Cyclic descents, matchings and Schur-positivity

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Abstract

A new descent set statistic on involutions, defined geometrically via their interpretation as matchings, is introduced in this paper, and shown to be equidistributed with the standard one. This concept is then applied to construct explicit cyclic descent extensions on involutions, standard Young tableaux and Motzkin paths. Schur-positivity of the associated quasisymmetric functions follows.

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Contents

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1 Introduction

The notion of descent set, for permutations as well as for standard Young tableaux, is well established. Klyachko [15] and Cellini [7] introduced a natural notion of cyclic descents for permutations. This notion was generalized to standard Young tableaux of rectangular shapes by Rhoades [20], and to other shapes and combinatorial sets in [3].

For a positive integer n, denote $[n] := \{1, \ldots, n\}$.

Definition 1. Let $\mathcal{T}$ be a finite set, equipped with any set map $\mathrm{Des} : \mathcal{T} \longrightarrow 2^{[n-1]}$. A *cyclic extension* of Des is a pair (cDes, p), where $\mathrm{cDes} : \mathcal{T} \longrightarrow 2^{[n]}$ is a map and $p : \mathcal{T} \longrightarrow \mathcal{T}$ is a bijection, satisfying the following axioms: for all T in $\mathcal{T}$,

$$\begin{array}{rl} \text{(extension)} & \mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T), \\ \text{(equivariance)} & \mathrm{cDes}(p(T)) = 1 + \mathrm{cDes}(T) \pmod n, \\ \text{(non-Escher)} & \varnothing \subsetneq \mathrm{cDes}(T) \subsetneq [n], \end{array}$$

where $1 + \mathrm{cDes}(T) \pmod n := \{(1+i) \pmod n : \ i \in \mathrm{cDes}(T)\}$. A pair (cDes, p), which satisfies the first two axioms but not the third is called an *Escherian cyclic descent extension.*

Example 2. Consider the symmetric group S_n on n letters and the standard *descent set* of a permutation $\pi = [\pi_1, \ldots, \pi_n]$,

$$\mathrm{Des}(\pi) := \{1 \leqslant i \leqslant n-1 \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n-1].$$

A corresponding *cyclic descent set* was defined by Cellini [7] as

$$\mathrm{CDES}(\pi) := \{1 \leqslant i \leqslant n \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n],$$

with the convention $\pi_{n+1} := \pi_1$. The pair (CDES, p), where $p : S_n \to S_n$ is the cyclic rotation defined by $p([\pi_1, \ldots, \pi_n]) := [\pi_n, \pi_1, \ldots, \pi_{n-1}]$, is a cyclic descent extension for S_n.

Cyclic descent extensions were introduced in the study of Lie algebras [15] and descent algebras [7]. Surprising connections of cyclic descent extensions to a variety of mathematical areas were found later. For connections of cyclic descents to Kazhdan-Lusztig theory see [20]; for topological aspects and connections to the Steinberg torus see [8]; for twisted Schützenberger promotion see [20, 13]; for cyclic quasisymmetric functions and Schur-positivity see [1, 2, 5]; for higher Lie characters see [2]; and for Postnikov's toric Schur functions and Gromov-Witten invariants see [3].

The question addressed in [2] was: which conjugacy classes in S_n carry a cyclic descent extension? Cellini's cyclic descent map does not provide a cyclic descent extension on most conjugacy classes. However, it turns out that most conjugacy classes carry such an extension.

Example 3. Consider the conjugacy class of transpositions in S_4

$$\{2134, 3214, 4231, 1324, 1432, 1243\}.$$

Cellini's cyclic descent sets are

$$\{1,4\}, \{1,2,4\}, \{1,3\}, \{2,4\}, \{2,3,4\}, \{3,4\},$$

respectively; and this collection is not closed under cyclic rotation. On the other hand, defining the cyclic descent sets by

$$\mathrm{cDes}(2134) = \{1,4\},\ \mathrm{cDes}(3214) = \{1,2\},\ \mathrm{cDes}(4231) = \{1,3\},$$

$$\mathrm{cDes}(1324) = \{2,4\},\ \mathrm{cDes}(1432) = \{2,3\},\ \mathrm{cDes}(1243) = \{3,4\},$$

and the map p by

$$3214 \to 1432 \to 1243 \to 2134\,(\to 3214), \qquad 4231 \to 1324\,(\to 4231),$$

yields a pair (cDes, p) which is a cyclic descent extension for this conjugacy class.

A full characterization of the conjugacy classes in S_n which carry a cyclic descent extension was given.

Theorem 4. [2, Theorem 1.4] *A conjugacy class of permutations of cycle type λ carries a cyclic descent extension if and only if λ is not equal to (r^s) for any square-free integer r.*

The proof of Theorem 4, presented in [2], is algebraic and not constructive.

Problem 5. [2, Problem 7.11] Find an explicit combinatorial description of the cyclic descent extension for conjugacy classes, whenever it exists.

In this paper we present a solution of this problem for the conjugacy classes of involutions. For $n \geqslant k \geqslant 0$ with $n-k$ even, let $\mathcal{I}_{n,k}$ be the conjugacy class of involutions with k fixed points in the symmetric group S_n. We present a purely combinatorial constructive proof of the following result.

Theorem 6. *For every $n > k > 0$ with $n-k$ even, $\mathcal{I}_{n,k}$ carries a cyclic descent extension. For $k = 0$ and $k = n$ there is only an Escherian cyclic extension.*

In order to construct an explicit cyclic descent extension for conjugacy classes of involutions with fixed points, we have to consider first the conjugacy class of fixed-point-free involutions. It will be shown that a certain geometrically-defined set-valued function on perfect matchings is equidistributed with the standard descent set on fixed-point-free involutions, leading to an Escherian cyclic descent extension for this conjugacy class of involutions ($k = 0$) and an ordinary (non-Escherian) extension for $0 < k < n$.

For $n \geqslant k \geqslant 0$ with $n-k$ even, let $\mathcal{M}_{n,k}$ be the set of partial matchings on n points, labeled by $[n] := \{1, \ldots, n\}$, with exactly k unmatched points.

Remark 7. An involution $(i_1, i_2) \cdots (i_{2r-1}, i_{2r}) \in \mathcal{I}_{n,k}$ (with $n = 2r + k$) may be naturally interpreted as the matching $m \in \mathcal{M}_{n,k}$ with matched pairs $\{i_1, i_2\}, \ldots, \{i_{2r-1}, i_{2r}\}$. This interpretation will be used frequently. Throughout this paper, the notations $\mathcal{I}_{n,k}$ and $\mathcal{M}_{n,k}$ are interchangeable.

Definition 8. The *standard descent set* of a matching $m \in \mathcal{M}_{n,k}$, denoted $\mathrm{Des}(m)$, is defined via the one-line notation of the corresponding involution in $\mathcal{I}_{n,k}$.

Definition 9. The *geometric descent set* of a matching $m \in \mathcal{M}_{n,k}$, denoted $\mathrm{GDes}(m)$, consists of the geometric descents of m, defined as follows. Draw the n points on a horizontal line in the real plane and label them by $1, \ldots, n$ from left to right. Indicate a matched pair $\{i, j\}$, with $i < j$, by drawing an arc in the upper half plane from the point labeled i to the point labeled j. An index $i \in [n-1]$ is a *geometric descent* of the matching $m \in \mathcal{M}_{n,k}$ if one of the following conditions holds:

1. $\{i, i+1\}$ is a matched pair in m.

2. The arc containing i intersects the arc containing $i+1$.

3. i is unmatched and $i+1$ is matched.

See Figure 1 for an example.

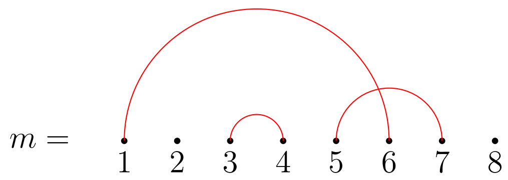

Figure 1: $m = (1,6)(3,4)(5,7) \in \mathcal{M}_{8,2}$, has $\mathrm{GDes}(m) = \{2,3,5,6\}$ and $\mathrm{Des}(m) = \mathrm{Des}([6,2,4,3,7,1,5,8]) = \{1,3,5\}$.

For a finite set of positive integers J let $\mathbf{x}^J := \prod_{j \in J} x_j$.

Lemma 10. *For every* $n \geqslant 0$

$$\sum_{m \in \mathcal{M}_{2n,0}} \mathbf{x}^{\mathrm{Des}(m)} \mathbf{y}^{\mathrm{GDes}(m)} = \sum_{m \in \mathcal{M}_{2n,0}} \mathbf{x}^{\mathrm{GDes}(m)} \mathbf{y}^{\mathrm{Des}(m)}.$$

For a matching $m \in \mathcal{M}_{n,k}$ let $\mathrm{cr}(m)$ and $\mathrm{ne}(m)$ be the crossing number and nesting number of m, respectively; see Definition 29 below. Using Lemma 10 we will prove the following.

Theorem 11. *For every* $n \geqslant k \geqslant 0$ *with* $n - k$ *even*

$$\sum_{m \in \mathcal{M}_{n,k}} q^{\mathrm{cr}(m)} \mathbf{x}^{\mathrm{GDes}(m)} = \sum_{m \in \mathcal{M}_{n,k}} q^{\mathrm{ne}(m)} \mathbf{x}^{\mathrm{Des}(m)}.$$

Bijective proofs of Lemma 10 and Theorem 11 will be described in Section 3.

Let $\mathcal{M}_n := \sqcup_{k=0}^{n} \mathcal{M}_{n,k}$ be the set of all matchings on n points, labeled by $1, \ldots, n$. Let $\mathrm{um}(m)$ be the number of unmatched points in a matching $m \in \mathcal{M}_n$. For a partition λ let $\mathrm{ht}(\lambda)$ be the number of parts in λ, let $\mathrm{oc}(\lambda)$ be the number of odd parts in the conjugate partition, and let s_λ be the corresponding Schur function. For $D \subseteq [n-1]$ let $F_{n,D}$ be the corresponding fundamental quasisymmetric function. For definitions and more details see Subsection 4.1. The following Schur-positivity phenomenon follows from the proof of Theorem 11.

Theorem 12. *For every $n \geqslant 0$*

$$\sum_{m \in \mathcal{M}_n} q^{\mathrm{um}(m)} t^{\mathrm{cr}(m)} F_{n,\mathrm{GDes}(m)} = \sum_{\lambda \vdash n} q^{\mathrm{oc}(\lambda)} t^{\lfloor \mathrm{ht}(\lambda)/2 \rfloor} s_\lambda.$$

The existence of cyclic descent extensions, on conjugacy classes of involutions with fixed points and other combinatorial sets, follows. To verify this observe, first, that there is a very natural cyclic extension of GDes on $\mathcal{M}_{n,k}$.

Definition 13. Draw n points on a circle and label them by $1, \ldots, n$ clockwise. Indicate a matched pair by drawing a chord between the corresponding points. A point $i \in [n]$ is a *cyclic geometric descent* of a matching $m \in \mathcal{M}_{n,k}$ if one of the following conditions holds (where addition is modulo n):

1. $\{i, i+1\}$ is a chord in m.
2. The chord containing i intersects the chord containing $i+1$.
3. i is unmatched and $i+1$ is matched.

The *cyclic geometric descent set* of m is denoted by $\mathrm{cGDes}(m)$.

See Figure 2 for an example.

The proof of Theorem 11 applies an explicit bijection $\hat{\iota} : \mathcal{I}_{n,k} \to \mathcal{M}_{n,k}$ for any $n \geqslant k \geqslant 0$, which satisfies

$$\mathrm{Des}(\pi) = \mathrm{GDes}(\hat{\iota}^{-1}(\pi)) \qquad (\forall \pi \in \mathcal{I}_{n,k}).$$

Using cGDes, define $\mathrm{cDes} : \mathcal{I}_{n,k} \to [n]$ by

$$\mathrm{cDes}(\pi) := \mathrm{cGDes}(\hat{\iota}^{-1}(\pi)) \qquad (\forall \pi \in \mathcal{I}_{n,k}).$$

Let $r : \mathcal{M}_{n,k} \to \mathcal{M}_{n,k}$ correspond to clockwise rotation by $2\pi/n$. Here is an explicit version of Theorem 6.

Proposition 14. *Assume that $n \geqslant k \geqslant 0$ with $n-k$ even.*

(a) If $0 < k < n$ then the pair $(\mathrm{cDes}, \hat{\iota}^{-1} \circ r \circ \hat{\iota})$ is a (non-Escherian) cyclic extension of Des *on $\mathcal{I}_{n,k}$.*

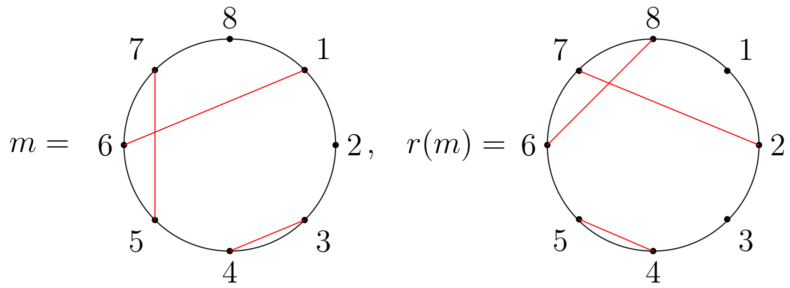

Figure 2: $m = (1,6)(3,4)(5,7) \in \mathcal{M}_{8,2}$ has $\mathrm{GDes}(m) = \{2,3,5,6\}$ and $\mathrm{cGDes}(m) = \{2,3,5,6,8\}$. Rotating m by $2\pi/8$ yields $r(m) = (2,7)(4,5)(6,8)$ with $\mathrm{GDes}(r(m)) = \mathrm{cGDes}(r(m)) = \{3,4,6,7,1\}$.

(b) If $k = 0$ or $k = n$ then the above pair is an Escherian cyclic extension of Des *on $\mathcal{I}_{n,k}$.*

The cyclic descent extension from Proposition 14 can be further refined to certain subsets of $\mathcal{I}_{n,k}$, yielding a combinatorial cyclic descent extension for sets of standard Young tableaux of bounded height with a given number of odd columns. Letting the height be at most 2 with all columns even, or height at most 3 with no further restrictions, give explicit cyclic descent extensions for the sets of Dyck paths and Motzkin paths of fixed length, respectively. These cyclic extensions coincide with those determined by Dennis White (as described in [17]) and Bin Han [12].

The structure of this paper is as follows. Section 2 contains some preliminary background. Section 3 contains bijective proofs of the equidistribution results, Lemma 10 and Theorem 11. Section 4 contains a proof of the Schur-positivity result, Theorem 12. Section 5 deals with cyclic descent extensions and proves Proposition 14, thus Theorem 6. Finally, Section 6 presents a non-constructive proof of a refinement of Theorem 11, based on a very recent unpublished result of Gessel.

2 Preliminaries

2.1 Permutations and tableaux

For $1 \leqslant k \leqslant n$ denote $[n] := \{1, 2, \ldots, n\}$ and $[k, n] := \{k, k+1, \ldots, n\}$. A partition of λ of a positive integer n, denoted $\lambda \vdash n$, is a sequence $\lambda = (\lambda_1, \ldots, \lambda_t)$ of weakly decreasing positive integers whose sum is n.

Let S_n denote the symmetric group consisting of all permutations of $[n]$. A permutation $\pi \in S_n$ will be represented by the one-row notation $\pi = [\pi_1, \ldots, \pi_n] \in S_n$, where $\pi_i := \pi(i)$ ($i \in [n]$); denote also $\mathrm{Fix}(\pi) := \{i \in [n] \,:\, \pi(i) = i\}$, the set of fixed points of π. Recall that the *descent set* of a permutation $\pi \in S_n$ is

$$\mathrm{Des}(\pi) := \{i \in [n-1] \,:\, \pi(i) > \pi(i+1)\}.$$

Another important family of combinatorial objects for which there is a well-studied notion of descent set is the set of standard Young tableaux (SYT). Let $\mathrm{SYT}(\lambda)$ denote the set of standard Young tableaux of shape λ, where λ is a partition of n. We draw tableaux in English notation, as in Figure 3. The *descent set* of $T \in \mathrm{SYT}(\lambda)$ is

$$\mathrm{Des}(T) := \{i \in [n-1] \,:\, i+1 \text{ is in a lower row than } i \text{ in } T\}.$$

For example, the descent set of the SYT in Figure 3 is $\{1, 3, 5, 6\}$.

1	3	5	9
2	4	6	
7	8		

Figure 3: A SYT of shape $\lambda = (4, 3, 2)$.

The Robinson-Schensted (RS) correspondence is a bijection $\pi \mapsto (P_\pi, Q_\pi)$ from permutations in S_n to pairs of standard Young tableaux (SYT) of the same shape and size n. The common shape λ of the *insertion tableau* P_π and the *recording tableau* Q_π is called the *shape* of the permutation π. We recall basic properties of the RS correspondence that will be used in the paper. For more details see, e.g., [22].

The *height* $\mathrm{ht}(\lambda)$ of a shape λ is the number of rows in λ.

Proposition 15. [23] *For every permutation $\pi \in S_n$, the height of the shape of π is equal to the maximal length of a decreasing subsequence in the one-line notation of π.*

Proposition 16. [4, Propositions 14.4.12 and 14.10.6]

1. $P_\pi = Q_{\pi^{-1}}$, *thus* $Q_\pi = P_{\pi^{-1}}$ *and* $P_\pi = Q_\pi$ *if and only if* $\pi \in S_n$ *is an involution.*
2. $\mathrm{Des}(Q_\pi) = \mathrm{Des}(\pi)$ *for all* $\pi \in S_n$.

Proposition 17. [24] *The number of columns of odd length in the shape of an involution with k fixed points is equal to k.*

Let SYT_n denote the set of all SYT of size n, and let $\mathrm{SYT}_{n,k}$ denote the set of standard Young tableaux of size n having k columns of odd length. Consider the map $Q : S_n \to \mathrm{SYT}_n$ defined by mapping $\pi \in S_n$ to the corresponding Robinson-Schensted recording tableau Q_π.

Corollary 18. *The map Q, restricted to involutions with a fixed number of fixed points, is a descent-set-preserving bijection from the set $\mathcal{I}_{n,k}$ of involutions in S_n with k fixed points to the set $\mathrm{SYT}_{n,k}$ of standard Young tableaux of size n with k odd columns.*

Proof. By Proposition 16.1, the map Q determines a bijection from the set of involutions $\mathcal{I}_n$ to the set of all SYT of size n. By Proposition 16.2, this map is descent-set-preserving and, by Proposition 17, the pre-image of $\mathrm{SYT}_{n,k}$ is $\mathcal{I}_{n,k}$. $\square$

Let X be a totally-ordered set of letters, and let U and V be disjoint finite subsets of X. Let S_U and S_V be the groups of permutations on U and V, respectively. For $\sigma \in S_U$ and $\tau \in S_V$, the *set of shuffles* of σ and τ, denoted by $\sigma \sqcup\!\sqcup \tau$, is the set of all permutations of the disjoint union $U \sqcup V$ in which the letters of U appear in the same order as in σ and the letters of V appear in the same order as in τ. For sets A and B of permutations on disjoint finite totally-ordered sets of letters U and V, respectively, denote by $A \sqcup\!\sqcup B$ the set of all shuffles of a permutation in A and a permutation in B. For example, if $A = \{12, 21\}$ and $B = \{43\}$, then $A \sqcup\!\sqcup B = \{1243, 1423, 1432, 4123, 4132, 4312, 2143, 2413, 2431, 4213, 4231, 4321\}$.

Observation 19. *By the definition of the RS correspondence, the smallest $n-k$ letters in P_π form a sub-tableau which depends only on their relative positions in π.*

In particular, letting σ be a permutation on $[k]$ and τ a permutation on $[n] \setminus [k]$, all $\pi \in \sigma \sqcup\!\sqcup \tau$ have a common sub-tableau of P_π consisting of the smallest k letters.

Proposition 20. *For every $\sigma \in \mathcal{I}_{n-k,0}$ and $\pi \in \sigma \sqcup\!\sqcup [n-k+1, \ldots, n]$, the number of odd columns in the RS shape of π is equal to k.*

Proof. Since $\sigma \in \mathcal{I}_{n-k,0}$, by Proposition 17 all the columns of its shape have even length. By Observation 19, the shape of the sub-tableau consisting of the smallest $n-k$ letters in P_π, which are the letters of σ, is the shape of σ. On the other hand, for every shuffle $\pi \in \sigma \sqcup\!\sqcup [n-k+1, \ldots, n]$ and every $n-k < i < n$, $i \notin \mathrm{Des}(\pi^{-1})$. By Proposition 16 this implies that $i \notin \mathrm{Des}(P_\pi)$ for all such i, so that the largest k letters in P_π belong to distinct columns and increase from left to right. They are therefore in the bottom cells of the odd columns of P_π, and the result follows. □

The proof of Proposition 20 implies the following.

Corollary 21. *For every $\sigma \in \mathcal{I}_{n-k,0}$ and $\pi \in \sigma \sqcup\!\sqcup [n-k+1, \ldots, n]$, the largest k letters in P_π appear in the bottom cells of the k odd columns of P_π, and they are increasing from left to right.*

2.2 Involutions and oscillating tableaux

Consider the Young lattice whose elements are all partitions, ordered by inclusion of the corresponding Young diagrams. A standard Young tableau of shape λ may be viewed as a maximal chain, in the Young lattice, from the empty partition to λ; see, e.g., [4, §14.2.5.1]. A variation of this description yields *oscillating tableaux*, which correspond to general paths in the Hasse diagram of the Young lattice, from the empty diagram to a diagram of shape λ. The *size* of the oscillating tableau is the length of the path, and its *shape* is λ. We focus on closed paths of length $2n$ from the empty diagram to itself; in other words, on oscillating tableaux of size $2n$ with an empty shape. The set of all such oscillating tableaux will be denoted by O_{2n}. A key tool in this paper is *Sundaram's bijection* $s : \mathcal{I}_{2n,0} \to \mathrm{O}_{2n}$, from the set $\mathcal{I}_{2n,0}$ of fixed-point-free involutions in S_{2n} to the set O_{2n} of oscillating tableaux of size $2n$ and empty shape; see [28]. We hereby describe this bijection.

Definition 22. (Sundaram's bijection [28]) Let $\pi \in \mathcal{I}_{2n,0}$. We start with $\lambda^0 = \varnothing$. For $1 \leqslant d \leqslant 2n$, define a standard Young tableau of shape λ^d, with letters forming a subset of $[2n]$, from a presumably-defined standard Young tableau of shape λ^{d-1}, as follows. Let $t_d = (i, j)$, $i < j$, be the unique transposition which affects d in the factorization of π into a product of n disjoint transpositions. If $d = i$, insert j into the tableau of shape λ^{d-1} using Robinson-Schensted insertion and get a tableau of shape λ^d. If $d = j$, delete j from the tableau of shape λ^{d-1} and apply jeu-de-taquin to get a tableau of shape λ^d. We get a sequence of $2n+1$ tableaux of shapes λ^d, $0 \leqslant d \leqslant 2n$. Ignoring the letters in the tableaux yields a sequence of shapes, which is the oscillating tableau corresponding to π.

Example 23. Let $\pi = (1,5)(2,4)(3,8)(6,7) \in \mathcal{I}_{8,0}$. The corresponding sequence of tableaux is

$$\varnothing\ ,\ \begin{array}{|c|}\hline 5 \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline\end{array}\ ,\ \begin{array}{|c|c|}\hline 4 & 8 \\ \hline 5 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline 5 & 8 \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline 8 \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline 7 \\ \hline 8 \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline 8 \\ \hline\end{array}\ ,\ \varnothing$$

Thus, the oscillating tableau corresponding to π is

$$s(\pi) = \left(\varnothing\ ,\ \begin{array}{|c|}\hline \ \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\ ,\ \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline \ \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline \ \\ \hline\end{array}\ ,\ \varnothing\right) \in \mathrm{O}_8.$$

A special case of [28, Theorem 5.3] is the following.

Theorem 24. *The map $s : \mathcal{I}_{2n,0} \to \mathrm{O}_{2n}$ defined above is a bijection.*

A characterization of the descents of π in the language of oscillating tableaux follows.

Observation 25. [14, Proof of Theorem 3.4] *For every $\pi \in \mathcal{I}_{2n,0}$, $i \in \mathrm{Des}(\pi)$ if and only if what we do in the i^{th} and $(i+1)^{st}$ steps of the corresponding oscillating tableau $s(\pi)$ is either*

1. *add a box in the i^{th} step and then delete a box in the next step; or*
2. *add a box in the i^{th} step and then add another box in a strictly lower row in the next step; or*
3. *delete a box in the i^{th} step and then delete another box in a strictly higher row in the next step.*

In all other cases, $i \notin \mathrm{Des}(\pi)$.

Definition 26. For an oscillating tableau $O = (D_0, D_1, \ldots, D_{2n}) \in O_{2n}$, define the *transpose* (or *conjugate*) tr $O := (D'_0, D'_1, \ldots, D'_{2n})$, where for each $0 \leqslant i \leqslant 2n$, the diagram D'_i is the transpose of the diagram D_i.

Another bijection $t : \mathcal{I}_{2n,0} \to \mathrm{O}_{2n}$, using growth diagrams, was described by Roby [21, §4.2]. This bijection applies a growth diagram algorithm to (a half of) the permutation matrix corresponding to a fixed-point-free involution $\pi \in \mathcal{I}_{2n,0}$, with empty boundary conditions, and reads an oscillating tableau $t(\pi)$ from the main diagonal. It relates to Sundaram's bijection via conjugation.

Proposition 27. [21, p. 69] *For every* $\pi \in \mathcal{I}_{2n,0}$

$$t(\pi) = \mathrm{tr}(s(\pi)).$$

Denote the longest permutation in S_{2n} by $w_0 := (1, 2n)(2, 2n-1)\cdots(n, n+1)$.

Corollary 28. *For every* $\pi \in \mathcal{I}_{2n,0}$*, the oscillating tableau* $s(w_0\pi w_0)$ *is the reverse of* $s(\pi)$*.*

Proof. Conjugating a permutation $\pi \in S_{2n}$ by w_0 corresponds to reflecting its permutation matrix about its vertical midline as well as about its horizontal midline. This is equivalent to a 180-degree rotation of the permutation matrix. An inspection of the algorithm shows that for $\pi \in \mathcal{I}_{2n,0}$ this simply reverses the oscillating tableau $t(\pi)$ on the main diagonal. Combining this with Proposition 27 completes the proof. $\square$

2.3 Matchings

Chen et al. [6] generalized Sundaram's bijection, described in Subsection 2.2 above, and applied it to the enumeration of crossings and nestings in perfect matchings and partitions.

Definition 29. Let $m \in \mathcal{M}_n$ be a matching.

1. The *crossing number* $\mathrm{cr}(m)$ of m is the maximal r such that there exist matched pairs $\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_r, j_r\}$ in m with $1 \leqslant i_1 < \cdots < i_r < j_1 < \cdots < j_r \leqslant n$.
2. The *nesting number* $\mathrm{ne}(m)$ of m is the maximal r such that there exist matched pairs $\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_r, j_r\}$ in m with $1 \leqslant i_1 < \cdots < i_r < j_r < \cdots < j_1 \leqslant n$.

Example 30. For $m \in \mathcal{M}_{8,2}$ as in Figure 1, $\{1, 6\}, \{5, 7\}$ is a maximal crossing, thus $\mathrm{cr}(m) = 2$. Also, $\{1, 6\}, \{3, 4\}$ is a maximal nesting, thus $\mathrm{ne}(m) = 2$.

Chen et al. introduced the involution $\iota : \mathcal{M}_{2n,0} \to \mathcal{M}_{2n,0}$, defined by

$$\iota := s^{-1} \circ \mathrm{tr} \circ s$$

Here s is Sundaram's bijection, $s : \mathcal{I}_{2n,0} \to \mathrm{O}_{2n}$, described in Definition 22, and tr is the transpose operation on oscillating tableaux, as in Definition 26. Following Remark 7, we identify perfect matchings in $\mathcal{M}_{2n,0}$ with involutions in $\mathcal{I}_{2n,0}$.

Example 31. Let $\pi = (1, 5)(2, 4)(3, 8)(6, 7) \in \mathcal{I}_{8,0}$ as in Example 23. Then

$$\mathrm{tr} \circ s(\pi) = \left(\varnothing\, ,\ \square\, ,\ \square\square\, ,\ \begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}\, ,\ \begin{smallmatrix}\square\\ \square\end{smallmatrix}\, ,\ \square\, ,\ \square\square\, ,\ \square\, ,\ \varnothing\right)$$

and $\iota(\pi) = s^{-1} \circ \mathrm{tr} \circ s(\pi) = (1, 4)(2, 7)(3, 5)(6, 8) \in \mathcal{I}_{8,0}$.

Theorem 32. [6] *For every* $m \in \mathcal{M}_{2n,0}$

$$\mathrm{cr}(m) = \mathrm{ne}(\iota(m)).$$

Thus

$$\sum_{m \in \mathcal{M}_{2n,0}} q^{\mathrm{cr}(m)} t^{\mathrm{ne}(m)} = \sum_{m \in \mathcal{M}_{2n,0}} q^{\mathrm{ne}(m)} t^{\mathrm{cr}(m)}.$$

3 Geometric versus standard descents: equidistribution results

The bijection of Chen et al., presented in Subsection 2.3, is applied in Subsection 3.1 to prove Lemma 10. This bijection serves as a component in the proof, in Subsection 3.2, of the following result.

Theorem 33. *For every $n \geqslant k \geqslant 0$ there exists an explicit bijection $\hat{\iota} : \mathcal{M}_{n,k} \to \mathcal{M}_{n,k}$, to be described in Definition 38, which satisfies*

$$\mathrm{GDes}(m) = \mathrm{Des}(\hat{\iota}(m)) \quad \text{and} \quad \mathrm{cr}(m) = \mathrm{ne}(\hat{\iota}(m)) \qquad (\forall m \in \mathcal{M}_{n,k}).$$

Theorem 11 follows. The bijection $\hat{\iota} : \mathcal{M}_{n,k} \to \mathcal{M}_{n,k}$ is used to prove Theorem 12 in Subsection 4.2, and to determine cyclic descents on involutions in Section 5.

3.1 Proof of Lemma 10

Recall the involution $\iota : \mathcal{M}_{2n,0} \to \mathcal{M}_{2n,0}$ introduced by Chen et al. [6], described in Subsection 2.3.

Proposition 34. *The involution $\iota : \mathcal{M}_{2n,0} \to \mathcal{M}_{2n,0}$ satisfies*

$$\mathrm{Des}(\iota(m)) = \mathrm{GDes}(m) \qquad (\forall m \in \mathcal{M}_{2n,0}).$$

Proof. Consider $m \in \mathcal{M}_{2n,0}$ as a fixed-point-free involution $\pi \in \mathcal{I}_{2n,0}$ (see Remark 7). The oscillating tableau $s(\hat{\pi})$, corresponding to the involution $\hat{\pi} := \iota(\pi)$, is the conjugate of the oscillating tableau $s(\pi)$: $s(\hat{\pi}) = \mathrm{tr}(s(\pi))$. Here $s : \mathcal{I}_{2n,0} \to \mathrm{O}_{2n}$ is Sundaram's bijection, described in Definition 22, and tr is the conjugation operation on oscillating tableaux, as in Definition 26.

We will show that $\mathrm{Des}(\hat{\pi}) = \mathrm{GDes}(\pi)$.

Fix $1 \leqslant i < 2n$. There are seven possible cases.

1. $(i, i+1)$ is a chord in π.
2. there exist $a < i$ and $b > i+1$, such that (a, i) and $(i+1, b)$ are chords in π.
3. there exist $a < i$ and $b > i+1$, such that (i, b) and $(a, i+1)$ are chords in π.
4. there exist $i+1 < a < b$ such that (i, a) and $(i+1, b)$ are chords in π.
5. there exist $i+1 < a < b$ such that $(i+1, a)$ and (i, b) are chords in π.
6. there exist $a < b < i$ such that (a, i) and $(b, i+1)$ are chords in π.
7. there exist $a < b < i$ such that $(a, i+1)$ and (b, i) are chords in π.

By Definition 9, $i \in \mathrm{GDes}(\pi)$ in cases (1), (3), (4) and (6) and $i \notin \mathrm{GDes}(\pi)$ in all other cases.

By Definition 22 of an oscillating tableau and basic properties of the insertion algorithm, what we do in the i^{th} and $(i+1)^{st}$ steps of the first 5 cases above is

1. add a box and then delete a box.
2. delete a box and then add a box.
3. add a box and then delete a box.
4. add a box and then add another box in a weakly higher row.
5. add a box and then add another box in a strictly lower row.

Cases (6) and (7) require more subtle analysis. Consider the involutions π and $\pi' := w_0\pi w_0$, and denote $i' := 2n - i$ (so that $i' + 1 = 2n + 1 - i$), $a' := 2n + 1 - b$ and $b' := 2n + 1 - a$. Then case (6) for π translates into

(6') there exist $b' > a' > i' + 1$ such that $(i' + 1, b')$ and (i', a') are chords in π',

namely case (4) for π'. Similarly, case (7) for π translates into

(7') there exist $b' > a' > i' + 1$ such that $(i' + 1, a')$ and (i', b') are chords in π',

namely case (5) for π'. By Corollary 28, $s(\pi')$ is the reverse of $s(\pi)$. We conclude that what we do in the i^{th} and $(i + 1)^{st}$ steps of cases (6) and (7) for π is

(6) delete a box and then delete another box in a weakly lower row.

(7) delete a box and then delete another box in a strictly higher row.

This is the description for π. For $\hat{\pi} = \iota(\pi)$ we have a conjugate oscillating tableau. The description for cases (1)-(3) remains the same, whereas case (4) is switched with case (5) and case (6) is switched with case (7). By Observation 25, this translates to $i \in \mathrm{Des}(\hat{\pi})$ in cases (1), (3), (4) and (6), but not in the other cases. This completes the proof. □

Remark 35. Arguments, similar to those used in the proof of Lemma 10, were used by Kim [14] to prove the symmetry of the Eulerian and Mahonian distributions on $\mathcal{I}_{2n,0}$.

The following refinement of Lemma 10 follows.

Corollary 36. *For every $n \geqslant 0$*

$$\sum_{m \in \mathcal{M}_{2n,0}} \mathbf{x}^{\mathrm{GDes}(m)} \mathbf{y}^{\mathrm{Des}(m)} q^{\mathrm{cr}(m)} t^{\mathrm{ne}(m)} = \sum_{m \in \mathcal{M}_{2n,0}} \mathbf{x}^{\mathrm{Des}(m)} \mathbf{y}^{\mathrm{GDes}(m)} q^{\mathrm{ne}(m)} t^{\mathrm{cr}(m)}.$$

Proof. By Proposition 34, the involution $\iota : \mathcal{M}_{2n,0} \to \mathcal{M}_{2n,0}$ satisfies $\mathrm{Des}(\iota(m)) = \mathrm{GDes}(m)$ for all $m \in \mathcal{M}_{2n,0}$. By Theorem 32, $\mathrm{ne}(\iota(m)) = \mathrm{cr}(m)$. Since ι is an involution, $\mathrm{Des}(m) = \mathrm{GDes}(\iota(m))$ and $\mathrm{ne}(m) = \mathrm{cr}(\iota(m))$. Thus

$$\sum_{m \in \mathcal{M}_{2n,0}} \mathbf{x}^{\mathrm{GDes}(m)} \mathbf{y}^{\mathrm{Des}(m)} q^{\mathrm{cr}(m)} t^{\mathrm{ne}(m)} = \sum_{m' \in \mathcal{M}_{2n,0}} \mathbf{x}^{\mathrm{Des}(m')} \mathbf{y}^{\mathrm{GDes}(m')} q^{\mathrm{ne}(m')} t^{\mathrm{cr}(m')},$$

where $m' := \iota(m)$. □

3.2 Proof of Theorem 11

In this subsection we describe a map

$$\hat{\iota} : \mathcal{M}_{n,k} \to \mathcal{M}_{n,k},$$

for any $0 \leqslant k \leqslant n$, which generalizes the bijection $\iota : \mathcal{M}_{2n,0} \to \mathcal{M}_{2n,0}$ used in the previous subsection. It will be shown that $\hat{\iota}$ is a bijection which maps the descent set to the geometric descent set and the crossing number to the nesting number, implying Theorem 11.

Recall that $\mathcal{M}_{n,k}$ is naturally identified with $\mathcal{I}_{n,k}$ (Remark 7). In the rest of this section it will be more convenient to consider involutions in $\mathcal{I}_{n,k}$, rather than matchings in $\mathcal{M}_{n,k}$, since the shuffle operation and the RS correspondence used here are defined in terms of permutations (in particular, involutions).

Remark 37. The bijection of Chen et al. is defined for involutions with fixed points as well. It is an involution which maps the crossing number to the nesting number and preserves the fixed point set. Unfortunately, for involutions with fixed points it does not map GDes to Des and vice versa. For example, Chen et al.'s involution maps $\pi = (1,4)(2,5)(3)$ to $\sigma = (1,5)(2,4)(3)$, but $\mathrm{Des}(\pi) = \{2,3\} \neq \mathrm{GDes}(\sigma) = \{3\}$ and also $\mathrm{Des}(\sigma) = \{1,2,3,4\} \neq \mathrm{GDes}(\pi) = \{1,3,4\}$.

Definition 38. Fix $n \geqslant k \geqslant 0$, with $n-k$ even.

1. For every $\pi \in \mathcal{I}_{n,k}$, let $\mathrm{res}(\pi)$ be the pair $(\mathrm{Fix}(\pi), \sigma)$, where $\mathrm{Fix}(\pi)$ is the set of fixed points of π, and σ is the fixed-point-free involution in S_{n-k} with the same relative order as that of π on $[n] \setminus \mathrm{Fix}(\pi)$.

2. For $(J,\sigma) \in \binom{[n]}{k} \times \mathcal{I}_{n-k,0}$ let $\mathrm{emb}(J,\sigma)$ be the permutation in the set of all shuffles $\mathcal{I}_{n-k,0} \sqcup [n-k+1, n-k+2, \dots n]$, for which the letters in $[n-k]$ are ordered as in σ, and set of positions of the increasing subsequence $[n-k+1,\dots n]$ is equal to J.

3. Define $\varphi : \mathcal{I}_{n,k} \longrightarrow \mathcal{I}_{n-k,0} \sqcup [n-k+1,\dots,n]$ by

$$\varphi :\ \mathcal{I}_{n,k} \xrightarrow{\mathrm{res}} \binom{[n]}{k} \times \mathcal{I}_{n-k,0} \xrightarrow{(id,\iota)} \binom{[n]}{k} \times \mathcal{I}_{n-k,0} \xrightarrow{\mathrm{emb}} \mathcal{I}_{n-k,0} \sqcup [n-k+1,\dots,n],$$

   where $(id,\iota)(J,\sigma) := (J,\iota(\sigma))$.

4. For $\tau \in \mathcal{I}_{n-k,0} \sqcup [n-k+1,\dots,n]$ let $q(\tau) \in \mathcal{I}_n$ be the RS preimage of (Q_τ, Q_τ), where Q_τ is the recording tableau of τ:

$$q :\ \mathcal{I}_{n-k,0} \sqcup [n-k+1,\dots,n] \xrightarrow{Q} \mathrm{SYT}_n \xrightarrow{\mathrm{diag}} \mathrm{SYT}_n \times \mathrm{SYT}_n \xrightarrow{\mathrm{RS}^{-1}} \mathcal{I}_n,$$

   where SYT_n denotes the set of standard Young tableaux of size n and $\mathcal{I}_n$ is the set of involutions in S_n. Note that $q(\tau)$ is an involution by Proposition 16.1.

5. Let $\hat{\imath} := q \circ \varphi$.

The following proposition implies Theorem 33.

Proposition 39. *The map $\hat{\imath} : \mathcal{I}_{n,k} \longrightarrow \mathcal{I}_n$ is a bijection from $\mathcal{I}_{n,k}$ onto itself, which satisfies*

$$\mathrm{GDes}(\pi) = \mathrm{Des}(\hat{\imath}(\pi)) \quad \text{and} \quad \mathrm{cr}(\pi) = \mathrm{ne}(\hat{\imath}(\pi)) \qquad (\forall \pi \in \mathcal{I}_{n,k}).$$

Example 40. Let $\pi = [4,2,6,1,5,3] = (1,4)(3,6)(2)(5) \in \mathcal{I}_{6,2}$. Then

$$\varphi:\ [4,2,6,1,5,3] \stackrel{\mathrm{res}}{\longmapsto} (\{2,5\}\,,\,(1,3)(2,4)) \stackrel{(id,\iota)}{\longmapsto} (\{2,5\}\,,\,(1,4)(2,3)) \stackrel{\mathrm{emb}}{\longmapsto} [4,5,3,2,6,1],$$

and thus

$$\hat{\imath}:\ [4,2,6,1,5,3] \stackrel{\varphi}{\longmapsto} [4,5,3,2,6,1] \stackrel{Q}{\longmapsto} \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3\\ \cline{1-1} 4\\ \cline{1-1} 6\\ \cline{1-1}\end{array}$$

$$\stackrel{\mathrm{diag}}{\longmapsto} \left(\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3\\ \cline{1-1} 4\\ \cline{1-1} 6\\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3\\ \cline{1-1} 4\\ \cline{1-1} 6\\ \cline{1-1}\end{array} \right) \stackrel{\mathrm{RS}^{-1}}{\longmapsto} [1,6,4,3,5,2].$$

Namely, $\hat{\imath}(\pi) = (2,6)(3,4)(1)(5) \in \mathcal{I}_{6,2}$. Indeed, $\mathrm{GDes}(\pi) = \mathrm{Des}(\hat{\imath}(\pi)) = \{2,3,5\}$ and $\mathrm{cr}(\pi) = \mathrm{ne}(\hat{\imath}(\pi)) = 2$.

To prove Proposition 39 we first generalize the concepts of crossing and nesting numbers, from matchings (or, equivalently, involutions) to shuffles of fixed-point-free involutions with increasing sequences.

Definition 41. For every $\tau \in \mathcal{I}_{n-k,0} \sqcup [n-k+1,\ldots,n]$ define $\mathrm{ne}(\tau) := \mathrm{ne}(\sigma)$ and $\mathrm{cr}(\tau) := \mathrm{cr}(\sigma)$, where $\sigma \in \mathcal{I}_{n-k,0}$ is obtained by deleting the letters $n-k+1,\ldots,n$ from τ.

Example 42. Let $\tau = [3,4,5,1,6,2] \in \mathcal{I}_{4,0} \sqcup [5,6]$. Then τ is not an involution, but deleting the letters 5 and 6 from τ gives a fixed-point-free involution $\sigma = [3,4,1,2] = (1,3)(2,4) \in \mathcal{I}_{4,0}$. By definition, $\mathrm{cr}(\tau) = \mathrm{cr}(\sigma) = 2$ and $\mathrm{ne}(\tau) = \mathrm{ne}(\sigma) = 1$.

Lemma 43. *For any $n \geqslant k \geqslant 0$ with $n-k$ even, the map*

$$\varphi : \mathcal{I}_{n,k} \to \mathcal{I}_{n-k,0} \sqcup [n-k+1, n-k+2, \ldots n]$$

is a bijection which satisfies

$$\mathrm{GDes}(\pi) = \mathrm{Des}(\varphi(\pi)) \qquad (\forall \pi \in \mathcal{I}_{n,k})$$

as well as

$$\mathrm{ne}(\pi) = \mathrm{cr}(\varphi(\pi)) \quad \text{and} \quad \mathrm{cr}(\pi) = \mathrm{ne}(\varphi(\pi)) \qquad (\forall \pi \in \mathcal{I}_{n,k}).$$

Proof. By Definition 38, φ is a bijection. To show that it maps GDes to Des, let $1 \leqslant i \leqslant n$ and consider the four possible cases.

Case 1 If $i, i+1 \in \mathrm{Fix}(\pi)$ then, by Definition 9, $i \notin \mathrm{GDes}(\pi)$; and, by Definition 38, $\varphi(\pi)(i) < \varphi(\pi)(i+1)$.

Case 2 If $i \notin \mathrm{Fix}(\pi)$ and $i+1 \in \mathrm{Fix}(\pi)$ then $i \notin \mathrm{GDes}(\pi)$ and $\varphi(\pi)(i) < \varphi(\pi)(i+1)$.

Case 3 If $i \in \mathrm{Fix}(\pi)$ and $i+1 \notin \mathrm{Fix}(\pi)$ then $i \in \mathrm{GDes}(\pi)$ and $\varphi(\pi)(i) > \varphi(\pi)(i+1)$.

Case 4 If $i, i+1 \notin \mathrm{Fix}(\pi)$ then, by Definition 38, we can ignore the fixed points and apply Proposition 34, which shows that $i \in \mathrm{GDes}(\pi) \iff i \in \mathrm{Des}(\varphi(\pi))$.

This proves the claim regarding GDes and Des. The claim about crossing and nesting numbers follows from Definition 38, Definition 41 and Theorem 32. $\square$

Corollary 44.

$$\sum_{\pi \in \mathcal{I}_{n,k}} \mathbf{x}^{\mathrm{GDes}(\pi)} q^{\mathrm{cr}(\pi)} t^{\mathrm{ne}(m)} = \sum_{\pi \in \mathcal{I}_{n-k,0} \sqcup [n-k+1,\ldots,n]} \mathbf{x}^{\mathrm{Des}(\pi)} q^{\mathrm{ne}(\pi)} t^{\mathrm{cr}(\pi)}.$$

To prove Proposition 39 we also need the following lemmas.

Lemma 45. *For every involution $\pi \in S_n$*

$$\mathrm{ne}(\pi) = \lfloor \mathrm{ht}(Q_\pi)/2 \rfloor,$$

where Q_π is the RS recording tableau of π.

Proof. By Proposition 15, $\mathrm{ht}(Q_\pi)$ is the length of the longest decreasing subsequence in the one-line notation of π. By Definition 29, if $\mathrm{ne}(\pi) = r$ then there exists a sequence $1 \leqslant i_1 < \cdots < i_r < i_{r+1} < \cdots < i_{2r} \leqslant n$ such that, for every $1 \leqslant j \leqslant 2r$, $\pi(i_j) = i_{2r+1-j}$. Then $(i_{2r}, \ldots, i_1)$ is a decreasing subsequence in the one-line notation of π, so that

$$\mathrm{ht}(Q_\pi) \geqslant 2r = 2\,\mathrm{ne}(\pi).$$

On the other hand, the fixed points of π form an increasing subsequence of its one-line notation; thus any decreasing subsequence contains at most one fixed point. Let $(\pi(i_1), \ldots, \pi(i_t))$ be a decreasing subsequence of maximal length in the one-line notation of π. Assume, first, that it contains no fixed points. Let $s := \max\{j :\ \pi(i_j) > i_j\}$. If $s > t/2$ then $(\pi(i_1), \ldots, \pi(i_s), i_s, \ldots, i_1)$ is a decreasing subsequence of length $2s > t$ in π, contradicting the maximality of t. Similarly, if $s < t/2$ then $(i_t, i_{t-1}, \ldots, i_{s+1}, \pi(i_{s+1}), \ldots, \pi(i_t))$ is a decreasing subsequence of length $2(t-s) > t$ in π, contradicting the maximality of t. We deduce that $s = t/2$. The sequence $(\pi(i_1), \ldots, \pi(i_s), i_s, \ldots, i_1)$ is a decreasing sequence of maximal length in π, and corresponds to a nesting. Thus

$$2\,\mathrm{ne}(\pi) \geqslant 2s = t = \mathrm{ht}(Q_\pi).$$

Finally, assume that the chosen decreasing subsequence $(\pi(i_1), \ldots, \pi(i_t))$ of maximal length in the one-line notation of π contains a fixed point, say $\pi(i_s) = i_s$. Then for all $1 \leqslant j \leqslant t$, $\pi(i_j) > i_j$ if and only if $j < s$. By the above argument, if either $2(s-1)+1 > t$ or $2(t-s)+1 > t$ then one can define a decreasing subsequence of length exceeding t, contradicting the maximality of t. Thus $t+1 \leqslant 2s \leqslant t+1$, namely $2s = t+1$. The sequence $(\pi(i_1), \ldots, \pi(i_{s-1}), i_{s-1}, \ldots, i_1)$ is decreasing subsequence corresponding to a nesting, so that

$$2\,\mathrm{ne}(\pi) \geqslant 2(s-1) = t-1 = \mathrm{ht}(Q_\pi) - 1.$$

This completes the proof. □

Remark 46. By Proposition 17, $\mathrm{ht}(Q_\pi)$ is even for every fixed-point-free involution $\pi \in \mathcal{I}_{2n,0}$. Hence, for fixed-point-free involutions, no floors are required in Lemma 45, i.e.,

$$\mathrm{ne}(\pi) = \mathrm{ht}(Q_\pi)/2 \qquad (\forall \pi \in \mathcal{I}_{2n,0}).$$

Corollary 47. *If $\sigma \in \mathcal{I}_{n-k,0}$ and $\tau \in \sigma \sqcup [n-k+1, \ldots, n]$ then*

$$\mathrm{ne}(\tau) = \lfloor \mathrm{ht}(Q_\tau)/2 \rfloor.$$

Proof. For $\pi \in S_n$ let $\ell(\pi)$ be the length of the longest decreasing subsequence in π. Observe that $\ell(\tau) - \ell(\sigma) \in \{0, 1\}$; hence, by Proposition 15,

$$\mathrm{ht}(Q_\tau) - \mathrm{ht}(Q_\sigma) \in \{0, 1\}.$$

By Definition 41 and Remark 46 we deduce

$$\mathrm{ne}(\tau) = \mathrm{ne}(\sigma) = \mathrm{ht}(Q_\sigma)/2 = \lfloor \mathrm{ht}(Q_\tau)/2 \rfloor.$$ □

Recall the map $Q : S_n \to \mathrm{SYT}_n$ sending each $\pi \in S_n$ to the corresponding RS recording tableau Q_π. Recall also the notation $\mathrm{SYT}_{n,k}$ for the set of all SYT of size n with k odd columns.

Lemma 48. *For any $n \geqslant k \geqslant 0$ with $n-k$ even, the map Q restricts to a descent-set-preserving bijection from the set of shuffles $\mathcal{I}_{n-k,0} \sqcup [n-k+1, \ldots, n]$ to $\mathrm{SYT}_{n,k}$.*

Proof. First, by Proposition 16.2, $\mathrm{Des}(Q_\tau) = \mathrm{Des}(\tau)$, so Q is descent-set-preserving.

Second, by Proposition 20, Q_τ has k odd columns for every $\tau \in \mathcal{I}_{n-k,0} \sqcup [n-k+1, \ldots, n]$, so Q maps $\mathcal{I}_{n-k,0} \sqcup [n-k+1, \ldots, n]$ into $\mathrm{SYT}_{n,k}$.

To prove that Q is a bijection, we will construct an inverse. Assuming that $\tau \in \mathcal{I}_{n-k,0} \sqcup [n-k+1, \ldots, n]$, it will be shown that τ can be reconstructed from its recording tableau Q_τ, which can be an arbitrary element of $\mathrm{SYT}_{n,k}$. Assume that $\tau \in \sigma \sqcup [n-k+1, \ldots, n]$, where $\sigma \in \mathcal{I}_{n-k,0}$. By Corollary 21, the largest k letters in P_τ appear in the bottom cells of the k odd columns of P_τ, and they are increasing from left to right. These cells can be identified from Q_τ, which has the same shape as P_τ. We want to recover the positions of these k largest letters in τ, i.e., the values $\tau^{-1}(i)$ for $n-k+1 \leqslant i \leqslant n$. Recall, from Proposition 16.1, that if τ corresponds (under RS) to the pair (P_τ, Q_τ) then τ^{-1} corresponds to $(P_{\tau^{-1}}, Q_{\tau^{-1}}) = (Q_\tau, P_\tau)$. Apply to Q_τ, k times, the inverse of the RS insertion algorithm, as in [22, proof of Theorem 3.1.1]. Here is the first step:

- Let $T_n := Q_\tau$. Assume that the bottom cell of the the rightmost odd column is in the r^{th} row. Let i_r be the entry in this cell.
- Let i_{r-1} be the largest letter smaller than i_r in the $(r-1)^{st}$ row. Delete i_r from the r^{th} row and replace i_{r-1} by i_r.
- Repeat this step until i_1, the largest letter smaller than i_2 in the first row, is replaced by i_2.
- The letter i_1 is the position of n in τ, namely $\tau^{-1}(n)$.

Apply the same procedure to the resulting tableau T_{n-1} (with $\tau^{-1}(n)$ removed) to find the position of $n-1$, namely $\tau^{-1}(n-1)$, and so on.

After k steps, the positions of the increasing subsequence of τ consisting of the largest k letters have been determined. At this stage we get T_{n-k}, which is the P tableau corresponding to the sequence $\tau^{-1}(1), \dots, \tau^{-1}(n-k)$. This sequence has the same relative order of letters as the sequence $\sigma^{-1}(1), \dots, \sigma^{-1}(n-k)$. Replacing the $n-k$ letters in T_{n-k} by $1, \dots, n-k$ with the same relative order therefore yields $P_{\sigma^{-1}} = Q_\sigma$. It is clear from the algorithm that all the columns of T_{n-k}, and therefore of Q_σ, have even lengths. By Corollary 18 (for $k = 0$) there is a unique fixed-point-free involution σ with this Q_σ as a Q tableau. Since σ is an involution, Proposition 16.1 implies that $P_\sigma = Q_\sigma$, and therefore σ is the RS preimage of (Q_σ, Q_σ). This completes the proof. □

Example 49. Let

$$Q_\tau = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 8 \\ \cline{1-3} 7 \\ \cline{1-1}\end{array} .$$

This tableau has $n = 8$ cells and $k = 2$ columns of odd length. Thus $\tau \in \sigma \sqcup [7, 8]$ for some $\sigma \in \mathcal{I}_{6,0}$.

Start with $T_8 = Q_\tau$. The bottom cell of the rightmost odd column in T_8 appears in the first row. Thus $r = 1$, the entry there is $i_r = i_1 = 6$, and therefore $\tau^{-1}(8) = 6$. The resulting tableau after deleting this letter is

$$T_7 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 8 \\ \hline 7 \\ \cline{1-1}\end{array} .$$

Now the bottom cell of the rightmost odd column appears in row $r = 3$. The entry there is $i_3 = 7$, and consequently $i_2 = 5$ and $i_1 = 4$. Thus $\tau^{-1}(7) = 4$. This yields

$$T_6 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 7 & 8 \\ \hline\end{array} .$$

Standartization (by mapping the letters in T_6 to $\{1, \dots, 6\}$ in a monotone increasing fashion) gives

$$Q_\sigma = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array} .$$

Hence $\sigma = \mathrm{RS}^{-1}(Q_\sigma, Q_\sigma) = [3, 5, 1, 6, 2, 4]$ and $\tau = [3, 5, 1, 7, 6, 8, 2, 4]$.

Recall the map q from Definition 38.4: for $\tau \in \mathcal{I}_{n-k,0} \sqcup [n-k+1, \ldots, n]$, the involution $q(\tau)$ is the RS preimage of (Q_τ, Q_τ), where Q_τ is the RS recording tableau of τ.

Corollary 50. *The map q is a descent set and nesting number preserving bijection from the set of shuffles $\mathcal{I}_{n-k,0} \sqcup [n-k+1, \ldots n]$ to the set of involutions $\mathcal{I}_{n,k}$.*

Proof. Let $\tau \in \mathcal{I}_{n-k,0} \sqcup [n-k+1, \ldots, n]$. First, by Proposition 20, Q_τ has k odd columns. Combining this with Propositions 16.1 and 17, the RS preimage of (Q_τ, Q_τ) is an involution with k fixed points, namely, $q(\tau) \in \mathcal{I}_{n,k}$. Moreover, by Lemma 48 together with Corollary 18, q is a descent set preserving bijection. Finally, by definition, $Q_\tau = Q_{q(\tau)}$, and by Lemma 45 for the involution $q(\tau)$,

$$\mathrm{ne}(q(\tau)) = \lfloor \mathrm{ht}(Q_{q(\tau)})/2 \rfloor.$$

Thus, by Corollary 47,

$$\mathrm{ne}(\tau) = \lfloor \mathrm{ht}(Q_\tau)/2 \rfloor = \lfloor \mathrm{ht}(Q_{q(\tau)})/2 \rfloor = \mathrm{ne}(q(\tau)),$$

so that q preserves nesting number as well. □

Proof of Proposition 39. By Lemma 43 and Corollary 50,

$$\hat{\imath} : \mathcal{I}_{n,k} \xrightarrow{\varphi} \mathcal{I}_{n-k,0} \sqcup [n-k+1, \ldots, n] \xrightarrow{q} \mathcal{I}_{n,k}$$

is a bijection which satisfies

$$\mathrm{GDes}(\pi) = \mathrm{Des}(\varphi(\pi)) = \mathrm{Des}(q(\varphi(\pi))) = \mathrm{Des}(\hat{\imath}(\pi))$$

as well as

$$\mathrm{cr}(\pi) = \mathrm{ne}(\varphi(\pi)) = \mathrm{ne}(q(\varphi(\pi))) = \mathrm{ne}(\hat{\imath}(\pi)).$$ □

4 Schur-positivity

4.1 Background

Schur functions indexed by partitions of n form a distinguished basis for Λ_n, the vector space of homogeneous symmetric functions of degree n; see, e.g., [26, Corollary 7.10.6]. A symmetric function in Λ_n is *Schur-positive* if all the coefficients in its expansion in the basis $\{s_\lambda : \lambda \vdash n\}$ of Schur functions are nonnegative.

For each $D \subseteq [n-1] = \{1, 2, \ldots, n-1\}$, define the *fundamental quasisymmetric function*

$$F_{n,D}(\mathbf{x}) := \sum_{\substack{i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Let A be a set of combinatorial objects, equipped with a set-valued function $D : A \to 2^{[n-1]}$. We say that A is *symmetric* (*Schur-positive*) with respect to D if

$$\mathcal{Q}_{A,D} := \sum_{\pi \in A} F_{n,D(\pi)}$$

is a symmetric (respectively, Schur-positive) function. Determining whether a given symmetric (quasisymmetric) function is Schur-positive is a major problem in contemporary algebraic combinatorics [27, §3].

The following theorem is due to Gessel.

Theorem 51. [26, Theorem 7.19.7] *For every partition* $\lambda \vdash n$,

$$\mathcal{Q}_{\mathrm{SYT}(\lambda),\mathrm{Des}} = s_\lambda.$$

Thus $\mathrm{SYT}(\lambda)$ *is symmetric and Schur-positive with respect to the standard descent set.*

We say that a statistic $f : A \to \mathbb{N} \cup \{0\}$ is Schur-positive on A with respect to the set-valued function D if

$$\sum_{\pi \in A} q^{f(\pi)} F_{n,D(\pi)},$$

is a Schur-positive symmetric function. Examples of Schur-positive statistics with respect to the standard descent set on permutations include

- Statistics on S_n which are invariant under conjugation; e.g., the cycle number and the number of fixed points. This follows from [11, Theorem 2.1].
- Statistics on S_n which are invariant under Knuth relations; e.g., the length of the longest increasing subsequence, the inverse descent number, and the inverse major index. This follows from Theorem 51 above together with Proposition 16.
- The inversion number on S_n (reduced to the inverse major index by Foata's bijection). For a far reaching generalization see [25, Theorem 6.3].

Theorem 12, to be proved in the following subsection, implies that on the set $\mathcal{M}_n$ of all matchings on n points, the pair $(\mathrm{cr}, \mathrm{um})$ of the crossing number and the number of unmatched points is Schur-positive with respect to the geometric descent set GDes.

4.2 Proof of Theorem 12

Recall from Section 1 the following notations: $\mathrm{um}(m)$ is the number of unmatched points in a matching $m \in \mathcal{M}_n$; and, for a partition λ, $\mathrm{ht}(\lambda)$ is the number of parts in λ and $\mathrm{oc}(\lambda)$ is the number of odd parts in the conjugate partition.

The following proposition follows from Theorem 11.

Proposition 52. *For every* $n \geqslant 0$

$$\sum_{m \in \mathcal{M}_n} q^{\mathrm{um}(m)} t^{\mathrm{cr}(m)} \mathbf{x}^{\mathrm{GDes}(m)} = \sum_{\lambda \vdash n} q^{\mathrm{oc}(\lambda)} t^{\lfloor \mathrm{ht}(\lambda)/2 \rfloor} \sum_{T \in \mathrm{SYT}(\lambda)} \mathbf{x}^{\mathrm{Des}(T)}. \tag{1}$$

Proof. We have

$$\begin{aligned}
\sum_{m\in\mathcal{M}_n} q^{\mathrm{um}(m)} t^{\mathrm{cr}(m)} \mathbf{x}^{\mathrm{GDes}(m)} &= \sum_{k=0}^{n} q^k \sum_{m\in\mathcal{M}_{n,k}} t^{\mathrm{cr}(m)} \mathbf{x}^{\mathrm{GDes}(m)} \\
&= \sum_{k=0}^{n} q^k \sum_{m\in\mathcal{M}_{n,k}} t^{\mathrm{ne}(m)} \mathbf{x}^{\mathrm{Des}(m)} \\
&= \sum_{m\in\mathcal{M}_n} q^{\mathrm{um}(m)} t^{\mathrm{ne}(m)} \mathbf{x}^{\mathrm{Des}(m)} \\
&= \sum_{\lambda\vdash n}^{n} q^{\mathrm{oc}(\lambda)} t^{\lfloor \mathrm{ht}(\lambda)/2\rfloor} \sum_{T\in\mathrm{SYT}(\lambda)} \mathbf{x}^{\mathrm{Des}(T)}.
\end{aligned}$$

The second equality follows from Theorem 11. The last equality is obtained from the interpretation of matchings as involutions, followed by the bijection to SYT via the Robinson-Schensted correspondence, using Corollary 18 and Lemma 45. $\square$

Proof of Theorem 12. Consider the equality in Proposition 52. Applying the vector space isomorphism from the ring of multilinear polynomials to the ring of quasisymmetric functions, defined by $\mathbf{x}^J \mapsto F_{n,J}$ for every subset $J \subseteq [n-1]$, one obtains

$$\begin{aligned}
\sum_{m\in\mathcal{M}_n} q^{\mathrm{um}(m)} t^{\mathrm{cr}(m)} F_{n,\mathrm{GDes}(m)} &= \sum_{\lambda\vdash n} q^{\mathrm{oc}(\lambda)} t^{\lfloor \mathrm{ht}(\lambda)/2\rfloor} \sum_{T\in\mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(T)} \\
&= \sum_{\lambda\vdash n} q^{\mathrm{oc}(\lambda)} t^{\lfloor \mathrm{ht}(\lambda)/2\rfloor} s_\lambda.
\end{aligned}$$

The last equality follows from Theorem 51. $\square$

5 Cyclic descent extensions

The above setting is applied in this section to construct a cyclic descent extension for conjugacy classes of involutions and their refinements, that is, involutions with fixed cycle structure and nesting number.

Let $\mathcal{M}_n$ be the set of matchings on n points on the circle, labeled clockwise by $1,\dots,n$. Let $r : \mathcal{M}_n \to \mathcal{M}_n$ be clockwise rotation by $2\pi/n$. Recall the definition of the geometric cyclic descent set map of a matching, $\mathrm{cGDes} : \mathcal{M}_n \mapsto 2^{[n]}$, from Definition 13.

Observation 53. *For every* $m \in \mathcal{M}_n$

$$\mathrm{cGDes}(m) \cap [n-1] = \mathrm{GDes}(m)$$

and

$$\mathrm{cGDes}(r(m)) = 1 + \mathrm{cGDes}(m),$$

where addition is modulo n*.*

In order to verify the non-Escher axiom for cGDes, we need the following lemma.

Lemma 54. *For $m \in \mathcal{M}_{n,k}$, where $n \geqslant k \geqslant 0$ with $n-k$ even,*

(a) $\mathrm{cGDes}(m) = \varnothing$ *if and only if $k = n$, namely, m contains no chords; and*

(b) $\mathrm{cGDes}(m) = [n]$ *if and only if $k = 0$ and* $\mathrm{cr}(m) = n/2$*, namely, n is even and m matches i with $i + n/2$ for any $1 \leqslant i \leqslant n/2$.*

Proof. Consider the possible values of k.

Case 1. If $k = n$ then all the points in m are unmatched, and therefore $\mathrm{cGDes}(m) = \varnothing$.

Case 2. If $0 < k < n$ then m has matched as well as unmatched points. There must be an umatched point followed by a matched one, and a matched point followed by an unmatched one. Therefore $\mathrm{cGDes}(m) \neq \varnothing, [n]$.

Case 3. If $k = 0$ then all the points are matched and n is even. If $\mathrm{cGDes}(m) = [n]$ then, in particular, $\mathrm{GDes}(m) = [n-1]$ and, by Proposition 34, $\mathrm{Des}(\iota(m)) = [n-1]$. It follows that $\iota(m) = w_0 = (1,n)(2,n-1)\cdots(n/2, n/2+1) \in \mathcal{I}_{n,0}$, hence m matches i with $i + n/2$ for any $1 \leqslant i \leqslant n/2$ and $\mathrm{cr}(m) = n/2$. For the opposite direction, if $\mathrm{cr}(m) = n/2$ then, by Definition 29, m matches i with $i + n/2$ for any $1 \leqslant i \leqslant n/2$ and $\mathrm{cGDes}(m) = [n]$. □

Denote now

$$\mathcal{I}_{n,k,j} := \{\pi \in \mathcal{I}_{n,k},\ \mathrm{ne}(\pi) = j\},$$

and recall the map $\hat{\iota} : \mathcal{I}_{n,k} \to \mathcal{I}_{n,k}$ from Definition 38.

Proposition 55. *Assume that $n \geqslant k \geqslant 0$ with $n-k$ even, and $0 \leqslant j \leqslant (n-k)/2$.*

(a) If $0 < k < n$, or $k = 0$ and $j \neq n/2$, then the pair

$$(\mathrm{cGDes} \circ \hat{\iota}^{-1}, \hat{\iota} \circ r \circ \hat{\iota}^{-1})$$

is a (non-Escherian) cyclic extension of Des *on $\mathcal{I}_{n,k,j}$.*

(b) If $k = n$ (and necessarily $j = 0$), or $k = 0$ and $j = n/2$, then the above pair is an Escherian cyclic extension of Des *on $\mathcal{I}_{n,k,j}$.*

Proof. The number of unmatched points is invariant under rotation and (by Proposition 39) also under $\hat{\iota}$, hence $\hat{\iota} \circ r \circ \hat{\iota}^{-1}(\pi) \in \mathcal{I}_{n,k}$ for every $\pi \in \mathcal{I}_{n,k}$. Furthermore,

$$\pi \in \mathcal{I}_{n,k,j} \Longrightarrow \hat{\iota} \circ r \circ \hat{\iota}^{-1}(\pi) \in \mathcal{I}_{n,k,j}$$

since

$$\mathrm{ne}(\hat{\iota} \circ r \circ \hat{\iota}^{-1}(\pi)) = \mathrm{cr}(r \circ \hat{\iota}^{-1}(\pi)) = \mathrm{cr}(\hat{\iota}^{-1}(\pi)) = \mathrm{ne}(\hat{\iota} \circ \hat{\iota}^{-1}(\pi)) = \mathrm{ne}(\pi).$$

Here we applied Proposition 39 and the fact that the crossing number (but not the nesting number!) is invariant under rotation.

Denote

$$\mathrm{cDes}(\pi) := \mathrm{cGDes}(\hat{\iota}^{-1}(\pi)) \qquad (\forall \pi \in \mathcal{I}_{n,k,j}).$$

By Proposition 39 and Observation 53 we have

$$\mathrm{cDes}(\pi) \cap [n-1] = \mathrm{cGDes}(\hat{\iota}^{-1}(\pi)) \cap [n-1] = \mathrm{GDes}(\hat{\iota}^{-1}(\pi)) = \mathrm{Des}(\pi)$$

and

$$\mathrm{cDes}(\hat{\iota} \circ r \circ \hat{\iota}^{-1}(\pi)) = \mathrm{cGDes}(r \circ \hat{\iota}^{-1}(\pi)) = 1 + \mathrm{cGDes}(\hat{\iota}^{-1}(\pi)) = 1 + \mathrm{cDes}(\pi)$$

for any $\pi \in \mathcal{I}_{n,k,j}$. This proves the extension and equivariance properties for every $0 \leqslant k \leqslant n$ and $0 \leqslant j \leqslant (n-k)/2$. Finally, by Lemma 54, the non-Escher property holds if and only if either $0 < k < n$ or $k = 0$ and $j \neq n/2$. □

Proof of Proposition 14. Follows from Proposition 55. □

Recall the map $Q : S_n \to \mathrm{SYT}_n$ sending each $\pi \in S_n$ to the corresponding RS recording tableau Q_π, and define $h : \mathcal{I}_{n,k} \mapsto \mathrm{SYT}_{n,k}$ by $h := Q \circ \hat{\iota}$. A cyclic descent extension on the set

$$\mathrm{SYT}_{n,k,j} := \{T \in \mathrm{SYT}_{n,k},\ 2j \leqslant \mathrm{ht}(T) \leqslant 2j+1\}.$$

is described in the following statement.

Proposition 56. *Assume that $n \geqslant k \geqslant 0$ with $n-k$ even, and $0 \leqslant j \leqslant (n-k)/2$.*

(a) If $0 < k < n$, or $k = 0$ and $j \neq n/2$, then the pair

$$(\mathrm{cGDes} \circ h^{-1}, h \circ r \circ h^{-1})$$

is a (non-Escherian) cyclic extension of Des *on* $\mathrm{SYT}_{n,k,j}$.

(b) If $k = n$ (and necessarily $j = 0$), or $k = 0$ and $j = n/2$, then the above pair is an Escherian cyclic extension of Des *on* $\mathrm{SYT}_{n,k,j}$.

Proof. Follows from Proposition 55, noting that, by Proposition 16 and Lemma 45, the restriction of Q to $\mathcal{I}_{n,k,j}$ is a descent set preserving bijection onto $\mathrm{SYT}_{n,k,j}$. □

Remark 57. Cyclic rotation of geometric configurations has been used before for the construction of cyclic descent extensions on standard Young tableaux of certain given shapes — rectangular shapes of height at most 3 [17] and flag shapes [16]. These results motivated our work, and some of them are indeed obtained as special cases:

- Letting $k = 0$ and $j = 1$ in Proposition 56 yields a cyclic descent extension on standard Young tableaux of shape (n, n), since $\mathrm{SYT}_{2n,0,1} = \mathrm{SYT}(n, n)$. One can verify that this cyclic extension coincides with the one determined by Dennis White, as described in [17, Theorem 1.4].

- Recalling that the number of Motzkin paths of length n is equal to the number of standard Young tableaux of size n and at most three rows [19, 9, 4], consider Proposition 56 on the union of $\cup_k \mathrm{SYT}_{n,k,1}$, namely $j = 1$ and k arbitrary. This determines a cyclic descent extension on Motzkin paths via Han's bijection [12], which coincides with Han's cyclic descent extension on Motzkin paths.

6 Equidistribution revisited

In an early version of this paper, the following conjecture was posed.

Conjecture 58. Let $\mu \vdash m$ and $\nu \vdash n$ be integer partitions with no common part. Let π and σ be permutations of cycle types μ and ν, respectively, with disjoint supports. Let $A_{\pi,\sigma}$ be the subset of the conjugacy class of cycle type $\mu \sqcup \nu \vdash m+n$ consisting of the permutations for which the relative order of the letters in the union of all cycles of μ is as in π, and the relative order of the letters in the union of all cycles of ν is as in σ. Then

$$\sum_{w \in A_{\pi,\sigma}} \mathbf{x}^{\mathrm{Des}(w)} = \sum_{\tau \in \pi \sqcup\!\sqcup \sigma} \mathbf{x}^{\mathrm{Des}(\tau)}.$$

Example 59. Let $\pi = (1,3,2)$ and $\sigma = (4)$. Then $A_{\pi,\sigma}$ is the following subset of the conjugacy class of cycle type $(3,1)$ in S_4:

$$A_{\pi,\sigma} = \{(1,3,2)(4),\ (1,4,2)(3),\ (1,4,3)(2),\ (2,4,3)(1)\} = \{[3124],\ [4132],\ [4213],\ [1423]\}$$

This set of permutations and the set

$$\pi \sqcup\!\sqcup \sigma = [312] \sqcup\!\sqcup [4] = \{[3124],\ [3142],\ [3412],\ [4312]\}$$

have the same distribution of the descent set.

Conjecture 58 was proved by Gessel.

Proposition 60. [10] *Conjecture 58 holds.*

The proof is partly algebraic and not bijective.

Proposition 60 implies the following.

Corollary 61. *There exists an (implicit) bijection*

$$\phi : \mathcal{I}_{n-k,0} \sqcup\!\sqcup [n-k+1,\ldots,n] \to \mathcal{I}_{n,k}.$$

preserving descent set, nesting number and crossing number.

Proof. Take, in Proposition 60, $\sigma \in \mathcal{I}_{n-k,0}$ and $\pi = [n-k+1,\ldots,n]$, the identity permutation in $S_{[n]\setminus[n-k]}$, the group of permutations on the letters $\{n-k+1,\ldots,n\}$. It follows that there exists a bijection

$$\phi : \mathcal{I}_{n-k,0} \sqcup\!\sqcup [n-k+1,\ldots,n] \to \mathcal{I}_{n,k}$$

which preserves the descent set and satisfies the following property: for every $\sigma \in \mathcal{I}_{n-k,0}$ and every $\tau \in \sigma \sqcup [n-k+1,\ldots,n]$, the relative order of the letters in the union of all 2-cycles in $\phi(\tau)$ is equal to the relative order of the letters in σ. By Definition 41, the nesting and crossing numbers of τ are the same as those of σ. Thus ϕ preserves nesting and crossing numbers as well. $\square$

Remark 62. The explicit bijection $q : \mathcal{I}_{n-k,0} \sqcup [n-k+1, n-k+2, \ldots n] \to \mathcal{I}_{n,k}$ from Lemma 50 preserves the descent set and the nesting number, but does not preserve the crossing number. The bijection $\phi : \mathcal{I}_{n-k,0} \sqcup [n-k+1, n-k+2, \ldots n] \to \mathcal{I}_{n,k}$, whose existence is claimed in Corollary 61, preserves the crossing number as well.

The following refinement of Theorem 11 follows.

Theorem 63. *For every $n \geqslant k \geqslant 0$ with $n-k$ even,*

$$\sum_{m \in \mathcal{M}_{n,k}} q^{\mathrm{cr}(m)} t^{\mathrm{ne}(m)} \mathbf{x}^{\mathrm{GDes}(m)} = \sum_{m \in \mathcal{M}_{n,k}} q^{\mathrm{ne}(m)} t^{\mathrm{cr}(m)} \mathbf{x}^{\mathrm{Des}(m)}.$$

Proof. Replace $\hat{\iota} := q \circ \varphi$ by $\eta := \phi \circ \varphi$ in the proof of Theorem 11, where again we use $\mathcal{I}_{n,k}$ instead of $\mathcal{M}_{n,k}$. Combining Lemma 43 with Corollary 61 implies that for any $n \geqslant k \geqslant 0$, the map

$$\eta : \mathcal{I}_{n,k} \to \mathcal{I}_{n,k}$$

is a bijection which satisfies

$$\mathrm{GDes}(\pi) = \mathrm{Des}(\eta(\pi)) \qquad (\forall \pi \in \mathcal{I}_{n,k})$$

as well as

$$\mathrm{ne}(\pi) = \mathrm{cr}(\eta(\pi)) \quad \text{and} \quad \mathrm{cr}(\pi) = \mathrm{ne}(\eta(\pi)) \qquad (\forall \pi \in \mathcal{I}_{n,k}).$$

This completes the proof. $\square$

Problem 64. Find an explicit bijective proof of Theorem 63.

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