The h^*-polynomial of the order polytope of the zig-zag poset

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Abstract

We construct a family of shellings for the canonical triangulation of the order polytope of the zig-zag poset. This gives a new combinatorial interpretation for the coefficients in the numerator of the Ehrhart series of this order polytope in terms of the swap statistic on alternating permutations. We also offer an alternate proof of this result using the techniques of rank selection. Finally, we show that the sequence of coefficients of the numerator of this Ehrhart series is symmetric and unimodal.

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1 Introduction and Preliminaries

The *zig-zag poset* $\mathcal{Z}_n$ on ground set $\{z_1, \ldots, z_n\}$ is the poset with exactly the cover relations $z_1 < z_2 > z_3 < z_4 > \ldots$. That is, this partial order satisfies $z_{2i-1} < z_{2i}$ and $z_{2i} > z_{2i+1}$ for all i between 1 and $\lfloor \frac{n-1}{2} \rfloor$. The order polytope of $\mathcal{Z}_n$, denoted $\mathcal{O}(\mathcal{Z}_n)$ is the set of all n-tuples $(x_1, \ldots, x_n) \in \mathbb{R}^n$ that satisfy $0 \leqslant x_i \leqslant 1$ for all i and $x_i \leqslant x_j$ whenever $z_i < z_j$ in $\mathcal{Z}_n$. In this paper, we introduce the "swap" permutation statistic on alternating permutations to give a new combinatorial interpretation of the numerator of the Ehrhart series of $\mathcal{O}(\mathcal{Z}_n)$.

The numerator of the Ehrhart series of a polytope, also known as its h^*-polynomial, can be computed in several different ways. In the present work, we compute the h^*-polynomial of $\mathcal{O}(\mathcal{Z}_n)$ both by shelling its canonical triangulation in Section 2 and by

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examining the descents in the elements of its Jordan-Hölder set in Section 3. While the latter approach may appear to be straightforward at first, it is significantly complicated by the fact that the elements of the Jordan-Hölder set in this case are neither alternating nor inverse alternating permutations.

The polytope $\mathcal{O}(\mathcal{Z}_n)$ is related to certain group-based hidden variable Markov models in mathematical phylogenetics. We began studying this problem in relation to combinatorial properties of the Cavender-Farris-Neyman model with a molecular clock (or CFN-MC model) [4]. We were interested in the polytope associated to the toric variety obtained by applying the discrete Fourier transform to the Cavender-Farris-Neyman model with a molecular clock on a given rooted binary phylogenetic tree. We call this polytope the CFN-MC polytope. In particular, the Ehrhart polynomial of $\mathcal{O}(\mathcal{Z}_n)$ is equal to that of the CFN-MC polytope of any rooted binary tree on $n+1$ leaves [4, Theorem 6.20]. Therefore, the Ehrhart series of $\mathcal{O}(\mathcal{Z}_n)$ is also equal to the Hilbert series of the toric ideal of phylogenetic invariants of the CFN-MC model on such a tree.

In the remainder of this section, we will give some preliminary definitions and key theorems regarding alternating permutations, order polytopes and Ehrhart theory. In Section 2, we prove our main result, Theorem 13, by giving a shelling of the canonical triangulation of the order polytope of the zig-zag poset. In Section 3, we give an alternate proof of Theorem 13 by counting chains in the lattice of order ideals of the zig-zag poset. This proof makes use of the theory of Jordan-Hölder sets of general posets developed in Chapter 2 of [9]. In Section 4, we discuss some combinatorial properties of the swap statistic and present some open problems.

1.1 Alternating Permutations

Definition 1. An *alternating permutation* on n letters is a permutation σ such that $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \dots$. That is, an alternating permutation satisfies $\sigma(2i-1) < \sigma(2i)$ and $\sigma(2i) > \sigma(2i+1)$ for $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$.

We denote by A_n the set of all alternating permutations. Notice that alternating permutations coincide with order-preserving bijections from $\mathcal{Z}_n$ to $[n]$ as we discuss in more detail at the end of this section.

The number of alternating permutations of length n is the nth *Euler zig-zag number* E_n. The sequence of Euler zig-zag numbers starting with E_0 begins $1, 1, 1, 2, 5, 16, 61, 272$. This sequence can be found in the Online Encyclopedia of Integer Sequences with identification number A000111 [6]. The exponential generating function for the Euler zig-zag numbers satisfies

$$\sum_{n \geqslant 0} E_n \frac{x^n}{n!} = \tan x + \sec x.$$

Furthermore, the Euler zig-zag numbers satisfy the recurrence

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$$

for $n \geqslant 1$ with initial values $E_0 = E_1 = 1$. A thorough background on the combinatorics of alternating permutations can be found in [8]. The following new permutation statistic on alternating permutations is central to our results.

Example 2. The fourth Euler zig-zag number E_4 is 5. The alternating permutations in S_4 are 1324, 1423, 2314, 2413 and 3412.

Definition 3. Let σ be an alternating permutation. The permutation statistic $\mathrm{swap}(\sigma)$ is the number of $i < n$ such that $\sigma^{-1}(i) < \sigma^{-1}(i+1) - 1$. Equivalently, this is the number of $i < n$ such that i is to the left of $i + 1$ and swapping i and $i + 1$ in σ yields another alternating permutation. The swap-set $\mathrm{Swap}(\sigma)$ is the set of all $i < n$ for which we can perform this operation. We say that σ *swaps to* τ if τ can be obtained from σ by performing this operation a single time.

Example 4. The permutation $\sigma = 34172856$ is an alternating permutation in A_8. Its swap-set is $\mathrm{Swap}(\sigma) = \{1, 4, 7\}$. Thus it has $\mathrm{swap}(\sigma) = \#\mathrm{Swap}(\sigma) = 3$. The permutation σ swaps to three different alternating permutations τ_1, τ_2 and τ_3 corresponding to the three elements of the swap set. They are

$$\tau_1 = 34271856, \qquad \tau_2 = 35172846 \qquad \text{and} \qquad \tau_3 = 34182756.$$

We will also make use of the following two features which can be defined for any permutation. Let $\sigma \in S_n$.

Definition 5. A *descent* of σ is an index $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$. An *inversion* of σ is any pair (i, j) for $1 \leqslant i < j \leqslant n$ such that $\sigma(j) < \sigma(i)$. The *inversion number* $\mathrm{inv}(\sigma)$ is the number of inversions of σ.

When we write σ in one-line notation, a descent is a position on σ where the value of σ drops. An inversion is any pair of positions in which a larger number appears before a smaller number in σ.

Example 6. The alternating permutation $\sigma = 34172856$ has descent set $\{2, 4, 6\}$. In fact, every alternating permutation of length n has descent set $\{2, 4, \ldots, n-2\}$ if n is even and $\{2, 4, \ldots, n-1\}$ if n is odd. The set of inversions of σ is

$$\{(1,3), (1,5), (2,3), (2,5), (4,5), (4,7), (4,8), (6,7), (6,8)\}.$$

So the inversion number $\mathrm{inv}(\sigma)$ is 9.

Recall that a *linear extension* of a poset P is a bijection $\lambda : P \to [n]$ such that if $x < y$ in P, then $\lambda(x) < \lambda(y)$. Note that a linear extension of $\mathscr{Z}_n$ can be viewed as a permutation of $[n]$ by considering its domain to be the set of subscripts of the elements z_i of $\mathscr{Z}_n$. We conclude this section with the observation linear extensions of $\mathscr{Z}_n$ are in bijection with alternating permutations. This proposition follows directly from the definition of the cover relations of the zig-zag poset and the definition of an alternating permutation.

Proposition 7. *A bijection* $\lambda : \mathscr{Z}_n \to [n]$ *is a linear extension of* $\mathscr{Z}_n$ *if and only if the map sending* i *to* $\lambda(i)$ *is an alternating permutation.*

1.2 Order Polytopes

To every finite poset on n elements one can associate a polytope in $\mathbb{R}^n$ by viewing the cover relations on the poset as inequalities on Euclidean space.

Definition 8. The *order polytope* $\mathcal{O}(P)$ of any poset P on ground set $p_1, \ldots, p_n$ is the set of all $\mathbf{v} \in \mathbb{R}^n$ that satisfy $0 \leqslant v_i \leqslant 1$ for all i and $v_i \leqslant v_j$ if $p_i < p_j$ is a cover relation in P.

Order polytopes for arbitrary posets have been the object of considerable study, and are discussed in detail in [7]. In the case of $\mathcal{O}(\mathcal{Z}_n)$, the facet defining inequalities are those of the form

$$-v_i \leqslant 0 \text{ for } i \leqslant n \text{ odd}$$
$$v_i \leqslant 1 \text{ for } i \leqslant n \text{ even}$$
$$v_i - v_{i+1} \leqslant 0 \text{ for } i \leqslant n - 1 \text{ odd, and}$$
$$-v_i + v_{i+1} \leqslant 0 \text{ for } i \leqslant n - 1 \text{ even.}$$

Note that the inequalities of the form $-v_i \leqslant 0$ for i even and $v_i \leqslant 1$ for i odd are redundant. The order polytope of $\mathcal{Z}_n$ is also the convex hull of all $(v_1, \ldots, v_n) \in \{0, 1\}^n$ that are indicator vectors of upper order ideals of $\mathcal{Z}_n$. These are also known as *filters*. Equivalently, the vertices are exactly the labelings of $\mathcal{Z}_n$ with 0 and 1 that are weakly consistent with the partial order on $\{p_1, \ldots, p_n\}$.

In [7], Stanley gives the following canonical unimodular triangulation of the order polytope of any poset P on ground set $\{p_1, \ldots, p_n\}$. Let $\sigma : P \to [n]$ be a linear extension of P. Denote by $\mathbf{e}_i$ the ith standard basis vector in $\mathbb{R}^n$. The simplex Δ^σ is the convex hull of $\mathbf{v}_0^\sigma, \ldots, \mathbf{v}_n^\sigma$ where $\mathbf{v}_0^\sigma$ is the all 1's vector and $\mathbf{v}_i^\sigma = \mathbf{v}_{i-1}^\sigma - \mathbf{e}_{\sigma^{-1}(i)}$. Letting σ range over all linear extensions of P yields a unimodular triangulation of $\mathcal{O}(P)$. Hence, the normalized volume of $\mathcal{O}(P)$ is the number of linear extensions of P. In particular, this means that the volume of $\mathcal{O}(\mathcal{Z}_n)$ is the Euler zig-zag number, E_n.

Example 9. Consider the case when $n = 4$. The zig-zag poset $\mathcal{Z}_4$ is pictured in Figure 1. The order polytope $\mathcal{O}(\mathcal{Z}_4)$ has facet defining inequalities

$$
\begin{aligned}
-v_1 &\leqslant 0 & v_2 &\leqslant 1 \\
-v_3 &\leqslant 0 & v_4 &\leqslant 1 \\
v_1 - v_2 &\leqslant 0 & -v_2 + v_3 &\leqslant 0 \\
v_3 - v_4 &\leqslant 0 &
\end{aligned}
$$

The vertices of $\mathcal{O}(\mathcal{Z}_4)$ are the columns of the matrix

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
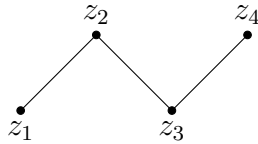0 & 0 & 1 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}.
$$

Figure 1: The zig-zag poset $\mathcal{Z}_4$

The alternating permutations on 4 elements, which correspond to linear extensions of $\mathcal{Z}_4$, are 1324, 1423, 2314, 2413, and 3412. Note that there are $E_4 = 5$ such alternating permutations, so the normalized volume of $\mathcal{O}(\mathcal{Z}_4)$ is 5. The simplex in the canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$ corresponding to 2413 is

$$\Delta^{2413} = \mathrm{conv} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

1.3 Ehrhart Theory

We turn our attention to the study of Ehrhart functions and series of lattice polytopes. Let $P \subset \mathbb{R}^n$ be any polytope with integer vertices. Recall that the *Ehrhart function*, $i_P(m)$, counts the integer points in dilates of P; that is,

$$i_P(m) = \#(\mathbb{Z}^n \cap mP),$$

where $mP = \{m\mathbf{v} \mid \mathbf{v} \in P\}$ denotes the mth dilate of P. The Ehrhart function is, in fact, a polynomial in m [1, Chapter 3]. We further define the *Ehrhart series* of P to be the generating function

$$\mathrm{Ehr}_P(t) = \sum_{m \geqslant 0} i_P(m) t^m.$$

The Ehrhart series is of the form

$$\mathrm{Ehr}_P(t) = \frac{h_P^*(t)}{(1-t)^{d+1}},$$

where d is the dimension of P and $h_P^*(t)$ is a polynomial in t of degree at most d. Often we just write $h^*(t)$ when the particular polytope is clear. The coefficients of $h^*(t)$ have an interpretation in terms of a *shelling* of a unimodular triangulation of P, if such a shellable unimodular triangulation exists.

Definition 10. Let $\mathcal{T}$ be the collection of maximal dimensional simplices in a pure simplicial complex of dimension d with $\#\mathcal{T} = s$. An ordering $\Delta_1, \Delta_2, \ldots, \Delta_s$ on the simplices in $\mathcal{T}$ is a *shelling order* if for all $1 < r \leqslant s$,

$$\bigcup_{i=1}^{r-1} \left(\Delta_i \cap \Delta_r \right)$$

is a union of facets of Δ_r.

Equivalently, the order $\Delta_1, \Delta_2, \ldots, \Delta_s$ is a shelling order if and only if for all $r \leqslant s$ and $k < r$, there exists an $i < r$ such that $\Delta_k \cap \Delta_r \subset \Delta_i \cap \Delta_r$ and $\Delta_i \cap \Delta_r$ is a facet of Δ_r. This means that when we build our simplicial complex by adding facets in the order prescribed by the shelling order, we add each simplex along its highest dimensional facets.

Example 11. Consider the order polytope $\mathcal{O}(\mathcal{Z}_4)$ with its canonical triangulation by alternating permutations

$$\Delta^{3412}, \Delta^{2413}, \Delta^{2314}, \Delta^{1423}, \Delta^{1324}.$$

This particular ordering of the facets in the canonical triangulation is a special case of the shelling order that will be established and proved in the next section. The fact that this is a shelling order can be checked directly in this example, for instance:

$$\Delta^{2314} \cap (\Delta^{3412} \cup \Delta^{2413}) = \mathbf{conv} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

which is a facet of Δ^{2314}.

Keeping track of the number of facets that each simplex is added along gives the following relationship between shellings of a triangulation of an integer polytope and the Ehrhart series of the polytope, which is proved in [1, Chapter 3].

Theorem 12. *Let P be a polytope with integer vertices. Let $\{\Delta_1, \ldots, \Delta_s\}$ be a unimodular triangulation of P using the integer points of P. Denote by h_j^* the coefficient of t^j in the h^* polynomial of P. If $\Delta_1, \ldots, \Delta_s$ is a shelling order, then h_j^* is the number of Δ_i that are added along j of their facets in this shelling. Equivalently,*

$$h^*(t) = \sum_{i=1}^{s} t^{a_i},$$

where $a_i = \#\{k < i \mid \Delta_k \cap \Delta_i \text{ is a facet of } \Delta_i\}$.

Example (Example 11, continued). Since Δ^{3412} comes first in the shelling, it is added along zero of its facets; hence it contributes a summand of 1 to $h_{\mathcal{O}(\mathcal{Z}_4)}^*(t)$. The intersections $\Delta^{2413} \cap \Delta^{3412}$, $\Delta^{2314} \cap (\Delta^{3412} \cup \Delta^{2413})$ and $\Delta^{1423} \cap (\Delta^{2314} \cup \Delta^{3412} \cup \Delta^{2413})$ each consist a single facet. So they each contribute a summand of t to $h_{\mathcal{O}(\mathcal{Z}_4)}^*(t)$. Finally $\Delta^{1324} \cap (\Delta^{1423} \cup \Delta^{2314} \cup \Delta^{3412} \cup \Delta^{2413})$ consists of two facets of Δ^{1324}. So it contributes a summand of t^2 to $h_{\mathcal{O}(\mathcal{Z}_4)}^*(t)$. Thus we have $h_{\mathcal{O}(\mathcal{Z}_4)}^*(t) = 1 + 3t + t^2$.

The goal of this paper is to prove the following theorem relating the h^*-polynomial of $\mathcal{O}(\mathcal{Z}_n)$ and the swap statistic.

Theorem 13. *The numerator of the Ehrhart series of $\mathcal{O}(\mathscr{Z}_n)$ is*

$$h^*_{\mathcal{O}(\mathscr{Z}_n)}(t) = \sum_{\sigma \in A_n} t^{swap(\sigma)}.$$

Example (Example 11, continued)**.** We verify Theorem 13 in the case of $\mathscr{Z}_4$. We have $swap(3412) = 0$, $swap(2413) = swap(2314) = swap(1423) = 1$ and $swap(1324) = 2$. So Theorem 13 holds in this case. In particular, note that in this example $swap(\sigma)$ is exactly the number of facets Δ^σ is added along in the shelling order.

In Section 2, we prove this result by giving a shelling of the canonical triangulation of $\mathcal{O}(\mathscr{Z}_n)$. Alternate formulas for the h^*-polynomial of the order polytope of a poset P exist, as described in [9, Chapter 3.13]. Many of these formulas refer to the *Jordan-Hölder set* of P and in particular, descents in the permutations in this set. In Section 3, we offer an alternate proof of Theorem 13 using this machinery. We now provide a review of the relevant theory.

1.4 Jordan-Hölder Sets and Rank Selection

Our second proof of Theorem 13 relies heavily on the concepts of rank selection and flag f-vectors developed for general posets in Sections 3.13 and 3.15 of [9]. We will focus our attention to the zig-zag poset, $\mathscr{Z}_n$. Denote by $J(\mathscr{Z}_n)$ the distributive lattice of order ideals in $\mathscr{Z}_n$ ordered by inclusion. Let $S = \{s_1, \ldots, s_k\} \subset [0, n]$, where $[0, n] = \{0, \ldots, n\}$. We always assume that $s_1 < s_2 < \ldots < s_k$. Denote by $\alpha_n(S)$ the number of chains of order ideals $I_1 \subsetneq \cdots \subsetneq I_k$ in $J(\mathscr{Z}_n)$ such that $\#I_j = s_j$ for all j. Define

$$\beta_n(S) = \sum_{T \subset S} (-1)^{\#(S-T)} \alpha_n(T).$$

By the Principle of Inclusion-Exclusion, or equivalently, via Möbius inversion on the Boolean lattice,

$$\alpha_n(S) = \sum_{T \subset S} \beta_n(T).$$

Example 14. Consider the case where $n = 5$ and $S = \{1, 3\}$. The chains of order ideals of $\mathscr{Z}_n$ of the form $I_1 \subsetneq I_2$ where $\#I_1 = 1$ and $\#I_2 = 3$ are:

$$\{z_1\} \subsetneq \{z_1, z_2, z_3\} \qquad\qquad \{z_3\} \subsetneq \{z_1, z_2, z_3\}$$
$$\{z_1\} \subsetneq \{z_1, z_3, z_5\} \qquad\qquad \{z_3\} \subsetneq \{z_1, z_3, z_5\}$$
$$\{z_5\} \subsetneq \{z_1, z_3, z_5\} \qquad\qquad \{z_3\} \subsetneq \{z_3, z_4, z_5\}$$
$$\{z_5\} \subsetneq \{z_3, z_4, z_5\}.$$

Thus $\alpha_5(\{1, 3\}) = 7$. Similarly, we have $\alpha_5(\{1\}) = 3$, $\alpha_5(\{3\}) = 3$ and $\alpha_5(\emptyset) = 1$. Thus, $\beta_5(\{1, 3\}) = 1 - 3 - 3 + 7 = 2$.

In Section 3.13 of [9], the function $\alpha_n : 2^{[0,n]} \to \mathbb{Z}$ is called the *flag f-vector* of $\mathscr{Z}_n$ and $\beta_n : 2^{[0,n]} \to \mathbb{Z}$ is called the *flag h-vector* of $\mathscr{Z}_n$. For any poset P of size n, let $\omega : P \to [n]$ be an order-preserving bijection (ie. a linear extension) that assigns a label to each element of P; in this case, ω is called a *natural labeling*. Then for any linear extension $\sigma : P \to [n]$, we may define a permutation in S_n by $\omega \circ \sigma^{-1}$. The *Jordan-Hölder set* $\mathcal{L}(P, \omega)$ is the set of all permutations obtained in this way. The following result for arbitrary finite posets can be found in Chapter 3.13 of [9].

Theorem 15 ([9], Theorem 3.13.1). *Let $S \subset [n-1]$. Then $\beta_n(S)$ is equal to the number of permutations $\tau \in \mathcal{L}(P, \omega)$ with descent set S.*

Example (Example 14, continued). We verify Theorem 15 in the case where $n = 5$ and $S = \{1, 3\}$. Consider the natural labeling $\omega : \mathscr{Z}_5 \to [5]$ given by $z_1 \mapsto 1$, $z_2 \mapsto 4$, $z_3 \mapsto 2$, $z_4 \mapsto 5$ and $z_5 \mapsto 3$. This is a linear extension of $\mathscr{Z}_5$ and corresponds to the alternating permutation 14253. Let $\sigma_1 = 23154$, which is an alternating permutation. We have $\sigma_1^{-1} = 31254$. So the element of the Jordan-Hölder set $\mathcal{L}(\mathscr{Z}_5, \omega)$ corresponding to σ_1 is $\omega \circ \sigma_1^{-1} = 21435$. Note that the descent set of $\omega \circ \sigma_1^{-1}$ is $\{1, 3\}$. Similarly, letting $\sigma_2 = 45231$, we have $\omega \circ \sigma_2^{-1} = 32514$ whose descent set is also $\{1, 3\}$. These are exactly the alternating permutations σ for which the descent set of $\omega \circ \sigma^{-1}$ is $\{1, 3\}$, and $\beta_5(\{1, 3\}) = 2$ as needed.

The *order polynomial* of a poset P, $\Omega_P(m)$ is the number of order preserving maps from P to $[m]$. The Ehrhart polynomial of $\mathcal{O}(\mathscr{Z}_n)$ evaluated at m is equal to the order polynomial of $\mathscr{Z}_n$ evaluated at $m+1$ [7]. We also have the following equality of generating functions from Theorem 3.15.8 of [9]. We restate the relevant special case of this theorem here.

Theorem 16 ([9], Theorem 3.15.8). *Let $\omega : P \to [n]$ be an order-preserving bijection. Then*

$$\sum_{m \geqslant 0} \Omega_P(m) x^m = \frac{\sum_{\sigma \in \mathcal{L}(P, \omega)} x^{1 + \mathrm{des}(\sigma)}}{(1 - x)^{n+1}}.$$

Therefore, since $i_{\mathcal{O}(\mathscr{Z}_n)}(m) = \Omega_{\mathcal{O}(\mathscr{Z}_n)}(m + 1)$, we have that

$$\mathrm{Ehr}_{\mathcal{O}(\mathscr{Z}_n)}(t) = \frac{\sum_{\sigma \in \mathcal{L}(\mathscr{Z}_n, \omega)} x^{\mathrm{des}(\sigma)}}{(1 - x)^{n+1}}.$$

It follows that the h^*-polynomial of $\mathcal{O}(\mathscr{Z}_n)$ is

$$h^*_{\mathcal{O}(\mathscr{Z}_n)}(t) = \sum_{S \subset [n-1]} \beta_n(S) t^{\#S}. \tag{1}$$

In the case of $\mathscr{Z}_n$, the elements of the Jordan-Hölder set are not alternating or inverse alternating permutations (as we see in Example 14) because they arise from linear extensions with respect to a natural labeling of $\mathscr{Z}_n$. Indeed, our labeling of $\mathscr{Z}_n$, with respect

to which linear extensions are the same as alternating permutations, is not natural since the map $z_i \mapsto i$ is not a linear extension of $\mathcal{Z}_n$. The elements of the Jordan-Hölder set do not have as nice of a combinatorial description as the alternating permutations, and there is not an obvious bijection between swaps in alternating permutations and descents in elements of the Jordan-Hölder set. Section 3 is largely devoted to showing that $\beta_n(S)$ is equal to the number of alternating permutations σ with $\mathrm{Swap}(\sigma) = S$, and the main theorem follows from this fact.

2 Shelling the Canonical Triangulation of the Order Polytope

In this section we describe a family of shelling orders on the simplices of the canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$. Let σ be an alternating permutation. We will denote by $\mathrm{vert}(\sigma)$ the set of all vertices of the simplex Δ^σ. Note that this is the set of all $0/1$ vectors $\mathbf{v}$ of length n that have $v_i \leqslant v_j$ whenever $\sigma(i) < \sigma(j)$.

Proposition 17. The simplices Δ^σ and Δ^τ are joined along a facet if and only if σ swaps to τ or τ swaps to σ.

Proof. Simplices Δ^σ and Δ^τ are joined along a facet if and only if $\mathrm{vert}(\sigma)$ and $\mathrm{vert}(\tau)$ differ by a single element. Since every simplex in the canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$ has exactly one vertex with the sum of its components equal to i for $0 \leqslant i \leqslant n$ and the all 0's and all 1's vector are in every simplex in this triangulation, this occurs if and only if there exists an i with $1 \leqslant i \leqslant n-1$ such that $\mathrm{vert}(\sigma) - \{\mathbf{v}_i^\sigma\} = \mathrm{vert}(\tau) - \{\mathbf{v}_i^\tau\}$. By definition of each $\mathbf{v}_j^\sigma$ and $\mathbf{v}_j^\tau$, this occurs if and only if $\sigma^{-1}(j) = \tau^{-1}(j)$ for all $j \neq i, i+1$ and $\mathbf{e}_{\sigma^{-1}(i)} + \mathbf{e}_{\sigma^{-1}(i+1)} = \mathbf{e}_{\tau^{-1}(i)} + \mathbf{e}_{\tau^{-1}(i+1)}$. This is true if and only if swapping the positions of i and $i+1$ in σ yields τ, as needed. $\qquad\square$

Example 18. Let $n = 5$ and consider the alternating permuation $\sigma = 35142$. We have $\mathrm{Swap}(\sigma) = \{1, 3\}$, so σ swaps to two permutations $\tau_1 = 35241$ and $\tau_2 = 45132$. Note that the vertex sets of Δ^σ and Δ^{τ_1} differ by a single vertex, and the same is true for τ_2. Indeed, the simplices Δ^σ and Δ^{τ_1} are

$$\Delta^\sigma = \mathbf{conv} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta^{\tau_1} = \mathbf{conv} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since τ_1 is obtained from σ by swapping the positions of 1 and 2, we have that $\mathbf{v}_1^\sigma \neq \mathbf{v}_1^{\tau_1}$, but all other vertices are equal. So these simplices intersect along a facet. Note that σ does not swap to $\tau_3 = 45231$ since it is obtained by swapping the positions of 1 and 2 and the positions of 3 and 4. And indeed, $\Delta^\sigma \cap \Delta^{\tau_3}$ is not a facet since $\mathbf{v}_1^\sigma \neq \mathbf{v}_1^{\tau_3}$ and $\mathbf{v}_3^\sigma \neq \mathbf{v}_3^{\tau_3}$.

Denote by $\mathrm{inv}(\sigma)$ the number of inversions of a permutation σ; that is, $\mathrm{inv}(\sigma)$ is the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$. We similarly define a *non-inversion* to be

a pair $i < j$ with $\sigma(i) < \sigma(j)$. We call an inversion or non-inversion (i, j) *relevant* if $i < j - 1$.

The motivation behind this definition is that all alternating permutations have at least $\lfloor (n-1)/2 \rfloor$ inversions, namely those of the form $(2i, 2i + 1)$ for $1 \leqslant i \leqslant \lfloor (n-1)/2 \rfloor$. These inversions are necessary in order for a permutation to be alternating. Similarly, all alternating permutations have at least $\lfloor n/2 \rfloor$ non-inversions of the form $(2i - 1, 2i)$ for $1 \leqslant i \leqslant \lfloor n/2 \rfloor$. Since these inversions and non-inversions are present in all alternating permutations, we consider them to be irrelevant. In the following proofs, the relevant inversions and non-inversions (that is, those that do not occur between consecutive positions) are the ones that give insight into the relationship between the swap statistic and the intersections of simplices in the triangulation of $\mathcal{O}(\mathcal{Z}_n)$.

Note that performing a swap on an alternating permutation always increases its inversion number by exactly one.

The following lemma relates relevant non-inversions to swaps in between them.

Lemma 19. *Let σ be an alternating permutation. Let $a, b \in [n]$ such that $(\sigma^{-1}(a), \sigma^{-1}(b))$ is a relevant non-inversion of σ. Then there exists a k with $a \leqslant k < b$ such that k is a swap of σ.*

Proof. We proceed by induction on $b - a$. If $b - a = 1$, then since $(\sigma^{-1}(a), \sigma^{-1}(b))$ is a relevant non-inversion, a is a swap in σ.

Let $b - a > 1$. Consider the position of $a + 1$ in σ. There are three cases. If $\sigma^{-1}(a + 1) < \sigma^{-1}(b) - 1$, then $(\sigma^{-1}(a + 1), \sigma^{-1}(b))$ is a relevant non-inversion, and we are done by induction. If $\sigma^{-1}(a+1) > \sigma^{-1}(b)$, then a is a swap in σ. If $\sigma^{-1}(a+1) = \sigma^{-1}(b) - 1$, then note that $\sigma^{-1}(a) < \sigma^{-1}(a + 1) - 1$ since otherwise, $a, a + 1, b$ would be an adjacent increasing sequence in σ, which would contradict that σ is alternating. So a is a swap in σ, as needed. $\square$

Example 20. Consider the alternating permutation $\sigma = 34172856$. We illustrate examples of the three cases described in the proof of Lemma 19. First, consider the relevant non-inversion $(1, 4)$, so that following the notation of the lemma, $a = 3$ and $b = 7$. Then $\sigma^{-1}(4) = 2 < \sigma^{-1}(7) - 1$, and $(2, 4)$ is a relevant non-inversion of σ. We are now done since 4 is a swap in σ. This then puts us in the second case of a proof, where 4 is itself a swap in σ. Finally, to illustrate the third case, consider the relevant non-inversion $(3, 6)$, so that $a = 1$ and $b = 8$. We have $\sigma^{-1}(2) = 5$, and indeed 1 is a swap in σ.

Theorem 13 follows as a corollary of Theorem 12, Proposition 17 and the following theorem.

Theorem 21. *Let $\sigma_1, \ldots, \sigma_{E_n}$ be an order on the alternating permutations such that if $i < j$ then $inv(\sigma_i) \geqslant inv(\sigma_j)$. Then the order $\Delta^{\sigma_1}, \ldots, \Delta^{\sigma_{E_n}}$ on the simplices of the canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$ is a shelling order.*

Note that since performing a swap increases inversion number by exactly one, the condition of Theorem 21 implies that if σ_j swaps to σ_i, then $i < j$. For any alternating

permutation σ, define the *exclusion set* of σ, $\mathrm{excl}(\sigma)$ to be the set of all $\mathbf{v}_k^\sigma \in \mathrm{vert}(\sigma)$ such that k is a swap in σ. In other words,

$$\mathrm{excl}(\sigma) = \{\mathbf{v} \mid \mathbf{v} \in \Delta^\sigma - \Delta^\tau \text{ for some } \tau \text{ such that } \sigma \text{ swaps to } \tau\}.$$

For example, we computed in Example 18 that $\mathrm{swap}(35142) = \{1, 3\}$. So $\mathrm{excl}(35142) = \{\mathbf{v}_1^{35142}, \mathbf{v}_3^{35142}\}$. In the proof of Theorem 21, we will show that Proposition 17 implies that in order to prove Theorem 21, it suffices to check that if $\mathrm{inv}(\sigma) \leqslant \mathrm{inv}(\tau)$, then $\mathrm{excl}(\sigma) \not\subset \mathrm{vert}(\tau)$. This fact follows from the next two propositions.

Proposition 22. Let σ be an alternating permutation. Then σ maximizes inversion number over all alternating permutations τ with $\mathrm{excl}(\sigma) \subset \mathrm{vert}(\tau)$.

Proof. Consider a vertex $\mathbf{v}_k^\sigma \in \mathrm{vert}(\sigma)$. Note that we may read all of the non-inversions (i, j) with $\sigma(i) \leqslant k < \sigma(j)$ from $\mathbf{v}_k^\sigma$ since these correspond to pairs of positions in $\mathbf{v}_k^\sigma$ with a 0 in the first position and a 1 in the second. That is to say, we have $\mathbf{v}_k^\sigma(i) = 0$, $\mathbf{v}_k^\sigma(j) = 1$, and $i < j$.

We claim that every relevant non-inversion of σ can be read from an element of $\mathrm{excl}(\sigma)$ in this way. By Lemma 19, there exists a swap k in σ with $\sigma(i) \leqslant k < \sigma(j)$, and the relevant non-inversion (i, j) can be read from $\mathbf{v}_k^\sigma$ in the manner described above.

Therefore, all relevant non-inversions in σ can be found as a non-adjacent $0 - 1$ pair in a vertex in $\mathrm{excl}(\sigma)$. In particular, we can count the number of relevant non-inversions in σ from the vertices in $\mathrm{excl}(\sigma)$. Furthermore, if $\mathrm{excl}(\sigma) \subset \mathrm{vert}(\tau)$, then all non-inversions in σ must also be non-inversions in τ, though τ can contain more non-inversions as well. So σ minimizes the number of non-inversions, and therefore maximizes the number of inversions, over all τ with $\mathrm{excl}(\sigma) \subset \mathrm{vert}(\tau)$. $\square$

Example 23. The alternating permutation $\sigma = 34172856$ whose inversion set is listed in Example 6 has $\mathrm{Swap}(\sigma) = \{1, 4, 7\}$. So the exclusion set of σ is

$$\mathrm{excl}(\sigma) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

In the proof of Proposition 22, we show that every relevant non-inversion corresponds to a pair of non-adjacent entries of some $\mathbf{v} \in \mathrm{excl}(\sigma)$ that contain a 0 before a 1. Indeed, these pairs in $\mathbf{v}_1^\sigma$ are $(3, 5), (3, 6), (3, 7)$ and $(3, 8)$. In $\mathbf{v}_4^\sigma$, they are $(1, 4), (1, 6), (1, 7), (1, 8), (2, 4)$, $(2, 6), (2, 7), (2, 8), (3, 6), (3, 7), (3, 8), (5, 7)$ and $(5, 8)$ and in $\mathbf{v}_7^\sigma$, they are $(1, 6), (2, 6), (3, 6)$ and $(4, 6)$. Note that the non-adjacent pairs that are *not* in any of these lists are exactly the relevant inversions of σ; namely, these are $(1, 3), (1, 5), (2, 5), (4, 5), (4, 7), (4, 8)$ and $(6, 8)$, as listed in Example 6. Let $\tau = 23164857$ and note that $\mathrm{excl}(\sigma) \subset \mathrm{vert}(\tau)$. And indeed, $\mathrm{inv}(\tau) = 6$ which is less than $\mathrm{inv}(\sigma)$, as guaranteed by Proposition 22.

Proposition 24. Let $S \subset \text{vert}(\mathcal{O}(\mathcal{Z}_n))$ be contained in $\text{vert}(\sigma)$ for some alternating σ. Then there exists a unique alternating $\hat{\sigma}$ that maximizes inversion number over all alternating permutations whose vertex set contains S.

Proof. Let $S = \{\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_r\}$ ordered by decreasing coordinate sum. We can assume that S contains both the all zeroes and all ones vectors since those vectors belong to the simplex Δ^σ for any alternating permutation σ. Since $S \subset \text{vert}(\sigma)$ for some alternating σ, if $\mathbf{s}_i(j) = 0$, then $\mathbf{s}_k(j) = 0$ for all $k > i$. For $i = 1, \ldots, r$, let m_i be the number of positions in $\mathbf{s}_i$ that are equal to zero, and let $n_i = m_i - m_{i-1}$ (with $n_1 = m_1$).

Let τ be any alternating permutation such that $S \subseteq \text{vert}(\tau)$. The 0-pattern of each $\mathbf{s}_i$ partitions the entries of all τ with $S \subset \text{vert}(\tau)$ as follows: For $1 \leqslant k \leqslant r$, the n_k positions j such that $\mathbf{s}_k(j) = 0$ and $\mathbf{s}_{k-1}(j) = 1$ are the positions of τ such that $\tau(j) \in \{m_{k-1} + 1, \ldots, m_k\}$.

The positions of inversions and non-inversions across these groups are fixed for all τ with $S \subset \text{vert}(\tau)$. We can build an alternating permutation $\hat{\sigma}$ that maximizes the inversions within each group as follows. For $1 \leqslant k \leqslant r$, let $j_1^k, \ldots, j_{n_k}^k$ be the positions of $\hat{\sigma}$ that must take values in $\{m_{k-1} + 1, \ldots, m_k\}$, as described above. We place these values in reverse; i.e. map j_l^k to $m_k - l + 1$. The permutation obtained in this way need not be alternating, so we switch adjacent positions that need to contain non-descents in order to make the permutation alternating. Note that we never need to make such a switch between groups, since the partition given by S respects the structure of an alternating permutation.

This permutation is unique because within the kth group, arranging the values in this way is equivalent to finding the permutation on n_k elements with some fixed non-descent positions that maximizes inversion number. To obtain this permutation, we begin with the permutation $m_k \ m_k - 1 \ldots m_{k-1} + 1$ and switch all the positions that must be non-descents. The alternating structure of the original permutation implies that none of these non-descent positions can be adjacent, so these transpositions commute and give a unique permutation. $\square$

Example 25. Let $n = 7$ and let

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

We will construct $\hat{\sigma}$, the alternating permutation that maximizes inversion number overall alternating permuations whose vertex set contains S. The second and third vertices in S are the only one that gives information about the position of each entry; we will denote them $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively. Since $\mathbf{w}_1$ has 0's in exactly the first, third and seventh

positions, we know that $1, 2$ and 3 are in these positions. We insert them into these positions in decreasing order, so that $\hat{\sigma}$ has the form

$$3 \_ 2 \_ \_ \_ 1.$$

The zeros added in $\mathbf{w}_2$ are in the fourth, fifth and sixth positions. Placing them in decreasing order yields the permutation

$$3 \_ 2\ 6\ 5\ 4\ 1.$$

However, this permutation cannot be alternating, since there must be an ascent from position 5 to position 6. To create this ascent, we switch the entries in these positions, yielding a permutation of the form

$$3 \_ 2\ 6\ 4\ 5\ 1.$$

Finally, the only entry missing is 7, which must go in the remaining space. This gives the permutation

$$\hat{\sigma} = 3\ 7\ 2\ 6\ 4\ 5\ 1.$$

Proof of Theorem 21. First, we claim that it suffices to show that for any alternating permutations σ and τ, if $\mathrm{inv}(\tau) \geqslant \mathrm{inv}(\sigma)$ then $\mathrm{excl}(\sigma) \not\subset \mathrm{vert}(\tau)$. Indeed, for any ρ with $\mathrm{inv}(\rho) \geqslant \mathrm{inv}(\sigma)$, by Proposition 17 we have that $\Delta_\sigma \cap \Delta_\rho$ is a facet of Δ_σ if and only if σ swaps to ρ. This is the case if and only if $\Delta_\sigma \cap \Delta_\rho = \Delta_\sigma \setminus \{\mathbf{v}_i\}$ for some $\mathbf{v}_i \in \mathrm{excl}(\sigma)$. So if $\Delta_\sigma \cap \Delta_\tau$ is not contained in $\Delta_\sigma \cap \Delta_\rho$ for any ρ such that $\mathrm{inv}(\rho) \geqslant \mathrm{inv}(\sigma)$ with $\Delta_\sigma \cap \Delta_\rho$ a facet of Δ_σ, then we must have $\mathrm{excl}(\sigma) \subset \mathrm{vert}(\tau)$. The contrapositive of this statement shows that if $\mathrm{excl}(\sigma) \not\subset \mathrm{vert}(\tau)$, then the given order on the facets of the triangulation is a shelling.

If $\mathrm{inv}(\tau) > \mathrm{inv}(\sigma)$, then since σ maximizes inversion number over all alternating permutations that contain the exclusion set of σ by Proposition 22, $\mathrm{excl}(\sigma) \not\subset \mathrm{vert}(\tau)$. Furthermore, Proposition 24 implies that if $\mathrm{inv}(\tau) = \mathrm{inv}(\sigma)$, then $\mathrm{excl}(\sigma) \not\subset \mathrm{vert}(\tau)$ because σ is the *unique* permutation that maximizes inversion number of all alternating permutation that contain its exclusion set. $\square$

Proof of Theorem 13. Let $\Delta^{\sigma_1}, \ldots, \Delta^{\sigma_{E_n}}$ be a shelling order as described in Theorem 21. Then by Proposition 17, each Δ^{σ_i} is added in the shelling along exactly $\mathrm{swap}(\sigma_i)$ facets. Therefore, by Theorem 12,

$$h^*_{\mathcal{O}(\mathcal{Z}_N)}(t) = \sum_{\sigma \in A_n} t^{\mathrm{swap}(\sigma)},$$

as needed. $\square$

We conclude this section by remarking that not all of the shellings described in Theorem 21 can be obtained from EL- or CL-labelings of the lattice of order ideals of $J(\mathcal{Z}_n)$. Saturated chains in $J(\mathcal{Z}_n)$ are in bijection with elements of A_n via the map that sends an alternating permutation σ to the chain of order ideals,

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_n$$

where $I_j = \{\sigma^{-1}(1), \ldots, \sigma^{-1}(j)\}$ [9, Chapter 3.5].
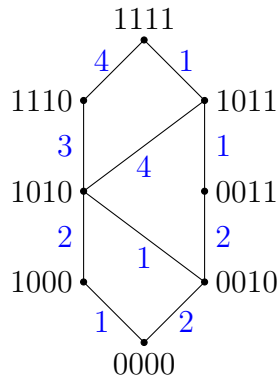
Figure 2: The distributive lattice of order ideals $J(\mathcal{Z}_4)$ with an EL-labeling in blue.

Definition 26. Let P be a graded bounded poset and let $E(P)$ be the set of cover relations of P. An *EL-labeling* of P is a labeling λ of $E(P)$ with integers such that

- each closed interval $[a, b]$ of P has a unique λ-increasing saturated chain, and

- this λ-increasing chain lexicographically precedes all other saturated chains from a to b.

A poset that has an EL-labeling is called *EL-shellable*.

For more details on poset shellability, we refer the reader to [11]. If P is EL-shellable with EL-labeling λ, then lexicographic order on the saturated chains of P with respect to λ gives a shelling of the order complex of P [2]. In the case of $J(\mathcal{Z}_n)$, its order complex is isomorphic to the canonical triangulation of the order polytope $\mathcal{O}(\mathcal{Z}_n)$ via the bijection described above.

Example 27. We consider the lattice of order ideals, $J(\mathcal{Z}_4)$ pictured in Figure 2. The elements of this poset are order ideals of $\mathcal{Z}_4$, which we denote by their indicator vector, and the labeling in blue is an EL-labeling. Hence this poset is EL-shellable. The lexicographic order on the labelings saturated chains is

$$(1, 2, 3, 4) < (1, 2, 4, 1) < (2, 1, 3, 4) < (2, 1, 4, 1) < (2, 2, 1, 1).$$

This corresponds to the order on alternating permutations,

$$1324, 1423, 2314, 2413, 3412,$$

which is indeed a shelling of the triangulation of $\mathcal{O}(\mathcal{Z}_n)$.

So finding EL-labelings of $J(\mathcal{Z}_n)$ is one way to construct shellings of the canonical triangulations of $\mathcal{O}(\mathcal{Z}_n)$. However, not all of the shellings described in Theorem 21 can be obtained in this way.

Proposition 28. There exist shelling orders on the canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$ given by the conditions of Theorem 21 that cannot be obtained from EL-labelings of $J(\mathcal{Z}_n)$.

Proof. For the sake of readability, we discuss these shellings on the level of alternating permutations. The "position of σ in a shelling order" is taken to mean the position of Δ^σ in that shelling order for the canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$. All of the shellings described in Theorem 21 begin with the unique alternating permutation $\bar{\sigma}$ that maximizes inversion number over all alternating permutations; this permutation exists by Proposition 24. Note that $\bar{\sigma}^{-1}(1)$ is $n-1$ or n, depending upon the parity of n.

We address the case where $n \geqslant 5$ is odd. Then $\bar{\sigma}$ is of the form

$$\bar{\sigma} = (n-1)\ n\ (n-3)\ (n-2)\ \ldots\ 4\ 5\ 2\ 3\ 1.$$

Let λ be an EL-labeling of the cover relations of $J(\mathcal{Z}_n)$ that induces a shelling with $\bar{\sigma}$ is its first element. (Note that if no such EL-labeling exists, the proposition holds trivially.)

The sets $\emptyset, \{z_n\}, \{z_{n-2}\}$ and $\{z_{n-2}, z_n\}$ are order ideals of $J(\mathcal{Z}_n)$ that comprise the interval $[\emptyset, \{z_{n-2}, z_n\}]$. The chain corresponding to $\bar{\sigma}$ begins with $\emptyset \lessdot \{z_n\} \lessdot \{z_{n-2}, z_n\}$. So this must must be the unique λ-increasing chain in the interval $[\emptyset, \{z_{n-2}, z_n\}]$. As such, it lexicographically precedes the chain $\emptyset \lessdot \{z_{n-2}\} \lessdot \{z_{n-2}, z_n\}$. This implies that any permutation σ with $\sigma^{-1}(1) = n$ and $\sigma^{-1}(2) = n-2$ will precede any permutation τ with $\tau^{-1}(2) = n$ and $\tau^{-1}(1) = n-2$.

In particular, let σ be obtained from $\bar{\sigma}$ by switching the positions of 3 and 4. Let τ be obtained from $\bar{\sigma}$ by switching the positions of 1 and 2. Then in any EL-shelling, σ will come before τ. However, σ and τ have the same inversion number and have exactly one swap position, so they are interchangeable in any order given by the conditions of Theorem 21.

An analogous argument works when $n \geqslant 6$ is even, and can be adapted for the case when $n = 4$. $\qquad\square$

This proposition and proof can also be adapted to show that not all shellings arising from Theorem 21 can be obtained from CL-labelings of $J(\mathcal{Z}_n)$.

3 The Swap Statistic Via Rank Selection

We now turn our attention to an alternate proof of Theorem 13 using the machinery of rank selection and Jordan-Hölder sets as introduced in Section 1.4. Theorem 13 will follow from Equation 1 and the following theorem, which is analogous to Theorem 3.13.1 in [9].

Theorem 29. *Let $S \subset [n-1]$. Then $\beta_n(S)$ is the number of alternating permutations ω with $Swap(\omega) = S$.*

To prove this theorem, for every $S = \{s_1, \ldots, s_n\} \subset [n-1]$, we will define a function ϕ_S that maps chains of order ideals of sizes $s_1, \ldots, s_k$ to alternating permutations whose swap set is contained in S. Let $I_1, \ldots, I_k$ be a chain of order ideals in $J(\mathcal{Z}_n)$ with sizes $\#I_j = s_j$. Let $\mathbf{w}_i$ be the vertex of $\mathcal{O}(\mathcal{Z}_n)$ that satisfies

$$\mathbf{w}_i(j) = \begin{cases} 0 & \text{if } j \in I_i \\ 1 & \text{if } j \notin I_i. \end{cases}$$

Define $\phi_S(I_1, \ldots, I_k)$ to be the unique alternating permutation that maximizes inversion number over all alternating permutations whose vertex set contains $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$. This map is well-defined by Proposition 24.

Let ψ_S be the map that sends an alternating permutation ω with $\mathrm{Swap}(\omega) \subset S$ to the chain of order ideals $(I_1, \ldots, I_k)$ where each $I_j = \{\omega^{-1}(1), \ldots, \omega^{-1}(s_j)\}$. Since every alternating permutation ω is a linear extension of $\mathcal{Z}_n$, each I_j obtained in this way is an order ideal. They form a chain by construction, so the map ψ_S is well-defined. We will show that ψ_S is the inverse of ϕ_S in the proof of Theorem 29.

Example 30. Consider the zig-zag poset on seven elements $\mathcal{Z}_7$ pictured in Figure 3. Let $S = \{3, 6\}$, and let $I_1 = \{a, c, g\}$ and $I_2 = \{a, c, d, e, f, g\}$ be the given order ideals of sizes 3 and 6 respectively. Then the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Notice that these are the same vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ as in Example 25. So the unique alternating permutation $\phi_S(I_1, I_2)$ that maximizes inversion number over all alternating permutations whose vertex set contains $\{\mathbf{w}_1, \mathbf{w}_2\}$ is the same permutation as in Example 25,

$$\phi_S(I_1, I_2) = 3\ 7\ 2\ 6\ 4\ 5\ 1.$$

Note that $\mathrm{Swap}(3726451) = \{3\} \subset \{3, 6\} = S$.

Now let $\omega = 3726451$. We will recover our original order ideals I_1 and I_2 by finding $\psi_S(\omega)$. For clarity, we will treat ω as a map from $\{a, \ldots, g\}$ to $\{1, \ldots, 7\}$. The first order ideal of $\psi_S(\omega)$ consists of the inverse images of $1, 2$, and 3 in ω. That is,

$$I_1 = \{\omega^{-1}(1), \omega^{-1}(2), \omega^{-1}(3)\} = \{a, c, g\}.$$

The second order ideal of $\psi_S(\omega)$ consists of the inverse images of 1 through 6 in ω. So we obtain

$$I_2 = \{\omega^{-1}(1), \ldots, \omega^{-1}(6)\} = \{a, c, d, e, f, g\}.$$

Note that this is, in fact, the chain of order ideals with which we began.

Proof of Theorem 29. Let $S = \{s_1, \ldots, s_k\} \subset [n-1]$. We will show that $\alpha_n(S)$ is the number of alternating permutations whose swap set is contained in S by showing that the map ϕ_S described above is a bijection. We suggest that the reader follow Example 30, which illustrates the fact that this is a bijection, alongside this proof.
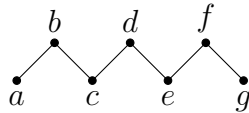
Figure 3: The zig-zag poset $\mathscr{Z}_7$

Let $I_1, \ldots, I_k$ be a chain of order ideals in $J(\mathscr{Z}_n)$ with sizes $\#I_j = s_j$. It is clear from the definitions of ϕ_S and ψ_S that

$$\psi_S(\phi_S(I_1, \ldots, I_k)) = (I_1, \ldots, I_k).$$

Since ϕ_S is injective, it suffices to show that ψ_S is also injective. We will show that $\phi_S(I_1, \ldots, I_k)$ is the only alternating permutation that maps to $(I_1, \ldots, I_k)$ under ψ_S.

Since $\omega = \phi_S(I_1, \ldots, I_k)$ is the unique alternating permuation that maximizes inversion number over all alternating permutations with $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ in their vertex sets, any other alternating permutation σ that maps to $(I_1, \ldots, I_k)$ under ψ_S must have fewer inversions than ω.

Let σ be such a permutation. Since each inversion between the sets I_1, $\mathscr{Z}_n - I_k$ and $I_j - I_{j-1}$ for all $1 < j \leqslant k$ are fixed, the additional non-inversion must be contained in one of these sets. Without loss of generality, let this be $R = I_j - I_{j-1}$. Denote by $\sigma|_R$ the restriction of σ to the domain R. Let $(\sigma^{-1}(a), \sigma^{-1}(b))$ be the non-inversion of $\sigma|_R$ that is *not* required by the alternating structure. Then by Lemma 19, there exists a k such that $a \leqslant k < b$ and k is a swap in σ. Since $a \leqslant k < b$, $\sigma^{-1}(k)$ and $\sigma^{-1}(k+1)$ are in R, so k is also a swap in $\sigma|_R$ as well. So the swap set of σ is not contained in S and we have reached a contradiction.

Therefore, ω is the only alternating permutation that can map to $(I_1, \ldots, I_k)$ under ψ_S, and ψ_S is the inverse map of ϕ_S. So $\alpha_n(S)$ is equal to the number of alternating permutations whose swap set is contained in S. By the Principle of Inclusion-Exclusion, $\beta_n(S)$ is the number of alternating permutations whose swap set is equal to S. $\qquad \square$

Theorem 13 follows as a corollary of Theorem 29.

Proof of Theorem 13. Equation 1 states that

$$h^*_{\mathcal{O}(\mathscr{Z}_n)}(t) = \sum_{S \subset [n-1]} \beta_n(S) t^{\#S}.$$

Theorem 29 tells us that $\beta_n(S)$ is the number of alternating permutations with swap set S. So the sum $\sum_{\#S=k} \beta_n(S)$ is the number of alternating permutations σ with $\mathrm{swap}(\sigma) = k$. So

$$h^*_{\mathcal{O}(\mathscr{Z}_n)}(t) = \sum_{\sigma} t^{\mathrm{swap}(\sigma)},$$

as needed. $\qquad \square$

We conclude this section with an equidistribution result that follows as a corollary of Theorem 13.

Corollary 31. Let ω be a natural labeling of $\mathcal{Z}_n$. Then

$$\sum_{\sigma \in A_n} t^{\mathrm{swap}(\sigma)} = \sum_{\sigma \in \mathcal{L}(\mathcal{Z}_n, \omega)} t^{\mathrm{des}(\sigma)}.$$

4 Combinatorial Properties of Swap Numbers

Let $s_n(k)$ denote the number of alternating permutations on n letters that have exactly k swaps. We call these numbers the *swap numbers*. Theorem 13 shows that the h^*-polynomial of $\mathcal{O}(\mathcal{Z}_n)$ is

$$\sum_{k=0}^{n-1} s_n(k) t^k.$$

We are interested in understanding these numbers. For example, it would be interesting to find an explicit formula for $s_n(k)$, though we have not been able to do this yet.

One straightforward property that becomes apparent looking at examples is that $s_n(n-1) = 0$. This is clear because it is not possible that every $k \in [n-1]$ is a swap. Indeed, otherwise k is to the left of $k+1$ for all $k \in [n-1]$ which implies that σ is the identity permutation, which is not alternating. Furthermore, $s_n(n-2) = 1$, since the unique alternating permutation with this many swaps is the one with $1, 2, \ldots, \lceil \frac{n}{2} \rceil$ in order in the odd numbered positions and $\lceil \frac{n}{2} \rceil + 1, \ldots, n$ in order in the even numbered positions. Similarly, $s_n(0) = 1$, because there is a unique alternating permutation with no swaps. It is the permutation $(n-1, n, n-3, n-2, n-5, n-4, \ldots)$.

Another property that is apparent from examples is summarized in the following:

Theorem 32. *The sequence* $s_n(0), s_n(1), \ldots, s_n(n-2)$ *is symmetric and unimodal.*

In fact, Theorem 32 and all the preceding properties will follow from the fact that $\mathcal{O}(\mathcal{Z}_n)$ is a Gorenstein polytope of index 3.

Definition 33. An integral polytope is *Gorenstein* if there is a positive integer m such that mP contains exactly one lattice point v in its relative interior, and for each facet-defining inequality $a^T x \leqslant b$, we have that $b - a^T v = 1$. The integer m is called the *index* of P.

See Lemma 4 (iii) in [3] for this characterization of Gorenstein polytopes. The following relevant theorem concerning the h^* polynomials of Gorenstein polytopes with unimodular triangulations is Theorem 1 in [3].

Theorem 34. *Suppose that P is a Gorenstein polytope of dimension d and index m. Then $h_P^*(t)$ is a polynomial of degree $d - m + 1$, whose coefficients form a symmetric sequence. Furthermore, the constant term of $h_P^*(t)$ is 1. If, in addition, P has a regular unimodular triangulation, then the coefficient sequence is unimodal.*

Proof of Theorem 32. It suffices to show that $\mathcal{O}(\mathcal{Z}_n)$ is a Gorenstein polytope of index three with a regular unimodular triangulation. The canonical triangulation of $\mathcal{O}(\mathcal{Z}_n)$ is a regular unimodular triangulation. This follows from the fact the triangulation is the initial complex of a Gröbner basis of the toric ideal associated to $\mathcal{O}(\mathcal{Z}_n)$. Indeed, this Gröbner basis is precisely the straightening law associated to the Hibi ring of the distributive lattice $L(\mathcal{Z}_n)$ [5]. Initial complexes of toric ideals always yield regular triangulations [10].

To see that $\mathcal{O}(\mathcal{Z}_n)$ satisfies the Gorenstein property with respect to $m = 3$, note that the defining inequalities for $3\mathcal{O}(\mathcal{Z}_n)$ are that $v_i \geqslant 0$ for i odd, $v_i \leqslant 3$ for i even, $v_{2i-1} \leqslant v_{2i}$ and $v_{2i+1} \leqslant v_{2i}$. The unique interior lattice point of $3\mathcal{O}(\mathcal{Z}_n)$ is the point v where $v_i = 1$ for i odd, and $v_i = 2$ for i even. Finally, this point has lattice distance 1 from each of the facet-defining inequalities. Hence $\mathcal{O}(\mathcal{Z}_n)$ is a Gorenstein polytope of index three with a regular unimodular triangulation and Theorem 34 can be applied to deduce that the coefficient sequence is symmetric and unimodal. $\square$

While general principles provide a proof of the symmetry and unimodality of the sequence $s_n(0), s_n(1), \ldots, s_n(n-2)$, it would be interesting to find explicit combinatorial arguments that would produce these results. In particular, we let $A_{n,k}$ denote the set of alternating permutations on n letters with exactly k swaps, then it would be interesting to solve the following problems.

Problem 35.   1. Find a bijection between $A_{n,k}$ and $A_{n,n-k-2}$.

2. For each $0 \leqslant k \leqslant \lfloor (n-4)/2 \rfloor$ find an injective map from $A_{n,k}$ to $A_{n,k+1}$.

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