

$\Rightarrow$

						1	
			1	2	3	4	5
1	2	3	4	5			

Therefore, $\mathcal{F}([i,2]) = \mathcal{F}([i-3,1]) = \mathcal{F}([i-6,0])$ for every $i \in \mathbb{Z}$. Similarly, $\mathcal{F}$ is periodic with period $[3,1]$. $\square$

Given a perfect coloring $\mathcal{F}$ of $G(\mathbb{Z}^2)$, we say that two subsets M and N of $\mathbb{Z}^2$ are colored *similarly* if $\varphi(M) = (N)$ and $\mathcal{F}(\varphi(\cdot)) \equiv \mathcal{F}(\cdot)$ for some automorphism φ of $G(\mathbb{Z}^2)$.

Proposition 41. *Let $\mathcal{F}$ be a perfect coloring of $G(\mathbb{Z}^2)$ with quotient $\{0,1\}$-matrix with zero diagonal and without equal rows. Then every two radius-2 balls centered in nodes of the same color are colored similarly.*

Proof. Consider a radius-2 ball and denote colors of its nodes as follows:

		e		
	d	B	a	
h	A	0	C	f
	c	D	b	
		g		

We consider different cases depending on the equalities that can take place between the colors $\boxed{a}$–$\boxed{f}$ (independently of equalities that can occur between the colors of nodes of different parities, for example, $\boxed{a} = \boxed{D}$ or $\boxed{e} = \boxed{C}$).

By Claim 37, two nodes of the same color cannot have difference in $\{\,[\pm1,\pm1], [0,\pm2], [\pm2,0], [\pm2,\pm2]\,\}$.

If there are two nodes of the same color with difference in $\{\,[\pm1,\pm3], [\pm3,\pm1]\,\}$, then the statement holds by Lemma 39.

So, we can assume that all colors $\boxed{a}$–$\boxed{f}$ are mutually different with maybe one or two exceptions from $\boxed{e} = \boxed{g}$, $\boxed{f} = \boxed{h}$.

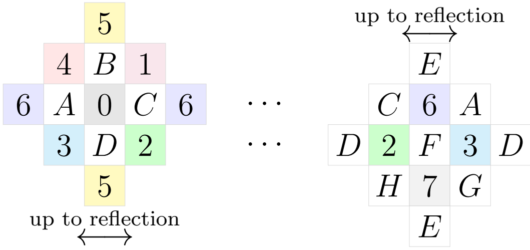
Now, if $\boxed{f} \neq \boxed{h}$ (the case $\boxed{e} \neq \boxed{g}$ is similar), then the statement is obvious: for any $\boxed{0}$-node, the $\boxed{A}$- and $\boxed{C}$-neighbors at distance 1 from it must be opposite, and the colors of the nodes at distance 2 from it are uniquely determined by the four $\{\boxed{A}, \boxed{B}, \boxed{C}, \boxed{D}\}$-neighbors.

Consider the last remaining case: assume that the colors $\boxed{a}$–$\boxed{f}$ are pairwise distinct with two exceptions $\boxed{e} = \boxed{g}$ and $\boxed{f} = \boxed{h}$. If the coloring $\mathcal{F}$ is bipartite, then it is one of two colorings in the statement of Lemma 38. If $\mathcal{F}$ is not bipartite, then the corresponding bipartite coloring $\overline{\mathcal{F}}$ (each color is subdivided into two subcolors in accordance with the parity of the node) satisfies the hypothesis of Lemma 38 and hence is one of the two colorings in the conclusion of Lemma 38. The original coloring $\mathcal{F}$ is obtained from $\overline{\mathcal{F}}$ by identifying each color from $\boxed{0}$–$\boxed{7}$ with one of $\boxed{A}$–$\boxed{H}$.

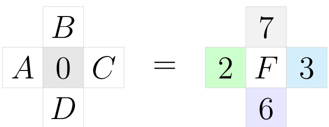
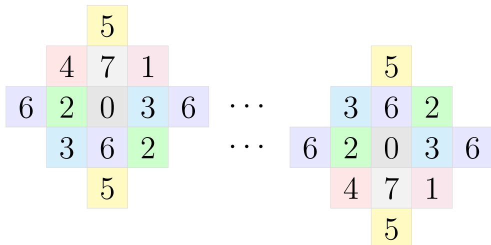
Consider the right coloring (in further discussion, we refer the picture in Lemma 38). Clearly, $\boxed{0}$ cannot be identified with one of its neighbor colors $\boxed{A}$, $\boxed{B}$, $\boxed{C}$, $\boxed{D}$. Next, $\boxed{0}$ cannot be identified with $\boxed{G}$ or $\boxed{H}$ (indeed, every $\{\boxed{G},\boxed{H}\}$-node has 7 different colors at distance 2 from it, while a $\boxed{0}$-node has only 6). We assume it is identified with $\boxed{F}$ (the case of $\boxed{E}$ is similar).

Consider the left coloring. Again, $\boxed{0}$ cannot be identified with $\boxed{A}$, $\boxed{B}$, $\boxed{C}$, or $\boxed{D}$. The remaining four cases ($\boxed{E}$, $\boxed{F}$, $\boxed{G}$, $\boxed{H}$) are similar, and we again assume that $\boxed{0}$ is identified with $\boxed{F}$.

The rest of the proof is common for the both colorings. For every $\boxed{i}$ from $\boxed{1}$ to $\boxed{6}$, it holds $d^2(\boxed{0},\boxed{i}) > 0$, while for $\boxed{i} = \boxed{7}$ this is not true: $d^2(\boxed{0},\boxed{7}) = 0$. Similarly, $\boxed{B}$ is a distinguished color with respect to $\boxed{F}$: $d^2(\boxed{F},\boxed{B}) = 0$, while $d^2(\boxed{F},\boxed{j}) > 0$ for any other color $\boxed{j}$ of the same parity. It follows that $\boxed{7}$ is identified with $\boxed{B}$. Now, let us take a look to the neighborhood of an $\boxed{F}$-node.



The neighbors of an $\boxed{F}$-node have colors $\boxed{2}$, $\boxed{6}$, $\boxed{3}$, $\boxed{7}$, and they should be identified with the neighbor colors of a $\boxed{0}$-node, i.e., with $\boxed{A}$, $\boxed{B}$, $\boxed{C}$, $\boxed{D}$, in some order. We already know that $\boxed{7}$ and $\boxed{B}$ are identified. There is only one possibility for the other three colors, taking into account that there is no two neighbor nodes of the same color. Therefore,


, and finally we have


So, we have convinced that the conclusion of the proposition holds for two $\boxed{0}$-nodes not only of the same parity (which is immediate from Lemma 38), but also of different parities. □

Proof of Theorem 1. The case when the coloring is bipartite and has twin colors is solved by the characterization in Theorem 2 and Corollary 5.

If the coloring, $\mathcal{F}$, is not bipartite and has two equal rows in the quotient matrix, then the corresponding bipartite coloring $\overline{\mathcal{F}}$ has twin colors and the quotient is a $\{0,1\}$-matrix. As in the paragraph above, $\overline{\mathcal{F}}$ has a binary diagonal, and hence $\mathcal{F}$ has a binary diagonal too.

Assume we have a coloring $\mathcal{F}$ satisfying the hypothesis of the theorem without two equal rows in the quotient matrix. We need to proof that the coloring is orbit, which, by the definition, is equivalent to the following assertion.

Claim 42. *For any two nodes $\bar{x}$ and $\bar{y}$ of the same color, there is an automorphism φ of $G(\mathbb{Z}^2)$ such that $\varphi(\bar{x}) = \bar{y}$ and for all $\bar{u} \in \mathbb{Z}^2$*

$$\mathcal{F}(\varphi(\bar{u})) = \mathcal{F}(\bar{u}). \tag{9}$$

We prove the claim by induction on the distance l between $\bar{x}$ and $\bar{u}$. By Proposition 41, it holds if $d(\bar{x}, \bar{u}) \leqslant 2$, with some automorphism φ. Suppose (9) holds with the same φ for every $\bar{u}$ at distance less than l from $\bar{x}$, where $l \geqslant 3$.

Consider $\bar{u}$ at distance l from $\bar{x}$. Consider a node $\bar{x}'$ between $\bar{x}$ and $\bar{u}$ such that $d(\bar{x}, \bar{x}') = l - 2$ and $d(\bar{x}', \bar{u}) = 2$. By the induction hypothesis, $\mathcal{F}(\varphi(\bar{x}')) = \mathcal{F}(\bar{x}')$ and, moreover

$$\mathcal{F}(\varphi(\bar{u}')) = \mathcal{F}(\bar{u}') \quad \text{for all } \bar{u}' \text{ at distance 1 from } \bar{x}', \tag{10}$$

because all such $\bar{u}'$ satisfy $d(\bar{x}, \bar{u}') \leqslant d(\bar{x}, \bar{x}') + d(\bar{x}', \bar{u}') = l - 1$. Now, consider the nodes $\bar{x}'$ and $\bar{y}' = \varphi(\bar{x}')$. They have the same color, and by Proposition 41 there is an automorphism ψ of $G(\mathbb{Z}^2)$ such that $\psi(\bar{x}') = \bar{y}'$ and for all $\bar{u} \in \mathbb{Z}^2$

$$\mathcal{F}(\psi(\bar{u}')) = \mathcal{F}(\bar{u}') \quad \text{for all } \bar{u}' \text{ at distance at most 2 from } \bar{x}'. \tag{11}$$

Now we see $\varphi(\bar{x}') = \psi(\bar{x}')$. Moreover, from (10) and (11) we find that both φ and ψ preserve the color of each neighbor of $\bar{x}'$. Since the neighbors of $\bar{x}'$ have pairwise different colors, it follows that the values of φ and ψ coincide on the neighborhood of $\bar{x}'$. Since an automorphism of the grid $G(\mathbb{Z}^2)$ is uniquely determined by its action on the neighborhood of a node, we have $\psi \equiv \varphi$. Then, substituting $\bar{u}' = \bar{u}$ in (11), we obtain (9), which completes the induction step and the proof of the theorem. □

5 Non-orbit perfect colorings

Here, we provide examples of non-orbit perfect colorings of $G(\mathbb{Z}^2)$, answering some natural questions. At first, we observe that in the colorings shown in Fig. M–P and T–W merging all groups of twin colors results in non-orbit bipartite perfect colorings without twin colors (for example in Fig. M, a $\boxed{1}$-node has two $\boxed{4}$-neighbors in opposite directions, which is not true for the two $\boxed{4}$-neighbors of a $\boxed{2}$-node; after merging, such nodes become of the same color but cannot be in the same orbit), providing an infinite number of such examples. One can ask if there are another examples (in particular, that cannot be obtained by merging twin colors). Up to now, we know a finite number of examples of non-orbit perfect colorings of $G(\mathbb{Z}^2)$ without a binary or single-color diagonal. Here we list the numbers of such colorings up to 9 colors, according to the catalogue [18]: 2-5, 2-7(1), 3-6, 3-8(1), 3-9(1), 3-14, 3-15, 3-17(2), 3-17(3), 3-20(1), 4-7, 4-13, 4-14(2), 4-16, 4-18, 4-29, 4-32(1), 4-32(2), 4-36, 4-37, 4-38(1), 4-41(1), 5-29, 5-36(1), 5-36(2), 5-42*, 5-43(1)*, 5-45, 6-22, 6-23, 6-36(1), 6-58(2), 6-67(1), 6-80(2)*, 6-81*, 6-83(1), 6-89(3)*, 8-32, 8-45, 8-49, 8-122, 8-123(2), 8-150(1)*, 9-127; the underlined numbers correspond to bipartite colorings, the symbol * indicates that the quotient matrix is a $\{0,1\}$-matrix. Each bipartite coloring from those examples is either of form $\overline{\mathcal{G}}$ for some non-bipartite perfect colorings $\mathcal{G}$, or obtained from Theorem 2 by merging (not necessarily all) twin colors. Below, we show two non-bipartite examples.

- 3-17(2,3): Examples of non-orbit perfect colorings with twin colors. Note that the quotient matrix (the same for the two colorings) has no equal rows, and hence the corresponding bipartite perfect colorings have no twin colors; so, the hypothetical future characterization of non-bipartite perfect colorings of $G(\mathbb{Z}^2)$ with twin colors cannot be derived straightforwardly from Theorem 2.

1	3	3	2	1	3	3	2	1	3	3	2
1	3	3	2	1	3	3	2	1	3	3	2
2	2	1	1	2	2	1	1	2	2	1	1
3	1	2	3	3	1	2	3	3	1	2	3
3	1	2	3	3	1	2	3	3	1	2	3
2	2	1	1	2	2	1	1	2	2	1	1
1	3	3	2	1	3	3	2	1	3	3	2
1	3	3	2	1	3	3	2	1	3	3	2

1	3	3	2	1	3	3	2	1	3	3	2
1	2	1	2	1	2	1	2	1	2	1	2
3	2	1	3	3	2	1	3	3	2	1	3
3	1	2	3	3	1	2	3	3	1	2	3
2	1	2	1	2	1	2	1	2	1	2	1
2	3	3	1	2	3	3	1	2	3	3	1
1	3	3	2	1	3	3	2	1	3	3	2
1	2	1	2	1	2	1	2	1	2	1	2

- 8-150(1,2): An example of two colorings with the same $\{0,1\}$-matrix, the first colorings is not orbit, the second one is orbit.

5	2	2	6	5	2	2	6	5	2	2	6
8	6	5	8	8	6	5	8	8	6	5	8
7	3	4	7	7	3	4	7	7	3	4	7
4	1	1	3	4	1	1	3	4	1	1	3
5	2	2	6	5	2	2	6	5	2	2	6
8	6	5	8	8	6	5	8	8	6	5	8
7	3	4	7	7	3	4	7	7	3	4	7
4	1	1	3	4	1	1	3	4	1	1	3
5	2	2	6	5	2	2	6	5	2	2	6
8	6	5	8	8	6	5	8	8	6	5	8

5	6	3	4	5	6	3	4	5	6	3	4
8	8	7	7	8	8	7	7	8	8	7	7
6	5	4	3	6	5	4	3	6	5	4	3
2	2	1	1	2	2	1	1	2	2	1	1
5	6	3	4	5	6	3	4	5	6	3	4
8	8	7	7	8	8	7	7	8	8	7	7
6	5	4	3	6	5	4	3	6	5	4	3
2	2	1	1	2	2	1	1	2	2	1	1
5	6	3	4	5	6	3	4	5	6	3	4
8	8	7	7	8	8	7	7	8	8	7	7

The number of colors (nine) for which the perfect colorings of $G(\mathbb{Z}^2)$ are characterized is relatively small to make convincing conjectures, but clearly, the problem of characterization of all such colorings includes the following question:

Question. Is the number of non-orbit perfect colorings of $G(\mathbb{Z}^2)$ without a binary or single-color diagonal finite?

Acknowledgements

The author thanks the reviewers for critically reading the manuscript and many substantial and constructive suggestions, which helped to improve and clarify this paper enormously, and Olga Yakovchenko for help with palettes.

References

[1] S. V. Avgustinovich, D. S. Krotov, and A. Yu. Vasil'eva. Completely regular codes in the infinite hexagonal grid. *Sib. Èlektron. Mat. Izv.*, 13:987–1016, 2016.

[2] S. V. Avgustinovich and A. Yu. Vasil'eva. Completely regular codes in the n-dimensional rectangular grid. *Sib. Èlektron. Mat. Izv.*, 19(2):861–869, 2022.

[3] S. V. Avgustinovich, A. Yu. Vasil'eva, and I. V. Sergeeva. Distance regular colorings of an infinite rectangular grid. *J. Appl. Ind. Math.*, 6(3):280–285, 2012. DOI: 10.1134/S1990478912030027 translated from *Diskretn. Anal. Issled. Oper.*, 18(2):3–10, 2011.

[4] M. A. Axenovich. On multiple coverings of the infinite rectangular grid with balls of constant radius. *Discrete Math.*, 268(1–3):31–48, 2003.

[5] J. Borges, J. Rifà, and V. A. Zinoviev. On completely regular codes. *Probl. Inf. Transm.*, 55(1):1–45, Jan. 2019. DOI: 10.1134/S0032946019010010 translated from *Probl. Peredachi Inf.*, 55(1):3–50, 2019.

[6] M. Braun, M. Kiermaier, and A. Wassermann. q-Analogs of designs: Subspace designs. In M. Greferath, M. O. Pavčević, N. Silberstein, and M. Á. Vázquez-Castro, editors, *Network Coding and Subspace Designs*, p. 171–211. Springer Int. Pub., Cham, 2018.

[7] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-Regular Graphs*. Springer-Verlag, Berlin, 1989.

[8] P. J. Cameron and R. A. Liebler. Tactical decompositions and orbits of projective groups. *Linear Algebra Appl.*, 46:91–102, 1982.

[9] C. J. Colbourn and R. Mathon. Steiner systems. In C. J. Colbourn and J. H. Dinitz, editors, *Handbook of Combinatorial Designs*, Discrete Mathematics and Its Applications, p. 102–108. Chapman & Hall/CRC, Boca Raton, London, New York, 2nd ed., 2006.

[10] J. H. Conway, H. Burgiel, and C. Goodman-Strauss. *The Symmetries of Things*. A K Peters, Wellesley, MA, 2008.

[11] B. De Bruyn and H. Suzuki. Intriguing sets of vertices of regular graphs. *Graphs Comb.*, 26(5):629–646, Sept. 2010.

[12] A. D. Keedwell and J. Dénes. *Latin Squares and Their Applications*. Elsevier, Amsterdam, 2nd edition, 2015.

[13] D. G. Fon-Der-Flaass. A bound on correlation immunity. *Sib. Èlektron. Mat. Izv.*, 4:133–135, 2007. Online: https://mi.mathnet.ru/eng/semr149

[14] E. M. Gabidulin. *Rank Codes*. TUM.University Press, Munich, 2021.

[15] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer-Verlag, New York, 2001.

[16] K. Golubev. Graphical designs and extremal combinatorics. *Linear Algebra Appl.*, 604:490–506, 2020.

[17] A. S. Hedayat, N. J. A. Sloane, and J. Stufken. *Orthogonal Arrays. Theory and Applications*. Springer Ser. Stat. Springer, New York, NY, 1999.

[18] D. S. Krotov. Perfect colorings of Z^2: Nine colors. arXiv:0901.0004.

[19] D. S. Krotov. On the binary codes with parameters of triply-shortened 1-perfect codes. *Des. Codes Cryptogr.*, 64(3):275–283, 2012.

[20] D. S. Krotov. On the OA(1536,13,2,7) and related orthogonal arrays. *Discrete Math.*, 343:111659/1–11, 2020.

[21] M. A. Lisitsyna, S. V. Avgustinovich, and O. G. Parshina. On perfect colorings of infinite multipath graphs. *Sib. Èlektron. Mat. Izv.*, 17:2084–2095, 2020.

[22] F. J. MacWilliams and N. J. A. Sloane. *The Theory of Error-Correcting Codes*. Amsterdam, Netherlands: North Holland, 1977.

[23] B. D. McKay and I. M. Wanless. A census of small Latin hypercubes. *SIAM J. Discrete Math.*, 22(2):719–736, 2008.

[24] D. P. Plotnikov. Perfect colorings of the vertices of a regular graph of degree three into three and more colors. Graduate thesis, Novosibirsk State University, 2005. In Russian.

[25] V. N. Potapov. On perfect 2-colorings of the q-ary n-cube. *Discrete Math.*, 312(6):1269–1272, 2012.

[26] V. N. Potapov and S. V. Avgustinovich. Combinatorial designs, difference sets, and bent functions as perfect colorings of graphs and multigraphs. *Sib. Math. J.*, 61(5):867–877, 2020. DOI: 10.1134/S0037446620050109 translated from Sib. Mat. Zh. 61(5):1087–1100, 2020.

[27] S. A. Puzynina. Periodicity of perfect colorings of the infinite rectangular grid. *Diskretn. Anal. Issled. Oper.*, Ser. 1, 11(1):79–92, 2004. In Russian. Online: https://www.mathnet.ru/eng/da98

[28] S. A. Puzynina. Perfect colorings of vertices of the graph $G(Z^2)$ in three colors. *Diskretn. Anal. Issled. Oper.*, Ser. 2, 12(1):37–54, 2005. In Russian. Online: https://www.mathnet.ru/eng/da85

[29] S. A. Puzynina. On periodicity of perfect colorings of the infinite hexagonal and triangular grids. *Sib. Math. J.*, 52(1):91–104, 2011. DOI: 10.1134/S0037446606010101 translated from *Sib. Mat. Zh.*, 52(1):115–132, 2011.

[30] A. Sayari, Y. Billiet, and H. Zarrouk. Complete list of subgroups and changes of standard setting of two-dimensional space groups. *Acta Crystallogr., Sect. A*, 34(4):553–555, 1978.

[31] A. Yu. Vasil'eva. Distance regular colorings of the infinite triangular grid. In *Int. Conference "Mal'tsev Meeting". Nov. 10–13, 2014. Collection of Abstracts*, p. 98, Novosibirsk, 2014. Sobolev Institute of Mathematics, Novosibirsk State University. Online: http://www.math.nsc.ru/conference/malmeet/14/Malmeet2014.pdf

[32] A. Yu. Vasil'eva. Completely regular codes in the triangular grid. *Proceedings of MIPT*, 14(2):14–42, 2022. In Russian.

A Appendix. Wallpaper groups and orbit colorings

The concept of orbit coloring plays a fundamental role in our study. Here, we briefly mention the classification of subgroups of the automorphism group of $G(\mathbb{Z}^2)$ of finite index (i.e., with finite number of orbits), mainly focused on subgroups whose orbit partitions fit the hypothesis and conclusion of Theorem 1. The set of nodes of $G(\mathbb{Z}^2)$ can be naturally treated as a set points of the Euclidean plain, forming a square lattice. Every automorphism of the square grid is continued to an isometry of the Euclidean plain. A group of isometries of the Euclidean plain whose translation subgroup is additively spanned by two linearly independent vectors (any group of automorphisms of $G(\mathbb{Z}^2)$ of finite index satisfies this condition) is called a *wallpaper group*. It is known that the wallpaper groups are divided into 17 families (see, e.g., [10, Ch. 2]), where the groups in the same family are conjugate to each other in the group of affine transformations of the plane. Of the 17 families, only 12 include subgroups of the automorphism group of $G(\mathbb{Z}^2)$ (the groups from the remaining 5 families contain a 120°-rotation, and hence cannot be such a subgroup). The groups are parameterized with three vectors $\bar{u}$, $\bar{v}$, and $\bar{o}$. The vectors $\bar{u}$ and $\bar{v}$ are periods that generate the translation subgroup; $\bar{o}$ is some point of the plain that defines the positions of the rotation centers and/or reflection (glide reflection) axes (some groups do not depend on $\bar{o}$ or depend only on one coordinate component of it).

There are three kinds of non-translation elements in wallpaper groups: rotations; reflections; glide reflections (a glide reflection is the composition of a reflection and a translation along the reflection axe).

The group of all automorphisms of $G(\mathbb{Z}^2)$ is a wallpaper group of the type called *p4m*. All its finite-index subgroups are listed in [30], as well as the wallpaper subgroups of each of the 17 wallpaper groups.

It is easy to see that if a group of automorphisms of $G(\mathbb{Z}^2)$ contains a rotation or a reflection, then there is an orbit of nodes with the minimum distance 1 or 2 between different elements. This means that the quotient matrix contains a non-zero element on the main diagonal or an element larger than 1 anywhere. Hence, such orbit coloring cannot be a covering of a simple graph. Below, we describe the two families (named *p1* and *pg*) of wallpaper groups that do not have reflections and rotations. The description of each group is followed (after a bullet) by the restrictions on the parameters that are necessary and sufficient for the group to be a subgroup of the automorphism group of $G(\mathbb{Z}^2)$.

p1. The group does not have non-translation elements.

- For a *p1*-group to consist of automorphisms of $G(\mathbb{Z}^2)$, the translations must be integer. For the orbit coloring of the nodes of $G(\mathbb{Z}^2)$ to be a covering, all translations must be different from $[0,\pm 1]$, $[0,\pm 2]$, $[\pm 1,\pm 1]$. The orbit coloring is bipartite if and only if all translations are even.

pg. The translation subgroup is generated by two perpendicular vectors $\bar{u}$ and $\bar{v}$; the non-translation elements are glide reflections with axes along $\bar{u}$ through the points $\bar{o}+\frac{j}{2}\bar{v}$, $j \in \mathbb{Z}$.

- There are two possibilities for a pg-group to consist of automorphisms of $G(\mathbb{Z}^2)$.

  (a) Diagonal glide reflections: $\bar{u} = [s,s]$, $\bar{v} = [t,-t]$ or $\bar{u} = [s,-s]$, $\bar{v} = [t,t]$, $s,t \in \mathbb{Z}^+$. For the orbit coloring of the nodes of $G(\mathbb{Z}^2)$ to be a covering, s must be at least 3 and t at least 2. There are additional restrictions:

  (a0) if s is even, then $\bar{o}$ is either integer or has two non-integer coordinate components (and hence the reflection axes contain integer points); the orbit coloring is bipartite in this case;

  (a1) if s is odd, then $\bar{o}$ has exactly one integer component (and hence the reflection axes do not contain integer points); the orbit coloring is not bipartite.

  (b) Horizontal or vertical reflections: $\bar{u} = [0,2s]$, $\bar{v} = [t,0]$ or $\bar{u} = [2s,0]$, $\bar{v} = [0,t]$, $s,t \in \mathbb{Z}^+$. Case (b) is divided into three subcases:

  (b0) all reflection axes contain integer points (even t);

  (b1) there are reflection axes both containing and not containing integer points (odd t);

  (b2) none of reflection axes contains an integer point (even t).

  Only subcase (b0) with even s and subcase (b2) with odd s correspond to bipartite orbit colorings. For the orbit coloring of the nodes of $G(\mathbb{Z}^2)$ to be a covering, t must be at least 3 and s at least 2 for (b0) and (b1) and at least 1 for (b2).

B Appendix. Perfect colorings with twin colors

Here we describe 23 types of perfect colorings that, according to Theorem 2, exhaust the bipartite perfect of $G(\mathbb{Z}^2)$ with twin colors. Each type is represented with the coloring shown in one of Fig. A–W, and the other colorings of the same type are obtained by applying automorphisms of $G(\mathbb{Z}^2)$, renaming colors, shifting binary diagonals, and/or merging groups of twin colors.

Fig. K–W show infinite parameterized series, where $n = m+1 = l+2 = \ldots = c+11$ is the number of colors. The arrows show the direction in which the colors increase with increment 2 (Fig. K–L), 5 (M–S), or 4 (T–W); the number of steps depends on n. Each of Fig. K–W is accompanied with an analytic description of the corresponding coloring.

To simplify some descriptions, we introduce three colorings with infinite number of colors. The coloring $\mathcal{F}_{23}$, $\mathcal{F}_{32}$, and $\mathcal{F}_{44}$ (partially illustrated in Fig. M–P, Fig. Q–S, and Fig. T–W, respectively) are defined by the period $[-4,4]$ for $\mathcal{F}_{23}$, $\mathcal{F}_{32}$ and $[4,4]$ for $\mathcal{F}_{44}$ and the following equalities for each $\alpha \in \{1,2,3,\ldots\}$, $\gamma \in \{0,1,2,3\}$:

$$\mathcal{F}_{23}([0,0]) = \mathcal{F}_{23}([-2,2]) = \boxed{1}, \qquad \mathcal{F}_{23}([-1,1]) = \mathcal{F}_{23}([-3,3]) = \boxed{2},$$

$$\mathcal{F}_{23}\big(\pm[\alpha+\gamma-1, \alpha-\gamma]\big) = \boxed{5\alpha - \tfrac{1}{2} - |\gamma - \tfrac{3}{2}|},$$

$$\mathcal{F}_{23}\big(\pm[\alpha+\gamma+1, \alpha-\gamma-1]\big) = \boxed{5\alpha + |2-\gamma|}.$$

$$\mathcal{F}_{32}([0,0]) = \mathcal{F}_{32}([-2,2]) = \boxed{1}, \qquad \mathcal{F}_{32}([-1,1]) = \mathcal{F}_{32}([-3,3]) = \boxed{2},$$

$$\mathcal{F}_{32}\big(\,[\tfrac{1}{2},-\tfrac{1}{2}] \pm [\alpha+\beta-\tfrac{3}{2}, \alpha-\beta+\tfrac{1}{2}]\,\big) = \boxed{5\alpha - |\gamma-2|},$$

$$\mathcal{F}_{32}\big(\,[\tfrac{1}{2},-\tfrac{1}{2}] \pm [\alpha+\beta-\tfrac{1}{2}, \alpha-\beta+\tfrac{1}{2}]\,\big) = \boxed{5\alpha + \tfrac{5}{2} - |\gamma-\tfrac{3}{2}|}.$$

$$\mathcal{F}_{44}([0,0]) = \mathcal{F}_{44}([2,2]) = \boxed{1}, \qquad \mathcal{F}_{44}([1,1]) = \mathcal{F}_{44}([3,3]) = \boxed{2},$$

$$\mathcal{F}_{44}([\alpha+\gamma,\gamma]) = \mathcal{F}_{44}([2+\gamma,\alpha+2+\gamma]) = \boxed{4\alpha+\gamma-1}.$$

9	8	4	7	9	7	4	8	9	8	4	7	9	7	4	8
7	5	3	6	8	5	2	6	7	5	3	6	8	5	2	6
4	2	1	2	4	3	1	3	4	2	1	2	4	3	1	3
8	6	3	5	7	6	2	5	8	6	3	5	7	6	2	5
9	7	4	8	9	8	4	7	9	7	4	8	9	8	4	7
8	5	2	6	7	5	3	6	8	5	2	6	7	5	3	6
4	3	1	3	4	2	1	2	4	3	1	3	4	2	1	2
7	6	2	5	8	6	3	5	7	6	2	5	8	6	3	5
9	8	4	7	9	7	4	8	9	8	4	7	9	7	4	8
7	5	3	6	8	5	2	6	7	5	3	6	8	5	2	6
4	2	1	2	4	3	1	3	4	2	1	2	4	3	1	3
7	6	2	5	8	6	3	5	7	6	2	5	8	6	3	5

	2	2						
1			1	1	1			
1			1	1	1			
	1	1				1	1	
	1	1				1	1	
	1	1				1	1	
			1	1	1			1
			1	1	1			1
						2	2	

Fig. A (Sunflower field). Twins: $\boxed{2}, \boxed{3}$; $\boxed{4}, \boxed{5}, \boxed{6}$; $\boxed{7}, \boxed{8}$. Periods: $[8,0]$, $[4,4]$.

9	7	4	8	10	8	4	7	9	7	4	8	10	8	4	7
7	5	3	6	8	6	2	5	7	5	3	6	8	6	2	5
4	2	1	2	4	3	1	3	4	2	1	2	4	3	1	3
8	6	3	5	7	5	2	6	8	6	3	5	7	5	2	6
10	8	4	7	9	7	4	8	10	8	4	7	9	7	4	8
8	6	2	5	7	5	3	6	8	6	2	5	7	5	3	6
4	3	1	3	4	2	1	2	4	3	1	3	4	2	1	2
7	5	2	6	8	6	3	5	7	5	2	6	8	6	3	5
9	7	4	8	10	8	4	7	9	7	4	8	10	8	4	7
7	5	3	6	8	6	2	5	7	5	3	6	8	6	2	5
4	2	1	2	4	3	1	3	4	2	1	2	4	3	1	3
7	5	2	6	8	6	3	5	7	5	2	6	8	6	3	5

	2	2							
1			1	1	1				
1			1	1	1				
	1	1				1	1		
	1	1				2	0		
	1	1				0	2		
			1	2	0			1	0
			1	0	2			0	1
						4	0		
						0	4		

Fig. B (Poppy field). Twins: $\boxed{2}, \boxed{3}$. Periods: $[8,0]$, $[4,4]$.

1	7	2	8	1	7	2	8	1	7	2	8	1	7	2	8	1	7
5	3	5	3	5	3	5	3	5	3	5	3	5	3	5	3	5	3
2	8	1	7	2	8	1	7	2	8	1	7	2	8	1	7	2	8
6	4	6	4	6	4	6	4	6	4	6	4	6	4	6	4	6	4
1	7	2	8	1	7	2	8	1	7	2	8	1	7	2	8	1	7
5	3	5	3	5	3	5	3	5	3	5	3	5	3	5	3	5	3
2	8	1	7	2	8	1	7	2	8	1	7	2	8	1	7	2	8
6	4	6	4	6	4	6	4	6	4	6	4	6	4	6	4	6	4
1	7	2	8	1	7	2	8	1	7	2	8	1	7	2	8	1	7
5	3	5	3	5	3	5	3	5	3	5	3	5	3	5	3	5	3

				1	1	1	1
				1	1	1	1
				2	0	1	1
				0	2	1	1
1	1	2	0				
1	1	0	2				
1	1	1	1				
1	1	1	1				

Fig. C (Mountain lake). Twins: $\boxed{1}, \boxed{2}$; $\boxed{7}, \boxed{8}$. Periods: $[4, 0]$, $[0, 4]$.
The quotient matrix is similar to that in Fig. J.

1	7	2	8	1	7	2	8	1	7	2	8	1	7	2	8	1	7
6	3	6	3	6	3	6	3	6	3	6	3	6	3	6	3	6	3
2	8	1	7	2	8	1	7	2	8	1	7	2	8	1	7	2	8
9	5	9	4	9	5	9	4	9	5	9	4	9	5	9	4	9	5
1	8	2	7	1	8	2	7	1	8	2	7	1	8	2	7	1	8
6	3	6	3	6	3	6	3	6	3	6	3	6	3	6	3	6	3
2	7	1	8	2	7	1	8	2	7	1	8	2	7	1	8	2	7
9	4	9	5	9	4	9	5	9	4	9	5	9	4	9	5	9	4
1	7	2	8	1	7	2	8	1	7	2	8	1	7	2	8	1	7
6	3	6	3	6	3	6	3	6	3	6	3	6	3	6	3	6	3

					1	1	1	1
					1	1	1	1
					2	1	1	0
					0	2	0	2
					0	0	2	2
1	1	2	0	0				
1	1	1	1	0				
1	1	1	0	1				
1	1	0	1	1				

Fig. D (Beach). Twins: $\boxed{1}, \boxed{2}$. Periods: $[4, 0]$, $[0, 8]$.

6	1	4	2	6	1	4	2	6	1	4	2	6	1	4	2	6	1
10	5	9	5	10	5	9	5	10	5	9	5	10	5	9	5	10	5
6	2	4	1	6	2	4	1	6	2	4	1	6	2	4	1	6	2
7	3	8	3	7	3	8	3	7	3	8	3	7	3	8	3	7	3
6	1	4	2	6	1	4	2	6	1	4	2	6	1	4	2	6	1
10	5	9	5	10	5	9	5	10	5	9	5	10	5	9	5	10	5
6	2	4	1	6	2	4	1	6	2	4	1	6	2	4	1	6	2
7	3	8	3	7	3	8	3	7	3	8	3	7	3	8	3	7	3
6	1	4	2	6	1	4	2	6	1	4	2	6	1	4	2	6	1
10	5	9	5	10	5	9	5	10	5	9	5	10	5	9	5	10	5

		1	1	1	1				
		1	1	1	1				
1	1					1	1	0	0
1	1					0	1	1	0
1	1					0	0	1	1
1	1					1	0	0	1
		2	0	0	2				
		2	2	0	0				
		0	2	2	0				
		0	0	2	2				

Fig. E (Spring). Twins: $\boxed{1}, \boxed{2}$. Periods: $[4, 0]$, $[0, 4]$.
The quotient matrix is similar to that in Fig. W($n = 10$).

6	4	7	3	8	2	6	4	7	3	8	2	6	4	7	3
4	5	4	8	1	8	4	5	4	8	1	8	4	5	4	8
7	4	6	2	8	3	7	4	6	2	8	3	7	4	6	2
2	8	3	7	4	6	2	8	3	7	4	6	2	8	3	7
8	1	8	4	5	4	8	1	8	4	5	4	8	1	8	4
3	8	2	6	4	7	3	8	2	6	4	7	3	8	2	6
6	4	7	3	8	2	6	4	7	3	8	2	6	4	7	3
4	5	4	8	1	8	4	5	4	8	1	8	4	5	4	8
7	4	6	2	8	3	7	4	6	2	8	3	7	4	6	2
2	8	3	7	4	6	2	8	3	7	4	6	2	8	3	7
8	1	8	4	5	4	8	1	8	4	5	4	8	1	8	4

				0	0	0	4
				0	1	1	2
				0	1	1	2
				1	1	1	1
0	0	0	4				
0	1	1	2				
0	1	1	2				
1	1	1	1				

Fig. F (Street basketball). Twins: $\boxed{2}, \boxed{3}$; $\boxed{6}, \boxed{7}$. Periods: $[6, 0]$, $[0, 6]$.

8	5	7	3	10	2	8	5	7	3	10	2	8	5	7	3
4	6	4	9	1	9	4	6	4	9	1	9	4	6	4	9
7	5	8	2	10	3	7	5	8	2	10	3	7	5	8	2
2	10	3	7	5	8	2	10	3	7	5	8	2	10	3	7
9	1	9	4	6	4	9	1	9	4	6	4	9	1	9	4
3	10	2	8	5	7	3	10	2	8	5	7	3	10	2	8
8	5	7	3	10	2	8	5	7	3	10	2	8	5	7	3
4	6	4	9	1	9	4	6	4	9	1	9	4	6	4	9
7	5	8	2	10	3	7	5	8	2	10	3	7	5	8	2
2	10	3	7	5	8	2	10	3	7	5	8	2	10	3	7
9	1	9	4	6	4	9	1	9	4	6	4	9	1	9	4

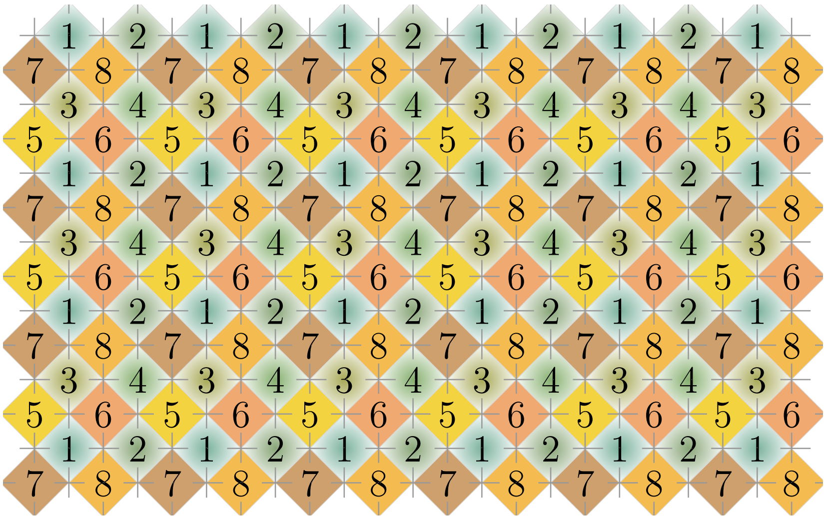
					0	0	0	2	2
					0	1	1	1	1
					0	1	1	1	1
					1	1	1	1	0
					1	1	1	0	1
0	0	0	2	2					
0	1	1	1	1					
0	1	1	1	1					
1	1	1	1	0					
1	1	1	0	1					

Fig. G (Beach volleyball). Twins: $\boxed{2}, \boxed{3}$; $\boxed{7}, \boxed{8}$. Periods: $[6, 0]$, $[0, 6]$.

10	5	10	3	8	2	10	5	10	3	8	2	10	5	10	3
5	9	5	11	1	11	5	9	5	11	1	11	5	9	5	11
10	5	10	2	8	3	10	5	10	2	8	3	10	5	10	2
2	11	3	7	4	7	2	11	3	7	4	7	2	11	3	7
8	1	8	4	6	4	8	1	8	4	6	4	8	1	8	4
3	11	2	7	4	7	3	11	2	7	4	7	3	11	2	7
10	5	10	3	8	2	10	5	10	3	8	2	10	5	10	3
5	9	5	11	1	11	5	9	5	11	1	11	5	9	5	11
10	5	10	2	8	3	10	5	10	2	8	3	10	5	10	2
2	11	3	7	4	7	2	11	3	7	4	7	2	11	3	7
8	1	8	4	6	4	8	1	8	4	6	4	8	1	8	4

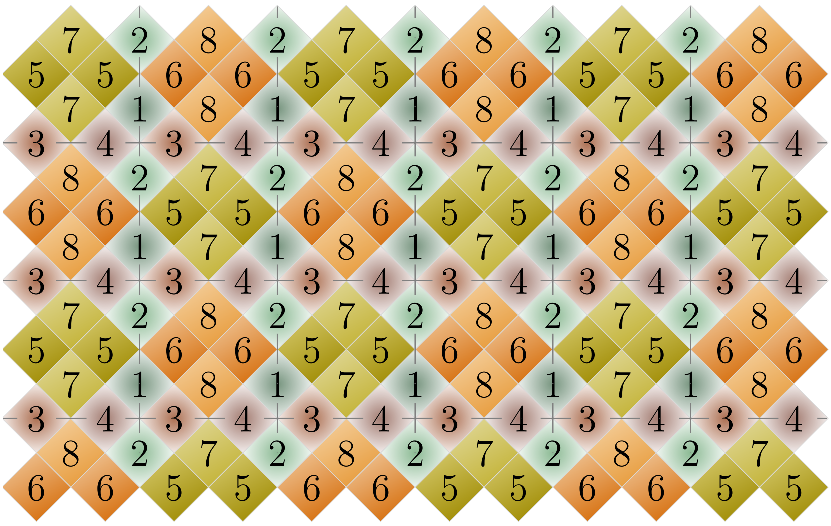
					0	0	2	0	0	2
					0	1	1	0	1	1
					0	1	1	0	1	1
					1	2	1	0	0	0
					0	0	0	1	2	1
0	0	0	4	0						
0	1	1	2	0						
1	1	1	1	0						
0	0	0	0	4						
0	1	1	0	1						
1	1	1	0	1						

Fig. H (Football). Twins: $\boxed{2}, \boxed{3}$. Periods: $[6, 0]$, $[0, 6]$.



				1	1	1	1
				1	1	1	1
				1	1	1	1
				1	1	1	1
1	1	1	1				
1	1	1	1				
1	1	1	1				
1	1	1	1				

Fig. I (Fall). Twins: 1, 2, 3, 4; 5, 6, 7, 8. Periods: $[2, 2]$, $[2, -2]$.



		1	1	1	1		
		1	1	1	1		
1	1					1	1
1	1					1	1
1	1					2	0
1	1					0	2
		1	1	2	0		
		1	1	0	2		

Fig. J (Apples & oranges). Twins: 1, 2; 3, 4. Periods: $[4, 0]$, $[0, 4]$.
The quotient matrix is similar to that in Fig. C.

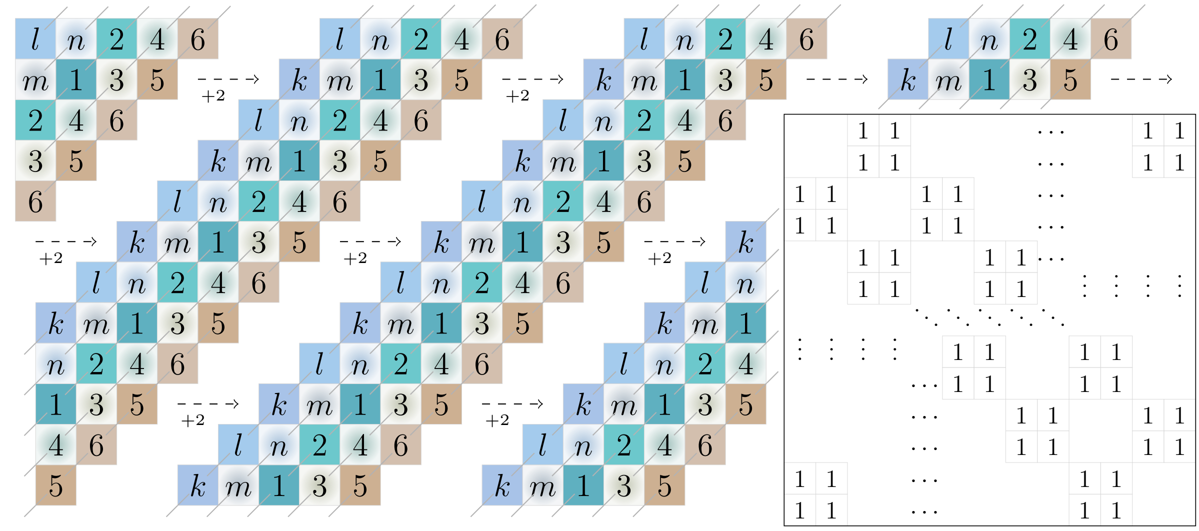


Figure K: $n = 4(\mathrm{I}), 8(\mathrm{I}), 12, 16, 20, \ldots$. Twins: $\boxed{1}, \boxed{2}$; $\boxed{3}, \boxed{4}$; $\boxed{5}, \boxed{6}$, ... Periods: $[2,2]$, $\left[\frac{n}{2}, 0\right]$.

$$\mathcal{F}([\alpha+\beta, \beta]) = \boxed{2\alpha+\beta+1}, \quad \alpha = 0, 1, \ldots, \tfrac{n}{2}-1, \ \beta = 0, 1.$$

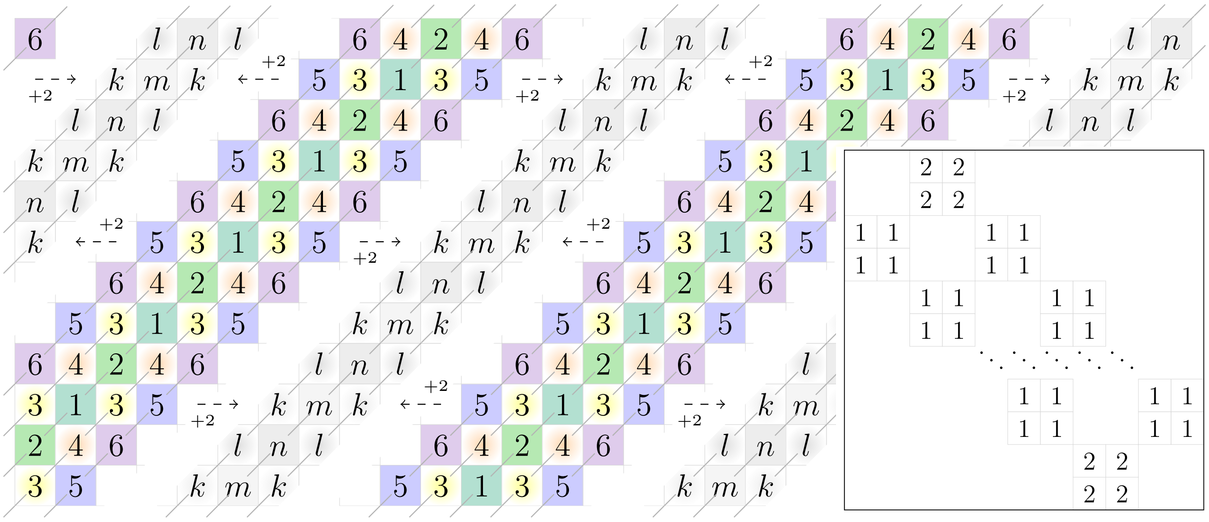


Figure L: $n = 4(\mathrm{I}), 6(\mathrm{I}), 8, 10, 12, \ldots$. Twins: $\boxed{1}, \boxed{2}$; $\boxed{3}, \boxed{4}$; $\boxed{5}, \boxed{6}$; .... Periods: $[2,2]$, $[n-2, 0]$.

$$\mathcal{F}([\pm\alpha+\beta, \beta]) = \boxed{2\alpha+\beta+1}, \quad \alpha = 0, 1, \ldots, \tfrac{n}{2}-1, \ \beta = 0, 1.$$

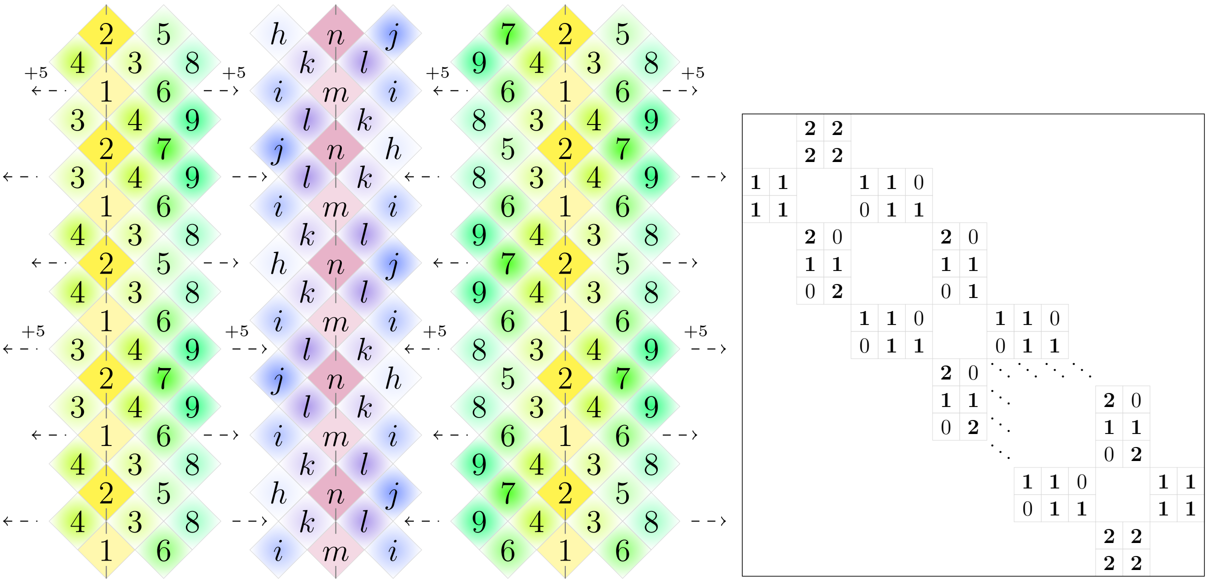


Figure M: $n = 6(\mathrm{I}), 11, 16, 21, 26, \ldots$. Twins: $\boxed{1}, \boxed{2}$; $\boxed{m}, \boxed{n}$. Periods: $[-4, 4]$, $\left[\frac{2n-2}{5}, \frac{2n-2}{5}\right]$.

$$\mathcal{F}([x,y]) = \mathcal{F}_{23}([x,y]),\ \ |x+y| < \tfrac{2n-2}{5},$$
$$\mathcal{F}([\tfrac{n-1}{5}-2z-\beta,\ \tfrac{n-1}{5}+2z+\beta]) = \boxed{n-1+\beta},\ \ z \in \mathbb{Z}, \beta \in \{0,1\}.$$

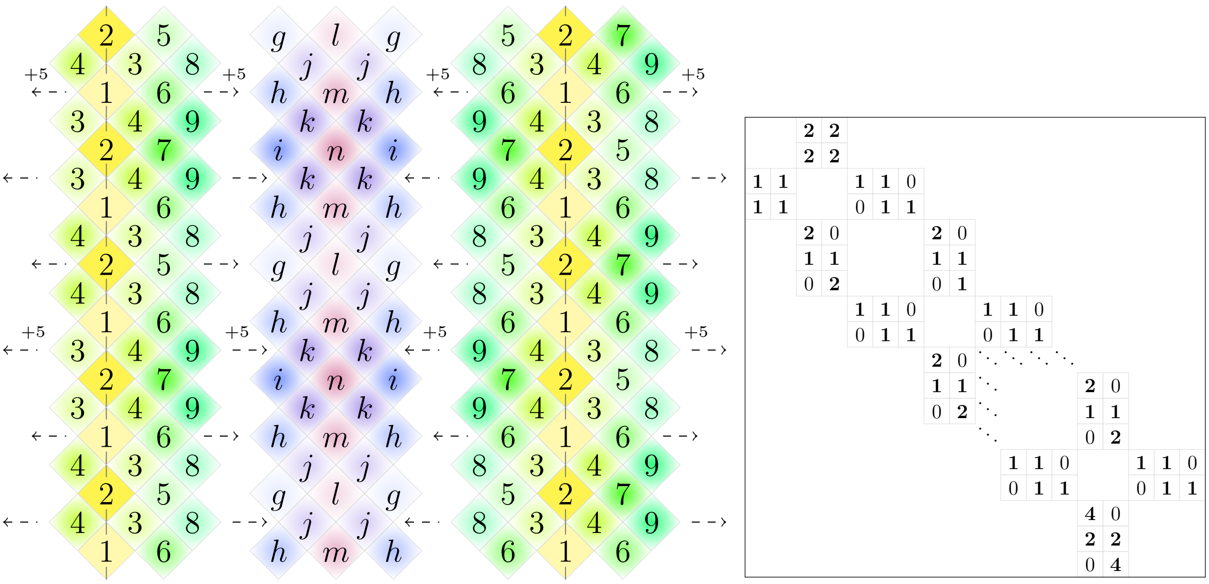


Figure N: $n = 7(\mathrm{A}), 12, 17, 22, 27, \ldots$. Twins: $\boxed{1}, \boxed{2}$. Periods: $[-4, 4]$, $\left[\frac{2n-4}{5} - 2, \frac{2n-4}{5} + 2\right]$.

$$\mathcal{F}([x,y]) = \mathcal{F}_{23}([x,y]),\ \ |x+y| \leqslant \tfrac{2n-4}{5}.$$

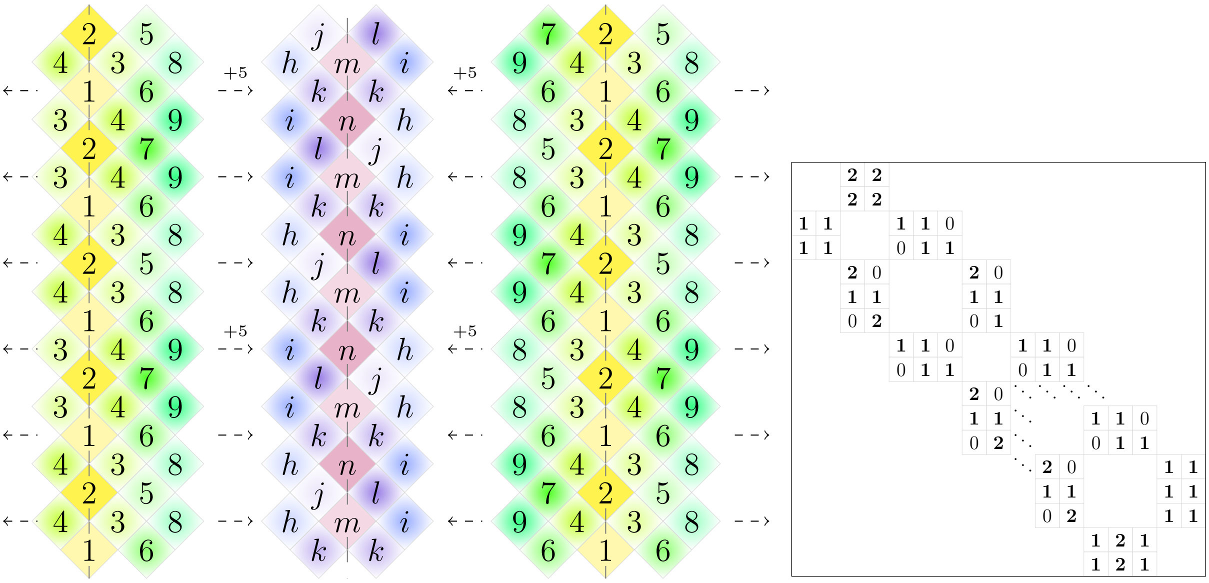


Figure O: $n = 4(\mathrm{I}), 9, 14, 19, 24, \ldots$. Twins: $\boxed{1}, \boxed{2}$; $\boxed{m}, \boxed{n}$. Periods: $[-4, 4]$, $\left[\frac{2n-3}{5}, \frac{2n-3}{5}\right]$.

$$\mathcal{F}([x,y]) = \mathcal{F}_{23}([x,y]), \ |x+y| < \tfrac{2n-3}{5},$$

$$\mathcal{F}([\tfrac{n-4}{5}-2z-\beta, \ \tfrac{n+1}{5}+2z+\beta]) = \boxed{n-1+\beta}, \ z \in \mathbb{Z}, \beta \in \{0,1\}.$$

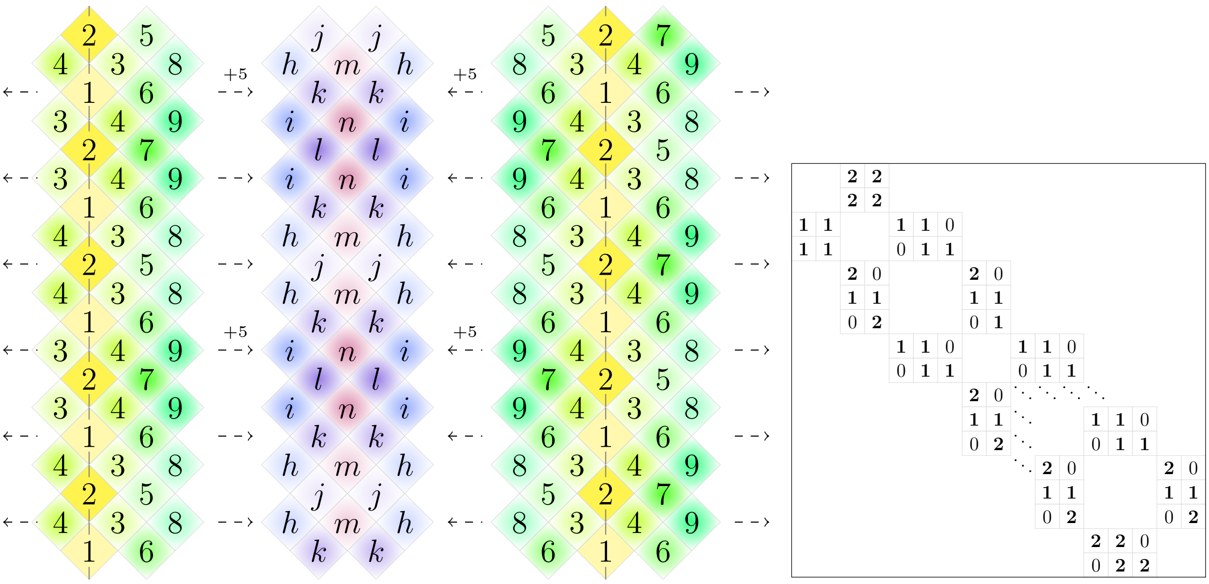


Figure P: $n = 4(\mathrm{I}), 9, 14, 19, 24, \ldots$. Twins: $\boxed{1}, \boxed{2}$. Periods: $[-4, 4]$, $\left[\frac{2n-3}{5} - 2, \frac{2n-3}{5} + 2\right]$.

$$\mathcal{F}([x,y]) = \mathcal{F}_{23}([x,y]), \ |x+y| \leqslant \tfrac{2n-3}{5}.$$

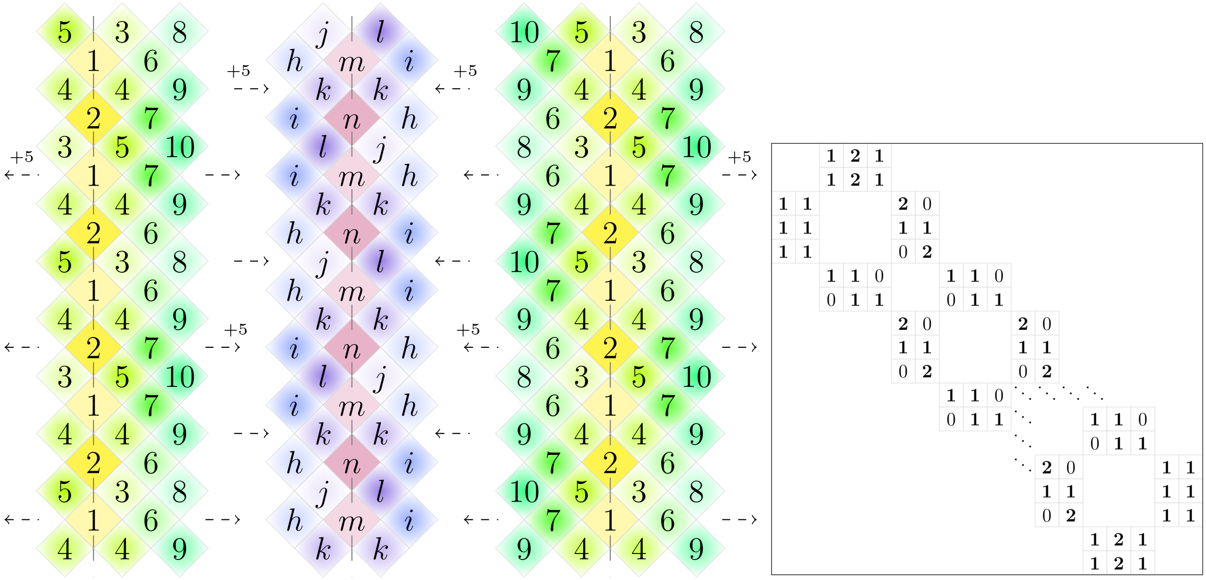


Figure Q: $n = 7(\mathrm{I}), 12, 17, 22, 27, \ldots$. Twins: $\boxed{1}, \boxed{2}$; $\boxed{m}, \boxed{n}$. Periods: $[-4, 4]$, $\left[\frac{2n-4}{5}, \frac{2n-4}{5}\right]$.

$$\mathcal{F}([x,y]) = \mathcal{F}_{32}([x,y]),\ |x+y| < \tfrac{2n-4}{5},$$

$$\mathcal{F}\big([\tfrac{n-2}{5}-2z-\beta,\ \tfrac{n-2}{5}+2z+\beta]\big) = \boxed{n-1+\beta},\ z \in \mathbb{Z}, \beta \in \{0,1\}.$$

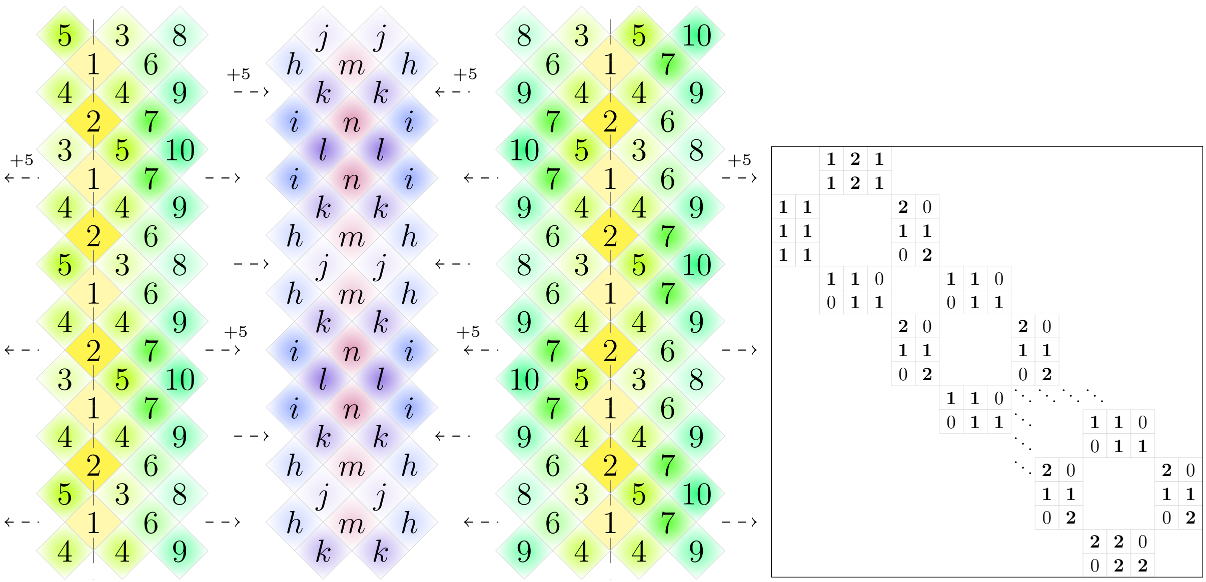


Figure R: $n = 7(\mathrm{J}), 12, 17, 22, 27, \ldots$. Twins: $\boxed{1}, \boxed{2}$. Periods: $[-4, 4]$, $\left[\frac{2n-4}{5} - 2, \frac{2n-4}{5} + 2\right]$.

$$\mathcal{F}([x,y]) = \mathcal{F}_{32}([x,y]),\ |x+y| \leqslant \tfrac{2n-4}{5}.$$

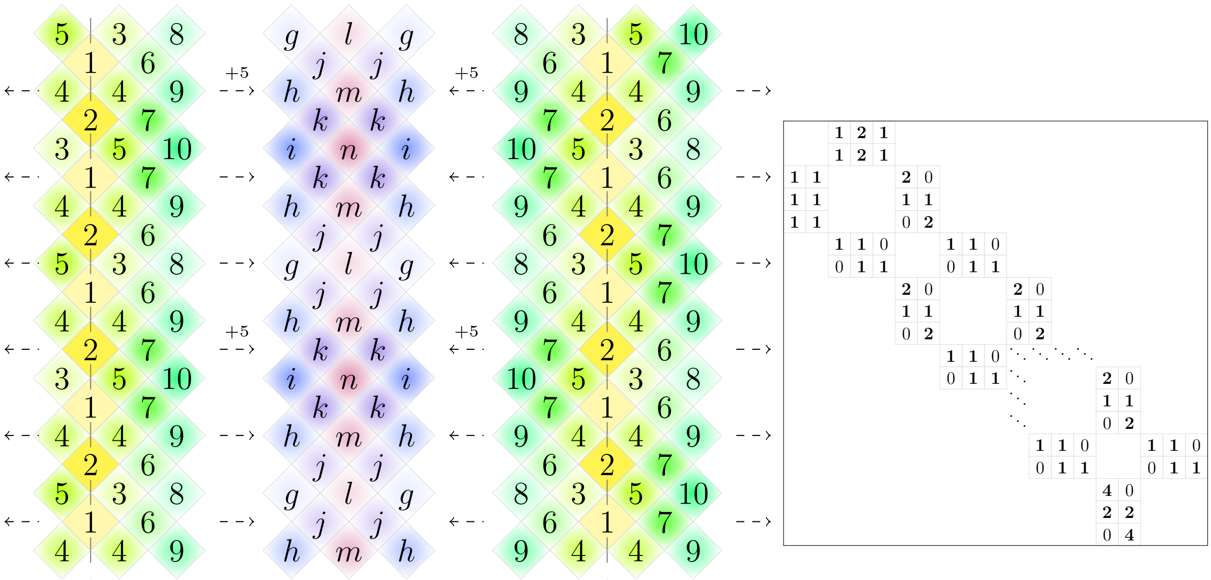


Figure S: $n = 5(\mathrm{I}), 10, 15, 20, 25, \ldots$. Twins: $\boxed{1}, \boxed{2}$. Periods: $[-4, 4]$, $\left[\frac{2n-5}{5} - 2, \frac{2n-5}{5} + 2\right]$.

$$\mathcal{F}([x, y]) = \mathcal{F}_{32}([x, y]),\ \ |x + y| \leqslant \tfrac{2n-5}{5}.$$

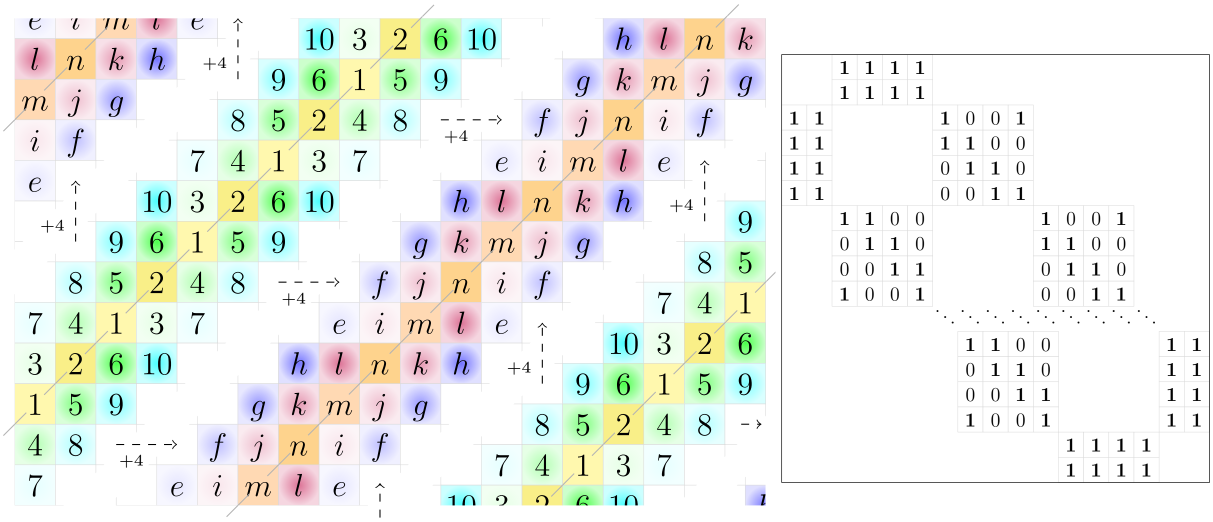


Figure T: $n = 4(\mathrm{I}), 8(\mathrm{I}), 12, 16, 20, \ldots$. Twins: $\boxed{1}, \boxed{2}$; $\boxed{m}, \boxed{n}$. Periods: $[4, 4]$, $\left[\frac{n}{4}, -\frac{n}{4}\right]$.

$$\mathcal{F}([x, y]) = \mathcal{F}_{44}([x, y]),\ \ |x - y| < \tfrac{n}{4},$$

$$\mathcal{F}\big(\left[\tfrac{n}{4}{+}2z{+}\beta,\ 2z{+}\beta\right]\big) = \boxed{n{-}1{+}\beta},\ \ z \in \mathbb{Z}, \beta \in \{0, 1\}.$$

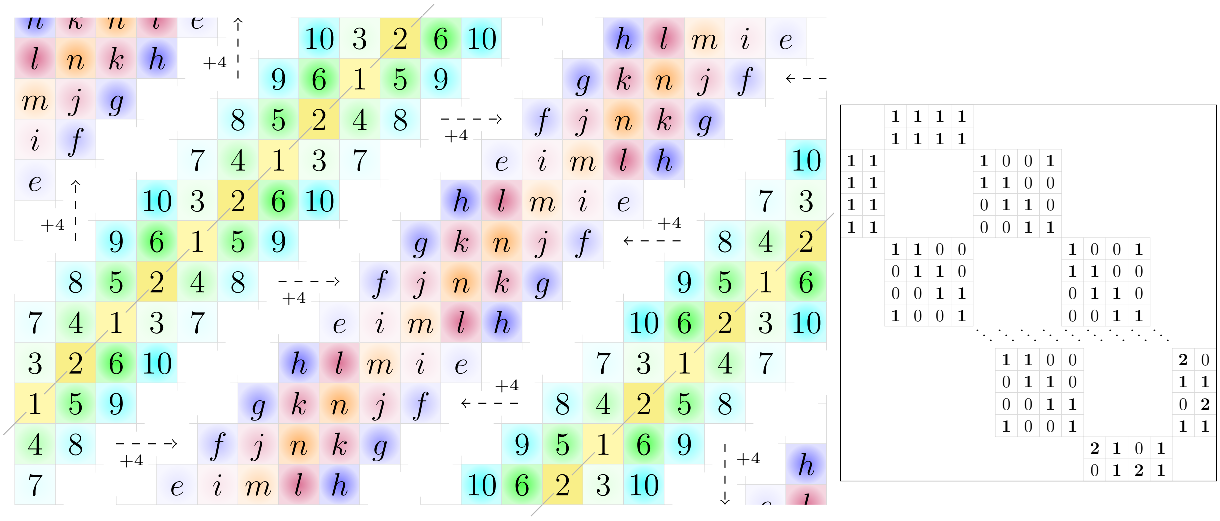


Figure U: $n = 4(\text{I}), 8(\text{J}), 12, 16, 20, \ldots$. Twins: $\boxed{1}, \boxed{2}$. Periods: $[4,4]$, $\left[\frac{n}{2}, -\frac{n}{2}\right]$.

$$\mathcal{F}([x,y]) = \mathcal{F}([\tfrac{n}{2}+3-x, 3-y]) = \mathcal{F}_{44}([x,y]),\ |x-y| < \tfrac{n}{4},$$

$$\mathcal{F}\big([\tfrac{n}{4}+4z+2+\gamma, 4z+2+\gamma]\big) = \mathcal{F}\big([4z+\gamma, \tfrac{n}{4}+4z+\gamma]\big) = \boxed{n-\tfrac{3}{2}+|\gamma-\tfrac{3}{2}|},\ z \in \mathbb{Z}, \gamma \in \{0,1,2,3\}.$$

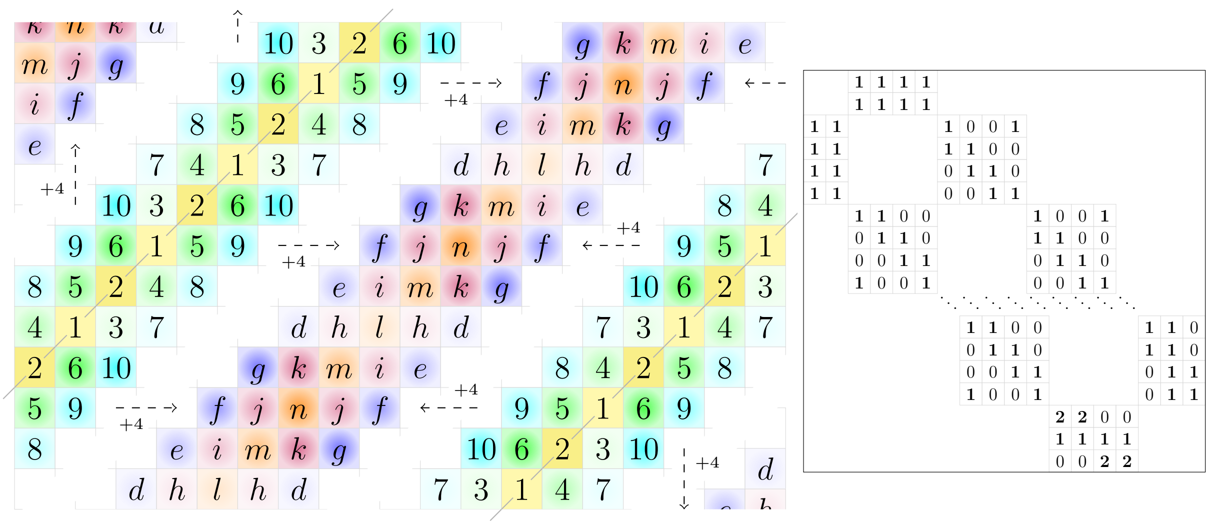


Figure V: $n = 5(\text{I}), 9(\text{A}), 13, 17, 21, \ldots$. Twins: $\boxed{1}, \boxed{2}$. Periods: $[4,4]$, $\left[\frac{n-1}{2}, -\frac{n-1}{2}\right]$.

$$\mathcal{F}([x,y]) = \mathcal{F}([\tfrac{n-1}{2}-x, -y]) = \mathcal{F}_{44}([x,y]),\ |x-y| < \tfrac{n-1}{4},$$

$$\mathcal{F}\big([\tfrac{n-1}{4}+4z+\gamma, 4z+\gamma]\big) = \mathcal{F}\big([4z+2+\gamma, \tfrac{n-1}{4}+4z+2+\gamma]\big) = \boxed{n-|2-\gamma|},$$

$$z \in \mathbb{Z}, \gamma \in \{0,1,2,3\}.$$

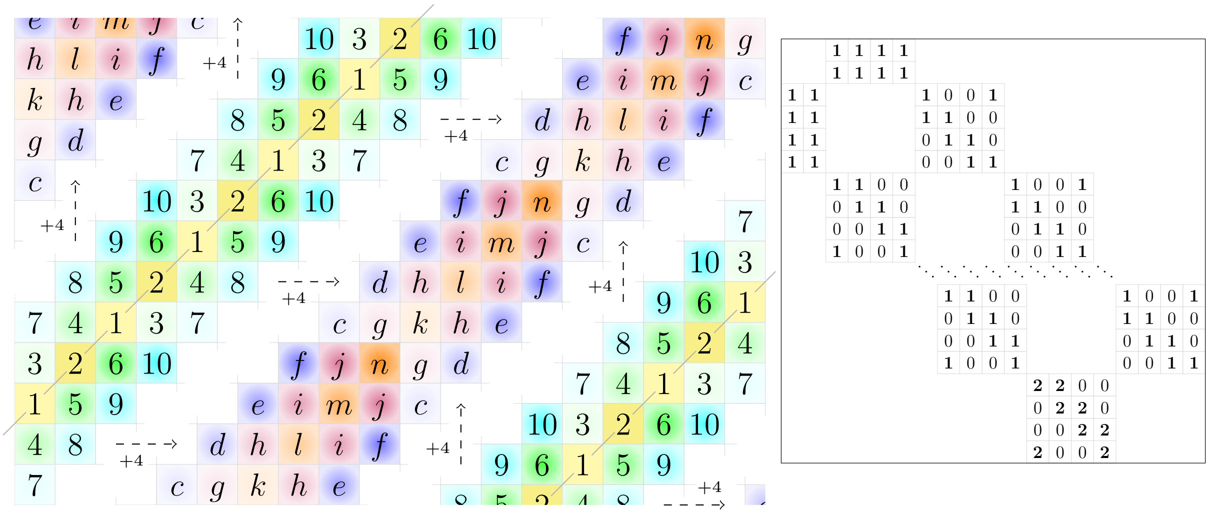


Figure W: $n = 6(\mathrm{I}), 10, 14, 18, \ldots$. Twins: $\boxed{1}, \boxed{2}$. Periods: $[4,4]$, $\left[2+\frac{n-2}{4}, 2-\frac{n-2}{4}\right]$.
$\mathcal{F}([x,y]) = \mathcal{F}_{44}([x,y]),\ |x-y| \leqslant \frac{n-2}{4}$.