On the minimum bisection of random 3-regular graphs

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Abstract

In this paper we give new bounds on the bisection width of random 3-regular graphs on n vertices. The main contribution is a new lower bound of $0.103295n$ based on a first moment method together with a structural analysis of the graph, thereby improving a 27-year-old result of Kostochka and Melnikov. We also give a complementary upper bound of $0.139822n$ by combining a result of Lyons with original combinatorial insights. Developping this approach further, we obtain a non-rigorous improved upper bound with the help of Monte Carlo simulations.

Mathematics Subject Classifications: 05C80, 68R10, 05D40, 05C30

1 Introduction

Given a graph $G = (V, E)$ with even number of vertices (denoted by n), a *bisection of* G is a partition of the vertex set of G into two equal parts, that is, subsets of equal sizes. A bisection is a *minimum bisection* if the number of edges having endpoints in different

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parts is minimized. The *bisection width* of a graph G, denoted by $bw(G)$, is the number of edges going between A and B in a minimum bisection.

The minimum bisection problem has received a lot of attention in mathematics, theoretical computer science and physics due to its applications in a number of graph layout and embedding problems such as the routing performance of a network [15]. The algorithmic problem of finding a minimum bisection is well known to be NP-complete in general [22] and even in the particular case of 3-regular graphs [7]. Moreover, even approximating the bisection width up to a constant factor is hard (see for example [29] and the references therein). On the positive side, $O(\log^2 n)$-approximation algorithms in polynomial time exist [18], and exact polynomial time algorithms are available for graphs of bounded treewidth [25].

In the context of random regular graphs, the minimum bisection problem has also received quite a bit of attention. In this setting, one usually assumes that the number of vertices tends to infinity, and the following results all hold only with probability tending to 1 as $n \to \infty$.

Regarding the bisection width of random 3-regular graphs, the first lower bound of $\frac{1}{11}n \approx 0.0909n$ was given by Bollobás [4], and this was later improved by Kostochka and Melnikov to $0.10101n$ [31]. Since then, during the last 27 years, to the best of our knowledge, no further improvements have been made. On the other hand, Kostochka and Melnikov [30] proved an upper bound of $\frac{1}{4}n + o(n)$ on the bisection width of any 3-regular graph, which was later improved to $\frac{1}{6}n$ for all sufficiently large 3-regular graphs by Monien and Preis [34]. For random 3-regular graphs, a slightly weaker but simpler algorithmic upper bound of $0.1740n$ was given by Díaz, Do, Serna and Wormald [14]. This bound was then improved by Lyons [33] to $0.16226n$.

Concerning random d-regular graphs, the first lower bound for fixed $d \geqslant 3$ was given by Bollobás [4] who showed that the bisection width is at least $(\frac{d}{4} - \frac{\sqrt{d \log 2}}{2})n$. Independently, Clark and Entringer [8] observed that the bisection width is at least $(\frac{d}{4} + o_d(d))n$ as $d \to \infty$. On the other hand, Alon [1] provided an upper bound of $(\frac{d}{4} - c\sqrt{d})n$ for some sufficiently small positive constant c and any $d \geqslant 3$. To our knowledge, the currently best known upper bound for d with $5 \leqslant d \leqslant 12$ is due to Díaz, Serna and Wormald [16], while for all other $d \geqslant 3$, Lyons [33] gave an upper bound of $\frac{d}{2\pi} \arccos(\frac{2\sqrt{d-1}}{d})n < (\frac{d}{4} - \frac{\sqrt{d}}{\pi})n$. The case $d \to \infty$ was recently settled by Dembo, Montanari and Sen [13]. Therein, the authors showed that the bisection width of a random d-regular graph is $(\frac{d}{4} - P_* \sqrt{\frac{d}{2}} + o_d(\sqrt{d}))n$ where $P_* \approx 0.7632$ denotes the ground state energy of the Sherington-Kirkpatrick model.

For the Erdős-Rényi graph $G(n,p)$ with $p = \frac{c}{n}$ for some constant $c > 0$, Luczak and McDiarmid [32] identified a phase transition: they showed that for $c < \log 4$, the largest component has size less than $\frac{1}{2}n$ and the bisection width of $G(n,p)$ is 0, whereas for $c > \log 4$, the bisection width is already $\Omega(n)$. For denser graphs with $p = \frac{c}{n}$ and $c \to \infty$, Dembo, Montanari and Sen [13] later showed that the bisection width of $G(n,p)$ has value $\left(\frac{c}{4} - P_* \sqrt{\frac{c}{4}} + o_c(\sqrt{c})\right) n$ with P_* as above.

Regarding more general results, it is well known (at least since the 70's, see Fiedler [19]) that $\frac{\lambda_2}{4}n$ is a lower bound for the bisection width for any graph where λ_2 is the second

eigenvalue of the Laplacian of the graph. For random d-regular graphs, using Friedman's result [20], this translates into a lower bound of $(\frac{d}{4} - \frac{\sqrt{d-1}}{2})n$ (in particular, giving $0.0428n$ for random 3-regular graphs). Later, several improvements using spectral techniques have been made: for example, Bezroukov, Elsässer, Monien, Preis and Tillich [3] gave a lower bound of $0.082n$ on the bisection width of 3-regular Ramanujan graphs.

A different line of research focused on estimating the bisection width in a planted bisection model. More precisely, given an unknown partition of the vertex set into two sets of equal sizes (corresponding to a planted bisection), add an edge between two vertices of the same part with probability p^+, and add an edge between two vertices of different parts with probability $p^- < p^+$, independently for different edges. An asymptotic formula for the bisection width in this setup was found by Coja-Oghlan, Cooley, Kang and Skubch [9] when the difference between p^+ and p^- is sufficiently large. This result was further extended by Sen [35] (for more references on the planted bisection problem, see [23]). Closely related to the minimum bisection problem on random 3-regular graphs is also the prominent conjecture of Zdeborová and Boettcher [38] who conjectured that the bisection width of a random 3-regular graph is equal to the number of edges not crossing a maximum cut up to $o(n)$. Unfortunately, our paper does not shed light on this conjecture. Weak indications in its favor come from the fact that the upper bound on the bisection width in Theorem 1 is the same as the upper bound for the edges not crossing a maximum cut from [21]. For recent advances on the maximum cut of random 3-regular graphs, see also [10].

In this paper we focus on improving the results on random 3-regular graphs. Denote by $\mathcal{G}_d(n)$ the set of all d-regular (multi)graphs. Note that graphs in $\mathcal{G}_d(n)$ can have loops and multiple edges; graphs without loops and without multiple edges are called *simple*. Denote also by $G(n,3)$ the random 3-regular graph with n vertices following the uniform distribution over the set $\mathcal{G}_3(n)$.

For a sequence of probability spaces $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)_{n\geqslant 1}$ and a sequence of events $(A_n)_{n\geqslant 1}$ where $A_n \in \mathcal{F}_n$ for every $n \geqslant 1$, we say that $(A_n)_{n\geqslant 1}$ happens *asymptotically almost surely* or *a.a.s.*, if $\lim\limits_{n\to+\infty} \mathbb{P}_n(A_n) = 1$. The sequence of events $(A_n)_{n\geqslant 1}$ itself is said to be *asymptotically almost sure* or again *a.a.s.*

Theorem 1. *A.a.s.* $0.103295n \leqslant bw(G(n,3)) \leqslant 0.139822n$.

Since the proof of the main theorem is rather cumbersome and relatively long, we provide a detailed overview with pointers to different observations in the introduction. In order to be precise, we first introduce the necessary notation.

1.1 Notation

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_{\geqslant 0}$ the set of all non-negative integers. For every $d \in \mathbb{N}$, we denote by $[d]$ the set of integers $\{i \mid 1 \leqslant i \leqslant d\}$.

For a graph $H = (V, E)$ and $U \subseteq V$, we denote by $H[U]$ the subgraph of H induced by the vertices in U. For a vertex $v \in V$, we denote by $N(v)$ the neighborhood of v in H, that is, $N(v) = \{u \in V : uv \in E\}$, and $N[v] = v \cup N(v)$. The lowercase letters u, v, w

will be reserved to denote vertices and the lowercase letters e, f will be reserved to denote edges, possibly with some lower or upper indices. The *order* of a graph H is the number of vertices in H, and the *size* of H is the number of edges of H. For $k \in \mathbb{N}$, the *k-core* $C_k(H)$ of a graph H is the (unique) largest subgraph of H with respect to inclusion in which every vertex is of degree at least k.

A *subdivision of an edge* $e = uv$ in a graph H is an operation of deleting the edge e and adding one new vertex w together with the edges uw and vw. Consecutive subdivisions of the edges of a graph H produce a new graph $\overline{H}$ from H. By abuse of terminology we call the graph $\overline{H}$ itself *subdivision of the graph* H. For a subdivision $\overline{H}$ of H, an edge e and a vertex w as above, we say that the vertex w *subdivides* the edge e in H. For a graph $H = (V, E)$, we say that three vertices $u, v, w \in V$ form a *cherry* with center v and endpoints u, w if $uv \in E$ and $vw \in E$. For a vertex v in H, we say that v is a *leaf* of H if v is of degree one in H. Moreover, given a vertex $v \in V$ and a non-negative integer r, we denote by $B_H(v, r)$ the ball with center v and radius r consisting of all vertices at (graph) distance at most r from v in H.

For a positive integer k, the *k-star* is the complete bipartite graph with parts of size 1 and k. A graph is a *star* if it is a k-star for some $k \geqslant 1$. In a subdivision of a star, a *branch* is a path that starts from its center and ends at some of the leaves.

A *cut* (V_1, V_2) of a graph $H = (V, E)$ is a partition of V into two non-empty sets with $V_1 \cup V_2 = V$. Finally, the *size* of a cut (V_1, V_2), denoted by $e(V_1, V_2)$, is the number of edges with one endvertex in V_1 and one endvertex in V_2.

1.2 Detailed overview of the proofs.

The main contribution of this paper is the lower bound of Theorem 1.

The most difficult part of the proof of the lower bound is dedicated to a detailed characterization of structural properties that a minimum bisection typically possesses. On a high level, we show that if certain (forbidden) substructures appear, then one could switch vertices between the parts of the bisection and thus reduce the total number of edges going between these two parts. We use this to do a refined first moment computation.

The structural characterization goes as follows. To begin with, we show that a.a.s. the random graph $G(n, 3)$ is *usual* meaning that its bisection width is at least $0.1n$ by the result of Kostochka and Melnikov (Theorem 7), and it has at most $\log n$ vertices in cycles of length at most 20 (we remark that in the above definition, any linear lower bound on the bisection width is sufficient for our purposes.) Given a minimum bisection (V_1, V_2) in a usual graph G, we show (via some elementary analysis of the local structure of the two parts) that for both $i = 1, 2$, V_i contains a linear number of vertex pairs u, v such that both vertices u, v are of degree 2 in $G[V_i]$, and u and v are at (graph) distance at most 4 from each other in $G[V_i]$ (Lemma 12). This allows us to deduce that up to $O(1)$ vertices, $G[V_i]$ and its 2-core coincide (Lemma 13). Exchanging the vertices outside the 2-cores provides a cut (U_1, U_2) which is almost a bisection (the differences of the two parts is $O(1)$), has minimum size among the cuts (W_1, W_2) with $|W_1| - |W_2| = |U_1| - |U_2|$ (Lemma 14 and Corollary 15), and both $G[U_1]$ and $G[U_2]$ coincide with their 2-cores. We will then deal with such minimum "almost" bisections with parts U_1 and U_2.

Once the cut (U_1, U_2) is defined, for both $i = 1, 2$, we encode the graph $G[U_i]$ by a weighted 3-regular graph $G_{3,i}$ where the weight of each edge corresponds to the number of times the edge was subdivided in the construction of $G[U_i]$ (see Definition 16, where we also define the subgraph $G^+_{3,i}$ of $G_{3,i}$ consisting of the edges of positive weight). By using that the cut (U_1, U_2) cannot be reduced by keeping the sizes of U_1 and U_2 fixed, we show in Observations 17 and 18 that two cases emerge.

In the first case, there is $i \in \{1, 2\}$ such that the sum of weights of the edges in $G_{3,i}$ having weight at least three is more than seven. Then, $G^+_{3,3-i}$ has $O(1)$ vertices of degree three and $O(1)$ edges of weight at least two. This gives rise to a first type of cuts (U_1, U_2) where $G^+_{3,3-i}$ essentially contains only paths and cycles of edges of weight one while the structure of $G_{3,i}$ remains unresticted.

In the second case, both $G^+_{3,1}$ and $G^+_{3,2}$ have $O(1)$ edges of weight at least three. For both $i = 1, 2$, we define the weighted graph $G^{\leqslant 2}_{3,i}$ as the subgraph of $G^+_{3,i}$ obtained by deleting these edges (again, see Definition 16). We call a vertex in $G^{\leqslant 2}_{3,i}$ *critical* if it either has degree three or is incident to an edge of weight two, and define $S_i \subseteq V(G[U_i])$ as the set of all critical vertices in $G^{\leqslant 2}_{3,i}$ (seen in $G[U_i]$) together with the vertices in $G[U_i]$ that subdivide the edges, which are incident to the critical vertices in $G^{\leqslant 2}_{3,i}$. At the end of Section 3 we show that for both $i = 1, 2$ and every $\ell \geqslant 2$, either $|S_i| \leqslant 52\ell + 1091$ or one may delete $O(1)$ edges from $G^{\leqslant 2}_{3,3-i}$ to ensure that no path of length $\ell/3$ and consisting of edges of weight 1 connects two critical vertices (Corollary 33). Finally, assuming without loss of generality that $|S_1| \leqslant |S_2|$, we study the cases $|S_1| \leqslant \log^2 n$ (bisections of type one, which also cover the first case described in the previous paragraph) and $|S_1| \geqslant \log^2 n$ (bisections of type two).

For both types of bisections, we perform a first moment method. In Section 4, we deal with bisections of type one. First, we count the number of possible skeletons for the graph $G_{pc,1} = G^+_{3,1} \setminus S_1$ of order βn for some $\beta \in [0.1, 0.5]$. Then, we fix one possible skeleton and label its vertices. Once this labeling is constructed, we bound from above the number of extensions of $G_{pc,1}$ to $G[U_1]$, and consequently to G. Finally, we optimize with respect to the parameter β to get that bisections of type one and size at most $0.1069n$ are a.a.s. not contained in $G(n, 3)$.

In Section 5, we deal with bisections of type two. In this (slightly harder) case, we first define an auxiliary graph that very much resembles $G^{\leqslant 2}_{3,i}$ for both $i = 1, 2$ and only consists of paths with at most one edge of weight two, cycles with edges of weight one, subdivisions of 3-stars with edges of weight one, and $o(n)$ further edges. Then, we apply the first moment method to count bisections according to the form of the auxiliary graph associated to $G^{\leqslant 2}_{3,1}$ and to $G^{\leqslant 2}_{3,2}$, respectively (which then can be transferred to a first moment method for the original graph). Since we want to count graphs instead of bisections, we also use an additional lemma (that may be of independent interest), which allows us to reduce the first moment: in Lemma 48, we show that in a bipartite graph H of maximal degree two with parts H_1 and H_2, there exists an independent set having roughly $|V(H_1)|/2$ vertices in H_1 and also $|V(H_2)|/2$ vertices in H_2. This allows us to derive that one choice of auxiliary graphs gives rise to many different bisections of the same size, and therefore, we may divide by the corresponding overcounting factor.

The complementary upper bound of Theorem 1 has two main ingredients. It is based on an idea of Lyons [33] using a result of Csóka, Gerencsér, Harangi and Virág [11]. Knowing that the sequence of random regular graphs $(G(n,3))_{n\geqslant 1}$ a.s. locally converges to T_3, Lyons used a sequence of Gaussian processes $(X_v^{(n)})_{v\in V(G(n,3))}$ that converges in distribution to the Gaussian wave function associated to the second largest eigenvalue of the transition operator of the (infinite) 3-regular tree T_3. Then, he defined a cut of the graph $G(n,3)$ (that a.a.s. bisects it up to $o(n)$ vertices) based on the sign of the variables $(X_v^{(n)})_{v\in V(G(n,3))}$, and used some local improvements to deduce a bisection of size approximately $0.16226n$. We refine this strategy by using more complicated local modifications based on the structural insights from the proof of the lower bound. More precisely, we define an auxiliary bipartite graph whose vertices are centers of *border cherries* (that is, cherries whose leaves both have the opposite sign to the center of the cherry) and whose edges cross the cut described above (see also Figure 17). Then, using Lemma 48 again, we show that we may switch certain centers of border cherries, thereby reducing the size of the constructed bisection. We conclude with a non-rigorous improvement obtained by switching vertices based on an even more complicated substructure (unfortunately we are not able to compute the associated integral, not even numerically).

1.3 Organization of the paper.

In Section 2 we introduce basic concepts and lemmas used in the proof of the lower bound. In Section 3 we describe several forbidden subgraphs that do not appear in a minimum bisection of a typical 3-regular graph, ending with a characterization of two types of candidates for a minimum bisection. In Section 4 we compute the expected number of minimum bisections of type one. In Section 5 we do the same for minimum bisections of type two, this time with an additional regrouping with respect to the underlying 3-regular graph they originate from. Section 6 is devoted to the proof of the upper bound.

2 Preliminaries

In this section we introduce a few basic concepts that will be used in the sequel.

Observation 2. *In a graph of maximum degree three, two cycles are vertex-disjoint if and only if they are edge-disjoint.*

Proof. Two vertex-disjoint cycles are clearly edge-disjoint. On the other hand, if two cycles have a common vertex but no common edge, then the degree of this common vertex must be at least four, thus contradicting the maximum degree condition in the statement. □

We now introduce the probability space we will be working in until the end of this paper. For two sequences of probability measures $(\mathbb{P}_n)_{n\geqslant 1}$ and $(\mathbb{Q}_n)_{n\geqslant 1}$ defined on sequence of spaces $(\Omega_n,\mathcal{F}_n)_{n\geqslant 1}$ respectively, we say that $(\mathbb{P}_n)_{n\geqslant 1}$ is *contiguous* to $(\mathbb{Q}_n)_{n\geqslant 1}$ if for every

sequence of measurable properties $(A_n)_{n\geqslant 1}$, $\lim_{n\to\infty} \mathbb{P}_n(A_n) = 0$ implies $\lim_{n\to\infty} \mathbb{Q}_n(A_n) = 0$.

Configuration model. Given positive integers d, n with dn even, consider dn points $(P_{i,j})_{1\leqslant i\leqslant d, 1\leqslant j\leqslant n}$ regrouped into n buckets according to their second index. The *configuration model* is the probability space of all perfect matchings of these dn points equipped with the uniform probability measure. It was introduced by Bender and Canfield [2] and further developed by Bollobás [6] and Wormald [36].

We call *configuration* a perfect matching of $(P_{i,j})_{i\in[d],j\in[n]}$. We also call *partial configuration* a matching of $(P_{i,j})_{i\in[d],j\in[n]}$ which is not necessarily perfect. In order to transition from configurations to graphs, identify the buckets with the n vertices of $G(n,d)$, and connect vertices v and v' by k edges if the configuration contains k pairs consisting of one point in the bucket v and one point in the bucket v' (in particular, one might add loops as well). It is well known that for any fixed value of d, this model is contiguous to the uniform distribution on simple d-regular graphs, see [26].

The following lemma is a standard result in the field of random graphs.

Lemma 3 ([37], Theorem 2.6 and [5])**.** *For every $\ell \geqslant 1$ and $d \geqslant 3$, the number of cycles of length ℓ in a random d-regular graph converges in distribution to a Poisson random variable.*

The next well-known observation explains how to algorithmically construct the 2-core of a graph. For the sake of completeness, we give the proof of it as well.

Observation 4. *The 2-core of a graph H is well-defined and may be obtained by consecutive deletions of the vertices of degree zero and one.*

Proof. In the end of the deletion process, one obviously obtains a subgraph H' of H of minumum degree at least two. On the other hand, suppose for the sake of contradiction that there is another graph $H'' \not\subseteq H'$ which has minimal degree at least two. Then, $H' \cup H''$ is also a subgraph of H of minimum degree at least two. Let v be the first vertex of $H'' \setminus H'$ that has been deleted throughout the construction of H'. At the moment of its deletion, since $v \in H''$, v had degree at least two, which is a contradiction. Thus, every subgraph of H of minimal degree at least two is contained in H', which proves the observation. □

Bounds on the number of partitions. We first state a weak version of the Hardy-Ramanujan theorem on the number of integer partitions that will be sufficient for our purposes.

Theorem 5 ([24])**.** *The number of partitions of an integer n is $\exp(\Theta(\sqrt{n}))$ as $n \to \infty$.*

The next lemma is an application of the previous theorem.

Lemma 6. *For every integer $M \geqslant 3$, the number of unlabeled forests on n vertices in which every connected component is a subdivision of a star with at most M leaves, is $\exp(o(n))$.*

Proof. To construct a forest, one may first partition its n vertices into subsets that induce the trees of the forest, and then decide on the structure of each tree separately. Since the number of partitions of n is $\exp(o(n))$ by Theorem 5, we only need to prove that, for a fixed partition, the number of ways to form a forest of the above type is at most $\exp(o(n))$. Observe that the number of unlabeled stars of order t with at most M leaves is given by the number of partitions of $t-1$ into at most M parts. This number is equal to the number of non-negative integer solutions of the equation $x_1 + \cdots + x_M = t-1$, which is at most $(t-1)^M < t^M$.

On the other hand, for every $t \geqslant 1$, let c_t be the number of sets of size t in the partition of n. First, consider the vertex sets of size at least $\log n$. The number of forests induced by these vertices is bounded above by $\prod_{t \geqslant \log n} t^{c_t M}$ where $\sum_{t \geqslant \log n} t c_t \leqslant n$. We conclude that

$$\prod_{t \geqslant \log n} t^{c_t M} = \exp\left(\sum_{t \geqslant \log n} (\log t) c_t M\right) \leqslant \exp\left(\left(\max_{t \geqslant \log n} \frac{M \log t}{t}\right) \sum_{t \geqslant \log n} t c_t\right) = \exp(o(n)). \tag{1}$$

Next, for the smaller parts, we need to refine the previous upper bound. As before, the number of stars on t vertices remains at most t^M. We count the number of ways to partition the c_t sets of size t into groups where one group consists of the vertex sets inducing a particular star. As every group has size between 0 and c_t, and there are at most t^M groups, the number of partitions as above is at most $(c_t+1)^{t^M}$. This means that the number of (unlabeled) forests induced by the vertices in the smaller parts is at most

$$\prod_{t=1}^{\log n} (c_t+1)^{t^M} \leqslant \prod_{t=1}^{\log n} \left(\frac{n}{t}+1\right)^{t^M} \leqslant (n+1)^{(\log n)^{M+1}} = \exp(o(n)). \tag{2}$$

Combining (1) and (2) finishes the proof. □

3 Structural properties of minimum bisections of the random 3-regular graph

The aim of this section is to give a detailed description of the structure of minimum bisections of the random 3-regular graph. In the proof, we use the already mentioned lower bound of Kostochka and Melnikov [31].

Theorem 7. *The bisection width of $G(n,3)$ is a.a.s. at least $\frac{10}{99}n \approx 0.101n$.*

First, we define some concepts used in the proof of Theorem 1. Let $G = (V,E)$ be a graph and let (V_1, V_2) be a cut of G. We will aim to decrease the size of the cut (V_1, V_2) while keeping the sizes of the sets V_1 and V_2 unchanged. For some $i \in \{1,2\}$ and $\ell \in \mathbb{N}$, a set $S \subseteq V_i$ is called (i,ℓ)-winning (with respect to the cut (V_1,V_2)) if $e(V_1,V_2) - e(V_i \setminus S, V_{3-i} \cup S) = \ell$. For $\ell \in \mathbb{N}$, a subset S of V is *ℓ-winning* or simply *winning* if there is $i \in \{1,2\}$ such that S is entirely contained in V_i and S is an (i,ℓ)-winning set. See Figure 1.

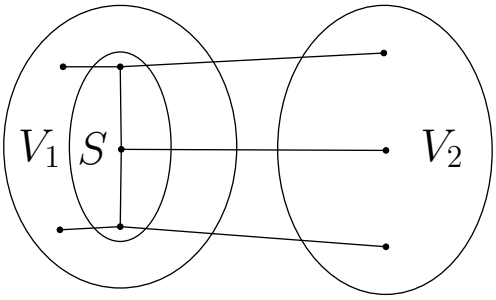


Figure 1: An illustration of a 1-winning set S.

A subset S of V is *indifferent* if there is $i \in \{1,2\}$, for which $S \subseteq V_i$ and $e(V_1,V_2) = e(V_i \setminus S, V_{3-i} \cup S)$. A set $S \subseteq V_i$ is (i,ℓ)-*losing* if $e(V_1,V_2) - e(V_i \setminus S, V_{3-i} \cup S) = -\ell$, and ℓ-*losing* (or just *losing*) if it is (i,ℓ)-losing for some $i \in \{1,2\}$ and $\ell \in \mathbb{N}$. Finally, an *improvement* of the cut (V_1,V_2) is an operation of exchanging two sets S_1 and S_2, with $|S_1| = |S_2|$, $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$, for which

$$e(V_1,V_2) - e(S_2 \cup (V_1 \setminus S_1), S_1 \cup (V_2 \setminus S_2)) \geqslant 1.$$

Thus, the operation of improvement of a given cut creates a new cut of smaller size. Moreover, it does not change the sizes of the two sets participating in the cut. In particular, if the cut is a bisection, improvements also produce a bisection.

In the sequel, we call a 3-regular graph G of order n *usual* if the following two conditions are both satisfied:

- the bisection width of G is at least $0.10n$,
- there are at most $\log n$ vertices in cycles of length at most 20.

We remark that the choice of 20 in the second point of the above definition is somehow arbitrary – it could be replaced by every large enough positive integer.

Observation 8. *A.a.s. $G(n,3)$ is usual.*

Proof. First, the bisection width of a uniformly chosen graph is a.a.s. at least $0.10n$ due to Theorem 7. Second, by Lemma 3, the number of cycles of fixed length converges in distribution to a Poisson random variable. Thus, the number of vertices in cycles of length at most 20 converges in distribution to a tight random variable, and is therefore less than $\log n$ a.a.s. Thus, a.a.s. both properties in the definition of a usual graph are satisfied. □

From now on, we fix a minimum bisection (V_1,V_2) in a usual graph G.

Observation 9. *For both $i = 1$ and $i = 2$, each of the following holds.*

1. *For every vertex u of degree two in $G[V_i]$ and for every $d \geqslant 0$, $|B_{G[V_i]}(u,d)| \leqslant 2^{d+1}-1$. Moreover, if all vertices at distance at most d from u have degree three in $G[V_i]$ and none of them participates in cycles of length at most $2d$, the above bound is sharp.*

2. *For every vertex u of degree one in $G[V_i]$ and for every $d \geqslant 0$, $|B_{G[V_i]}(u,d)| \leqslant 2^d$. Moreover, if all vertices at distance at most d have degree three in $G[V_i]$ and none of them participates in cycles of length at most $2d$, the above bound is sharp.*

Proof. The first point follows from a strong induction showing that for every $i \geqslant 1$, there are at most 2^i vertices at distance at most i from u in $G[V_i]$ with equality if the i-th neighborhood of u in $G[V_i]$ is a binary tree of height i. The second point follows in a similar way. □

The previous lemma yields the following corollary.

Corollary 10. *For both $i = 1$ and $i = 2$ and for every large enough n, there are two vertices of degree at most two and at distance at most eight in $G[V_i]$.*

Proof. We argue by contradiction. By assumption, the balls of radius four around the vertices of degree one or two in $G[V_i]$ are disjoint. Since G is usual, first, there are at least $0.05n$ vertices of degree one or two in $G[V_i]$, and second, the number of vertices participating in cycles of length at most 8 is at most $\log n$. Therefore, the balls of radius four around at least $0.05n - 8\log n$ of the vertices of degree one or two in $G[V_i]$ contain at least $\min(2^5 - 1, 2^4) = 16$ vertices. In total, this shows that the vertices in $G[V_i]$ are at least $16(0.05n - 8\log n) = 0.8n - o(n)$, which is a contradiction. The corollary is proved. □

Corollary 11. *For both $i = 1$ and $i = 2$ and for every large enough n, the number of vertices of degree 0 or 1 in $G[V_i]$ is at most 19.*

Proof. Fix $i \in \{1, 2\}$. Since (V_1, V_2) is a minimum bisection of G, there is no improvement of (V_1, V_2) in G. By Corollary 10 there is a path $(v_j)_{j=0}^s$ in $G[V_{3-i}]$ where $s \leqslant 8$ and v_0, v_s are both of degree at most 2 in $G[V_{3-i}]$. Then, the vertices $(v_j)_{j=0}^s$ form a $(3-i, s-1)$-losing set. We show that there cannot exist $s+1$ vertices in V_i of degree at most one in $G[V_i]$ outside the neighborhood of the path $(v_j)_{j=0}^s$ in G. Indeed, such a set would be (i, ℓ)-winning for some $\ell \geqslant s+1$, and exchanging it with the vertices $(v_j)_{j=0}^s$ in V_{3-i} would lead to an improvement of (V_1, V_2) in G. On the other hand, $(v_j)_{j=0}^s$ have at most $s+3$ neighbors in V_i in G (at most one for each of $(v_j)_{j=1}^{s-1}$, and at most two for v_0 and v_s), so in total the number of vertices of degree one in $G[V_i]$ is at most $s + (s+3) \leqslant 19$. □

Lemma 12. *For every large enough n and for both $i = 1$ and $i = 2$, there are at least $\frac{n}{4000}$ disjoint pairs of vertices of degree 2 in $G[V_i]$ at distance at most 4.*

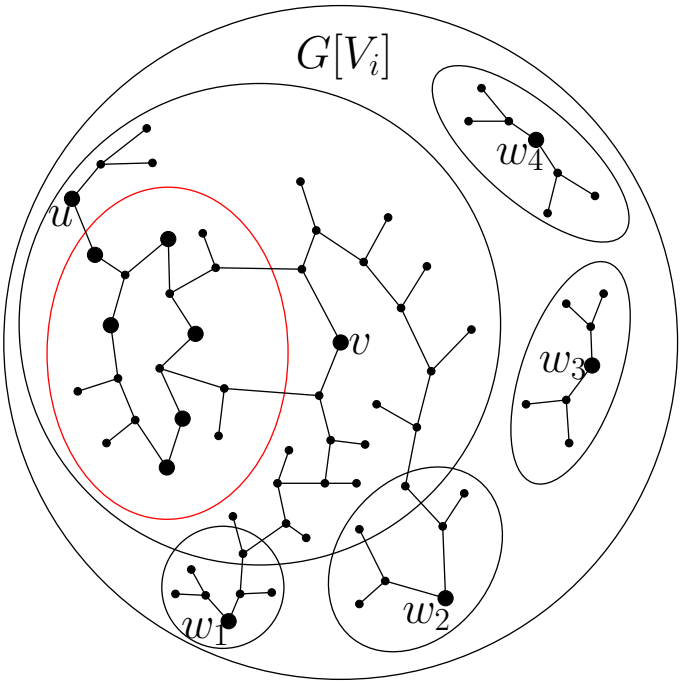


Figure 2: In the figure, the thick black vertices are the ones of degree two in $G[V_i]$ and the others are of degree one or three in $G[V_i]$. Note that some edges of $G[V_i]$ are not drawn to preserve the clarity of the figure. The vertices of V_i' are the ones contained in the smallest of the three nested regions (encircled in red). The vertices u and v are at distance less than five from V_i', but do not participate in V_i'. The vertices w_1, w_2, w_3 and w_4 are at distance at least five from V_i' and therefore the balls of radius two in $G[V_i]$ around them are disjoint.

Proof. We argue by contradiction. Call a pair of vertices in $G[V_i]$ *good* if both have degree 2 and are at distance at most 4 in $G[V_i]$, and suppose that there are no $\frac{n}{4000}$ disjoint good pairs. Let us construct a set $V_i' \subset V_i$ as follows: as long as $V_i \setminus V_i'$ contains a good pair u, v, add both u and v to V_i'. In particular, $|V_i'| \leqslant \frac{n}{2000}$ and for every good pair, at least one of its vertices belongs to V_i'. Moreover, by maximality of V_i', every vertex $v \in V_i$ of degree 2 and at distance at least 5 from V_i' in $G[V_i]$ is such that $B_{G[V_i]}(v, 4)$ contains only vertices of degree 3 in $G[V_i]$. We conclude that the balls of radius 2 around vertices of degree 2 in $G[V_i]$, which are at distance at least 5 from V_i', are disjoint, see Figure 2. Since G is a usual 3-regular graph and (V_1, V_2) is a minimum bisection of G by Corollary 11 the total number of vertices of degree 2 in $G[V_i]$ is at least $\frac{n}{10} - 3 \cdot 19$ (every vertex participates in at most 3 edges between V_1 and V_2) and out of these vertices of degree 2 at least $\frac{n}{10} - 3 \cdot 19 - (1+2) \cdot 19 = \frac{n}{10} - 114$ are at distance at least 3 from all leaves in $G[V_i]$. Moreover, G contains at most $\log n$ vertices in cycles of length at most 4. By Observation 9, the number of vertices in $V_i \setminus V_i'$ should be at least

$$\left(\frac{n}{10} - 114 - 4\log n - |V_i'|(2^5 - 1)\right)(2^3 - 1) \geqslant \frac{7n}{10} - \frac{217n}{2000} - 28\log n - 798 \geqslant \frac{55n}{100}$$

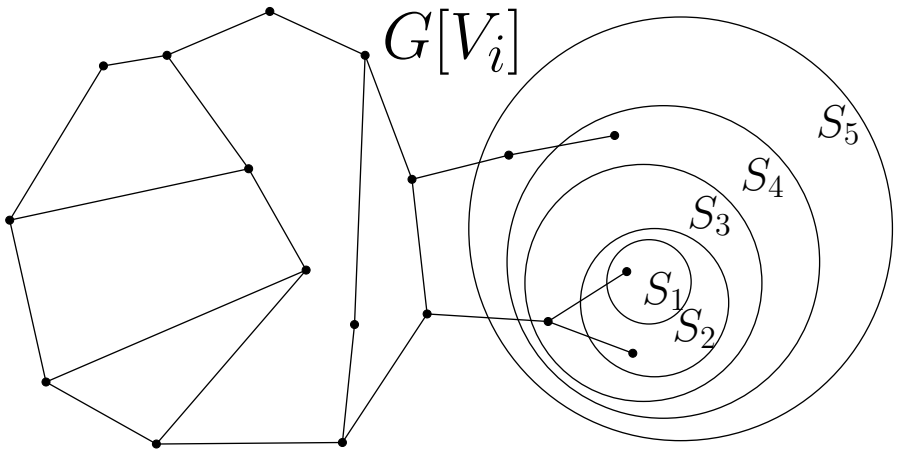


Figure 3: A possible choice of sets S_1, S_2, S_3, S_4, S_5 from the proof of Lemma 13 for a given graph $G[V_i]$

for every large enough n. This is a contradiction since $|V_i| = \frac{n}{2}$. □

Next, we show that for both $i \in \{1, 2\}$, almost all vertices in V_i belong to the 2-core of $G[V_i]$.

Lemma 13. *For both $i = 1$ and $i = 2$ and for all sufficiently large n, the graph $G[V_i] \setminus C_2(G[V_i])$ contains at most five vertices.*

Proof. We argue by contradiction. Starting from the leaves of $G[V_i]$, we consecutively construct sets $S_0 = \emptyset \subset S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ such that:

- $\forall j \in [5]$, $|S_j| = j$,
- for every $j \in [5]$ there is $\ell \geqslant j$, for which S_j is an (i, ℓ)-winning set with respect to (V_1, V_2).

More precisely, for every $j \in [5]$, we construct S_j from S_{j-1} by adding a leaf or an isolated vertex of $G[V_i] \setminus S_{j-1}$. Notice that this construction is possible for every $j \in [5]$ by our assumption and Observation 4, see Figure 3.

Now, note that there are at most $5(3^4 - 1) = O(1)$ vertices at distance at most 4 from S_5 in G. Thus, by Lemma 12, for every sufficiently large n there is a pair of vertices u, v of degree 2 and at distance $d \leqslant 4$ in $G[V_{3-i}]$ such that no vertex on a shortest path between u and v is connected to S_5 in G. Denoting by P the set of vertices on a shortest path between u and v, we see that P is a $(3 - i, d - 1)$-losing set without edges towards S_{d+1}. Hence, exchanging P and S_{d+1} leads to an improvement, a contradiction. □

Now, given a minimum bisection (V_1, V_2) of G, let us first move all vertices in $G[V_1] \setminus C_2(G[V_1])$ to V_2, thus forming a set $\widetilde{V}_2$. Then, move the vertices in $G[\widetilde{V}_2] \setminus C_2(G[\widetilde{V}_2])$ back to $V \setminus \widetilde{V}_2$, thus forming a cut (U_1, U_2) of G.

Note that $G[U_2]$ is a graph with minimum degree 2. Moreover, $G[U_1]$ may be obtained from $C_2(G[V_1])$ by consecutively attaching vertices of degree at least 2 (corresponding to consecutive deletions of vertices of degree 0 or 1 in $G[\widetilde{V}_2]$). Thus, both $G[U_1]$ and $G[U_2]$ coincide with their 2-cores. Another easy observation is that $||U_1| - |U_2|| \leqslant 10$: indeed, the vertices in $\widetilde{V}_2$ outside the 2-core of $G[\widetilde{V}_2]$ either belong to V_1 (and were sent to V_2 during the first exchange) or belong to V_2 but remain outside the 2-core of $G[V_2]$. By Lemma 13, this number of vertices is at most 10.

The following lemma shows that, roughly speaking, the minimality assumption for the size of the bisection (V_1, V_2) remains correct for the cut (U_1, U_2).

Lemma 14. *For every large enough n, the cut (U_1, U_2) obtained from (V_1, V_2) has minimal size among the family of cuts $\{(W_1, W_2) \,|\, |W_1| - |W_2| = |U_1| - |U_2|\}$.*

Proof. We argue by contradiction. Let (W_1, W_2) be a cut with $|W_1| - |W_2| = |U_1| - |U_2|$ and of smaller size than (U_1, U_2). Assume without loss of generality $|U_1| \geqslant |U_2|$, and note that moving a vertex from V_1 to V_2 or from $\widetilde{V}_2$ back to the first part (strictly) decreases the size of the cut. Hence, using that at least $(|U_1| - |U_2|)/2$ vertices changed their part during the exchange, the size of (U_1, U_2) is at most $e(V_1, V_2) - (|U_1| - |U_2|)/2$. Notice that the cut (W_1, W_2) contains at least $n/10 - 15$ edges since otherwise sending any set of $(|W_1| - |W_2|)/2 \leqslant 5$ vertices from W_1 to W_2 would lead to two parts with the same number of vertices, which form a bisection of size less than $n/10$ in the usual graph G. We conclude that there is a set of at least $n/30 - 5$ vertices in W_1 with an edge to W_2 in G. Notice that every vertex with an edge in the cut is of degree at most two in the graph induced by its part. Therefore, by Observation 9, there is a set of at least $n/90 - 2$ non-neighboring vertices of degree at most two in $G[W_1]$. Sending any $(|W_1| - |W_2|)/2$ of them to W_2 produces a bisection of size $e(W_1, W_2) + (|W_1| - |W_2|)/2 < e(U_1, U_2) + (|U_1| - |U_2|)/2 \leqslant e(V_1, V_2)$. This is a contradiction with the minimality of the size of the bisection (V_1, V_2), which proves the lemma. □

Corollary 15. *The cut (U_1, U_2) of the graph G does not admit improvements.* □

Since any constant difference between the sizes of the two parts will not alter subsequent ideas and calculations, we abuse terminology and call bisections these "almost balanced" cuts as well. However, we will keep the notation (U_1, U_2) for such almost balanced cuts and (V_1, V_2) for true bisections.

Definition 16. For both $i = 1$ and $i = 2$, we define:

- $G_{3,i}$ as the unique 3-core whose edges may be subdivided to obtain the graph $G[U_i]$. Moreover, we define a weight for every edge of $G_{3,i}$ equal to the number of times this edge should be subdivided in the construction of $G[U_i]$. The weight of the edge e in $G_{3,i}$ will be denoted by $p(e)$, see Figure 4.
- $G^+_{3,i}$ as the graph obtained from $G_{3,i}$ by deleting the edges of weight 0.

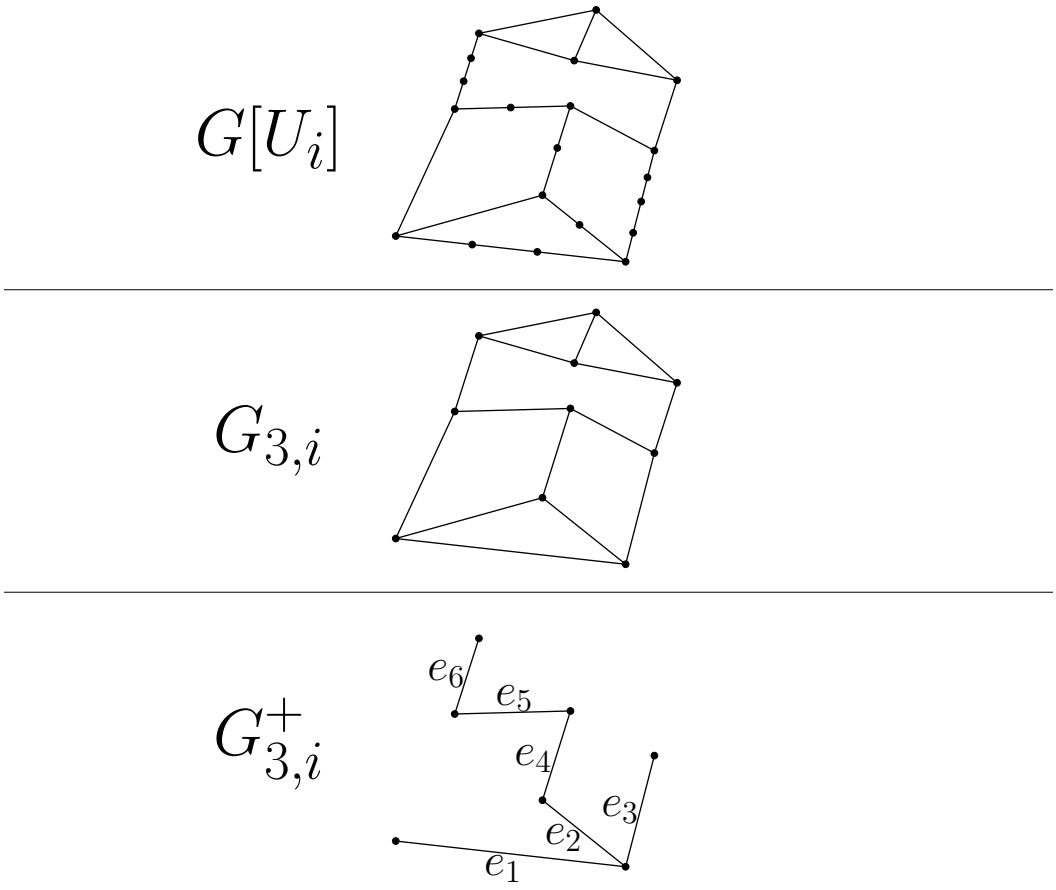


Figure 4: Top figure: An example of the graph $G[U_i]$. Middle figure: the corresponding graph $G_{3,i}$. Bottom figure: the graph $G^+_{3,i}$. In it, the weight of the edges e_2, e_4, e_5 is one, the weight of e_1 and e_6 is two and the weight of e_3 is three.

- $G^{\leqslant 2}_{3,i}$ as the graph obtained from $G^{+}_{3,i}$ by deleting all edges of weight more than 2.

Observation 17. *For both $i = 1$ and $i = 2$, at most one of the following happens:*

1. *The sum of the weights of the edges*

$$\{e \in E(G_{3,i}) \mid p(e) \geqslant 3\}$$

   is more than five.

2. *$G_{3,3-i}$ contains at least nine edges of weight at least two.*

Proof. We argue by contradiction. Suppose that each of the above two events happens for some $i \in \{1, 2\}$. We consider two cases: either there are two edges e_1, e_2 in $G_{3,i}$ with $p(e_1) \geqslant 3$ and $p(e_2) \geqslant 3$ or there is an edge e_3 in $G_{3,i}$ with $p(e_3) \geqslant 6$. We define $(w_i)_{1\leqslant i\leqslant 6}$ as follows:

- In the first case, $(w_i)_{1\leqslant i\leqslant 3}$ are consecutive vertices subdividing the edge e_1 and $(w_i)_{4\leqslant i\leqslant 6}$ are consecutive vertices subdividing the edge e_2 in $G_{3,i}$ (see the left part of Figure 5).
- In the second case, $(w_i)_{1\leqslant i\leqslant 6}$ are consecutive vertices subdividing the edge e_3 in $G_{3,i}$.

By assumption there are edges $(f_j)_{1\leqslant j\leqslant 9}$ in $G_{3,3-i}$, each of weight at least two. Moreover, there are at least three of these edges that are subdivided by vertices, none of which is adjacent to any of $(w_i)_{1\leqslant i\leqslant 6}$. Let f'_1, f'_2, f'_3 be one such choice of edges among $(f_j)_{1\leqslant j\leqslant 9}$. Then, exchanging the set of vertices $(w_i)_{1\leqslant i\leqslant 6}$ in U_i and three pairs of neighboring vertices of degree two in $G[U_{3-i}]$ subdividing f'_1, f'_2 and f'_3 respectively, leads to an improvement of the cut (U_1, U_2). This is a contradiction with Corollary 15, which proves the observation. □

Observation 18. *For both $i = 1$ and $i = 2$, at most one of the following happens:*

1. *The sum of the weights of the edges*

$$\{e \in E(G_{3,i}) \mid p(e) \geqslant 3\}$$

   is more than seven.

2. *$G^{+}_{3,3-i}$ contains at least 21 vertices of degree three.*

Proof. Suppose that each of the two events above happens for some $i \in \{1, 2\}$. Let $(e_j)_{1\leqslant j\leqslant s}$ be a set of at most three edges in $G_{3,i}$ of weights $p(e_1) \geqslant \ldots \geqslant p(e_s) \geqslant 3$ and

$$\sum_{1\leqslant j\leqslant s} p(e_l) \geqslant 8.$$

There are three cases.

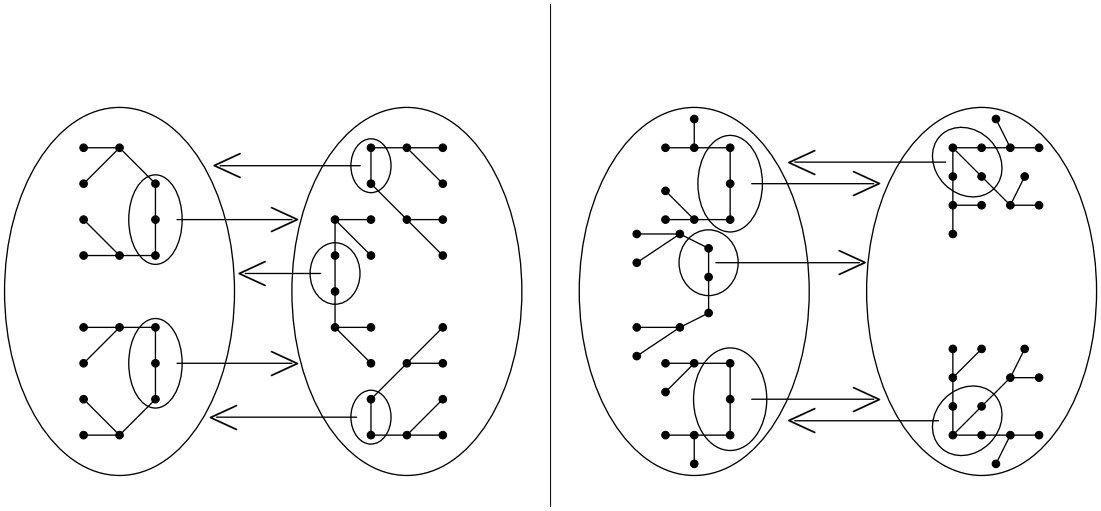


Figure 5: To the left: the first case from the proof of Observation 17. To the right: the first case from the proof of Observation 18. Only the edges contained in the two parts of the bisection are given. Edges between the two parts are not depicted, however, in both cases there is no edge between the two sets of vertices exchanged between U_1 and U_2.

- In the first case, $s = 3$ and there are three connected sets of vertices $(w_i)_{1\leqslant i\leqslant 3}$, $(w_i)_{4\leqslant i\leqslant 6}$ and $(w_i)_{7\leqslant i\leqslant 8}$ in $G[U_i]$, subdividing e_1, e_2 and e_3 respectively in $G_{3,i}$ (see the right part of Figure 5).
- In the second case, $s = 2$ and there are two connected sets of vertices, regrouped either as $(w_i)_{1\leqslant i\leqslant 4}$ and $(w_i)_{5\leqslant i\leqslant 8}$ or as $(w_i)_{1\leqslant i\leqslant 5}$ and $(w_i)_{6\leqslant i\leqslant 8}$, subdividing e_1 and e_2 respectively in $G_{3,i}$.
- In the third case, $s = 1$ and there is a connected set of vertices $(w_i)_{1\leqslant i\leqslant 8}$, subdividing e_1.

Let also $(v_i)_{1\leqslant i\leqslant 21}$ be vertices of degree three in $G^+_{3,3-i}$. Out of these 21 vertices, at least five are at distance at least three from the set $(w_i)_{1\leqslant i\leqslant 8}$ in G. Indeed, every vertex of degree three in $G[U_{3-i}]$ at distance at most two from the set $(w_i)_{1\leqslant i\leqslant 8}$ in G must be adjacent to some of the eight neighbors of the vertices $(w_i)_{1\leqslant i\leqslant 8}$ in U_{3-i}. Moreover, out of this subset of five vertices there are two vertices v'_1 and v'_2, which are not adjacent in $G^+_{3,3-i}$.

Let u_1, u_2, u_3 and u_4, u_5, u_6 be the vertices at distance one to v'_1 and v'_2 in $G[U_{3-i}]$, respectively. Exchanging $(w_i)_{1\leqslant i\leqslant 8}$ and $(u_i)_{1\leqslant i\leqslant 6} \cup \{v'_1, v'_2\}$ between U_1 and U_2 would lead to an improvement of (U_1, U_2) – contradiction with Corollary 15, which proves the observation. □

If for $i = 1$ or for $i = 2$, the first point from the statement of Observation 18 holds, then $G^+_{3,3-i}$ must contain predominantly chains and cycles with edges of weight one. We deal with this case in Section 4. There, we show that the proportion of graphs possessing

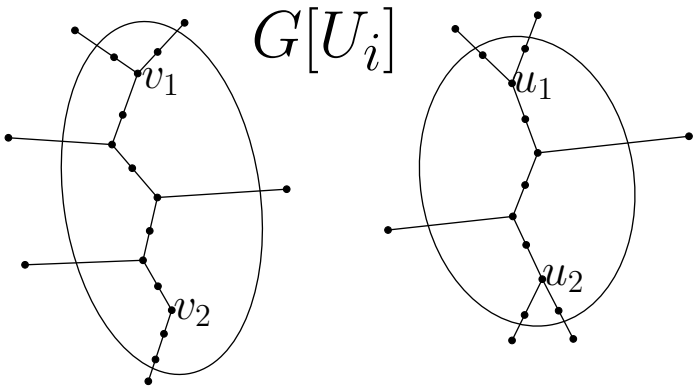


Figure 6: The set S from the proof of Observation 19 is the union of the vertices in the two encircled regions.

a bisection of this type of size between $0.1n$ and $0.1069n$ tends to zero as n tends to infinity.

From now on we concentrate on the setting, in which the sum of the weights of the edges $\{e \in E(G_{3,i}) \mid p(e) \geqslant 3\}$ is at most seven for both $i = 1$ and $i = 2$. In particular, there are at most two edges of weight at least three in both $G^+_{3,1}$ and $G^+_{3,2}$.

For both $i = 1$ and $i = 2$, recall the graph $G^{\leqslant 2}_{3,i}$ from Definition 16. We call a vertex in $G^{\leqslant 2}_{3,i}$ *critical* if it is either of degree three in $G^{\leqslant 2}_{3,i}$, or if it is incident to an edge of weight two. For example, in Figure 4 the graph $G^{\leqslant 2}_{3,i}$ is obtained from $G^+_{3,i}$ by deleting the leaf in $G^+_{3,i}$ incident to the edge e_3, and the critical vertices in $G^{\leqslant 2}_{3,i}$ are the endvertices of the edges e_1 and e_6.

Observation 19. *For both $i = 1$ and $i = 2$ and for every $\ell \geqslant 2$, at most one of the following happens:*

1. *In $G^{\leqslant 2}_{3,i}$, there are two (not necessarily disjoint) pairs of critical vertices, (v_1, v_2) and (u_1, u_2), connected by two paths p_1 and p_2 satisfying the following conditions:*
   - *The sum of the lengths of p_1 and p_2 is at most ℓ.*
   - *Both paths contain only edges of weight one in $G^{\leqslant 2}_{3,i}$.*
2. *There are at least $3\ell + 27$ edges in $G^{\leqslant 2}_{3,3-i}$ of weight two.*

Proof. We argue by contradiction. Suppose that each of the above two events happens for some $i \in \{1, 2\}$. By choosing the two paths in $G^{\leqslant 2}_{3,i}$ satisfying the above conditions and with the smallest sum of lengths, one may assume that p_1 and p_2 intersect in at most one vertex. Moreover, if p_1 and p_2 have a common vertex, it must be an endvertex for each of them.

Let S' be the set of vertices in $G[U_i]$ that subdivide the edges in $G^{\leqslant 2}_{3,i}$, which are incident to some of the vertices v_1, v_2, u_1, u_2 from the first statement. Define $S \subseteq V(G[U_i])$ as the

union of S' and the set of vertices contained in the subdivisions of p_1 and p_2 in $G[U_i]$. See Figure 6. Of course, the subdivisions of the two paths contain both vertices of degrees two and three in $G[U_i]$. Then S is an (i, ℓ')-winning set of size $|S| \leqslant 2\ell + 18$, where $\ell' \geqslant 2$. Indeed, the subdivisions of the paths p_1 and p_2 contain at most $2\ell + 2$ vertices and every endvertex is incident in $G_{3,i}^{\leqslant 2}$ to at most two edges of weight at most two. Then, among the $3\ell + 27$ edges of weight two in $G_{3,3-i}^{\leqslant 2}$ there are $\ell + 9$, for which the pairs of vertices, which subdivide these edges in $G[U_{3-i}]$, contain no vertex incident to a vertex in S. Then, an improvement of the bisection (U_1, U_2) is given by exchanging S in $G[U_i]$ with

- $|S|/2$ of the above pairs of vertices in $G[U_{3-i}]$, if $|S|$ is even.
- $(|S| - 1)/2$ of the above pairs of vertices in $G[U_{3-i}]$ together with some additional vertex of degree two in $G[U_{3-i}]$, not connected to the set S by an edge in G, if $|S|$ is odd.

This is a contradiction, which proves the observation. □

Observation 20. *For any two cycles c_1 and c_2 in a graph H containing a common edge, there are cycles c_1' and c_2' contained in $c_1 \cup c_2$, for which $c_1' \cap c_2'$ is a path.*

Proof. If some of the cycles c_1 and c_2 has length two, the claim is trivial. Otherwise, let p be a shortest path with endvertices $u, v \in c_1 \cap c_2$ contained in c_2 and sharing no common edges with c_1. Then, $c_1 \cup p$ consists of three disjoint paths p, p_1, p_2 between u and v. Choosing $c_1' = p \cup p_1$ and $c_2' = p \cup p_2$ finishes the proof. □

Observation 21. *For both $i = 1$ and $i = 2$ and for every $\ell \geqslant 2$, at most one of the following happens:*

1. *There are at least two cycles c_1 and c_2 in $G_{3,i}^{\leqslant 2}$, whose union contains at least two vertices of degree three in $G_{3,i}^{\leqslant 2}$, and the sum of whose lengths is at most ℓ.*
2. *There are at least $3\ell + 27$ edges in $G_{3,3-i}^{\leqslant 2}$ of weight two.*

Proof. Suppose that each of the above two events happens for some $i \in \{1, 2\}$ and some $\ell \geqslant 2$. If the two cycles have a common edge, by Observation 20 one can find two cycles c_1' and c_2' in $G_{3,i}^{\leqslant 2}$ whose intersection is a path p with endvertices u and v, see the left part of Figure 7. Then, one may find without difficulty two paths (possibly some of them of length zero, for example when all edges in $c_1' \cup c_2'$ have weight two) between critical vertices in $G_{3,i}^{\leqslant 2}$ without common interior vertices and containing only edges of weight one, one starting from v and contained in $c_1' \setminus p$ and one starting from u and contained in $c_2' \setminus p$. We obtain a contradiction by Observation 19.

If the cycles are edge-disjoint, by Observation 2 they are vertex-disjoint as well. If one cycle, say c_1, contains at least two vertices of degree three, we directly apply Observation 19 (again, paths of length zero may occur) for the subdivision of c_1 in $G_{3,i}^{\leqslant 2}$. If both cycles contain exactly one vertex of degree three and one of them, say c_1, contains an edge of weight two in $G_{3,i}^{\leqslant 2}$, then, again, we directly apply Observation 19 for the subdivision of

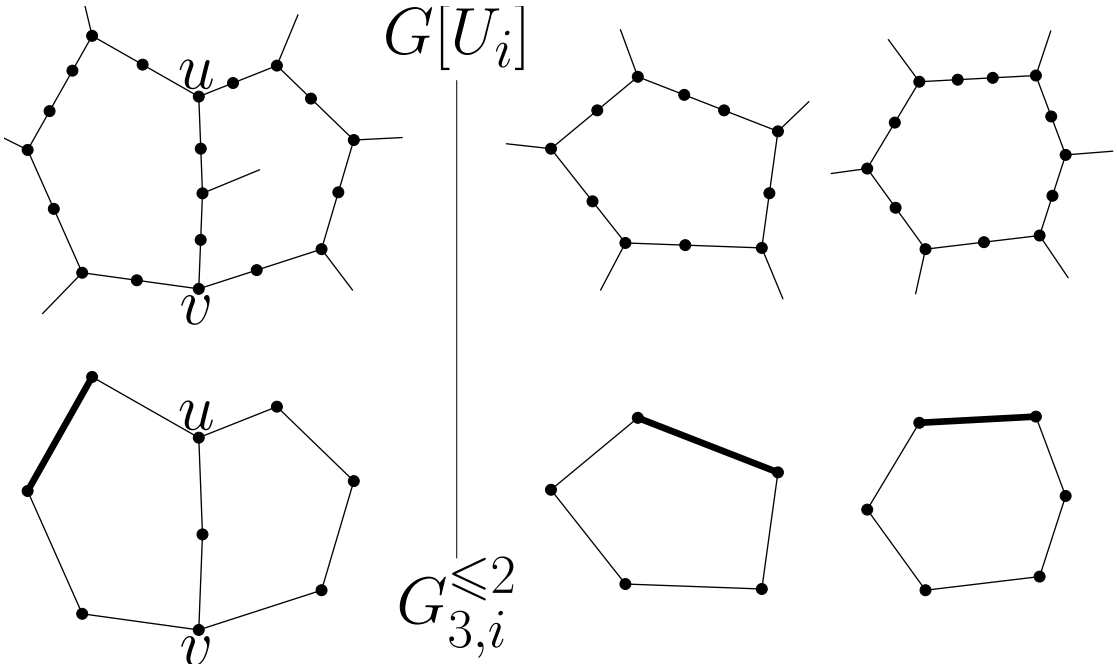


Figure 7: The cycles in the proofs of Observation 21 (on the left) and Observation 22 (on the right). The edges of weight two in $G_{3,i}^{\leqslant 2}$ are thickened.

c_1. It remains the case when c_1 and c_2 are disjoint and each contains exactly one vertex of degree three and no edges of weight two. Then, the set S of vertices in $G[U_i]$ contained in the subdivisions of the two cycles is 2-winning and has size $2\ell + 2$. Thus, there are $\ell + 1$ pairs of vertices of degree two in $G_{3,3-i}^{\leqslant 2}$, none of which is adjacent to S. Exchanging S with this set of pairs leads to an improvement of (U_1, U_2) in G – contradiction. □

Observation 22. *For both $i = 1$ and $i = 2$ and for every $\ell \geqslant 2$, at most one of the following happens:*

1. *There are two cycles c_1 and c_2 in $G_{3,i}^{\leqslant 2}$, whose union contains at least two edges of weight two, and the sum of whose lengths is at most ℓ.*

2. *There are at least $3\ell + 27$ edges in $G_{3,3-i}^{\leqslant 2}$ of weight two.*

Proof. We argue by contradiction. If the two cycles have a common vertex, they also have a common edge by Observation 2 and therefore they contain at least two vertices of degree three in $G_{3,i}^{\leqslant 2}$. One may directly apply Observation 21 in this case.

If the cycles are edge-disjoint and some of them contains at least two edges of weight two, then we directly apply Observation 19 (again, paths of length zero may occur). If both of them contain exactly one edge of weight two, then the set S of vertices in $G[U_i]$ contained in the subdivisions of the two cycles is winning and has even size, which is at most $2\ell + 2$, see the right part of Figure 7. Thus, there are $\ell + 1$ pairs of vertices of degree

two in $G^{\leqslant 2}_{3,3-i}$, none of which is adjacent to S. Exchanging S with this set of pairs leads to an improvement of (U_1, U_2) in G – contradiction. □

Remark 23. The same conclusion holds in the case of two cycles containing at least one vertex of degree three in $G^{\leqslant 2}_{3,i}$ and at least one edge of weight two. □

Corollary 24. *For both $i = 1$ and $i = 2$ and for every $\ell \geqslant 2$, at most one of the following happens:*

1. *There are two cycles c_1 and c_2 in $G^{\leqslant 2}_{3,i}$, each of length at most $\ell/2$ and each containing either an edge of weight two or a vertex of degree three, or both.*

2. *There are at least $3\ell + 27$ edges in $G^{\leqslant 2}_{3,3-i}$ of weight two.*

Corollary 25. *Suppose that there are at least $3\ell + 27$ edges in $G^{\leqslant 2}_{3,3-i}$ of weight two. Then, by deleting at most six edges in $G^{\leqslant 2}_{3,i}$, one may construct a graph $G''_{3,i}$, which does not contain two critical vertices connected by a path of length at most $\ell/2$, which contains only edges of weight one.*

Proof. If there is a cycle of length at most $\ell/2$, containing either a a vertex of degree three, or an edge of weight two, then:

1. in the first case, delete one of the edges incident to the vertex of degree three and participating in the cycle.

2. in the second case, delete the edge of weight two participating in the cycle.

By Corollary 24 one may conclude that after the deletion, in each of the two cases, no cycle of length at most $\ell/2$ contains a vertex of degree three or an edge of weight two.

If after the deletion there is no path of length at most $\ell/2$ between two critical vertices containing only edges of weight one, then we are done. In any other case, let p be a path of minimal length between some pair of critical vertices v_1, v_2. By minimality of p, p contains only vertices of degree two in $G^{\leqslant 2}_{3,i}$ and only edges of weight one, and has length at most $\ell/2$. Deleting all edges in $G^{\leqslant 2}_{3,i}$ outside p that are incident to v_1 and v_2, and one edge from p, ensures that there remains no pair of critical vertices at distance at most $\ell/2$. Indeed, deleting the edges incident to v_1 and v_2 outside p disconnects the path p from the rest of $G^{\leqslant 2}_{3,i}$. Deleting further one edge in p means that if there is a path of length at most $\ell/2$ between two critical vertices in the new graph, it would be disjoint from p in $G^{\leqslant 2}_{3,i}$, which would contradict Observation 19. The corollary is proved. □

Note that Observations 19, 21, 22, Remark 23 and Corollary 24 all deal with edges of weight two in $G^{\leqslant 2}_{3,i}$, whose subdivisions in either $G[U_1]$ or in $G[U_2]$ play the role of minimal indifferent sets in the bisection (U_1, U_2). These results have natural counterparts for the other minimal indifferent sets in $G[U_1]$ and $G[U_2]$ – the 3-stars. We continue by presenting these analogous results.

Observation 26. *In a graph of maximal degree three that contains at least m vertices of degree three, one may find an independent set I of at least $m/4$ vertices of degree three.*

Proof. Such an independent set can be constructed by consecutively adding a vertex of degree three to I, which does not yet have a neighbor in I. At every step, this decreases the number of vertices that could be added to I by at most four. □

For a graph H and a vertex v in H, we define the *(closed) degree two neighborhood* $N_H^{\leqslant 2}[v]$ as the set of vertices that may be attained from v by a path in H, containing no vertex of degree more than two except possibly v itself.

Observation 27. *For both $i = 1$ and $i = 2$ and for every $\ell \geqslant 2$, at most one of the following happens:*

1. *There are three pairs of critical vertices in $G_{3,i}^{\leqslant 2}$, $(v_j^1, v_j^2)_{1\leqslant j\leqslant 3}$, connected by three paths $(p_j)_{1\leqslant j\leqslant 3}$ such that:*
   - *the sum of their lengths is at most ℓ,*
   - *containing only edges of weight one in $G_{3,i}^{\leqslant 2}$.*
2. *There is an independent set I of at least $5\ell/2+55$ vertices of degree three in $G_{3,3-i}^{\leqslant 2}$.*

Proof. We argue by contradiction. Suppose that each of the above two events happens for some $i \in \{1, 2\}$ and some $\ell \geqslant 2$. Notice that up to choosing the paths p_1, p_2 and p_3 such that the sum of their lengths is minimal, we may assume that p_1, p_2 and p_3 do not share common interior vertices since otherwise one may always shorten at least one of the three paths. We remark that the subdivision of $p_1 \cup p_2 \cup p_3$ together with $\bigcup_{1\leqslant j\leqslant 3} \left(N_{G[U_i]}^{\leqslant 2}[v_j^1] \cup N_{G[U_i]}^{\leqslant 2}[v_j^2]\right)$ in $G[U_i]$ forms an ℓ'-winning set S_i for some $\ell' \geqslant 3$ – this follows directly by an elementary case by case analysis of the positions of the endvertices of the paths p_1, p_2 and p_3: Indeed, together p_1, p_2 and p_3 may form:

- two cycles, which intersect in a common path;
- one cycle containing two of the paths and intersecting the third path;
- one cycle containing two of the paths that is disjoint from the third path;
- a subdivision of a 3-star;
- a longer path consisting of p_1, p_2 and p_3, composed one after the other in some order in a sequential manner;
- a path consisting of two of p_1, p_2 and p_3, composed one after the other in some order in a sequential manner, and a third disjoint path; this is the case from Figure 8;
- three disjoint paths.

Now, on the one hand, the number of vertices in S_i is at most $(\ell+6\cdot 2\cdot 2)+(\ell+3)=2\ell+27$. On the other hand, there are at most $2(\ell+6\cdot 2\cdot 2)=2\ell+48$ vertices in I, which may have a neighbor in $G[U_{3-i}]$, adjacent to a vertex in S_i. Indeed, every vertex in $G[U_{3-i}]$ with a neighbor in S_i must have degree at most two in $G[U_{3-i}]$. There remain at least $\ell/2+7$ vertices in I at distance at least three from S_i in G, for which the balls of radius one in $G[U_{3-i}]$ are two by two disjoint. Thus, one may choose $\lfloor |S_i|/4 \rfloor$ of these balls together with one, two or three vertices from another ball in $G[U_{3-i}]$ with center in I to form an indifferent, a 1-losing or a 2-losing set $S_{3-i}\subseteq U_{3-i}$ with $|S_i|=|S_{3-i}|$ and no edge between S_i and S_{3-i} in G. Exchanging S_i and S_{3-i} between U_i and U_{3-i} leads to an improvement of the bisection (U_1,U_2) – contradiction. The observation is proved. □

We directly deduce the following corollary of Observation 27:

Corollary 28. *For both $i=1$ and $i=2$ and for every $\ell\geqslant 2$, at most one of the following happens:*

1. *There are three connected components in $G^{\leqslant 2}_{3,i}$, each of them containing a path:*
   - *of length at most $\ell/3$,*
   - *containing only edges of weight one in $G^{\leqslant 2}_{3,i}$,*
   - *starting and ending with critical vertices in $G^{\leqslant 2}_{3,i}$.*
2. *There is an independent set I containing at least $5\ell/2+55$ vertices of degree three in $G^{\leqslant 2}_{3,3-i}$.* □

The next observation is in the same spirit.

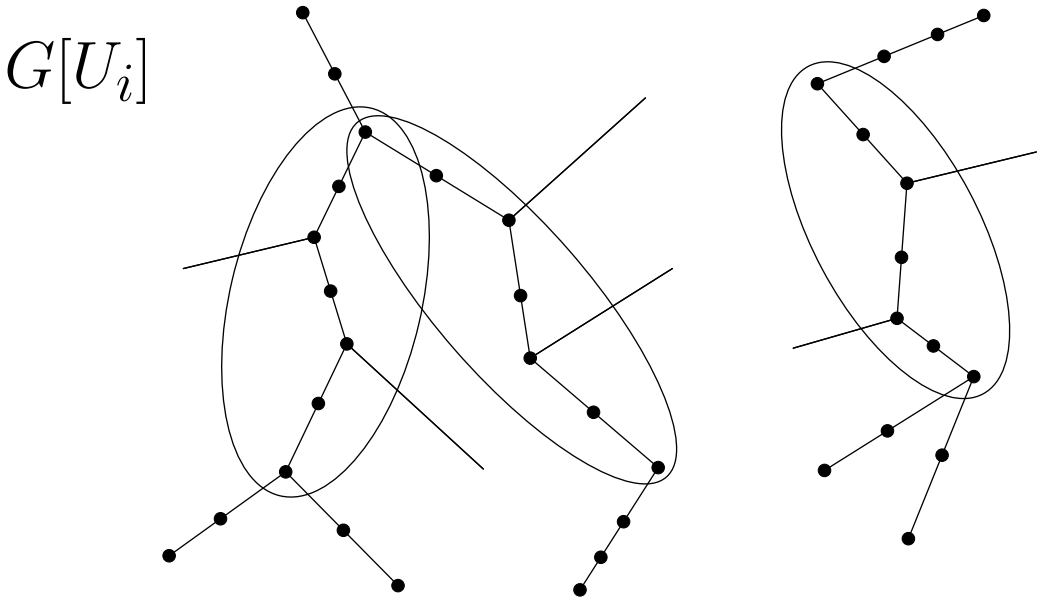
Observation 29. *For both $i=1$ and $i=2$ and for every $\ell\geqslant 2$, at most one of the following happens:*

1. *There are three vertex-disjoint cycles in $G^{\leqslant 2}_{3,i}$, each of length at most $\ell/3$ and each containing a critical vertex in $G^{\leqslant 2}_{3,i}$.*
2. *There is an independent set I containing at least $5\ell/2+55$ vertices of degree three in $G^{\leqslant 2}_{3,3-i}$.*

Proof. The proof is analogous to the one of Observation 27 by choosing $v_i^1=v_i^2$ for every $i\in[3]$. □

Corollary 30. *For both $i=1$ and $i=2$ and for every $\ell\geqslant 2$, at most one of the following happens:*

1. *There are two cycles in $G^{\leqslant 2}_{3,i}$, each of length at most $\ell/3$, which share a common edge.*
2. *There is an independent set I containing at least $5\ell/2+55$ vertices of degree three in $G^{\leqslant 2}_{3,3-i}$.*



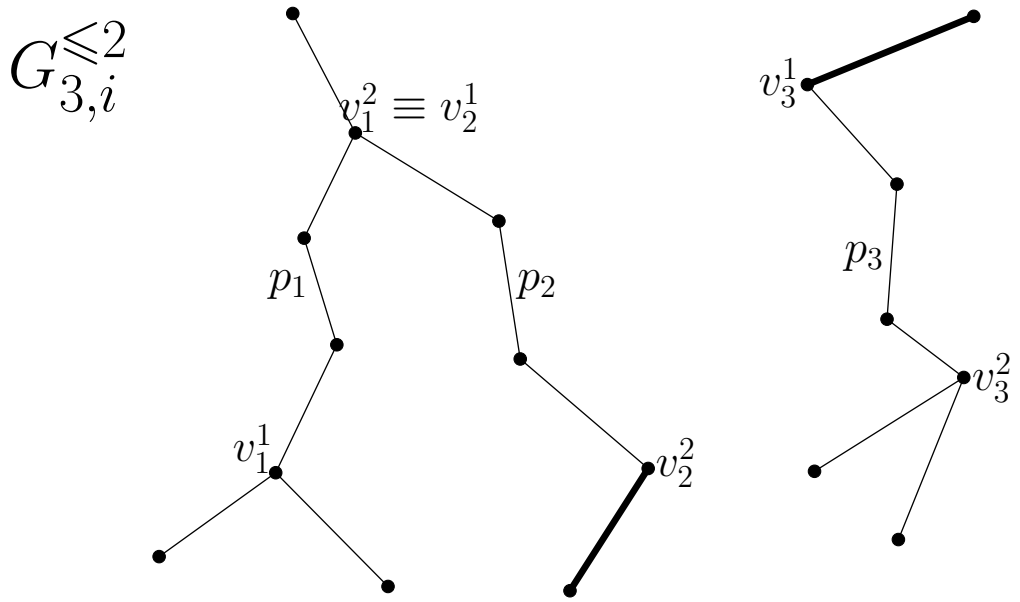


Figure 8: The three paths p_1, p_2 and p_3 from the proof of Observation 27. The edges of weight two are thickened.

Proof. We argue by contradiction. Suppose that each of the above two events happens for some $i \in \{1,2\}$ and some $\ell \geqslant 2$. Let c_1 and c_2 be two cycles as described in the first assertion. By applying Observation 20 one deduces that there are two cycles c_1' and c_2' such that $c_1' \cup c_2' \subseteq c_1 \cup c_2$ and $p = c_1' \cap c_2'$ is a path. Then each of the paths $p, c_1' \setminus p$ and $c_2' \setminus p$ in $G_{3,i}^{\leqslant 2}$ is of length at most $\ell/3$, and they share common endvertices, which are therefore critical in $G_{3,i}^{\leqslant 2}$ (in case the cycles contain edges of weight two, one may shorten the paths so that all three of them contain only edges of weight one. Note that this may possibly lead to paths of length zero.). This is a contradiction by Observation 27. The observation is proved. □

Corollary 31. *Suppose that there are at least* $10\ell + 220$ *vertices of degree three in* $G_{3,3-i}^{\leqslant 2}$. *Then, by deleting at most* 20 *edges in* $G_{3,i}^{\leqslant 2}$ *one may construct a graph* $G_{3,i}''$, *which does not contain two critical vertices, connected by a path of length at most* $\ell/3$ *of edges of weight one.*

Proof. We argue by contradiction. By Observation 26 there is an independent set of at least $5\ell/2+55$ vertices of degree three in $G_{3,3-i}^{\leqslant 2}$. Then, by Observation 29 and Corollary 30 there are at most two cycles of length at most $\ell/3$ in $G_{3,i}^{\leqslant 2}$, which contain a critical vertex in $G_{3,i}^{\leqslant 2}$, and, if present, they must be (vertex-)disjoint. Moreover, in each of the cycles there are at most:

- two edges of weight two and no vertex of degree three in $G_{3,i}^{\leqslant 2}$, or
- one edge of weight two and one vertex of degree three in $G_{3,i}^{\leqslant 2}$, or
- two vertices of degree three in $G_{3,i}^{\leqslant 2}$.

Indeed, if a cycle contains at least, say, two edges of weight two and a vertex of degree three in $G_{3,i}^{\leqslant 2}$, one may find three edge-disjoint paths between critical vertices (some of which may be reduced to a single vertex), containing only edges of weight one in $G_{3,i}^{\leqslant 2}$ – contradiction with Observation 27. All other cases are treated analogously. We conclude that one may delete at most eight edges – at most four edges to disconnect the cycles containing critical vertices from the rest of $G_{3,i}^{\leqslant 2}$, and at most four more edges to disconnect all paths of positive length between critical vertices in $G_{3,i}^{\leqslant 2}$ containing only edges of weight one. See Figure 9.

It remains to deal with the paths of length at most $\ell/3$ between critical vertices, which consist of edges of weight one. There are at most two paths p_1 and p_2 of this type by Observation 27. Notice also that the paths p_1 and p_2 cannot have common interior vertices and may only share common endvertices – otherwise one would be able to decompose p_1 and p_2 into at least three paths of length at most $\ell/3$ of edges of weight one, contradicting Observation 27. For the same reason all interior vertices of these paths are of degree two in $G_{3,i}^{\leqslant 2}$. Suppose that the paths p_1 and p_2 have endpoints the critical vertices (v_1, v_2) and (u_1, u_2). Then, by deleting all (at most twelve) edges $G_{3,i}^{\leqslant 2}$, incident to v_1, v_2, u_1 and u_2 we obtain a graph without critical vertices at distance $\ell/3$. Indeed, if there remains some

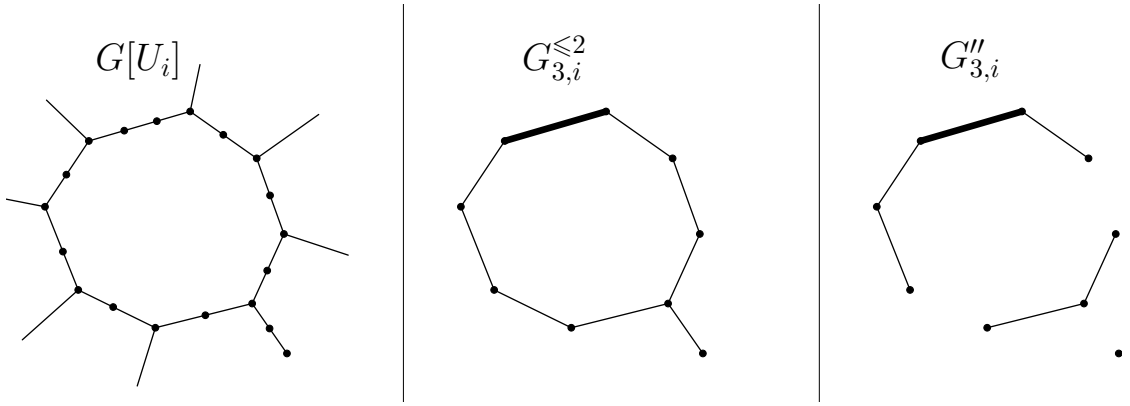


Figure 9: Deleting edges incident to cycles, containing critical vertices, in the proof of Corollary 31. Edges of weight two in $G^{\leqslant 2}_{3,i}$ and $G''_{3,i}$ are thickened. In the figure, three edges are sufficient to disconnect the paths of edges of weight one between critical vertices.

path of length at most $\ell/3$ between two critical vertices in the obtained graph, containing only edges of weight one, it will be disjoint from the first two and this would contradict Observation 27. The observation is proved since we deleted in total at most $8 + 12 = 20$ edges in $G^{\leqslant 2}_{3,i}$. □

Lemma 32. *Suppose that there are at least* $13\ell + 273$ *critical vertices in* $G^{\leqslant 2}_{3,3-i}$*. Then, one may delete at most* 20 *edges in* $G^{\leqslant 2}_{3,i}$ *to obtain a graph* $G''_{3,i}$*, which does not contain a path of length at most* $\ell/3$ *between two critical vertices in* $G^{\leqslant 2}_{3,i}$*, which contains only edges of weight one.*

Proof. Out of these at least $13\ell + 273$ critical vertices, either at least $6\ell + 54$ are incident to edges of weight two and thus the number of these edges is at least $3\ell + 27$ in $G^{\leqslant 2}_{3,3-i}$, or there are at least $10\ell + 220$ vertices of degree three in $G^{\leqslant 2}_{3,i}$. In the first case, we apply Corollary 25. In the second case, we apply Corollary 31. The lemma follows. □

The results in this section suggest the following idea. For both $i = 1$ and $i = 2$, let

$$S_i = \bigcup_{v \text{ is critical in } G^{\leqslant 2}_{3,i}} N^{\leqslant 2}_{G[U_i]}[v]. \tag{3}$$

In words, S_i is the union of the set of critical vertices in $G^{\leqslant 2}_{3,i}$ together with the vertices in $G[U_i]$, which subdivide the edges, incident to the critical vertices in $G^{\leqslant 2}_{3,i}$. See Figure 10.

Corollary 33. *Suppose that* $|S_{3-i}| \geqslant 52\ell + 1092$*. Then, one may delete at most* 20 *edges in* $G^{\leqslant 2}_{3,i}$ *to obtain a graph* $G''_{3,i}$*, which does not contain a path of length at most* $\ell/3$ *between two critical vertices in* $G^{\leqslant 2}_{3,i}$*, which contains only edges of weight one.*

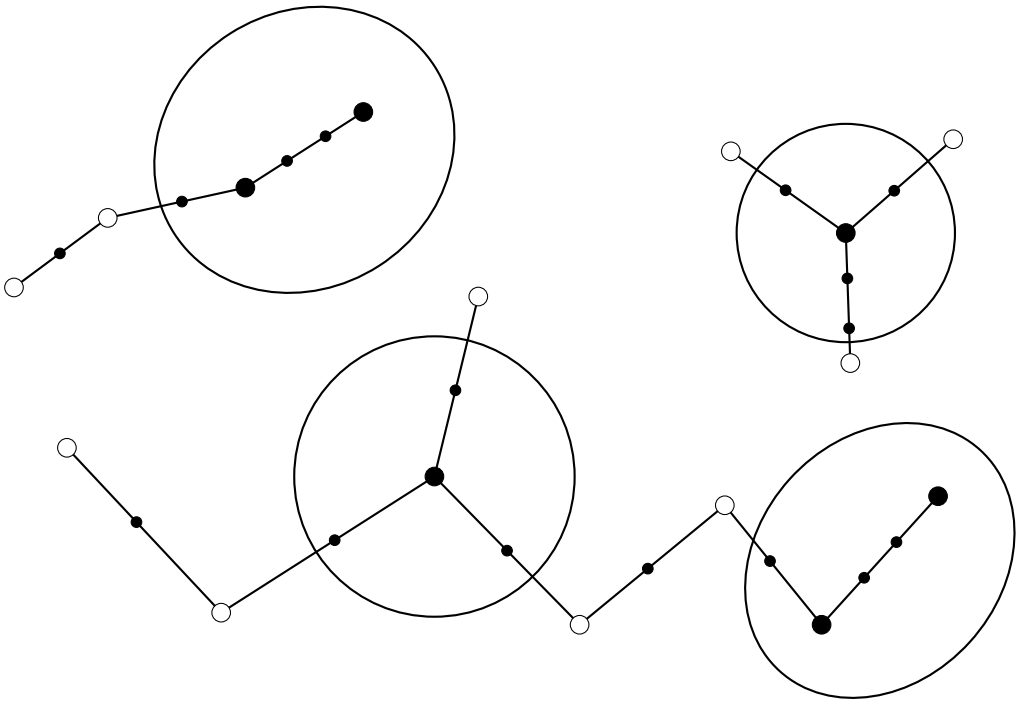


Figure 10: The figure depicts an example of the subdivision of the graph $G_{3,i}^{\leqslant 2}$ included in $G[U_i]$. The big black vertices are the critical ones in $V(G_{3,i}^{\leqslant 2}) \subseteq V(G[U_i])$. The big white vertices are the non-critical ones in $V(G_{3,i}^{\leqslant 2}) \subseteq V(G[U_i])$. The small black vertices are the ones that subdivide the edges of $G_{3,i}^{\leqslant 2}$ in $G[U_i]$. Finally, the set S_i consists of the vertices in the union of the encircled regions.

Proof. Every vertex in S_{3-i}, which subdivides an edge in $G_{3-i}^{\leqslant 2}$, is at distance one to at least one critical vertex. On the other hand, every critical vertex in $G_{3-i}^{\leqslant 2}$ is at distance one in $G[U_{3-i}]$ to at most three vertices in S_{3-i}. Thus, the number of critical vertices in $G_{3,3-i}^{\leqslant 2}$ is at least $|S_{3-i}|/4 \geqslant 13\ell + 273$. It remains to apply Lemma 32. □

In the rest of the paper we assume without loss of generality that $|S_1| \leqslant |S_2|$.

Lemma 34. *For both $i = 1$ and $i = 2$, at most one of the following happens:*

1. *The sum of the weights of the edges*

$$\{e \in E(G_{3,i}) \mid p(e) \geqslant 3\}$$

   is at least 34.

2. $|S_{3-i}| \geqslant 129$.

Proof. We argue by contradiction. Suppose that each of the above events happens for some $i \in \{1, 2\}$. Then, by Observation 17 and Observation 18 we know that there are at most eight edges of weight two and at most 20 vertices of degree three.

Suppose that there is an edge e in $G^+_{3,3-i}$ of weight at least three and let u, v, w be three consecutive vertices subdividing this edge in $G[U_{3-i}]$. Then, by the pigeonhole principle, either there are four edges in $G_{3,i}$ of weight at least three or there is one edge in $G_{3,i}$ of weight at least twelve. In both cases, one may find three consecutive vertices u', v', w' in $G[U_i]$, subdividing an edge of $G^+_{3,i}$ and none of them being adjacent to any of u, v and w. Therefore, one may exchange u, v, w in U_{3-i} with u', v', w' in U_i and thus improve the bisection (U_1, U_2), which is a contradiction. See Figure 11. Therefore, all edges in $G_{3,3-i}$ have weight at most two. In total, this would mean that $|S_{3-i}| \leqslant 8 \times 2 \times 3 + 20 \times 4 = 128$, which is a contradiction. The observation is proved. □

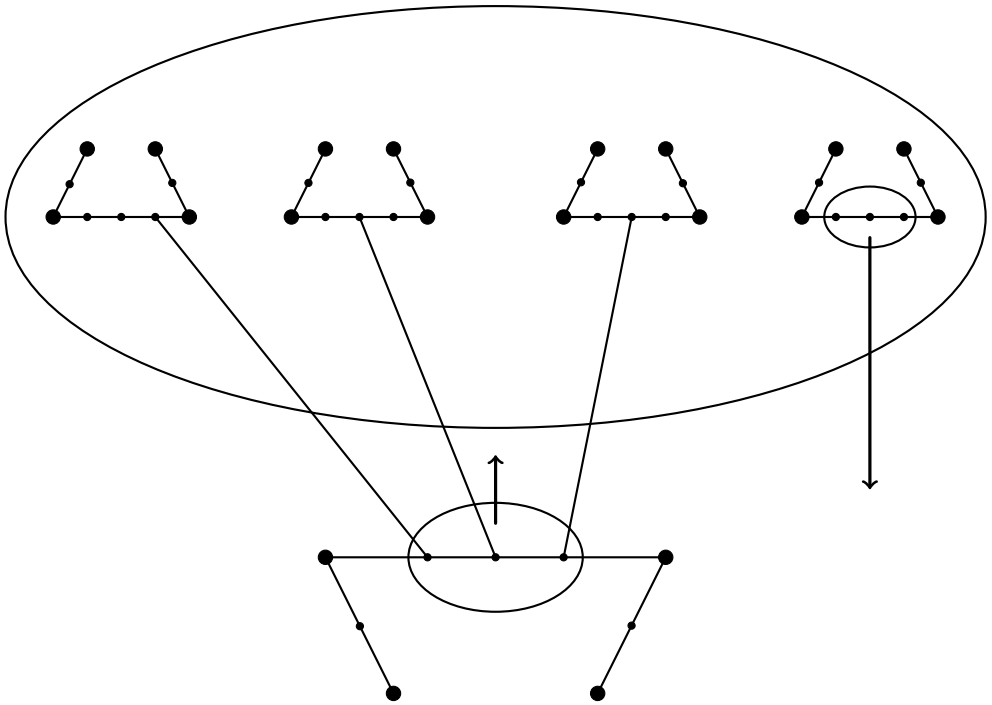


Figure 11: The improvement from the proof of Lemma 34. The large vertices are the ones in $G^{\leqslant 2}_{3,1} \cup G^{\leqslant 2}_{3,2}$. The small vertices are the subdivision vertices of the edges of $G^{\leqslant 2}_{3,1} \cup G^{\leqslant 2}_{3,2}$ in $G[U_1] \cup G[U_2]$.

Corollary 35. *If the sum of the weights of the edges*

$$\{e \in E(G_{3,1}) \mid p(e) \geqslant 3\}$$

is at least 34*, then* $|S_1| \leqslant 128$.

Proof. This follows from Lemma 34 and the assumption that $|S_1| \leqslant |S_2|$. □

We partially characterized the structure of a minimum bisection. In the next two sections we do first moment computations for two types of bisections (U_1, U_2) depending on $|S_1|$:

1. Bisections (U_1, U_2) of type one: $|S_1| \leqslant \log^2 n$. This case is treated in Section 4.
2. Bisections (U_1, U_2) of type two: $\log^2 n \leqslant |S_1|$. This case will be considered in Section 5.

4 Bisections of type one

In this case, we count bisections with $|S_1| \leqslant \log^2 n$ with S_1 defined in (3).

We define the *skeleton* $Sk(H)$ of a labeled graph H to be the unlabeled graph obtained from H by deleting the labels of the vertices of H.

Recall that by Observation 8, a.a.s. $G(n, 3)$ is a usual graph, so the results from Section 3 hold a.a.s. for $G(n, 3)$. Thus, our aim in this section is to count the number of bisections (U_1, U_2) of type one in usual 3-regular graphs. We begin by counting the number of possible skeletons of $G^+_{3,1}$. Then, we count the ways to give labels to the vertices in $G[U_1]$ in the subdivisions of these skeletons. Finally, we count the extensions of these subdivisions of $G^+_{3,1}$ to $G[U_1]$, and consequently to G.

Let $\beta \geqslant 0.1$ be such that $e(U_1, U_2) = \beta n$. Then, $G[U_1]$ contains βn vertices of degree two. Since $|S_1| \leqslant \log^2 n$, all but at most $\log^2 n$ of the edges of $G^+_{3,1}$ have weight one, and at most $\log^2 n$ vertices in $G^+_{3,1}$ have degree three.

Observation 36. *By deleting the vertices of S_1 in $G^+_{3,1}$, we obtain a graph whose connected components are paths and cycles with edges of weight one.*

Proof. The maximum degree in this graph is two, and all vertices, incident to edges of weight more than one, have been deleted. The observation follows. □

Observation 37. *The number of unlabeled graphs containing at most βn edges and of maximal degree two is $\exp(o(n))$ as $n \to \infty$.*

Proof. Any graph of the above type consists of paths and cycles. Thus, for any $t \leqslant \beta n$ and $k, \ell \in \mathbb{N}$ with $k + \ell = t$, the number of graphs with k edges in paths and ℓ edges in cycles is given by $\exp(o(k) + o(l))$ by Theorem 5. Summing over all pairs (k, ℓ) with $k + \ell \leqslant \beta n$, we obtain an upper bound of $n^2 \exp(o(n)) = \exp(o(n))$ on the number of unlabeled graphs of maximal degree two and at most βn edges. The lemma is proved. □

The main object in this section will be the graph

$$G_{pc,1} = G^+_{3,1} \setminus S_1. \tag{4}$$

Let also $G_{pc,1}$ have $\beta' n \geqslant \beta n - 3\log^2 n$ edges. Our goal is to bound from above the number of possibilities for the graph G in which (U_1, U_2) is a minimal bisection of size βn. To do

this, we first construct the skeleton of $G_{pc,1}$ and label its vertices. Then, we bound from above the number of extensions of $G_{pc,1}$ to $G[U_1]$ and consequently to G. After that, we optimize with respect to the parameter β.

Let $t_i = t_i(n)$ be the number of paths of length i in $G_{pc,1}$ and let $c_i = c_i(n)$ be the number of cycles of length i edges in $G_{pc,1}$.

Lemma 38. *The number of automorphisms of the graph $G_{pc,1}$ are*

$$\left(\prod_{i\geqslant 1} 2^{t_i} t_i!\right) \times c_1! \times 2^{c_2} c_2! \times \prod_{i\geqslant 3} (2i)^{c_i} c_i!.$$

Proof. First, the paths (respectively the cycles) of the same length are indistinguishable. Second, a path has two symmetries and a cycle has one symmetry, if it is of length one, two symmetries, if it is of length two, and $2i$ symmetries, if it is of length $i \geqslant 3$. This proves the lemma. □

Now, we count the number of bisections (U_1, U_2) of size βn. By Observation 37, the number of possible (unlabeled) graphs for $G_{pc,1}$ is subexponential. We conclude that the sum over all possibilities for $(t_i)_{i\geqslant 1}$ and $(c_i)_{i\geqslant 1}$ must be dominated, up to a subexponential factor, by the number of extensions of the unlabeled graph $G_{pc,1}$, which has the largest number of extensions among all unlabeled graphs of maximum degree two on at most βn vertices. Thus, maximizing over $(t_i)_{i\geqslant 1}$ and $(c_i)_{i\geqslant 1}$ will give us the correct exponential order of growth of the number of extensions in general. This is what we do in the sequel. For the same reason, we ignore the fact that the size of U_1 is between $n/2-5$ and $n/2+5$, since summing over all possible sizes of U_1 does not make a difference on an exponential scale.

We now explain our counting procedure step by step. Since G is a 3-regular graph on n vertices, n is even and therefore $n/2 \in \mathbb{N}$.

1. Choose in $\binom{n}{n/2}$ ways the labels of the vertices participating in U_1 and the labels of the vertices participating in U_2.

2. Choose the βn labels of the vertices of degree two in $G[U_1]$ in $\binom{0.5n}{\beta n} = \exp(o(n))\binom{0.5n}{\beta' n}$ ways. Out of these, choose in $\binom{\beta n}{\beta' n} = \exp(o(n))$ ways which of these vertices must subdivide the edges of $G_{pc,1}$.

3. Assign the labels to the vertices in $(\beta' n)!(\beta n - \beta' n)! = \exp(o(n))(\beta' n)!$ ways.

4. Choose $Tn := \sum_{i\geqslant 1}(i+1)t_i$ labels for the vertices of degree three in U_1 participating in the paths in $G_{pc,1}$ and $Cn := \sum_{i\geqslant 1} ic_i$ labels for the vertices of degree three in U_1 participating in the cycles in $G_{pc,1}$ in

$$\binom{(0.5-\beta)n}{Tn, Cn, (0.5-T-C-\beta)n} = \exp(o(n))\binom{(0.5-\beta')n}{Tn, Cn, (0.5-T-C-\beta')n}$$

ways. Here, $T = T(n)$ and $C = C(n)$ are functions of n.

5. Assign the labels to the vertices in $(Tn)!(Cn)!$ ways.

6. Divide by the size of the automorhpism group of $G_{pc,1}$, which by Lemma 38 is

$$\left(\prod_{i\geqslant 1} 2^{t_i} t_i!\right) \times c_1! \times 2^{c_2} c_2! \times \prod_{i\geqslant 3} (2i)^{c_i} c_i!,$$

since different ways to distribute the labels might lead to the same final (labeled) graph.

7. Extend the labeled copy of the optimal skeleton in

$$\frac{((1.5-4\beta'-\beta)n)!!}{2^{2(T+C-\beta')n+(\beta-\beta')n}6^{(0.5-(\beta+T+C))n}} = \exp(o(n))\frac{((1.5-5\beta')n)!!}{2^{2(T+C-\beta')n}6^{(0.5-(\beta'+T+C))n}}$$

ways to a graph $G[U_1]$. The exponent of 2 in the formula comes from the fact that the number of paths in $G_{pc,1}$ is exactly $(T+C-\beta')n$ since, first, every cycle contains the same number of edges and vertices, and second, every path contains one vertex more than edges. Moreover, every path of length at least 1 contains two vertices of degree one.

8. Choose βn labels for the vertices of degree two in U_2 in $\binom{0.5n}{\beta n} = \exp(o(n))\binom{0.5n}{\beta' n}$ ways.

9. Form the matching between the vertices of degree two in $G[U_1]$ and $G[U_2]$ in $(\beta n)! = \exp(o(n))(\beta' n)!$ ways.

10. Construct the graph $G[U_2]$ in

$$\frac{((1.5-\beta)n)!!}{2^{\beta n}6^{(0.5-\beta)n}} = \exp(o(n))\frac{((1.5-\beta')n)!!}{2^{\beta' n}6^{(0.5-\beta')n}}.$$

11. Multiply by 6^n to count configurations instead of graphs.

12. Finally, divide by the total number of 3-regular configurations $(3n-1)!!$ to find an upper bound on the proportion of bisections of type one.

The final formula is given by

$$\frac{\mathrm{e}^{o(n)}\binom{n}{0.5n}\binom{0.5n}{\beta' n}(\beta' n)!\binom{(0.5-\beta')n}{Tn,Cn,(0.5-\beta'-T-C)n}(Tn)!(Cn)!\frac{((1.5-5\beta')n)!!}{2^{2(T+C-\beta')n}6^{(0.5-(\beta'+T+C))n}}\binom{0.5n}{\beta' n}(\beta' n)!\frac{((1.5-\beta')n)!!}{2^{\beta' n}6^{(0.5-\beta')n}}6^n}{\left(\prod_{i\geqslant 1} 2^{t_i} t_i!\right) \times c_1! \times 2^{c_2} c_2! \times \left(\prod_{i\geqslant 3}(2i)^{c_i} c_i!\right)(3n-1)!!}.$$

Before proceeding with explicit optimization computation, we observe that the numerator depends only on $T+C$ as a function of T and C. We define a *cycle transformation* of some unlabeled graph H of degree at most two to be the unlabeled graph containing the exact same multiset of paths as H and in which each of the remaining vertices participates in one common cycle. Remark that neither the number of vertices in H nor the number

of edges changes by applying this transformation and therefore both $T+C$ and β' remain unchanged. Then, the denominator becomes

$$\left(\prod_{i\geqslant 1} 2^{t_i} t_i!\right) \times 2(\beta' n - \sum_{i\geqslant 1} i t_i) \times (3n-1)!!.$$

Since the term $2(\beta' n - \sum_{i\geqslant 1} i t_i) = \exp(o(n))$, we may include it in the $\exp(o(n))$ term in the beginning of the formula and thus simplify the expression, leaving only the terms in $(t_i)_{i\geqslant 1}$ and T. Moreover, taking one edge out of the large cycle (which decreases $\beta' n$ by one) and transforming it into a path changes the number of extensions by at most $\Theta(n)$. Therefore, one may suppose in this optimization part that we optimize over the unlabeled graphs H, which consist of a multiset of paths. The final formula simplifies to

$$\frac{\exp(o(n))\binom{n}{0.5n}\binom{0.5n}{\beta' n}(\beta' n)!\binom{(0.5-\beta')n}{Tn}(Tn)!\dfrac{((1.5-5\beta')n)!!}{2^{2(T-\beta')n}6^{(0.5-(\beta'+T))n}}\binom{0.5n}{\beta' n}(\beta' n)!\dfrac{((1.5-\beta')n)!!}{2^{\beta' n}6^{(0.5-\beta')n}}6^n}{\left(\prod_{i\geqslant 1} 2^{t_i} t_i!\right)(3n-1)!!}. \tag{5}$$

Now we maximize the above formula over the parameters $(t_i)_{i\geqslant 1}$ under the conditions

1. $$Tn - \sum_{i\geqslant 1} t_i = \beta' n \iff \sum_{i\geqslant 1} t_i = (T-\beta')n.$$

2. $$\sum_{i\geqslant 1} (i+1) t_i = Tn \iff \sum_{i\geqslant 1} i t_i = \beta' n,$$

where the second equivalence uses the left equality in the first one. Our first main goal is the minimize the product

$$\prod_{i\geqslant 1} t_i!$$

itself for fixed T and β' since $\prod_{i\geqslant 1} 2^{t_i} = 2^{(T-\beta')n}$.

Lemma 39. *There is a sequence $(t_i^{(n)})_{i\geqslant 1}$, which minimizes the function $(t_i)_{i\geqslant 1} \mapsto \prod_{i\geqslant 1} t_i!$ under the conditions 1 and 2, such that for all but at most one $i \geqslant 1$ we have $t_i^{(n)} \geqslant t_{i+1}^{(n)}$. Moreover, for this exceptional i we may only have $t_i^{(n)} = 0$, $t_{i+1}^{(n)} = 1$ and $t_{i+2}^{(n)} = 0$.*

Proof. Let $(t_i)_{i\geqslant 0}$ be a minimizing sequence for $\prod_{i\geqslant 1} t_i!$ under the two conditions above. Let $(t'_i)_{i\geqslant 0}$ be a sequence such that $\{t'_i\}_{i\geqslant 1} \equiv \{t_i\}_{i\geqslant 1}$ as multisets of non-negative integers and $(t'_i)_{i\geqslant 1}$ is decreasing.

Then, first, $\sum_{i\geqslant 1} t_i = \sum_{i\geqslant 1} t'_i = (T-\beta')n$ and $\prod_{i\geqslant 1} t_i! = \prod_{i\geqslant 1} t'_i!$. Moreover, let $m = \max\{i \in \mathbb{N}, t'_i \geqslant 1\}$. Now, define $(t_i^{(n)})_{i\geqslant 1}$ as follows:

$$t_i^{(n)} = \begin{cases} t'_i & \text{if } i \notin \{m, m+\beta' n - \sum_{i\geqslant 1} i t'_i\} \\ t'_i - 1 & \text{if } i = m, \\ 1 & \text{if } i = m+\beta' n - \sum_{i\geqslant 1} i t'_i. \end{cases}$$

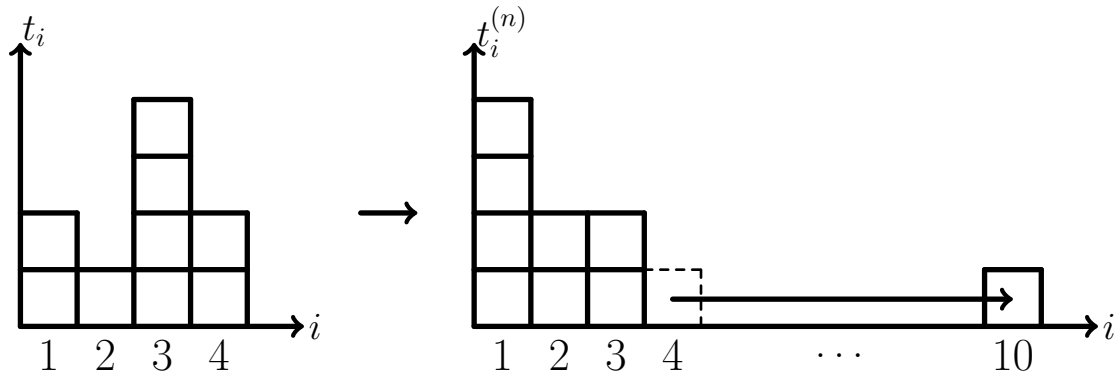


Figure 12: An example of a sequence $(t_i)_{i\geqslant 1}$ being transformed into $(t_i^{(n)})_{i\geqslant 1}$ (in this example $\beta' n - \sum_{i\geqslant 1} i t'_i = 6$).

See Figure 12. One may easily verify that the product $\prod_{i\geqslant 1} t_i^{(n)}!$ can only decrease and this time $(t_i^{(n)})_{i\geqslant 1}$ satisfies both conditions given above. □

Observation 40. *In the sequence $(t_i^{(n)})_{i\geqslant 1}$, the second-largest $i \geqslant 1$, for which $t_i^{(n)} \geqslant 1$, is less than $\sqrt{2\beta' n}$.*

Proof. Notice that all indices $j \leqslant i$ to the left of this second-largest i satisfy $t_j^{(n)} \geqslant 1$. For this second-largest i we have

$$\beta' n \geqslant \sum_{1\leqslant j\leqslant i} j t_j^{(n)} \geqslant \frac{i(i+1)}{2},$$

so $2\beta' n \geqslant i^2$. □

Corollary 41. $t_1^{(n)} \geqslant \frac{(T-\beta')n-1}{\sqrt{2\beta' n}}$.

Proof. Let i be the second-largest index, for which $t_i^{(n)} \geqslant 1$. We have that

$$\sqrt{2\beta' n}\, t_1^{(n)} \geqslant i t_1^{(n)} \geqslant \sum_{1\leqslant j\leqslant i} t_j^{(n)} = (T-\beta')n - 1.$$ □

We distinguish two cases. First, let us treat the sequences $(t^{(n)})_{i\geqslant 1}$, for which $t_1^{(n)} = 1$ (and therefore for every $i \geqslant 1$ one has $t_i^{(n)} \in \{0,1\}$). In this case, by the proof of Corollary 41, we have that $\sqrt{2\beta' n} + 1 \geqslant (T-\beta')n$. Then, one may rewrite (5) as

$$\frac{\exp(o(n))\binom{n}{0.5n}\binom{0.5n}{\beta' n}(\beta' n)!\binom{(0.5-\beta')n}{\beta' n}(\beta' n)!\frac{((1.5-5\beta')n)!!}{6^{(0.5-2\beta')n}}\binom{0.5n}{\beta' n}(\beta' n)!\frac{((1.5-\beta')n)!!}{2^{\beta' n}6^{(0.5-\beta')n}}6^n}{(3n-1)!!}.$$

By Stirling's formula we deduce that the above expression can be rewritten as

$$\exp(o(n))\left(\frac{(1.5-5\beta')^{\frac{1.5-5\beta'}{2}}(1.5-\beta')^{\frac{1.5-\beta'}{2}}6^{3\beta'}}{2^{\beta'}3^{1.5}(0.5-\beta')^{0.5-\beta'}(0.5-2\beta')^{0.5-2\beta'}}\right)^n.$$

One may easily check that the maximum of the function

$$\beta' \in [0, 0.25) \mapsto \frac{(1.5-5\beta')^{\frac{1.5-5\beta'}{2}}(1.5-\beta')^{\frac{1.5-\beta'}{2}}6^{3\beta'}}{2^{\beta'}3^{1.5}(0.5-\beta')^{0.5-\beta'}(0.5-2\beta')^{0.5-2\beta'}}$$

over the interval $[0, 0.25)$ is strictly less than one. We deduce that at most an exponentially small fraction of the configurations have bisections with $t_1^{(n)} = 1$.

We now treat the second case, in which $t_1^{(n)} \geqslant 2$.

Observation 42. *For every $j \geqslant \lceil 2\sqrt{2n}+2\rceil$, we have that $t_j^{(n)} = 0$.*

Proof. We argue by contradiction. Suppose that $t_j^{(n)} \geqslant 1$ for some $j \geqslant \lceil 2\sqrt{2n}+2\rceil$. Then, define $(s_\ell^{(n)})_{\ell\geqslant 0}$ as follows:

$$s_\ell^{(n)} = \begin{cases} t_\ell^{(n)} & \text{if } \ell \notin \{1, 1+\lceil\sqrt{2n}\rceil, j-\lceil\sqrt{2n}\rceil, j\} \\ t_1^{(n)}-1 & \text{if } \ell = 1, \\ 1 & \text{if } \ell = 1+\lceil\sqrt{2n}\rceil, \\ 1 & \text{if } \ell = j-\lceil\sqrt{2n}\rceil, \\ 0 & \text{if } \ell = j. \end{cases}$$

Using Lemma 39 and Observation 40, one may easily verify that

$$\frac{\prod_{\ell\geqslant 1} t_\ell^{(n)}!}{\prod_{\ell\geqslant 1} s_\ell^{(n)}!} = t_1^{(n)} \geqslant 2,$$

which is a contradiction, since $(s_\ell^{(n)})_{\ell\geqslant 1}$ satisfies both conditions 1 and 2 and $(t_\ell^{(n)})_{\ell\geqslant 1}$ is a sequence minimizing the function $(t_i)_{i\geqslant 1} \mapsto \prod_{i\geqslant 1} t_i!$ with these properties. □

Since for our purposes a subexponential factor in the formula (5) does not matter, by abuse we forget about this largest isolated positive term of the sequence $(t_i^{(n)})_{i\geqslant 1}$, if it exists. Indeed, by Observation 42, it contributes at most $\lceil 2\sqrt{2n}+2\rceil$ to the sum $\sum_{i\geqslant 1} i t_i^{(n)}$ and at most one to the sum $\sum_{i\geqslant 1} t_i^{(n)}$. From now on, we consider the sequence $(t_i^{(n)})_{i\geqslant 1}$ to be decreasing.

Let $m = m(n) = \max\{i \in \mathbb{N}, t_i^{(n)} \geqslant 1\}$. By Observation 42 we have that $m(n) \leqslant 2\sqrt{2n}+2$. For the terms $(t_i^{(n)})_{1\leqslant i\leqslant m}$, we bound $t_i^{(n)}!$ from below by

$$\left(\frac{t_i^{(n)}}{e}\right)^{t_i^{(n)}} \sqrt{2\pi t_i^{(n)}}.$$

We remark that the product of the terms $\sqrt{2\pi t_i^{(n)}}$ for $i \in [m]$ is at most $(\sqrt{2\pi n})^m \leqslant (\sqrt{2\pi n})^{2\sqrt{2n}+2} = \exp(o(n))$, so it is absorbed by the $\exp(o(n))$ term in the beginning of the formula. Since $\prod_{1\leqslant i\leqslant m} \exp(-t_i^{(n)}) = \exp((\beta' - T)n)$ depends only on $\sum_{i=1}^m t_i^{(n)}$ but not on any of the individual terms $t_i^{(n)}$, we need to minimize the quantity

$$\prod_{1\leqslant i\leqslant m} {t_i^{(n)}}^{t_i^{(n)}}.$$

under the constraints 1 and 2. We rewrite this as

$$n^{(\sum_{i\geqslant 1} t_i^{(n)})} \left(\prod_{1\leqslant i\leqslant m} \left(\frac{t_i^{(n)}}{n} \right)^{\frac{t_i^{(n)}}{n}} \right)^n = n^{(\beta'-T)n} \left(\prod_{1\leqslant i\leqslant m} \left(\frac{t_i^{(n)}}{n} \right)^{\frac{t_i^{(n)}}{n}} \right)^n. \tag{6}$$

Define the function

$$f_m : (t_i)_{1\leqslant i\leqslant m} \in [0,1]^m \mapsto \sum_{1\leqslant i\leqslant m} t_i \ln(t_i) \in \mathbb{R}.$$

By extending $t \in (0,1] \mapsto t\log t \in \mathbb{R}$ at zero by continuity to the value zero, f_m may be seen as a projection of the function

$$f : (t_i)_{i\geqslant 1} \in [0,1]^{\mathbb{N}} \mapsto \sum_{1\leqslant i\leqslant m} t_i \ln(t_i) \in \mathbb{R} \cup \{-\infty\}$$

onto its first m coordinates.

Clearly under the conditions

$$\sum_{i\geqslant 1} t_i = T - \beta' \tag{7}$$

and

$$\sum_{i\geqslant 1} i t_i = \beta', \tag{8}$$

the minimum of the function $f_m = f_{m(n)}$ is larger than the minimum of the function f. On the other hand, $f^{|\mathbb{R}}_{|(0,1]^{\mathbb{N}}}$ is a convex and infinitely differentiable function on an infinite-dimensional Banach space, and therefore by ([12], Theorem 1) and [17] we know that if there is some critical point in the interior of the domain, it must be unique and it must be a global minimum for the function f. This allows us to apply the method of Lagrange multipliers for the function f under the constraints (7) and (8) for any fixed n in the infinite-dimensional setting.

Let

$$F((t_i)_{i\geqslant 1}, \lambda_1, \lambda_2) := f((t_i)_{i\geqslant 1}) - \lambda_1 \sum_{i\geqslant 1} t_i - \lambda_2 \sum_{i\geqslant 1} i t_i.$$

Differentiating with respect to t_i and setting the derivative to zero gives

$$1 + \ln(t_i) - \lambda_1 - i\lambda_2 = 0 \iff t_i = \exp(\lambda_1 - 1 + i\lambda_2).$$

Therefore, we solve the following system to find λ_1 and λ_2:

$$\begin{cases} \dfrac{\exp(\lambda_1 + \lambda_2 - 1)}{1 - \exp(\lambda_2)} = T - \beta', \\ \dfrac{\exp(\lambda_1 + \lambda_2 - 1)}{(1 - \exp(\lambda_2))^2} = \beta'. \end{cases}$$

Solving this system gives

$$\begin{cases} \exp(\lambda_1) = \dfrac{e(T-\beta')^2}{2\beta' - T}, \\ \exp(\lambda_2) = \dfrac{2\beta' - T}{\beta'}. \end{cases}$$

Thus, we have that

$$\forall i \geqslant 1, t_i = \frac{(T-\beta')^2}{2\beta' - T} \left(\frac{2\beta' - T}{\beta'}\right)^i$$

is the argument where the absolute minimum of f is attained. The value of this minimum is

$$f((t_i)_{i\geqslant 1}) = \left(\log\left(\frac{(T-\beta')^2}{\beta'}\right) - \frac{2\beta - T}{\beta'} \log\left(\frac{(T-\beta')^2}{2\beta' - T}\right)\right)\beta'.$$

Plugging in this minimum consecutively into (6) and then into (5) leads to the formula

$$\exp(o(n)) \left(\frac{(1.5-5\beta')^{\frac{1.5-5\beta'}{2}} 2^{4\beta'-2T} 3^{2\beta'+T-1.5} (1.5-\beta')^{\frac{1.5-\beta'}{2}}}{(0.5-\beta)^{0.5-\beta'} (0.5-\beta'-T)^{0.5-\beta'-T} \exp\left(\beta'\left(\log\left(\frac{(T-\beta')^2}{\beta'}\right) - \frac{2\beta'-T}{\beta'}\log\left(\frac{(T-\beta')^2}{2\beta'-T}\right)\right)\right)} \right)^n.$$

Taking the logarithm of the $n-$th root of the entire formula and letting $n \to \infty$, we obtain the following expression, which remains to be maximized as a function of β' and T:

$$\left.\begin{aligned} 0.5(1.5-5\beta')\ln(1.5-5\beta') + (4\beta'-2T)\ln(2) + (2\beta'+T-1.5)\ln(3) \\ +0.5(1.5-\beta')\ln(1.5-\beta') - (0.5-\beta')\ln(0.5-\beta') - (0.5-\beta'-T)\ln(0.5-\beta'-T) \\ -\beta\left(\log\left(\frac{(T-\beta')^2}{\beta'}\right) - \frac{2\beta'-T}{\beta'}\log\left(\frac{(T-\beta')^2}{2\beta'-T}\right)\right) \end{aligned}\right\} \tag{9}$$

One may easily observe that in the range $\beta' \in [0.1, 0.1069]$ the above function is well defined for every $T \in (\beta', 2\beta')$ and may be extended by continuity at the values $T = \beta'$ and $T = 2\beta'$. Maximizing this function in the range $\{(\beta', T) : \beta \in [0.10, 0.1069],\ \beta' \leqslant T \leqslant 2\beta'\}$ gives a maximum of -3.713×10^{-5}, which is attained at the point $(\beta', T) = (0.1069, 0.1802)$. This calculation is confirmed by the graphing calculator Desmos - see Figure 13. It proves that the proportion of 3-regular graphs containing a bisection of size $\beta \leqslant 0.1069n$ of type one is exponentially small.

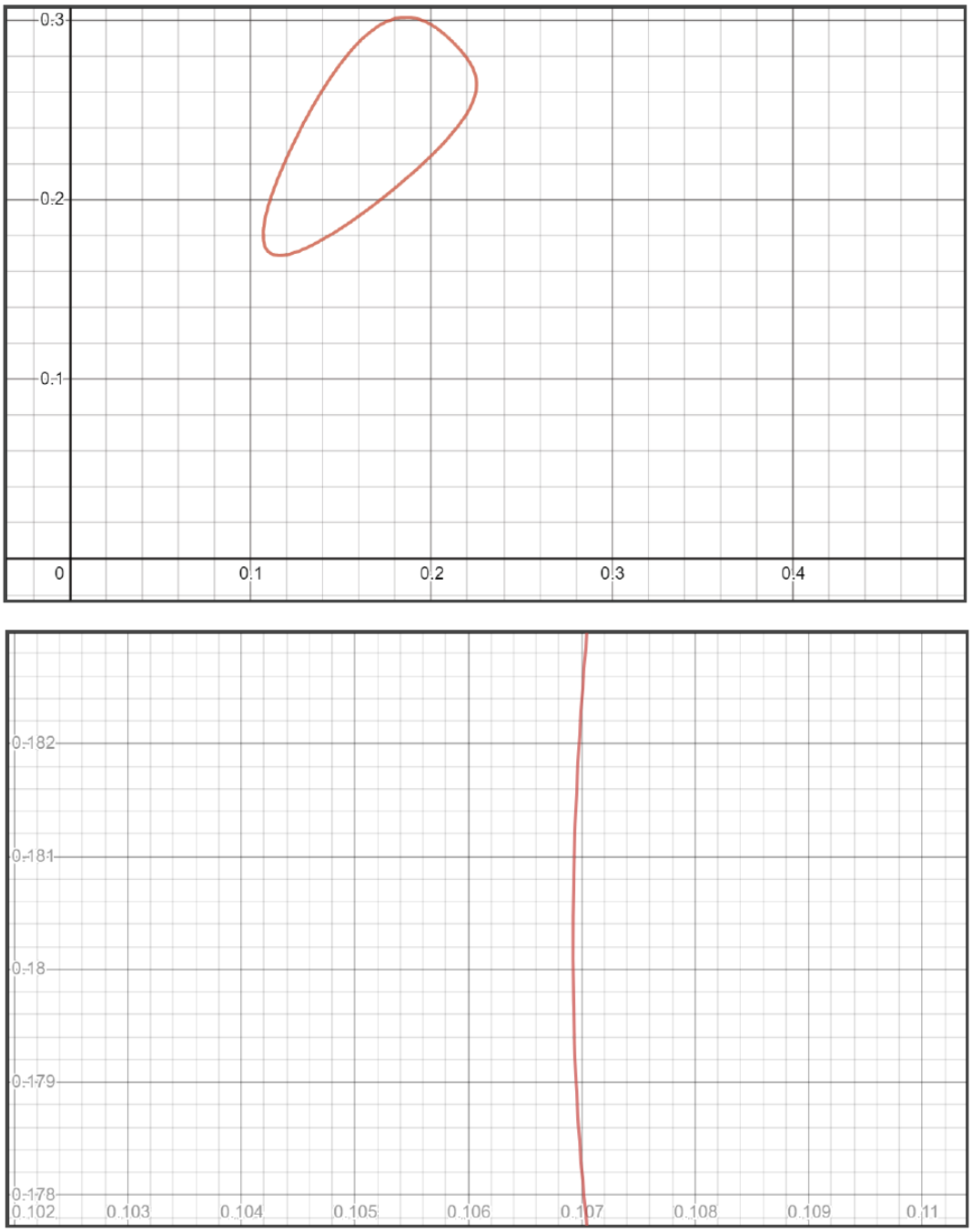


Figure 13: In both figures, the horizontal axis represents the β'-coordinate, and the vertical axis stands for the T-coordinate. In the top figure, the curve represents the set of coordinates (β', T), for which the expression (9) is equal to zero. The exterior of the encircled region is the set where (9) is negative, and its interior corresponds to the set where (9) is positive. The bottom figure is a zoomed copy of the top one around the point on the curve with minimal β'.

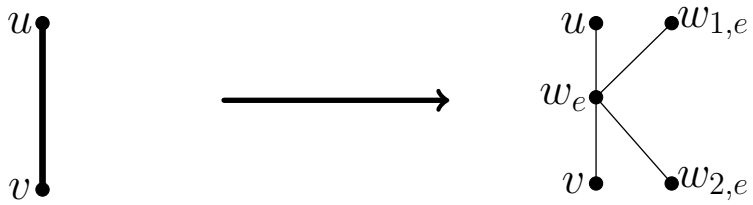


Figure 14: The transformation of an edge of weight two into a 4-star

5 Bisections of type two

This section is dedicated to counting the bisections of type two. Recall S_1 and S_2 from (3). In this section, we suppose that $\log^2 n \leqslant |S_1| \leqslant |S_2|$.

Lemma 43. *For both $i = 1$ and $i = 2$, the number of edges in $E(G^+_{3,i}) \setminus E(G^{\leqslant 2}_{3,i})$ is at most 12. Moreover, the number of extensions of the skeleton of the graph $G^{\leqslant 2}_{3,i}$ to the skeleton of the graph $G^+_{3,i}$ is $\exp(o(n))$.*

Proof. By Lemma 34 one may conclude that, for both $i = 1$ and $i = 2$ and for every large enough n, the sum of the weights of the edges

$$\{e \in E(G_{3,i}) \,|\, p(e) \geqslant 3\}$$

is at most 34. This means in particular that, for both $i = 1$ and $i = 2$, $E(G^+_{3,i}) \setminus E(G^{\leqslant 2}_{3,i})$ contains less than $\lceil 34/3 \rceil = 12$ edges and therefore the number of ways to extend $G^{\leqslant 2}_{3,i}$ to $G^+_{3,i}$ is less than

$$\sum_{0 \leqslant i \leqslant 12} n^{2i} = \exp(o(n)).$$

Indeed, knowing $G^{\leqslant 2}_{3,i}$, the number of ways to add a new edge is always at most n^2. The lemma is proved. □

Since our counting strategy will be similar to that in Section 4, polynomial factors will not be of any importance for us. Hence, we may and do assume that $|U_1| = |U_2| = n/2$ (recall that $||U_1| - |U_2|| \leqslant 10$ in general).

By Corollary 33 we know that, for both $i = 1$ and $i = 2$, by deleting at most 20 edges from $G^{\leqslant 2}_{3,i}$ we can obtain a graph in which the minimal length of a path of edges of weight one between critical vertices is at least $\frac{\log^2 n - 1092}{3 \cdot 52}$. We call this graph $G_{ld,i}$. This graph inherits the weights of the edges that come from $G^{\leqslant 2}_{3,i}$.

Definition 44 (The graphs $\boldsymbol{G'_{ld,i}}$ and $\boldsymbol{G''_{ld,i}}$)**.** For $i \in \{1, 2\}$, define the graph $G'_{ld,i}$ from $G_{ld,i}$ as follows: for every edge $e = uv$ of $G_{ld,i}$ of weight two, delete e and add three vertices $w_e, w_{1,e}, w_{2,e}$ together with the edges $w_e v, w_e u, w_e w_{1,e}, w_e w_{2,e}$.

By definition, the distance between every pair of vertices of degree three or four in $G'_{ld,i}$ is at least $\frac{\log^2 n - 1092}{3 \cdot 52}$. Construct the graph $G''_{ld,i}$ by deleting every edge of weight one

in $G'_{ld,i}$ with endvertices at distance exactly $\left\lfloor \frac{\log^2 n - 1092}{2 \cdot 3 \cdot 52} \right\rfloor - 2$ and $\left\lfloor \frac{\log^2 n - 1092}{2 \cdot 3 \cdot 52} \right\rfloor - 1$ from a vertex of degree three or four in $G'_{ld,i}$.

Roughly speaking, the construction of $G'_{ld,i}$ replaces every edge of weight two in $G_{ld,i}$ by a 4-star (see Figure 14), while the construction of $G''_{ld,i}$ ensures that every connected component contains at most one vertex of degree more than two.

Observation 45. *Given the skeleton of the graph $G'_{ld,i}$, there is a unique weighted unlabeled graph $H = Sk(G_{ld,i})$ such that $Sk(G'_{ld,i})$ is obtained from H by the operation from Definition 44.*

Proof. Consider a vertex of degree four in $G'_{ld,i}$ and denote its four neighbors by w_1, w_2, w_3, w_4. We consider three cases.

- If two of w_1, w_2, w_3, w_4, say w_1 and w_2, are of degree at least two, then the star came from an edge w_1w_2 of weight two.
- If only one of w_1, w_2, w_3, w_4 is of degree at least two, say w_1, then the star came from an edge of weight two with endvertices w_1 and some leaf in $G_{ld,i}$. Since skeletons are unlabeled graphs, this leaf may be an arbitrary vertex among w_2, w_3 or w_4.
- If none of w_1, w_2, w_3, w_4 is of degree at least two, then the star came from an isolated edge in $G_{ld,i}$ of weight two.

Finally, there is a unique way to reconstruct $G'_{ld,i}$ in each of the cases, as desired. □

Observation 46. *For every large enough n, the number of deleted edges of $G'_{ld,i}$ in the construction of $G''_{ld,i}$ is at most $\frac{624n}{\log^2 n}$.*

Proof. For every edge in $G'_{ld,i}$ to be deleted in the construction of $G''_{ld,i}$, associate to it the shortest path from some of its endvertices to a vertex of degree three or four. Notice that these paths are well defined and edge-disjoint since, first, in $G'_{ld,i}$ all pairs of vertices of degree three or four are at distance at least $\frac{\log^2 n - 1092}{3 \cdot 52}$, and second, each of these paths has length exactly $\left\lfloor \frac{\log^2 n - 1092}{2 \cdot 3 \cdot 52} \right\rfloor - 1$ (the edge to be deleted is counted as an edge of the path). We conclude that for every large enough n, the total number of deleted edges must be at most

$$\frac{3n/2}{\left\lfloor \frac{\log^2 n - 1092}{2 \cdot 3 \cdot 52} \right\rfloor - 1} \leqslant \frac{2 \cdot 3 \cdot 52 \cdot 2n}{\log^2 n} = \frac{624n}{\log^2 n}.$$

The observation is proved. □

Corollary 47. *The number of possible skeletons for the graph $G''_{ld,i}$ is $\exp(o(n))$.*

Proof. First, recall that all connected components in $G''_{ld,i}$ contain at most one vertex of degree three or four. Moreover, by construction $Sk(G''_{ld,i})$ is the union of a forest F on $N \leqslant n$ vertices, and a 2-regular graph H. By Lemma 6 with $M = 4$, there are $\exp(o(N)) = \exp(o(n))$ choices for F. Moreover, Theorem 5 implies that there are $\exp(\Theta(\sqrt{n - N})) = \exp(o(n))$ choices for H. Since $Sk(G''_{ld,i})$ is determined by the pair (F, H) (which can also be chosen in $\exp(o(n))$ ways), the corollary follows. □

By applying the reverse transformation of the graph $G''_{ld,i}$ of 4-stars into edges of weight two (see Figure 14), we conclude that the connected components in the skeleton of the graph $G_{ld,i}$ can be:

- paths with at most one edge of weight two,
- cycles with edges of weight one,
- subdivisions of 3-stars with edges of weight one, and
- at most $\frac{624n}{\log^2 n} + 12$ more edges due to Lemma 43 and Observation 46.

We will base our first moment computation on this decomposition. Since we are in fact interested in counting graphs containing a bisection of size βn and not bisections themselves, we will divide by a factor, which ensures that the same graph is not counted separately for too many bisections. In the next lemma (that might be of independent interest), let H be a bipartite graph with parts H_1 and H_2 of maximal degree two.

Lemma 48. *H contains an independent set I, which contains at least $\left\lceil \frac{|V(H_1)|}{2} \right\rceil - 1$ vertices in H_1 and at least $\left\lceil \frac{|V(H_2)|}{2} \right\rceil - 1$ vertices in H_2.*

Proof. Since isolated vertices of H may always be added to any independent set, suppose without loss of generality that there are none. The components of the graph H are paths and even cycles. Let $p_1, p_2, \ldots, p_{k'}$ be the paths of even length containing one more vertex of H_1 than of H_2 with $2 \leqslant |p_1| \leqslant |p_2| \leqslant \ldots \leqslant |p_{k'}|$ and also let $q_1, q_2, \ldots, q_{k''}$ be the paths of even length containing one more vertex of H_2 than of H_1 with $2 \leqslant |q_1| \leqslant |q_2| \leqslant \ldots \leqslant |q_{k''}|$. Suppose without loss of generality that $k' \geqslant k''$. Since the property from the statement of the lemma is decreasing, one may add edges to H to form cycles from the paths of odd length and also to form one long cycle by consecutively joining the endvertices of the paths $p_{k'-k''+1}, q_1, p_{k'-k''+2}, q_2, \ldots, p_{k'}, q_{k''}, p_{k'-k''+1}$ in this order while keeping the sets H_1 and H_2 independent (see Figure 15). Let the new bipartite graph be called G with parts H_1 and H_2 and define $k = k' - k''$. Let I be the empty set in the beginning. We consider two cases:

- The number of vertices of H_1 in $\bigcup_{1 \leqslant i \leqslant k} p_i$ is at most $\left\lceil \frac{|V(H_1)|}{2} \right\rceil - 1$. Add these to I and then start exploring the cycles vertex by vertex in the following way: First, choose a cycle $c = v_0 v_1 \ldots v_{2s-1} v_0$, a vertex (say v_0) in $H_1 \cap c$ and a direction. Then, start adding to I the vertices $v_2, v_4, \ldots$ consecutively. If the vertex v_{2s-2} is added to I, and I still contains less than $\left\lceil \frac{|V(H_1)|}{2} \right\rceil - 1$ vertices of H_1, then choose another cycle and iterate. When this number of vertices of H_1 in I is reached, then either jump over the next vertex of H_1 in the cycle we are just exploring and start in the same way adding vertices of H_2 to I, if possible, or go to another cycle and do the same (for H_2). This ensures that in total the number of vertices from H_2 added to I is at least
$$\frac{|V(G)|-2}{2} - \left(\left\lceil \frac{|V(H_1)|}{2} \right\rceil - 1 \right) \geqslant \left\lceil \frac{|V(H_2)|}{2} \right\rceil - 1,$$

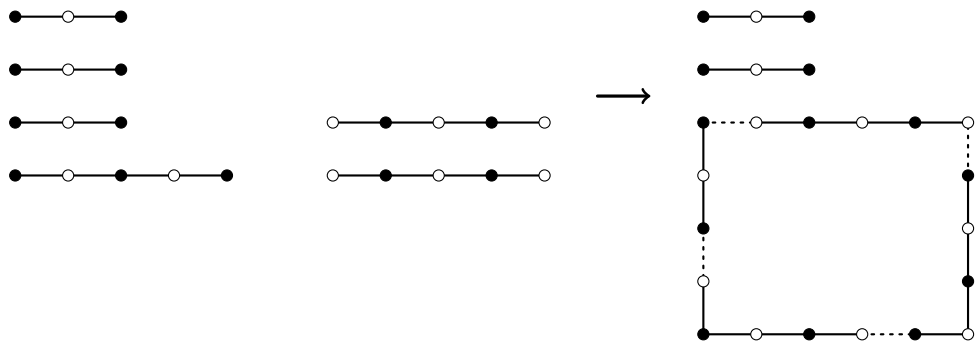


Figure 15: Joining the paths of even length in the proof of Lemma 48. The vertices of H_1 are black, and the vertices of H_2 are white. The dashed edges are added to form cycles.

which is enough to conclude.

- The number of vertices of H_1 in $\bigcup_{1\leqslant i\leqslant k} p_i$ is more than $\left\lceil \frac{|V(H_1)|}{2} \right\rceil - 1$. Then, start exploring the paths of $p_1, p_2, \ldots, p_k$ in this order. For a path p, add to I one of the endvertices of p and then continue adding vertices of H_1 in order until either we explore p to the other endvertex (in which case we redo the exploration process with the next path) or the number of vertices of H_1 in I reaches $\left\lceil \frac{|V(H_1)|}{2} \right\rceil - 1$. We add the vertices of H_2 outside the neighborhood of $I \cap H_1$ in G to I and prove that this independent set I satisfies the condition of the lemma. Let $\ell_1, \ell_2, \ldots, \ell_k$ be the lengths of the paths $p_1, p_2, \ldots, p_k$ respectively, and suppose that there are exactly ℓ edges in cycles. Suppose that, for some $i \in [k]$, for every $j \leqslant i-1$, all vertices from H_1 in the path p_j are in I, and there are b vertices from H_1 in the path p_i in I. On the one hand, the number of vertices in H_2 outside the neighborhood of $H_1 \cap I$ is exactly $\frac{1}{2}(\ell + \ell_{i+1} + \ldots \ell_k + \max\{0, \ell_i - 2b\})$. On the other hand, since at most half of the vertices in H_1 have been added to I,

$$\sum_{1\leqslant j\leqslant i-1} \left(\frac{\ell_j}{2} + 1\right) + b \leqslant \frac{1}{2}\left(\frac{\ell}{2} + \sum_{1\leqslant j\leqslant k} \left(\frac{\ell_j}{2} + 1\right)\right).$$

We deduce that

$$\sum_{1\leqslant j\leqslant i-1} (\ell_j + 2) + 2b \leqslant \ell + \sum_{i+1\leqslant j\leqslant k} (\ell_j + 2) + \ell_i + 2 - 2b.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}(\ell + \ell_{i+1} + \cdots + \ell_k + \max\{0, \ell_i - 2b\}) \\
\geqslant\ &\frac{1}{2}(\ell + \ell_{i+1} + \cdots + \ell_k + \ell_i - 2b) \\
\geqslant\ &\frac{1}{2}(\ell_1 + \cdots + \ell_{i-1} + 2b + 2(i-1) - 2(k-i+1)) \\
=\ &\frac{1}{2}(\ell_1 + \cdots + \ell_{i-1} + 2b + 2(i - (k-i) - 2)).
\end{aligned}$$

However, there are at most $\frac{1}{2}(\ell_1 + \cdots + \ell_{i-1} + 2b)$ vertices in the neighborhood of $H_1 \cap I$ that belong to H_2. Moreover, $i \geqslant k - i$, since by definition the number of vertices from H_1 in the union of the paths $p_1, p_2, \ldots, p_i$ plus one must be at least as large as the number of vertices from H_1 in the union of the paths $p_{i+1}, \ldots, p_k$ (recall that $2 \leqslant |p_1| \leqslant |p_2| \leqslant \ldots \leqslant |p_k|$). Therefore, the number of vertices from H_2 added to I is at least the number of vertices from H_2 in the neighborhood of $H_1 \cap I$ minus two.

This finishes the proof of the lemma. □

We will be interested in an upper bound on the number of 3-regular graphs, containing a bisection of size βn, coming from a first moment computation. As we shall see below, as in Section 4, only the exponential order in this upper bound will be of any importance. Therefore, by Corollary 47 it is sufficient to count the extensions of the skeleton of $G''_{ld,i}$, which contains the largest number of them.

To characterize this particular skeleton, we do a new transformation, which was already presented in Section 4. We merge all cycles in this unlabeled graph into one very large cycle. This transformation changes neither the number of vertices of any degree in $G''_{ld,i}$ nor the number of edges in $G''_{ld,i}$. We conclude that the number of extensions of the newly obtained graph to $G_{3,i}$ remains the same as the number of extensions of the original graph $G''_{ld,i}$ itself. Moreover, as already observed, a constant number of edges contributes $\exp(o(n))$ to the counting given by the first moment. Therefore, after merging the cycles in $G''_{ld,i}$ one may delete one edge from the obtained very long cycle, and thus we are left only with paths and subdivisions of stars. By abuse of notation, we denote by $G''_{ld,i}$ the graph after the transformation as well.

Let $\beta'_1 n$ (respectively $\beta'_2 n$) be the number of vertices of degree two in $G[U_1]$ (respectively in $G[U_2]$), which subdivide the paths and the 3-stars in $G''_{ld,1}$ (respectively in $G''_{ld,2}$), and let βn be the total number of vertices of degree two on both sides. Note that β, β'_1 and β'_2 are functions of n with $\max\{(\beta - \beta'_1)n, (\beta - \beta'_2)n\} \leqslant \frac{624n}{\log^2 n} + 12$. For $i, j, \ell \in \mathbb{N}$ with $\ell \leqslant j \leqslant i$, let x'_i be the number of paths containing no edge of weight two of length i in $G''_{ld,1}$, $x'_{j,i}$ be the number of paths with an edge of weight two in position $j \leqslant \frac{i+1}{2}$ in $G''_{ld,1}$ and of length i, and $y'_{i,j,l}$ be the number of stars with branches of length i, j, ℓ in $G''_{ld,1}$. Let also x''_i be the number of paths containing no edge of weight two of length i in

$G''_{ld,2}$, $x''_{j,i}$ be the number of paths with edge of weight two in position $j \leqslant \frac{i+1}{2}$ in $G''_{ld,2}$ and of length i, and $y''_{i,j,l}$ be the number of stars with branches of length i, j, l in $G''_{ld,1}$.

Now, for any given β, we give an upper bound, up to $\exp(o(n))$, of the number of graphs that contain a bisection of type two of size βn. To this end, we find the skeleton of $G''_{ld,i}$, whose subdivision admits a maximal number of extensions, up to a $\exp(o(n))-$factor, to $G[U_i]$ by maximizing the first moment. We now explain the factors appearing in the counting in this first moment, as in the previous section.

1. Choose the $n/2$ labels of the vertices in U_1 (and respectively in U_2) in

$$\binom{n}{n/2}$$

ways.

2. Choose βn labels for the vertices in U_1, which are going to be vertices of degree two in $G[U_1]$, and another βn labels for the vertices in U_2, which are going to be vertices of degree two in $G[U_2]$, in

$$\binom{n/2}{\beta n}^2 = \exp(o(n))\binom{n/2}{\beta_1 n}\binom{n/2}{\beta_2 n}$$

ways. Furthermore, choose the labels of the vertices of degree two in the subdivision of $G''_{ld,1}$ in $\binom{\beta n}{\beta'_1 n} = \exp(o(n))$ ways and choose labels of the the vertices in the subdivision of $G''_{ld,2}$ in $\binom{\beta n}{\beta'_2 n} = \exp(o(n))$ ways. Here we use that for every constant $c > 0$, by Stirling's formula

$$\left\lfloor \frac{cn}{\log^2 n} \right\rfloor ! = o\left(\left(\frac{cn}{\log^2 n}\right)^{\frac{cn}{\log^2 n}}\right) = \exp(o(n)).$$

3. Distribute the labels of the vertices of degree two on the edges of the unlabeled graph $G''_{ld,1}$ in

$$((\beta - \beta'_1)n)!(\beta'_1 n)! = \exp(o(n))(\beta'_1 n)!$$

ways. Also, distribute the labels of the vertices of degree two on the edges of the unlabeled graph $G''_{ld,2}$ in

$$((\beta - \beta'_2)n)!(\beta'_2 n)! = \exp(o(n))(\beta'_2 n)!$$

ways.

4. Divide by the product of the sizes of the automorphism groups of $G''_{ld,1}$ and $G''_{ld,2}$,

which is given by

$$\prod_{i\geqslant 1} 2^{x'_i}(x'_i)!2^{x''_i}(x''_i)! \prod_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} (x'_{j,i})!(x''_{j,i})! \prod_{i\geqslant 1,i \text{ odd}} 2^{x'_{\frac{i+1}{2},i}} \prod_{i\geqslant 1,i \text{ odd}} 2^{x''_{\frac{i+1}{2},i}}$$
$$\prod_{l\geqslant j\geqslant i\geqslant 1} (y'_{i,j,l})! \prod_{l>j\geqslant 1} 2^{y'_{j,j,l}} \prod_{j>i\geqslant 1} 2^{y'_{i,j,j}} \prod_{j\geqslant 1} 6^{y'_{j,j,j}}$$
$$\prod_{\ell\geqslant j\geqslant i\geqslant 1} (y''_{i,j,l})! \prod_{l>j\geqslant 1} 2^{y''_{j,j,l}} \prod_{j>i\geqslant 1} 2^{y''_{i,j,j}} \prod_{j\geqslant 1} 6^{y''_{j,j,j}}.$$

5. For $i = 1, 2$, we introduce the parameters t_i, the number of connected components in $G''_{ld,i}$, k_i, the number of edges of weight two in $G''_{ld,i}$, and l_i, the number of leaves in $G''_{ld,i}$. Thus we have

$$\sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} x'_{j,i} = k_1 n;$$
$$\sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} x''_{j,i} = k_2 n;$$
$$\sum_{i\geqslant 1} x'_i + \sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} x'_{j,i} + \sum_{l\geqslant j\geqslant i\geqslant 1} y'_{i,j,l} = t_1 n;$$
$$\sum_{i\geqslant 1} x''_i + \sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} x''_{j,i} + \sum_{l\geqslant j\geqslant i\geqslant 1} y''_{i,j,l} = t_2 n;$$
$$\sum_{i\geqslant 1} ix'_i + \sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} (i+1)x'_{j,i} + \sum_{l\geqslant j\geqslant i\geqslant 1} (i+j+l)y'_{i,j,l} = \beta'_1 n;$$
$$\sum_{i\geqslant 1} ix''_i + \sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} (i+1)x''_{j,i} + \sum_{l\geqslant j\geqslant i\geqslant 1} (i+j+l)y''_{i,j,l} = \beta'_2 n;$$
$$\sum_{i\geqslant 1} 2x'_i + \sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} 2x'_{j,i} + \sum_{l\geqslant j\geqslant i\geqslant 1} 3y'_{i,j,l} = \ell_1 n;$$
$$\sum_{i\geqslant 1} 2x''_i + \sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} 2x''_{j,i} + \sum_{l\geqslant j\geqslant i\geqslant 1} 3y''_{i,j,l} = \ell_2 n.$$

6. Express the number of vertices of degree three in $G[U_1]$ participating in $G''_{ld,1}$ (the same computation holds for $G''_{ld,2}$ and $G[U_2]$, respectively). Note that this number is equal to the number of vertices of degree two in the subdivision of $G''_{ld,1}$, plus a "correction" of one vertex per connected component without edge of weight two. In total this yields $(\beta'_1 + t_1 - k_1)n$ vertices in $G''_{ld,1}$ and $(\beta'_2 + t_2 - k_2)n$ vertices in $G''_{ld,2}$. Thus, choose the labels of these vertices on the two sides in

$$\binom{(0.5-\beta)n}{(\beta'_1+t_1-k_1)n}\binom{(0.5-\beta)n}{(\beta'_2+t_2-k_2)n} = \exp(o(n))\binom{(0.5-\beta'_1)n}{(\beta'_1+t_1-k_1)n}\binom{(0.5-\beta'_2)n}{(\beta'_2+t_2-k_2)n}$$

ways and distribute them in the skeleton in

$$((\beta'_1 + t_1 - k_1)n)!((\beta'_2 + t_2 - k_2)n)!$$

ways.

7. Choose the remaining edges completing the graphs $G[U_1]$ and $G[U_2]$ in

$$\left(\frac{((1.5-4\beta_1'-\beta+2k)n)!!}{2^{(\ell_1+\beta-\beta_1')n}6^{(0.5-\beta-\beta_1'-(t_1-k_1))n}}\right)\left(\frac{((1.5-4\beta_2'-\beta+2k)n)!!}{2^{(\ell_2+\beta-\beta_2')n}6^{(0.5-\beta-\beta_2'-(t_2-k_2))n}}\right)$$
$$=\exp(o(n))\left(\frac{((1.5-5\beta_1'+2k_1)n)!!}{2^{\ell_1 n}6^{(0.5-2\beta_1'-(t_1-k_1))n}}\right)\left(\frac{((1.5-5\beta_2'+2k_2)n)!!}{2^{\ell_2 n}6^{(0.5-2\beta_2'-(t_2-k_2))n}}\right)$$

ways.

8. Multiply by $(\beta n)! = \exp(o(n))\sqrt{(\beta_1' n)!(\beta_2' n)!}$ for the matching between the two parts.

9. Multiply by 6^n to count configurations instead of graphs.

10. Divide by the total number of $(3n-1)!!$ configurations to transform the counting into a probabilistic upper bound.

11. Up to now, we counted only bisections and did not take into consideration the fact that these may come from the same graph. Now, we make one step further, partially regrouping the bisections according to the 3-regular graph G they come from.

    Consider the vertices of degree two in $G[U_1]$ and $G[U_2]$, which have a neighbor of degree two on the same side. These vertices are exactly the ones which subdivide in $G[U_1]$ and $G[U_2]$ the edges of weight at least two in $G^+_{3,1}$ and $G^+_{3,2}$, respectively. We concentrate on the pairs of vertices subdividing edges of weight two in $G''_{ld,1}$ and $G''_{ld,2}$. Contracting each of these pairs to a single vertex and considering the graph induced by the edges in the cut, which are incident to at least one vertex in the contracted pairs, leads to a bipartite graph of maximal degree two, see Figure 16. We also know the number of vertices in each part of this graph, these are $k_1 n$ and $k_2 n$, respectively. By Lemma 48 one may form an independent set in this graph with $\left\lceil \frac{k_1 n}{2} \right\rceil - 1$ vertices from the first part and $\left\lceil \frac{k_2 n}{2} \right\rceil - 1$ vertices from the second part. Now, notice that to form a bisection of G, it is sufficient to put the vertices in any $\left\lceil \frac{k_1 n}{2} \right\rceil - 1$ of the contracted pairs into U_1 and the remaining pairs into U_2 to form a minimal bisection of G. This makes a total of

$$\binom{\lceil k_1 n/2 \rceil + \lceil k_2 n/2 \rceil - 2}{\lceil k_1 n/2 \rceil - 1}$$

    choices, which correspond to different bisections of the same size coming from the same graph.

Using that $\ell_1 = 2t_1 + \sum_{l \geqslant j \geqslant i \geqslant 1} y_{i,j,l}$ and $\ell_2 = 2t_2 + \sum_{l \geqslant j \geqslant i \geqslant 1} y_{i,j,l}$, the final formula

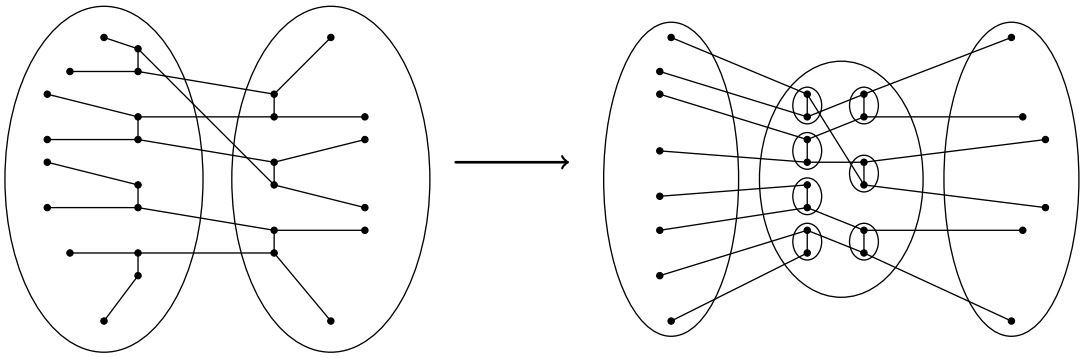


Figure 16: The formation of the bipartite graph of maximal degree two from the pairs of vertices subdividing the edges of weight two in $G''_{ld,1}$ and $G''_{ld,2}$. The only edges in the cut on the left that are depicted, are the ones that participate in the bipartite graph.

reads

$$\begin{aligned}
&\exp(o(n))\binom{n}{n/2}\binom{n/2}{\beta'_1 n}\binom{n/2}{\beta'_2 n}(\beta'_1 n)!(\beta'_2 n)!\\
&\times\left(\prod_{i\geqslant 1}2^{x'_i}(x'_i)!\prod_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1}(x'_{j,i})!\prod_{i\geqslant 1,i\text{ odd}}2^{x'_{\frac{i+1}{2},i}}\prod_{l\geqslant j\geqslant i\geqslant 1}(y'_{i,j,l})!\prod_{l>j\geqslant 1}2^{y'_{j,j,l}}\prod_{j>i\geqslant 1}2^{y'_{i,j,j}}\prod_{j\geqslant 1}6^{y'_{j,j,j}}\right)^{-1}\\
&\times\left(\prod_{i\geqslant 1}2^{x''_i}(x''_i)!\prod_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1}(x''_{j,i})!\prod_{i\geqslant 1,i\text{ odd}}2^{x''_{\frac{i+1}{2},i}}\prod_{l\geqslant j\geqslant i\geqslant 1}(y''_{i,j,l})!\prod_{l>j\geqslant 1}2^{y''_{j,j,l}}\prod_{j>i\geqslant 1}2^{y''_{i,j,j}}\prod_{j\geqslant 1}6^{y''_{j,j,j}}\right)^{-1}\\
&\times\binom{(0.5-\beta'_1)n}{(\beta'_1+t_1-k_1)n}\binom{(0.5-\beta'_2)n}{(\beta'_2+t_2-k_2)n}((\beta'_1+t_1-k_1)n)!((\beta'_2+t_2-k_2)n)!\\
&\times\left(\frac{((1.5-5\beta'_1+2k_1)n)!!}{2^{\ell_1}6^{(0.5-2\beta'_1-(t_1-k_1))n}}\right)\left(\frac{((1.5-5\beta'_2+2k_2)n)!!}{2^{\ell_2}6^{(0.5-2\beta'_2-(t_2-k_2))n}}\right)\\
&\times\sqrt{(\beta'_1 n)!(\beta'_2 n)!}\frac{6^n}{(3n-1)!!}\binom{\lfloor k_1 n/2\rfloor+\lfloor k_2 n/2\rfloor-3}{\lfloor k_1 n/2\rfloor-1}^{-1}
\end{aligned}$$

Similarly to Section 4, we use the method of Lagrange multipliers to minimize the product of the order of the automorphism group of $G''_{ld,1}$ and $2^{\ell_1}=2^{2t_1+\sum_{l\geqslant j\geqslant i\geqslant 1}y'_{i,j,l}}$ in the first case, and the product of the order of the automorphism group of $G''_{ld,2}$ and $2^{\ell_2}=2^{2t_2+\sum_{l\geqslant j\geqslant i\geqslant 1}y''_{i,j,l}}$ in the second case, under the constraints given in point 5 of the counting procedure above. The calculation is the same for $(x'_i)_{i\geqslant 1},(x'_{j,i})_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1},(y'_{i,j,l})_{l\geqslant j\geqslant i\geqslant 1}$ and for $(x''_i)_{i\geqslant 1},(x''_{j,i})_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1},(y''_{i,j,l})_{l\geqslant j\geqslant i\geqslant 1}$, so we only do it for the first set of variables.

Let

$$x_i = \frac{x'_i}{n}, x_{j,i} = \frac{x'_{j,i}}{n} \text{ and } y_{i,j,l} = \frac{y'_{i,j,l}}{n}.$$

Stirling's formula and the constraints in point 5 above directly imply that one needs to maximize the expression

$$\prod_{i\geqslant 1} 2^{x_i} x_i^{x_i} \prod_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} x_{j,i}^{x_{j,i}} \prod_{i\geqslant 1, i \text{ odd}} 2^{x_{\frac{i+1}{2},i}} \prod_{\ell\geqslant j\geqslant i\geqslant 1} y_{i,j,l}^{y_{i,j,l}} \prod_{l>j\geqslant 1} 2^{y_{j,j,l}} \prod_{j>i\geqslant 1} 2^{y''_{i,j,j}} \prod_{j\geqslant 1} 6^{y''_{j,j,j}}$$
$$\times \prod_{i\geqslant 1} 2^{2x_i} \prod_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} 2^{2x_{j,i}} \prod_{l\geqslant j\geqslant i\geqslant 1} 2^{3y_{i,j,l}}$$

as a function of $(x_i)_{i\geqslant 1}, (x_{j,i})_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1}, (y_{i,j,l})_{l\geqslant j\geqslant i\geqslant 1}$.

Let f be a logarithm of the above expression, that is,

$$\begin{aligned}
&f((x_i)_{i\geqslant 1}, (x_{j,i})_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1}, (y_{i,j,l})_{l\geqslant j\geqslant i\geqslant 1}) = \\
&\sum_{i\geqslant 1} x_i \ln(2) + x_i \ln(x_i) + \sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} x_{j,i} \ln(x_{j,i}) + \sum_{i\geqslant 1, i \text{ odd}} x_{\frac{i+1}{2},i} \ln(2) + \\
&+ \sum_{l\geqslant j\geqslant i\geqslant 1} y_{i,j,l} \ln(y_{i,j,l}) + \sum_{l>j\geqslant 1} y_{j,j,l} \ln(2) + \sum_{j>i\geqslant 1} y_{i,j,j} \ln(2) + \sum_{j\geqslant 1} y_{j,j,j} \ln(6) + \\
&+ \sum_{i\geqslant 1} 2x_i \ln(2) + \sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} 2x_{j,i} \ln(2) + \sum_{l\geqslant j\geqslant i\geqslant 1} 3y_{i,j,l} \ln(2).
\end{aligned}$$

Viewing $t_1, t_2, k_1, k_2, \beta'_1, \beta'_2$ as parameters, note that f contains all terms depending on the variables of x_i, $x_{j,i}, y_{i,j,l}$. Moreover, f is a strictly convex and infinitely differentiable function over its feasible domain. Therefore, by ([12], Theorem 1) and [17] the method of Lagrange multipliers may be applied to find the global minimum of f (which is its unique critical point by convexity) even in infinite-dimension.

The optimization will be performed under the constraints

$$\begin{cases}
\Lambda'_1 : \sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} x_{j,i} = k_1; \\
\Lambda'_2 : \sum_{i\geqslant 1} x_i + \sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} x_{j,i} + \sum_{l\geqslant j\geqslant i\geqslant 1} y_{i,j,l} = t_1; \\
\Lambda'_3 : \sum_{i\geqslant 1} i x_i + \sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} (i+1) x_{j,i} + \sum_{l\geqslant j\geqslant i\geqslant 1} (i+j+l) y_{i,j,l} = \beta'_1.
\end{cases}$$

The Lagrange objective function is

$$
\begin{aligned}
&F((x_i)_{i\geqslant 1},(x_{j,i})_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1},(y_{i,j,l})_{l\geqslant j\geqslant i\geqslant 1})\\
=&f((x_i)_{i\geqslant 1},(x_{j,i})_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1},(y_{i,j,l})_{l\geqslant j\geqslant i\geqslant 1})\\
&-\lambda_1'\left(\sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} x_{j,i}-k_1\right)\\
&-\lambda_2'\left(\sum_{i\geqslant 1} x_i+\sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1} x_{j,i}+\sum_{l\geqslant j\geqslant i\geqslant 1} y_{i,j,l}-t_1\right)\\
&-\lambda_3'\left(\sum_{i\geqslant 1} i x_i+\sum_{i\geqslant 1,(i+1)/2\geqslant j\geqslant 1}(i+1)x_{j,i}+\sum_{l\geqslant j\geqslant i\geqslant 1}(i+j+l)y_{i,j,l}-\beta_1'\right).
\end{aligned}
$$

Direct calculation gives

$$
\begin{aligned}
&3\ln(2)+1+\ln(x_i)-\lambda_2'-i\lambda_3'=0\Rightarrow x_i=\frac{\exp(\lambda_2'+i\lambda_3')}{8e},\\
&\text{for } j\neq\frac{i+1}{2}: 2\ln(2)+1+\ln(x_{j,i})-\lambda_1'-\lambda_2'-(i+1)\lambda_3'=0\Rightarrow x_{j,i}=\frac{\exp(\lambda_1'+\lambda_2'+(i+1)\lambda_3')}{4e},\\
&\text{for } j=\frac{i+1}{2}: 1+3\ln(2)+\ln(x_{j,i})-\lambda_1'-\lambda_2'-(i+1)\lambda_3'=0\Rightarrow x_{j,i}=\frac{\exp(\lambda_1'+\lambda_2'+(i+1)\lambda_3')}{8e},\\
&\text{for } l>j>i: 1+3\ln(2)+\ln(y_{i,j,l})-\lambda_2'-(i+j+l)\lambda_3'=0\Rightarrow y_{i,j,l}=\frac{\exp(\lambda_2'+(i+j+l)\lambda_3')}{8e},\\
&\text{for } l=j>i: 1+4\ln(2)+\ln(y_{i,j,j})-\lambda_2'-(i+2j)\lambda_3'=0\Rightarrow y_{i,j,j}=\frac{\exp(\lambda_2'+(i+2j)\lambda_3')}{16e},\\
&\text{for } l>j=i: 1+4\ln(2)+\ln(y_{j,j,l})-\lambda_2'-(2j+l)\lambda_3'=0\Rightarrow y_{j,j,l}=\frac{\exp(\lambda_2'+(2j+l)\lambda_3')}{16e},\\
&\text{for } l=j=i: 1+3\ln(2)+\ln(6)+\ln(y_{j,j,j})-\lambda_2'-3j\lambda_3'=0\Rightarrow y_{j,j,j}=\frac{\exp(\lambda_2'+3j\lambda_3')}{48e}.
\end{aligned}
$$

Inversing the system this time is more difficult, but at the same time we would like to have β_1' as a parameter and not as a function of λ_1', λ_2' and λ_3'. There is an easy solution – we keep λ_1', λ_3' and β_1' as parameters and express t_1, k_1 and λ_2' as functions of these parameters. This is not difficult since each of $x_i, x_{j,i}$ and $y_{i,j,l}$ can be represented as $\exp(\lambda_2')g(\lambda_1', \lambda_3')$ for some function g depending on the term. An analogous computation may be done for t_2, k_2, β_2' with Lagrange multipliers $\lambda_1'', \lambda_2'', \lambda_3''$. A numerical optimization with Maple shows that, optimizing over $\lambda_1', \lambda_1'', \lambda_3', \lambda_3''$ in the interval $(\beta_1', \beta_2') \in [0.1, 0.103295]$, the maximum is attained at $\lambda_1' = \lambda_1'' = 0.002428$, $\lambda_3' = \lambda_3'' = -1.412768$, yielding the value of 0.999996. We conclude that the proportion of 3-regular graphs of order n with bisection of size at most $0.103295n$ tends to zero exponentially fast with n (the maple code for the maximization can be found on the authors' webpages). Together with the results of the previous section, the lower bound of Theorem 1 follows.

6 An upper bound

In this section we show a complementary asymptotically almost sure upper bound on the bisection width of a random 3-regular graph. The approach is very much inspired by the works of [11] and [33], together with combinatorial observations from the previous sections. In order to explain it, we need a few definitions: first, define a *rooted* graph as a couple (G, ρ), with G some graph and ρ a distinguished vertex of G called *root*. A finite rooted graph (G, ρ) is said to be *uniformly rooted* if ρ is a vertex chosen uniformly at random among the vertices of G. Thus, one may consider (G, ρ) as a probability space equipped with the uniform measure. A sequence of uniformly rooted graphs $(G_n, \rho_n)_{n\geqslant 1}$ is said to *converge locally* (in the Benjamini-Schramm sense) if for every fixed $r > 0$ and every fixed graph H, $\lim_{n\to\infty} \mathbb{P}(B_{G_n}(\rho_n, r) \cong H)$ exists. Indeed, local convergence for sequences of graphs is a particular case of weak convergence of probability measures (of course, formally, one needs to embed the sequence of probability spaces into a common probability space and then argue on the basis of it). Since one often comes up with a simple description of the limit measure (let us call it μ) in terms of the empirical frequency of the balls of any radius in a given infinite graph G_∞, μ is usually abusively identified with G_∞.

The following lemma is well known (and follows from the observation that there are a.a.s. at most $o(\log n)$ cycles of any constant length):

Lemma 49. *Let G_n be a random 3-regular graph on n vertices, uniformly rooted at ρ_n. The sequence $(G_n, \rho_n)_{n\geqslant 1}$ converges locally to the infinite 3-regular tree T_3.*

In order to use the previous lemma, we first define the concept of a *factor of iid process*. Suppose that, for a (possibly infinite) graph G, a family of standard normal random variables $(Z_v)_{v\in V(G)}$ is assigned to the vertices of G. A *factor of iid process* on a graph G is a family of random variables $(X_v)_{v\in V(G)}$ such that:

1. for every $v \in V(G)$, X_v is a measurable function of the random variables $(Z_v)_{v\in V(G)}$, and

2. the joint distribution of the family $(X_v)_{v\in V(G)}$ is invariant under permutation of the indices induced by the action of any automorphism on $V(G)$.

Let $\mathcal{F}$ be the class of factors of iid processes $(\phi_v)_{v\in V(T_3)}$ on T_3 with values $\phi_v \in \{0, 1\}$, for which $\mathbb{P}(\phi_\rho = 0) = \frac{1}{2}$. The proof of Theorem 4.1 in [33] shows that

$$\limsup_{n\to\infty} \{bw(G_n)/|V(G_n)|\} \leqslant \inf_{\phi\in\mathcal{F}} \mathbb{E}(|\{v \sim \rho : \phi_v \neq \phi_\rho)\}|)/2, \tag{10}$$

where (G_n, ρ_n) is any sequence of graphs converging locally to T_3. By Lemma 49, in particular, this also applies to the random 3-regular graph. Hence, for an upper bound on the scaled bisection width it suffices to find a suitable factor of iid process ϕ on T_3 having values in $\{0, 1\}$, and for which the probability that the root obtains value 0 is $1/2$.

We say that a process $(X_v)_{v\in V(G)}$ is *a linear factor of iid process* $(Z_v)_{v\in V(G)}$ on a graph G if there exist $\alpha_0, \alpha_1, \ldots$ such that $X_v = \sum_{u\in V(G)} \alpha_{d_G(v,u)} Z_u$ for all $v \in V(G)$.

Note that X_v is always a centered Gaussian random variable. We call a collection of random variables $(Y_v)_{v\in V(G)}$ a *Gaussian process* on G if they are jointly Gaussian and Y_v is centered for every $v \in V(G)$. We say that a Gaussian process $(Y_v)_{v\in V(G)}$ is *invariant* if the distribution of the family $(Y_v)_{v\in V(G)}$ is invariant under the action of any automorphism of G on the index set $V(G)$. We use the following theorem, proven in [11]:

Theorem 50. *For any real number λ with $|\lambda| \leqslant 3$ there exists a non-trivial invariant Gaussian process $(Y_v)_{v\in V(T_3)}$ on T_3 that satisfies the eigenvector equation with eigenvalue λ, i.e. (with probability 1), for every vertex v it holds*

$$\sum_{u\in N(v)} Y_u = \lambda Y_v.$$

Moreover, the joint distribution of such a process is unique under the additional condition that, for every $v \in V(T_3)$, the variance of Y_v is 1. We will refer to this (essentially unique) process as the Gaussian wave function corresponding to the eigenvalue λ.

Let $((G_n, \rho_n))_{n\geqslant 1}$ be any sequence of uniformly rooted graphs converging locally to T_3. It was shown by Kesten (see [28]) that the spectrum of the transition operator for the simple random walk on T_3 is contained in the interval $[-2\sqrt{2}, 2\sqrt{2}]$. Moreover, Csóka, Gerencsér, Harangi and Virág showed in [11] the following theorem:

Theorem 51. *For any real number λ with $|\lambda| \leqslant 2\sqrt{2}$, there exists a sequence of linear factors of iid processes $(X_v^{(n)})_{v\in V(G_n)}$ that converges in distribution to the Gaussian wave function $(Y_v)_{v\in V(T_3)}$ corresponding to the eigenvalue λ.*

As shown in [11], the joint distribution of a Gaussian process on T_3 corresponding to the eigenvalue λ with $|\lambda| \leqslant 2\sqrt{2}$ is completely determined by its covariances $(\mathrm{Cov}(Y_u, Y_v))_{u,v\in V(T_3)}$ and satisfies $\mathrm{Cov}(Y_u, Y_v) = \sigma_{d_{T_3}(u,v)}$, where

$$\sigma_0 = 1, 3\sigma_1 - \lambda\sigma_0 = 0 \text{ and } 2\sigma_{k+1} - \lambda\sigma_k + \sigma_{k-1} = 0 \text{ for every } k \geqslant 1, \tag{11}$$

and so in particular $\sigma_1 = \lambda/3$ and $\sigma_2 = \frac{\lambda^2-3}{6}$. By the general theory of recursive sequences, the covariances decay exponentially to zero as $d_{T_3}(u, v) \to \infty$. Note that due to this decay of covariances together with Theorem 51 we may (and do) choose the process $(X_v^{(n)})_{v\in V(G_n)}$ to be of the form

$$X_v^{(n)} = \sum_{i=0}^{\lfloor \log\log n\rfloor} \sum_{u\in V(G_n),\, d_{G_n}(u,v)=i} \sigma_i Z_u, \tag{12}$$

where $(Z_v)_{v\in V(G_n)}$ is a family of independent standard normal random variables.

We consider the largest eigenvalue for which the theorems apply, that is, $\lambda = 2\sqrt{2}$, as already done in [33]. The idea is that the corresponding eigenvector has positive correlation between neighbors in T_3 (since $\lambda > 0$), and thus the cut between the set of vertices $\{v \,|\, Y_v \geqslant 0\}$ and the set of vertices $\{v \,|\, Y_v < 0\}$ has relatively few edges going across. More formally, consider the Gaussian wave $(Y_v)_{v\in V(T_3)}$ on T_3 given in Theorem 50

associated to the eigenvalue $\lambda = 2\sqrt{2}$. As before, let $((G_n, \rho_n))_{n\geqslant 1}$ be any sequence of rooted graphs converging locally to T_3. By Theorem 51, there exists a linear factor of iid process $(X_v^{(n)})_{v\in V(G_n)}$ on G_n that converges in distribution to the Gaussian wave $(Y_v)_{v\in V(T_3)}$. Since every variable in the Gaussian wave is centered, by setting for each vertex v of T_3, $\phi_v = 0$ in case $Y_v < 0$ and setting $\phi_v = 1$ in case $Y_v \geqslant 0$, we see that $\phi_v \in \{0, 1\}$ and $\mathbb{P}(\phi_\rho = 0) = \frac{1}{2}$. Hence, $\phi \in \mathcal{F}$, and by (10),

$$\limsup_{n\to\infty}\{bw(G_n)/|V(G_n)|\} \leqslant \mathbb{E}(|\{v \sim \rho : \phi_v \neq \phi_\rho)\}|)/2.$$

Now, consider the sequence $((G_n, \rho_n))_{n\geqslant 1}$. We first explain the approach of [33] using the linear factor of iid process $(X_v^{(n)})_{v\in V(G(n,3))}$ (denoted by $(X_v^{(n)})$ in the sequel to simplify notation) described above. We consider events regarding cherries. Recall that a cherry (v_1, v, v_2) consists of three vertices, $v, v_1, v_2 \in V(T_3)$, connected by edges vv_1 and vv_2. Here v is said to be the center of the cherry, and v_1 and v_2 are its endpoints. We also call a cherry (v_1, v, v_2) a *border cherry* if the sign of X_v is different from both the sign of X_{v_1} and the sign of X_{v_2}.

Consider the event that a given cherry with center at v is such that $X_v^{(n)}$ has the same sign as both $X_{v_1}^{(n)}$ and $X_{v_2}^{(n)}$. The probability of this event, in the limit as $n \to \infty$, is given by

$$\begin{aligned}&\lim_{n\to+\infty} \mathbb{P}(X_v^{(n)} \geqslant 0, X_{v_1}^{(n)} \geqslant 0, X_{v_2}^{(n)} \geqslant 0) + \mathbb{P}(X_v^{(n)} < 0, X_{v_1}^{(n)} < 0, X_{v_2}^{(n)} < 0)\\ = &\lim_{n\to+\infty} 2\mathbb{P}(X_v^{(n)} \geqslant 0, X_{v_1}^{(n)} \geqslant 0, X_{v_2}^{(n)} \geqslant 0) = 2\left(\frac{1}{8} + \frac{1}{4\pi}\left(2\arcsin r_1 + \arcsin r_2\right)\right),\end{aligned}$$

with

$$r_1 = \mathrm{Corr}(Y_v, Y_{v_1}) = \mathrm{Corr}(Y_v, Y_{v_2}) = \mathrm{Cov}(Y_v, Y_{v_2})/\sqrt{\mathrm{Var}(Y_v)\mathrm{Var}(Y_{v_2})}$$

and

$$r_2 = \mathrm{Corr}(Y_{v_1}, Y_{v_2}) = \mathrm{Cov}(Y_{v_1}, Y_{v_2})/\sqrt{\mathrm{Var}(Y_{v_1})\mathrm{Var}(Y_{v_2})}.$$

Here, we abusively view (v_1, v, v_2) as a cherry in T_3 as well. For the calculation, see for example (6.22) of [27], or also Subsection 4.1 in [11]. Plugging in the values of $\sigma_1 = \frac{2\sqrt{2}}{3}$ and $\sigma_2 = \frac{5}{6}$ we get that

$$\lim_{n\to+\infty} 2\mathbb{P}(X_v^{(n)} \geqslant 0, X_{v_1}^{(n)} \geqslant 0, X_{v_2}^{(n)} \geqslant 0) = 2\left(\frac{1}{8} + \frac{1}{4\pi}\left(2\arcsin\frac{2\sqrt{2}}{3} + \arcsin\frac{5}{6}\right)\right) \approx 0.798611.$$

Hence, we may conclude that, as $n \to \infty$, with probability tending to 0.798611 none of the edges of a cherry participates in the cut with parts $\{v \in G(n, 3) \mid X_v^{(n)} \geqslant 0\}$ and $\{v \in G(n, 3) \mid X_v^{(n)} < 0\}$. Similarly, as $n \to \infty$, the probability of a cherry (v_1, v, v_2)

having exactly one endpoint with a sign, different from the sign of $X_v^{(n)}$, is given by

$$\begin{aligned}
&\lim_{n\to+\infty} \mathbb{P}(X_v^{(n)} \geqslant 0, X_{v_1}^{(n)} \geqslant 0, X_{v_2}^{(n)} < 0) + \mathbb{P}(X_v^{(n)} < 0, X_{v_1}^{(n)} < 0, X_{v_2}^{(n)} \geqslant 0) \\
&\quad + \mathbb{P}(X_v^{(n)} \geqslant 0, X_{v_1}^{(n)} < 0, X_{v_2}^{(n)} \geqslant 0) + \mathbb{P}(X_v^{(n)} < 0, X_{v_1}^{(n)} \geqslant 0, X_{v_2}^{(n)} < 0) \\
= &\lim_{n\to+\infty} 4\mathbb{P}(X_v^{(n)} \geqslant 0, X_{v_1}^{(n)} \geqslant 0, X_{v_2}^{(n)} < 0) \\
= &\quad 4\left(\frac{1}{8} + \frac{1}{4\pi}\left(\arcsin\frac{2\sqrt{2}}{3} + \arcsin\frac{-2\sqrt{2}}{3} + \arcsin\frac{-5}{6}\right)\right) \approx 0.186429.
\end{aligned}$$

By analogy, the probability to obtain a border cherry, as $n \to \infty$, tends to

$$\begin{aligned}
&\lim_{n\to+\infty} \mathbb{P}(X_v^{(n)} \geqslant 0, X_{v_1}^{(n)} < 0, X_{v_2}^{(n)} < 0) + \mathbb{P}(X_v^{(n)} < 0, X_{v_1}^{(n)} \geqslant 0, X_{v_2}^{(n)} \geqslant 0) \\
= &\lim_{n\to+\infty} 2\mathbb{P}(X_v^{(n)} \geqslant 0, X_{v_1}^{(n)} < 0, X_{v_2}^{(n)} < 0) \\
= &\quad 2\left(\frac{1}{8} + \frac{1}{4\pi}\left(\arcsin\frac{-2\sqrt{2}}{3} + \arcsin\frac{-2\sqrt{2}}{3} + \arcsin\frac{5}{6}\right)\right) \approx 0.0149586.
\end{aligned}$$

We define a partition $(V_1^{(n)}, V_2^{(n)})$ of $V(G(n,3))$, where $V_1^{(n)} = \{v \in V(G(n,3)) \mid X_v \geqslant 0\}$ and $V_2^{(n)} = V(G(n,3)) \setminus V_1^{(n)}$.

Observation 52 (Theorem 4.1 in [33])**.** *The minimum bisection in a random* 3*-regular graph is of size a.a.s. at most*

$$(0.186429 + 2 \times 0.0149586)\frac{3n}{4} \approx 0.16226n.$$

Sketch of proof. By summing over all $3n$ cherries and taking into account that each edge is counted in four cherries, we get that, as $n \to \infty$, the expected size of the cut $(V_1^{(n)}, V_2^{(n)})$ is equal to $(0.186429 + 2 \times 0.0149586)\frac{3n}{4} \approx 0.16226n$. While this cut is not necessarily a bisection, by (12), $(X_v^{(n)})$ is a process depending on the set of standard normal variables corresponding to vertices at distance at most $\log\log n$ from v in $G(n,3)$ only. Hence, since two uniformly chosen vertices in $G(n,3)$ are at distance $\Omega(\log n)$ from each other (which follows by a direct computation), by a standard second moment argument, a.a.s. $|V_1^{(n)}| - |V_2^{(n)}| = o(n)$. Concentration of the number of border cherries, as well as the number of cherries (v_1, v, v_2) with $X_{v_1}^{(n)}$ and $X_{v_2}^{(n)}$ being of different signs, around their expected values follows from a similar second moment computation. □

We now improve on Lyons' result of Observation 52.

Observation 53. *The number of vertices* v *in* $G(n,3)$*, which are centers of at least two border cherries, is a.a.s.* $o(n)$ *as* $n \to +\infty$*.*

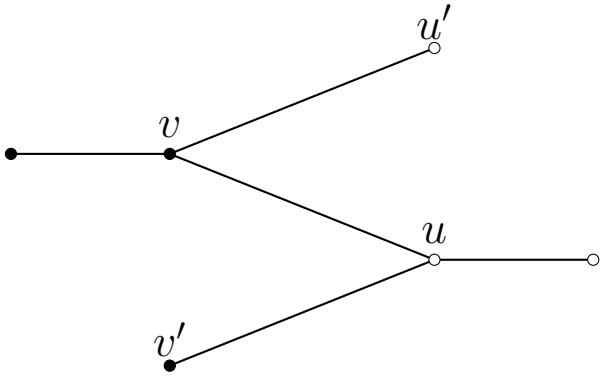


Figure 17: The construction of the bipartite graph $H^{(n)}$. The vertices in black belong to $V_1^{(n)}$, and the vertices in white belong to $V_2^{(n)}$. Since (u', v, u) is a cherry with black center and white endpoints, and (v, u, v') is a cherry with white center and black endpoints, the vertex v (u, respectively) is included in $H_1^{(n)}$ ($H_2^{(n)}$, respectively), and the edge uv is included in $H^{(n)}$.

Proof. A center v of at least two border cherries with leaves v_1, v_2, v_3 in $G(n, 3)$ must be attributed the value $X_v^{(n)}$ whose sign is different from each of $X_{v_1}^{(n)}, X_{v_2}^{(n)}, X_{v_3}^{(n)}$. Since the distribution of the process $(X_v^{(n)})_{v \in V(G(n,3))}$ approximates the one of the Gaussian wave on T_3 for the eigenvalue $\lambda = 2\sqrt{2}$, for a uniformly random vertex v of $G(n, 3)$ with neighbors v_1, v_2 and v_3 and for every $\varepsilon > 0$, a.a.s.

$$|2\sqrt{2}X_v^{(n)} - (X_{v_1}^{(n)} + X_{v_2}^{(n)} + X_{v_3}^{(n)})| \leqslant \varepsilon. \tag{13}$$

Thus, for v to be a center of at least two border cherries, we must have that either $|X_v^{(n)}| \leqslant \varepsilon$ or that (13) does not hold. By choosing ε sufficiently small, the probability of either event may be made arbitrarily close to 0 as $n \to \infty$, which shows that the expected number of centers of at least two border cherries in $G(n, 3)$ is $o(n)$. The proof is completed by Markov's inequality. □

Proof of the upper bound of Theorem 1. We construct the bipartite graph $H^{(n)}$ with parts $W_1^{(n)} \subseteq V_1^{(n)}$ and $W_2^{(n)} \subseteq V_2^{(n)}$. Its vertices are the centers of border cherries. Its edges are the ones crossing the cut $(V_1^{(n)}, V_2^{(n)})$, whose endvertices are both centers of border cherries (see Figure 17). Also, recall from the (sketch of) the proof of Observation 52 that a.a.s. $|V_1^{(n)}| - |V_2^{(n)}| = o(n)$.

Now, for a given cherry, the probability that it is a border cherry with center in $V_1^{(n)}$ is $\frac{0.0149595}{2} + o(1)$, and there are a.a.s. $3n/2 + o(n)$ cherries with centers in $V_1^{(n)}$. Thus, there are a.a.s. at least $0.0149595 \cdot \frac{3n}{2} + o(n) \approx 0.022438n + o(n)$ vertices in $W_1^{(n)}$ (concentration around the expected values follows as before). Notice that here, we use Observation 53 to say that a.a.s. $|V(H^{(n)})|$ is, up to $o(n)$, equal to the total number of centers of border cherries. We conclude by Lemma 48 that a.a.s. there exists an independent set I_n having

at least $\xi_n = \frac{0.022438n}{2} + o(n) = 0.011219n + o(n)$ vertices of $W_1^{(n)}$ and ξ_n vertices of $W_2^{(n)}$. Sending the vertices in $I_n \cap W_1^{(n)}$ to $V_2^{(n)}$ and sending back the vertices in $I_n \cap W_2^{(n)}$ to $V_1^{(n)}$ leads a.a.s. to a cut $(U_1^{(n)}, U_2^{(n)})$ with $|U_1^{(n)}| = |U_2^{(n)}| = n/2 - o(n)$ and size at most $0.16226n + o(n) - 2\xi_n = 0.139822n + o(n)$. After exchanging another $o(n)$ vertices, $(U_1^{(n)}, U_2^{(n)})$ may be transformed into a bisection, which finishes the proof. □

We remark that the value obtained is exactly the same as the one given in [21] for the number of edges not going through a maximum cut of a random 3-regular graph (the authors therein prefer to give only two digits after the decimal point). Their result is a corollary of the paper [11]. It uses a similar approach to our paper to find a maximum cut based on the eigenvalue $\lambda_{min} = -2\sqrt{2}$ of the transition operator of the random 3-regular graph. The numbers appearing in both calculations are the same. From a combinatorial point of view, the rough intuitive explanation of this phenomenon is the following: in [21] the authors pick two maximal independent sets on both sides of the cut (of equal sizes) and then add the remaining vertices in greedy order to the part where they have less neighbors. From an algorithmic point of view, this is equivalent to choosing arbitrarily the part of every vertex outside of the two initial independent sets and then switching vertices having at most one neighbor on the other side, as long as this is possible. We pick a minimum bisection given by the largest eigenvector and switch vertices having at least two neighbors on the other side, as long as this is possible. Despite the fact that this is far from a proof, as mentioned in the introduction, this phenomenon was observed before – Zdeborová and Boettcher conjectured in [38] that the number of edges crossing a minimum bisection in a random 3-regular graph is a.a.s. (up to $o(n)$) equal to the number of edges not crossing a maximum cut in such a graph.

Non-rigorous improvement. The previous argument does not take into account that vertices of degree zero in the bipartite graph $(H_1^{(n)}, H_2^{(n)})$ can be freely switched from one side to the other. Indeed, having once identified the number of these vertices on each side, they can be switched to reduce the cut, and in the remaining graph consisting of vertices of degree one or two, Lemma 48 can be applied. In order to compute the probability that one vertex v_4 in, say, $V_2^{(n)}$ with neighbors v_3, v_5 in $V_1^{(n)}$ has degree zero in $H^{(n)}$, we must have for the other two neighbors v_1, v_2 of v_3 and for the other two neighbors v_6, v_7 of v_5 that $X_{v_1}^{(n)} \geqslant 0, X_{v_2}^{(n)} \geqslant 0, X_{v_6}^{(n)} \geqslant 0$ and $X_{v_7}^{(n)} \geqslant 0$ (see Figure 18). Analogous computations hold if the vertex v_4 is in $V_1^{(n)}$ and all remaining vertices are in $V_2^{(n)}$.

Now, let us compute

$$\mathbb{P}(X_{v_1}^{(n)} > 0, X_{v_2}^{(n)} > 0, X_{v_3}^{(n)} > 0, -X_{v_4}^{(n)} > 0, X_{v_5}^{(n)} > 0, X_{v_6}^{(n)} > 0, X_{v_7}^{(n)} > 0). \tag{14}$$

We proceed as before. Using (11), we obtain that the covariance matrix of the random vector

$$(X_{v_1}^{(n)}, X_{v_2}^{(n)}, X_{v_3}^{(n)}, -X_{v_4}^{(n)}, X_{v_5}^{(n)}, X_{v_6}^{(n)}, X_{v_7}^{(n)})^T$$

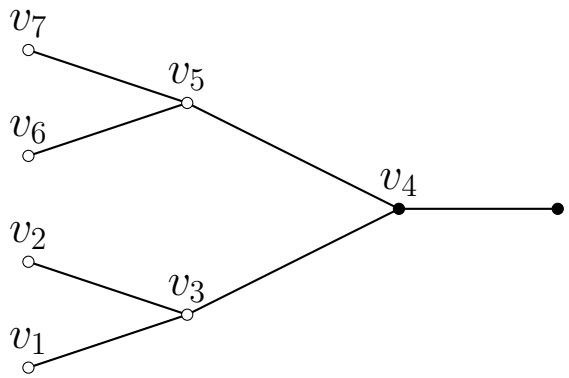


Figure 18: The vertices in black belong to $V_1^{(n)}$, and the vertices in white belong to $V_2^{(n)}$. In the figure v_4 is an isolated vertex in $H^{(n)}$, since neither v_3, nor v_5 participate in a border cherry.

converges to the matrix B, given by

$$\begin{pmatrix} 1 & 5/6 & 2\sqrt{2}/3 & -5/6 & 1/\sqrt{2} & 7/12 & 7/12 \\ 5/6 & 1 & 2\sqrt{2}/3 & -5/6 & 1/\sqrt{2} & 7/12 & 7/12 \\ 2\sqrt{2}/3 & 2\sqrt{2}/3 & 1 & -2\sqrt{2}/3 & 5/6 & 1/\sqrt{2} & 1/\sqrt{2} \\ -5/6 & -5/6 & -2\sqrt{2}/3 & 1 & -2\sqrt{2}/3 & -5/6 & -5/6 \\ 1/\sqrt{2} & 1/\sqrt{2} & 5/6 & -2\sqrt{2}/3 & 1 & 2\sqrt{2}/3 & 2\sqrt{2}/3 \\ 7/12 & 7/12 & 1/\sqrt{2} & -5/6 & 2\sqrt{2}/3 & 1 & 5/6 \\ 7/12 & 7/12 & 1/\sqrt{2} & -5/6 & 2\sqrt{2}/3 & 5/6 & 1 \end{pmatrix}$$

Using the eigenvalue decomposition of B, we find the matrix V satisfying $VV^T = B$ with approximate entries

$$\begin{pmatrix} 0 & 0 & -0.2887 & 0 & -0.4082 & 0.0676 & 0.8634 \\ 0 & 0 & 0.2887 & 0 & -0.4082 & 0.0676 & 0.8634 \\ 0 & 0 & 0 & 0 & -0.2887 & -0.0215 & 0.9572 \\ 0 & 0 & 0 & 0 & 0 & 0.1962 & -0.9806 \\ 0 & 0 & 0 & 0 & 0.2887 & -0.0215 & 0.9572 \\ 0 & 0 & 0 & -0.2887 & 0.4082 & 0.0676 & 0.8634 \\ 0 & 0 & 0 & 0.2887 & 0.4082 & 0.0676 & 0.8634 \end{pmatrix}.$$

In particular, applying V on the left to the standard normal vector

$$(Z_{v_1}, Z_{v_2}, Z_{v_3}, Z_{v_4}, Z_{v_5}, Z_{v_6}, Z_{v_7})^T$$

gives us the desired joint distribution of the seven-dimensional vector

$$(Y_{v_1}, Y_{v_2}, Y_{v_3}, -Y_{v_4}, Y_{v_5}, Y_{v_6}, Y_{v_7})^T$$

(by abuse of notation we assume again that the vertices $(v_i)_{i\geqslant 1}$ are included in $G(n,3)$ as well as in T_3). Indeed, recall that the linear factors of iid $(X_v^{(n)})_{v\in V(G(n,3))}$ converge

in distribution to the Gaussian wave $(Y_v)_{v\in V(T_3)}$. More precisely, in order for all seven coordinates of this resulting vector to be positive, the following inequalities must hold:

$$\begin{aligned}
Z_{v_7} &> -(V_{1,3}Z_{v_3} + V_{1,5}Z_{v_5} + V_{1,6}Z_{v_6})/V_{1,7} && (15)\\
Z_{v_7} &> -(V_{2,3}Z_{v_3} + V_{2,5}Z_{v_5} + V_{2,6}Z_{v_6})/V_{2,7} && (16)\\
Z_{v_7} &> -(V_{3,5}Z_{v_5} + V_{3,6}Z_{v_6})/V_{3,7} && (17)\\
Z_{v_7} &\leqslant V_{4,6}Z_{v_6}/(-V_{4,7}) && (18)\\
Z_{v_7} &> -(V_{5,5}Z_{v_5} + V_{5,6}Z_{v_6})/V_{5,7} && (19)\\
Z_{v_7} &> -(V_{6,4}Z_{v_4} + V_{6,5}Z_{v_5} + V_{6,6}Z_{v_6})/V_{6,7} && (20)\\
Z_{v_7} &> -(V_{7,4}Z_{v_4} + V_{7,5}Z_{v_5} + V_{7,6}Z_{v_6})/V_{7,7} && (21)
\end{aligned}$$

Defining LB to be the maximum of all lower bounds corresponding to inequalities (15, 16, 17, 19, 20, 21) for X_{v_7}, and $UB = \max\{LB,$ the right hand side of (18)$\}$, computing the desired probability (14) is thus equivalent to computing the following integral:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{LB}^{UB} e^{-\frac{x_3^2}{2}-\frac{x_4^2}{2}-\frac{x_5^2}{2}-\frac{x_6^2}{2}-\frac{x_7^2}{2}}\frac{1}{(2\pi)^{5/2}}dx_7dx_6dx_5dx_4dx_3.$$

Unfortunately, we are not able to compute this integral (not even numerically). Performing a Monte Carlo simulation with 3×10^7 iterations (see the authors' wegpages for the corresponding Python code) and checking in each round whether the new sample of $Z_{v_3}, Z_{v_4}, Z_{v_5}, Z_{v_6}, Z_{v_7}$ satisfies all inequalities, we obtain an estimated value of 0.002818666. Therefore, the total number of border cherries having their centers in $H_2^{(n)}$ and of degree zero in $H^{(n)}$ is $3\times 0.00281866n = 0.008456n + o(n)$ a.a.s. (concentration around the expected value follows as before). By symmetry, this clearly holds also for border cherries having their centers in $H_1^{(n)}$ and of degree zero in $H^{(n)}$.

We can therefore improve the previous argument in the following way: first switch the $2\times 0.008456n+o(n)$ centers of border cherries having degree zero in $H^{(n)}$ ($0.008456n+o(n)$ in both parts of the graph) to the other side of the cut (that is, change the associated value ϕ_v from 0 to 1 and vice versa), yielding a total a.a.s. gain of $0.016912n + o(n)$ edges. These centers of cherries "disappear" as vertices of $H^{(n)}$ since they are no longer centers of border cherries. We apply Lemma 48 to the remaining bipartite graph with parts of size $0.022438n - 0.0084912n + o(n) = 0.013982n + o(n)$ to obtain an independent set of size at least $2\times\frac{0.013982n}{2} + o(n)$. Switching these to the other side gives another gain of $0.013982 + o(n)$, thus yielding a total cut $(U_1^{(n)}, U_2^{(n)})$ size of at most $0.16226n - 0.016912n - 0.013982n + o(n) = 0.131366n + o(n)$ with $|U_1^{(n)}| - |U_2^{(n)}| = o(n)$ a.a.s. As above, this cut may be transformed into a bisection by switching the sides of at most $o(n)$ vertices of $G(n,3)$. We remark that this numerical value is also close to the one in Remark 4.1 given in [11] for numerical calculation of the number of edges, not included in a maximum cut of $G(n,3)$, thus giving another indication that the conjecture of [38] holds true.

Clearly, taking into account different substructures (such as, for example, paths v_1, $v_2, \ldots, v_k$ of three or more consecutive vertices, for which $(X_{v_i})_{1 \leqslant i \leqslant k}$ have all the same sign, and each of $(v_i)_{1 \leqslant i \leqslant k}$ having exactly one neighbor $(u_i)_{1 \leqslant i \leqslant k}$, respectively, such that for every $i \in [k]$, X_{v_i} and X_{u_i} have different signs, or other structures that were observed in the proof of the lower bound), we could further improve the previous bound. Since we cannot evaluate the corresponding integrals numerically, we stopped as this point.

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