On the Running Time of Hypergraph Bootstrap Percolation

Jonathan A. Noel*

Department of Mathematics and Statistics
University of Victoria
Victoria, B.C., Canada

noelj@uvic.ca

Arjun Ranganathan†

Department of Mathematics
Indian Institute of Science Education and Research (IISER), Pune
Pune, India

arjun.ranganathan@students.iiserpune.ac.in

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Abstract

Given $r \geqslant 2$ and an r-uniform hypergraph F, the *F-bootstrap process* starts with an r-uniform hypergraph H and, in each time step, every hyperedge which "completes" a copy of F is added to H. The maximum running time of this process has been recently studied in the case that $r = 2$ and F is a complete graph by Bollobás, Przykucki, Riordan and Sahasrabudhe [Electron. J. Combin. 24(2) (2017), Paper No. 2.16], Matzke [arXiv:1510.06156v2] and Balogh, Kronenberg, Pokrovskiy and Szabó [arXiv:1907.04559v1]. We consider the case that $r \geqslant 3$ and F is the complete r-uniform hypergraph on k vertices. Our main results are that the maximum running time is $\Theta\left(n^r\right)$ if $k \geqslant r+2$ and $\Omega\left(n^{r-1}\right)$ if $k = r+1$. For the case $k = r+1$, we conjecture that our lower bound is optimal up to a constant factor when $r = 3$, but suspect that it can be improved by more than a constant factor for large r.

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1 Introduction

Bootstrap percolation is a model of propagation phenomena in discrete structures which originated in the statistical physics literature to study the dynamics of ferromagnetism [20, 1, 2, 53]. It has since inspired the introduction of many related models that have been the subject of intense research in sociology [26, 54], computer science [24, 22] and, especially, mathematics [17, 9, 30, 14, 13, 49, 31, 5, 44, 45, 10, 18, 6, 43, 27, 34, 28, 7, 52, 35, 8, 19, 33, 39, 21, 29].

Our focus in this paper is on *hypergraph bootstrap percolation* which generalizes the notion of *graph bootstrap percolation* introduced by Bollobás [16] in 1968, originally under the name "weak saturation." For a set S and $r \geqslant 0$, let $S^{(r)}$ denote the collection of all subsets of S of cardinality r. Given an r-uniform hypergraph F, the *F-bootstrap process* starts with an r-uniform hypergraph H_0. Then, at each time step $t \geqslant 1$, the hypergraph H_t consists of all hyperedges of H_{t-1} as well as each $e \in V(H_0)^{(r)}$ for which there exists a copy F' of F containing e such that $E(F') \subseteq E(H_{t-1}) \cup \{e\}$ (that is, e "completes" a copy of F when added to H_{t-1}). The hypergraph H_0 is referred to as the *initial infection* and, for $t \geqslant 0$, the hyperedges of H_t are said to be *infected at time t*.

The most well studied problem related to the F-bootstrap process is to determine the minimum number of hyperedges in an initial infection H_0 on n vertices such that every element of $V(H_0)^{(r)}$ is eventually infected. When F is the complete r-uniform hypergraph on k vertices, which we denote by K_k^r, this is equivalent to the famous Skew Two Families Theorem proved by Frankl [25] using a beautiful exterior algebraic approach of Lovász [41]; see also [37, 4]. Alon [3] proved a multipartite version of the Skew Two Families Theorem, which was extended further by Moshkovitz and Shapira [46]; the results of both of these papers can be thought of in terms of hypergraph bootstrap percolation in a multipartite "host" hypergraph. Pikhurko [47, 48] solved the problem for several classes of non-complete hypergraphs F using the exterior algebraic method of [41, 25] and a related matroid-theoretic approach of Kalai [37]. For results on graphs, i.e. the case $r = 2$, see [16, 40, 15, 45, 36, 37].

Another major focus in bootstrap percolation has been on analyzing the maximum number of time steps that a given bootstrap percolation model can take before it stabilizes [17, 42, 14, 13, 49, 31, 5]. In the context of the F-bootstrap process with initial infection H_0, the *running time* is

$$M_F(H_0) := \min\{t : H_t = H_{t+1}\}$$

and the *maximum running time* among all hypergraphs H_0 with n vertices is denoted by $M_F(n)$. If $F = K_k^r$ for some k, then we write $M_F(H_0)$ and $M_F(n)$ as $M_k^r(H_0)$ and $M_k^r(n)$, respectively. We omit the superscript in the case $r = 2$.

Our goal is to bound $M_k^r(n)$ for fixed r and k asymptotically as a function of n. This problem was first studied in the case $r = 2$ independently by Bollobás, Przykucki, Riordan and Sahasrabudhe [17] and Matzke [42]. Clearly, $M_k^r(n) \leqslant \binom{n}{r}$ for all n, r and k as at least one hyperedge must become infected in each step. It is an easy exercise to show that the maximum running time for the K_3-bootstrap process, i.e. $M_3(n)$, is precisely $\lceil \log_2 (n-1) \rceil$. Interestingly, for $k=4$, the maximum running time jumps from logarithmic to linear.

Theorem 1 (Bollobás et al. [17], Matzke [42]). *$M_4(n) = n-3$ for all $n \geqslant 3$.*

Bollobás et al. [17] conjectured that $M_k(n) = o(n^2)$ for all $k \geqslant 5$. This was disproved for all $k \geqslant 6$ by Balogh, Kronenberg, Pokrovskiy and Szabó [11]. Since the inequality $M_{k+1}(n+1) \geqslant M_k(n)$ can be shown by simply taking a construction on n vertices, adding a vertex and infecting every edge incident to it (see [11, Proposition 10] or Lemma 8 in this paper), the critical case is $k=6$.

Theorem 2 (Balogh et al. [11]). *$M_6(n) \geqslant \frac{n^2}{2500}$ for all n sufficiently large.*

The growth rate of the maximum running time for the K_5-bootstrap process is still unknown. In fact, it is an interesting open problem to determine whether or not it is quadratic [17, 11]. To date, the best known lower bound is given by Balogh et al. [11] by exploiting connections to additive combinatorics; their result improved on a bound of $n^{13/8-o(1)}$ due to [17]. Let $r_3(n)$ be the largest cardinality of a subset of $[n]$ without a 3-term arithmetic progression. While the asymptotics of $r_3(n)$ are not known precisely, Roth's Theorem [50] implies that $r_3(n) = o(n)$ (see [51, 38] for recent quantitative bounds) and the Behrend Construction [12] yields $r_3(n) \geqslant n^{1-O\left(1/\sqrt{\log(n)}\right)}$.

Theorem 3 (Balogh et al. [11]). *$M_5(n) \geqslant \frac{n r_3(n)}{1200}$.*

Here, we initiate the study of $M_k^r(n)$ for $r \geqslant 3$. Our main contributions are constructions of initially infected hypergraphs yielding lower bounds. Our first result concerns the case that $k = r+1$. It would be interesting to know whether this bound is tight up to a constant factor, especially in the case $r=3$ and $k=4$; see Conjecture 25 and Question 26.

Theorem 4. *Let $r \geqslant 3$. If $k = r+1$, then $M_k^r(n) = \Omega(n^{r-1})$.*

In contrast, for $k \geqslant r+2$, we show that the trivial upper bound $\binom{n}{r}$ is tight up to a constant factor.

Theorem 5. *Let $r \geqslant 3$. If $k \geqslant r+2$, then $M_k^r(n) = \Theta(n^r)$.*

The rest of the paper is organized as follows. In the next section, we build up some basic notation and terminology and establish a few preliminary lemmas. In particular, these lemmas will be used to reduce Theorems 4 and 5 to finding constructions with some additional properties for $r=3$ and $k \in \{4,5\}$. In Section 3, we introduce a key construction, which we call the "beachball hypergraph". This hypergraph has running time that is only linear with respect to its number of vertices, but plays a key role in the proofs of both of our main theorems. In the same section, we show how linearly many beachball hypergraphs can be "chained together" to prove Theorem 4. In Section 4, we use the beachball construction in a different way to prove Theorem 5. We conclude the paper in Section 5 with two open problems.

2 Preliminaries

Given r-uniform hypergraphs F and H, say that a copy F' of F is *susceptible* to H if there exists a hyperedge $e \notin E(H)$ and $e \in E(F')$ such that $E(F') \subseteq E(H) \cup \{e\}$. Say that H is *F-stable* if there are no copies of F that are susceptible to H. We make the following simple observation.

Observation 6. *Let H_0 be an initial infection for the F-bootstrap process. Then, for $t \geqslant 1$, $E(H_t)$ is the union of $E(H_{t-1})$ and the hyperedge sets of all copies of F which are susceptible to H_{t-1}.*

The following very straightforward definition and the lemma that follows it allow us to use a lower bound on $M_k^r(n)$ to get a lower bound on $M_{k+1}^r(n)$ which is only slightly worse; c.f. [11, Proposition 10].

Definition 7. Given an r-uniform hypergraph H and $w \notin V(H)$, let $H \vee w$ be the hypergraph with vertex set $V(H) \cup \{w\}$ and hyperedge set

$$E(H) \cup \left\{ e \cup \{w\} : e \in V(H)^{(r-1)} \right\}.$$

Lemma 8. *Let H_0 be an initial infection for the K_k^r-bootstrap process and, for $w \notin V(H_0)$, let $H_0' = H_0 \vee w$ be an initial infection for the K_{k+1}^r-bootstrap process. Then $H_t' = H_t \vee w$ for all $t \geqslant 0$.*

Proof. Suppose not and let t be the minimum time that the equality is violated. Since $H_0' = H_0 \vee w$ by definition, we must have $t \geqslant 1$. Observe that no hyperedge containing w becomes infected in any step $t \geqslant 1$ since all such hyperedges are already in H_0.

First, suppose that there is $e \in E(H_t \vee w)$ such that $e \notin E(H_t')$. As noted above, we may assume that $e \in E(H_t)$ since $w \notin e$. Let F be the corresponding copy of K_k^r that is susceptible to H_{t-1}. Then, by minimality of t, all hyperedges of $F \vee w$ except for e are present in H_{t-1}'. Thus, $e \in E(H_t')$, which is a contradiction. The proof of the other direction is similar. □

The analysis of the running time of H_0 tends to become unwieldy if there are time steps t in which there is more than one copy of F that is susceptible to H_t. For this reason, the constructions in this paper will be designed with a specific goal of avoiding this situation; a similar approach is taken for graphs in [11, 17]. This motivates the following definitions.

Definition 9. Let H_0 be an initial infection for the F-bootstrap process. Say that H_0 is *F-tame* if there is at most one copy of F which is susceptible to H_t for all $t \geqslant 0$.

Definition 10. Let H_0 be F-tame. The *trajectory* of H_0 is the sequence

$$(F_0, e_1, F_1, \ldots, e_{T-1}, F_{T-1}, e_T)$$

where $T = M_F(H_0)$ and, for all $0 \leqslant t \leqslant T-1$, F_t is the unique copy of F that is susceptible to H_t and e_{t+1} is the unique hyperedge of $E(F_t) \setminus E(H_t)$.

The following proposition is an easy consequence of Lemma 8; we omit the proof.

Proposition 11. *If H_0 is K_k^r-tame with trajectory $(F_0, e_1, F_1, \ldots, F_{T-1}, e_T)$, then $H_0 \vee w$ is K_{k+1}^r-tame with trajectory $(F_0 \vee w, e_1, F_1 \vee w, \ldots, F_{T-1} \vee w, e_T)$.*

In our constructions, it will be especially useful to restrict our attention to F-tame hypergraphs with limited interaction between the elements of their trajectories.

Definition 12. Let $(F_0, e_1, \ldots, F_{T-1}, e_T)$ be the trajectory of an F-tame hypergraph H_0 and let $e_0 \in E(F_0)$. We say that H_0 is *F-civilized* with respect to e_0 if the following two conditions hold:

(a) $E(F_j) \cap \{e_0, e_1, \ldots, e_T\} = \{e_j, e_{j+1}\}$ for any $0 \leqslant j \leqslant T-1$ and

(b) $H_0 \setminus \{e_0\}$ is F-stable.

In other words, an F-tame hypergraph H_0 is F-civilized with respect to e_0 if every copy of F in its trajectory has exactly two hyperedges missing from $H_0 \setminus \{e_0\}$ and $H_0 \setminus \{e_0\}$ is F-stable. A useful property of an F-civilized hypergraph is that its trajectory can be reversed by swapping e_0 and e_T.

Lemma 13. *Let H_0 be a hypergraph which is F-civilized with respect to e_0 with trajectory $(F_0, e_1, \ldots, F_{T-1}, e_T)$ and let $H_0' = H_0 \setminus \{e_0\} \cup \{e_T\}$. Then H_0' is F-civilized with respect to e_T with trajectory $(F_{T-1}, e_{T-1}, \ldots, F_0, e_0)$.*

Proof. To show that H_0' is F-civilized with respect to e_T, it suffices to show that it is F-tame with the correct trajectory since the additional conditions of Definition 12 will follow from the fact that H_0 is F-civilized with respect to e_0.

For $0 \leqslant t < T$, let F' be a copy of F which is susceptible to H_t' and let e' be the unique hyperedge of F' that is not in H_t'. We claim that $F' = F_{T-t-1}$ and $e' = e_{T-t-1}$. For the sake of contradiction, suppose that at least one of these equalities does not hold and let t be the minimum time for which such an F' and e' exist. By minimality of t, we have that

$$H_t' = H_0' \cup \{e_{T-t}, \ldots, e_{T-1}\}.$$

In particular, H_t' is a subhypergraph of H_T. Now, observe that e' must be contained in H_T; if not, then F' would be susceptible to H_T which contradicts the assumption that $(F_0, e_1, \ldots, F_{T-1}, e_T)$ is the trajectory of H_0. Next, we claim that F' is not a subhypergraph of H_0. If it were, then, since e' is not contained in H_t', we must have that $e' = e_0$ and all other hyperedges of F' are contained in H_0. However, this would contradict condition 12 of Definition 12. Therefore, F' is not a subhypergraph of H_0.

So, H_0 does not contain every hyperedge of F' but H_T does. Thus, we can let $0 \leqslant j < T$ be the maximum index such that there is a hyperedge of F' that is not contained in H_j. So, F' is susceptible to H_j, which, by the assumption of the lemma, implies that $F' = F_j$. Since H_0 is F-civilized with respect to e_0, the only hyperedges of F_j missing from $H_0 \setminus \{e_0\}$ are e_j and e_{j+1}. Since F_j is susceptible to $H_t' = H_0' \cup \{e_{T-t}, \ldots, e_{T-1}\}$ and also to H_j, but not to H'_{t-1}, the only possibility is that $e_{T-t} = e_{j+1}$ and $e' = e_j$. So, $j+1 = T-t$ which implies that $F' = F_j = F_{T-t-1}$ and $e_j = e_{T-t-1}$, as we wanted.

Thus, F_{T-t-1} is the only copy of F that can be susceptible to H'_t. The last thing to show is that F_{T-t-1} is, indeed, susceptible to H'_t. To see this, note that H_0 is F-civilized with respect to e_0, and so e_{T-t-1} is the unique hyperedge of F_{T-t-1} that is not in H'_t. Thus, H'_0 is F-tame with the desired trajectory and the proof is complete. $\square$

Next, we define a natural "step up" construction for converting constructions for the K_k^r-bootstrap process into constructions for the K_{k+1}^{r+1}-bootstrap process. Later, in Lemma 16, we will see how Lemma 13 can be applied to "chain together" these step up constructions to transform a K_k^r-civilized hypergraph into a K_{k+1}^{r+1}-civilized hypergraph with a linear number of extra vertices in such a way that the running time is boosted by a linear factor.

Definition 14. Let $r \geqslant 2$, let H be an r-uniform hypergraph and let $w \notin V(H)$. Define H^{+w} to be the $(r+1)$-uniform hypergraph with vertex set $V(H) \cup \{w\}$ and hyperedges

$$V(H)^{(r+1)} \cup \{e \cup \{w\} : e \in E(H)\}.$$

Next, we show that H_0^{+w} behaves in essentially the same way with respect to the K_{k+1}^{r+1}-bootstrap process as H_0 does with respect to the K_k^r-bootstrap process.

Lemma 15. *For $k \geqslant r \geqslant 2$, let H_0 be an initial infection for the K_k^r-bootstrap process, let $w \notin V(H_0)$, and let $H'_0 = H_0^{+w}$ be the initial infection for the K_{k+1}^{r+1}-bootstrap process. Then $H'_t = (H_t)^{+w}$ for all $t \geqslant 0$.*

Proof. Suppose that the lemma is false and let t be the smallest index such that $H'_t \neq (H_t)^{+w}$. By definition, we have $H'_0 = H_0^{+w}$ and so $t \geqslant 1$.

First, suppose that there is a hyperedge which is contained in $(H_t)^{+w}$ but not in H'_t. Since both of these hypergraphs contain all of $V(H_0)^{(r+1)}$, this hyperedge must have the form $e \cup \{w\}$ where $e \in E(H_t)$. By minimality of t, we must have $e \notin E(H_{t-1})$ and so there must be a copy F of K_k^r containing e which is susceptible to H_{t-1}. As F is a copy of K_k^r containing e, this means that F^{+w} must be a copy of K_{k+1}^{r+1} containing $e \cup \{w\}$. However, by minimality of t, we have that $H'_{t-1} = (H_{t-1})^{+w}$, and hence $e \cup \{w\} \notin H'_{t-1}$. Since H_{t-1} contains every hyperedge of F except e, we conclude that H'_{t-1} must contain every hyperedge of F^{+w} except for $e \cup \{w\}$. Therefore, F^{+w} is susceptible to H'_{t-1} and so $e \cup \{w\}$ is contained in H'_t, which is a contradiction.

Now, suppose that there is a hyperedge e' that is contained in H'_t but not in $(H_t)^{+w}$. As in the previous case, this hyperedge must contain w. By minimality of t, we must have $e' \notin E\left(H'_{t-1}\right)$ and so there must be a copy F' of K_{k+1}^{r+1} containing e' which is susceptible to H'_{t-1}. However, by minimality of t, we have $H'_{t-1} = (H_{t-1})^{+w}$ and so H_{t-1} contains all hyperedges of the form $e'' \setminus \{w\}$ for $e'' \in E(F') \setminus \{e'\}$. Thus, the copy of K_k^r on vertex set $V(F) \setminus \{w\}$ is susceptible to H_{t-1} which implies that e' is in $(H_t)^{+w}$. This contradiction completes the proof. $\square$

As alluded to earlier, we now apply Lemmas 13 and 15 to show that step up constructions for K_k^r-civilized hypergraphs can be chained together to yield K_{k+1}^{r+1}-civilized hypergraph with very long running time.

Lemma 16. *For $r \geqslant 3$ and $k \geqslant r+1$, let H_0 be a K_k^r-civilized hypergraph. Then, for any $m \geqslant 1$, there exists a K_{k+1}^{r+1}-civilized hypergraph H_0' with $|V(H_0)|+m+(m-1)(k-r-1)$ vertices such that*

$$M_{k+1}^{r+1}(H_0') = m \cdot M_k^r(H_0) + m - 1.$$

Proof. Let e_0 be a hyperedge of H_0 such that H_0 is K_k^r-civilized with respect to e_0 and let $(F_0, e_1, \ldots, F_{T-1}, e_T)$ be the trajectory of H_0. Let us describe the construction of H_0'. First, let $w_1, \ldots, w_m$ be distinct vertices which are not in $V(H_0)$. For any $e \in V(H_0)^{(r)}$ and $1 \leqslant j \leqslant m$, let $e^{+j} = e \cup \{w_j\}$. Let H_0^1 be $H_0^{+w_1}$ and, for $2 \leqslant j \leqslant m$, let H_0^j be $H_0^{+w_j} \setminus \{e_0^{+j}\}$. For each $1 \leqslant j \leqslant m-1$, let $X_j := \{x_j^1, \ldots, x_j^{k-r-1}\}$ be a set of vertices disjoint from the set of vertices introduced so far. For $1 \leqslant j \leqslant m-1$, let $f(j) = T$ if j is odd and $f(j) = 0$ if j is even. For $1 \leqslant j \leqslant m-1$, let C_j be the $(r+1)$-uniform hypergraph with vertex set $\{w_j, w_{j+1}\} \cup X_j \cup e_{f(j)}$ containing all hyperedges in this set, except for $e_{f(j)}^{+j}$ and $e_{f(j)}^{+(j+1)}$. Finally,

$$H_0' := \left(\bigcup_{j=1}^{m} H_0^j\right) \cup \left(\bigcup_{j=1}^{m-1} C_j\right).$$

Intuitively, the way to think of this is as follows. For each $1 \leqslant j \leqslant m$, the hypergraph H_0^j will emulate the K_k^r-bootstrap process with initial infection H_0, either in the forward or backwards (as in Lemma 13) direction, depending on the parity of j. The hypergraph C_j for $1 \leqslant j \leqslant m-1$ is used to link the process on H_0^j to the process H_0^{j+1} in such a way that the termination of the former triggers the start of the latter.

Let us formalize this. First, we show that, if $x \in X_j$ for some $1 \leqslant j \leqslant m-1$, then, for all $t \geqslant 0$, every hyperedge e of H_t' containing x is contained in $V(C_j)$. Suppose that this is not the case, let t be the minimum time for which it fails and let z be a vertex of e that is not in $V(C_j)$. Clearly, by construction of H_0', we must have $t \geqslant 1$. By minimality of t, there must be a copy F of K_{k+1}^{r+1} which is susceptible to H_{t-1}' and contains e. Since $k+1 > r+1$, we can let u be a vertex of $V(F) \setminus e$. Since $r+1 \geqslant 4$, we can choose y to be a vertex of $e \setminus \{x, z\}$. Now, consider the set $e' = e \setminus \{y\} \cup \{u\}$. This is a hyperedge of F which contains both of x and z. However, all hyperedges of F other than e are in H_{t-1}', and so e' is in H_{t-1}' which contradicts the minimality of t.

Next, observe that, for any $t \geqslant 0$, the hypergraph H_t' does not contain any hyperedge which includes w_j and $w_{j'}$ for $|j-j'| > 1$. Note that H_0' has no such hyperedge. Consider the first time $t \geqslant 1$ that such a hyperedge appears, and observe that the relevant copy of K_{k+1}^{r+1} that was allegedly susceptible to H_{t-1}' is missing all other hyperedges containing w_j and $w_{j'}$ by minimality of t, which is a contradiction.

Now, let us show that, for $1 \leqslant j \leqslant m-1$, for every $t \geqslant 0$, every hyperedge e of H_t' containing w_j and w_{j+1} is contained in $V(C_j)$. If not, let t be the minimum time that it is violated and let z be the offending vertex of e. As usual, $t \geqslant 1$ by construction of H_0'. Take a copy F of K_{k+1}^{r+1} which is susceptible to H_{t-1}' and contains e. Since $k+1 > r+1$, we can let u be a vertex of $V(F) \setminus e$. Since $r+1 \geqslant 4$, we can pick $y \in e \setminus \{w_j, w_{j+1}, z\}$. Now, consider $e' = e \setminus \{y\} \cup \{u\}$. This is a hyperedge of F other than e which contains w_j, w_{j+1} and z, which contradicts the minimality of t.

Putting all of this together, we see that the only hyperedges that can become infected during the K_{k+1}^{r+1}-bootstrap process starting with H_0' are those which are contained within $V\left(H_0^j\right)$ for some $1 \leqslant j \leqslant m$ or within $V(C_j)$ for some $1 \leqslant j \leqslant m-1$. The hyperedges of H_0' within $V(C_j)$ form nothing more than a copy of K_{k+1}^{r+1} with two hyperedges removed. The same is true for the hyperedges within $V(F_i) \cup \{w_j\}$ for any copy F_i of K_k^r in the trajectory of H_0 and any $1 \leqslant j \leqslant m$ with the exception of the case $i = 0$ and $j = 1$, in which case the hyperedges form a copy of K_{k+1}^{r+1} with one hyperedge removed. Therefore, the trajectory of H_0' follows that of $H_0^{+w_1}$ for the first $M_k^r(H_0)$ steps. At that point, the hyperedge e_T^{+1} has become infected, which causes C_1 to become susceptible. Thus, e_T^{+2} becomes infected. This triggers the process in H_0^2 to start running in the reverse direction as in Lemma 13, and so on. Putting all of this together, the fact that H_0 is K_k^r-civilized with respect to e_0 now implies that H_0' is K_{k+1}^{r+1}-civilized with respect to e_0^{+1}. The running time includes $M_k^r(H_0)$ steps for every $1 \leqslant j \leqslant m$, plus one extra step for each $1 \leqslant j \leqslant m-1$. This completes the proof. □

Using the results of this section, we show that, in order to prove Theorems 4 and 5, it suffices to find K_k^r-civilized constructions with long running time for $r = 3$ and $k \in \{4, 5\}$.

Corollary 17. *If there is a K_4^3-civilized hypergraph H_0 on n vertices with $M_4^3(H_0) = \Omega(n^2)$, then Theorem 4 holds. Likewise, if there is a K_5^3-civilized hypergraph H_0' on n vertices with $M_5^3(H_0') = \Omega(n^3)$, then Theorem 5 holds.*

Proof. For any fixed $r \geqslant 3$ and k such that $k = r+1$, applying Lemma 16 with $m = \Theta(n)$ exactly $r-3$ times to the hypergraph H_0 in the statement of the corollary yields an r-uniform hypergraph with $O(n)$ vertices whose running time with respect to the K_k^r-bootstrap process is $\Omega(n^{r-1})$. For $r \geqslant 3$ and $k \geqslant r+2$, apply Proposition 11 to H_0' exactly $k-r-2$ times, and then Lemma 16 with $m = \Theta(n)$ exactly $r-3$ times to get an r-uniform hypergraph with $O(n)$ vertices whose running time with respect to the K_k^r-bootstrap process is $\Omega(n^r)$. Trivially, $M_k^r(n) \leqslant \binom{n}{r} = O(n^r)$ and so $M_k^r(n) = \Theta(n^r)$. □

3 Getting the Beachball Rolling

The following definition provides a simple gadget that will be used in both of our main constructions. See Figure 1 for an illustration.

Definition 18. For $n \geqslant 1$ and distinct vertices $v_1, v_2, \ldots, v_n, u_1$ and u_2, the *beachball hypergraph* $B(v_1, \ldots, v_n, u_1, u_2)$ is the 3-uniform hypergraph with vertex set $\{v_1, \ldots, v_n\} \cup \{u_1, u_2\}$ and hyperedges $\{u_1, v_i, v_{i+1}\}$ and $\{u_2, v_i, v_{i+1}\}$ for $1 \leqslant i \leqslant n-1$, as well as the hyperedge $\{u_1, u_2, v_1\}$.

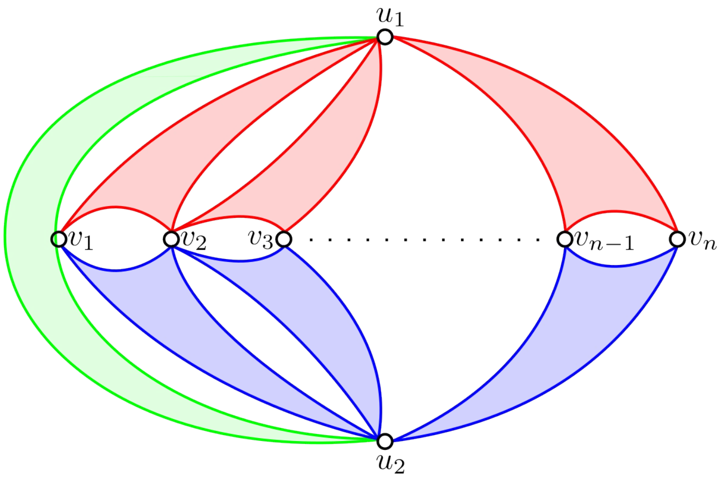

Figure 1: An illustration of the hypergraph $B(v_1, \ldots, v_n, u_1, u_2)$.

We call u_1 and u_2 the *top* and *bottom* vertices of $B(v_1, \ldots, v_n, u_1, u_2)$, respectively. The vertices $v_1, v_2, \ldots, v_n$ are the *middle* vertices. A key property of beachball hypergraphs is that they behave in a very straightforward way with respect to the K_4^3-process.

Lemma 19. *For $n \geqslant 2$, the hypergraph $B(v_1, \ldots, v_n, u_1, u_2)$ is K_4^3-civilized with respect to $e_0 = \{u_1, u_2, v_1\}$ with trajectory $(F_0, e_1, \ldots, F_{n-2}, e_{n-1})$ where*

$$V(F_i) = \{u_1, u_2, v_{i+1}, v_{i+2}\}, \qquad e_{i+1} = \{u_1, u_2, v_{i+2}\}$$

for all $0 \leqslant i \leqslant n-2$.

Proof. For the purposes of this proof, denote $B(v_1, \ldots, v_n, u_1, u_2)$ by B_0^n. Say that a hyperedge e is *flat* if it is contained in $\{v_1, \ldots, v_n\}$ and *wide* if it contains exactly one of u_1 or u_2 and is not an element of $E(B_0^n)$.

We claim that B_t^n contains no flat or wide hyperedges for any $t \geqslant 0$. For $t = 0$, this is true by definition of B_0^n. Now, consider the minimum t for which the statement does not hold and let $e \in B_t^n$ be a hyperedge which is either flat or wide. Then there must exist a copy F of K_4^3 with $e \in E(F)$ which is susceptible to B_{t-1}^n. Let v be the unique vertex of $V(F) \setminus e$. If e is flat, then we can write $e = \{v_i, v_j, v_k\}$ for $1 \leqslant i < j < k \leqslant n$. If $v \in \{v_1, \ldots, v_n\}$, then B_{t-1}^n must contain the hyperedge $e \setminus \{v_i\} \cup \{v\}$. However, this hyperedge is flat, which contradicts the minimality of t. Likewise, if $v \in \{u_1, u_2\}$, then B_{t-1}^n contains the hyperedge $\{v_i, v_k, v\}$, which is wide, and so we get another contradiction. Similarly, if e is wide, then, depending on whether $v \in \{v_1, \ldots, v_n\}$ or $v \in \{u_1, u_2\}$, we find either a flat or a wide hyperedge in B_{t-1}^n, respectively. This proves the claim.

Thus, the only hyperedges which can become infected after any number of steps are those of the form $\{u_1, u_2, v_i\}$ for some $i \in \{2, \ldots, n\}$. This implies that the only copies of K_4^3 that can be susceptible at any time are those whose vertex sets have the form $\{u_1, u_2, v_i, v_{i+1}\}$ for some $i \in \{1, \ldots, n-1\}$.

Now, we claim that, for all $0 \leqslant t \leqslant n-2$, a hyperedge of the form $\{u_1, u_2, v_i\}$ for $1 \leqslant i \leqslant n$ is contained in B_t^n if and only if $i \leqslant t+1$. This is clearly true for $t = 0$

by construction. Consider the minimum $t \geqslant 1$ for which this statement is false and let $j > t+1$ such that $e = \{u_1, u_2, v_j\} \in B_t^n$. Then, by the result of the previous paragraph, the only possible copies of K_4^3 containing e which could be susceptible at time $t-1$ are the ones with vertex set $\{u_1, u_2, v_{j-1}, v_j\}$ or $\{u_1, u_2, v_j, v_{j+1}\}$. However, this implies that either $\{u_1, u_2, v_{j-1}\}$ or $\{u_1, u_2, v_{j+1}\}$ is in $E\left(B_{t-1}^n\right)$; in either case, this contradicts the minimality of t.

Therefore, B_0^n is K_4^3-tame with the trajectory described in the statement in the lemma. Let us argue that B_0^n is K_4^3-civilized with respect to $e_0 = \{u_1, u_2, v_1\}$. Condition 12 of Definition 12 follows from the construction of the beachball hypergraph, since all of the sets $\{u_1, u_2, v_{i+1}, v_{i+2}\}$ for $0 \leqslant i \leqslant n-2$ form a copy of K_4^3 with two hyperedges missing from $B_0^n \setminus \{e_0\}$. For the same reason, condition 12 also holds. This completes the proof. $\square$

In our construction for the K_4^3-bootstrap process, we will also require the hypergraph $B'(v_1, \ldots, v_n, u_1, u_2)$ obtained from $B(v_1, \ldots, v_n, u_1, u_2)$ by adding the hyperedge $\{u_1, v_1, v_n\}$. We call this an *augmented beachball hypergraph*. Next, we show that the addition of this hyperedge extends the running time by one step without disrupting any of the steps that came before it. In fact, this hypergraph is still K_4^3-civilized, albeit with respect to a different hyperedge $e_0' \neq e_0$.

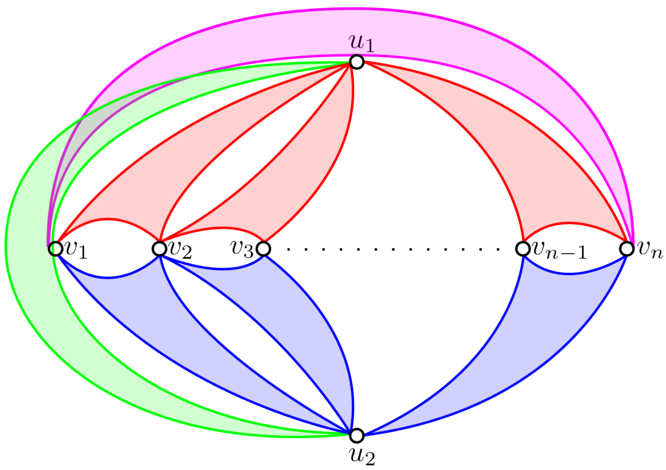

Figure 2: An illustration of the hypergraph $B'(v_1, \ldots, v_n, u_1, u_2)$.

Lemma 20. *For $n \geqslant 4$, the hypergraph $B'(v_1, \ldots, v_n, u_1, u_2)$ is K_4^3-civilized with respect to $e_0' = \{u_1, v_1, v_2\}$ with trajectory $(F_0, e_1, \ldots, F_{n-1}, e_n)$ where $(F_0, e_1, \ldots, F_{n-2}, e_{n-1})$ is the trajectory of $B(v_1, \ldots, v_n, u_1, u_2)$ and*

$$V(F_{n-1}) = \{u_1, u_2, v_1, v_n\}, \qquad e_n = \{u_2, v_1, v_n\}.$$

Proof. Let us denote $B(v_1, \ldots, v_n, u_1, u_2)$ by B_0^n and $B'(v_1, \ldots, v_n, u_1, u_2)$ by ${B'}_0^n$. By Lemma 19, for every $0 \leqslant t \leqslant n-2$, the only hyperedge of B_t^n contained within $\{u_1, u_2, v_1, v_n\}$ is $\{u_1, u_2, v_1\}$. Therefore, if we additionally infect the hyperedge $\{u_1, v_1, v_n\}$

at time zero, as is the case in ${B'_0}^n$, then, for $0 \leqslant t \leqslant n-2$, the copy of K_4^3 with this vertex set has at most two infected hyperedges at time t, and is therefore not susceptible at that time. Any other copy of K_4^3 containing $\{u_1, v_1, v_n\}$ has at most one infected hyperedge. Thus, we have that $B'^n_t = B^n_t \cup \{\{u_1, v_1, v_n\}\}$ for $0 \leqslant t \leqslant n-1$.

Given that the hyperedge $\{u_1, v_1, v_n\}$ is infected in ${B'_0}^n$, we see that the copy of K_4^3 with vertex set $\{u_1, u_2, v_1, v_n\}$ is susceptible to B'^n_{n-1}. Note that, by Lemma 19, $\{u_1, v_1, v_n\}$ is the unique wide hyperedge (to borrow terminology from the proof of Lemma 19) that is infected at time $n-1$. Thus, there is only one susceptible copy of K_4^3 at time $n-1$ and we have that ${B'_n}^n = B'^n_{n-1} \cup \{\{u_2, v_1, v_n\}\}$. Let us show that ${B'_n}^n$ is K_4^3-stable. Note that the only wide or flat infected hyperedges at time n are precisely $\{u_1, v_1, v_n\}$ and $\{u_2, v_1, v_n\}$, both of which are wide. Any K_4^3 containing both of u_1 and u_2 is either fully infected in ${B'_n}^n$ or contains two healthy wide hyperedges. Any K_4^3 containing exactly one of u_1 or u_2 contains at least one healthy wide hyperedge (here, we use that $n \geqslant 4$) and exactly one healthy flat hyperedge. Finally, any K_4^3 consisting containing neither u_1 nor u_2 consists only of flat hyperedges and therefore none of its edges are in ${B'_n}^n$. Thus, ${B'_0}^n$ is K_4^3-tame. Note that the only $0 \leqslant j \leqslant n-1$ for which F_j contains the hyperedge e'_0 is F_0. Also, $\{u_1, u_2, v_1, v_n\}$ does not contain any of the hyperedges $e'_0, e_1, \ldots, e_{n-2}$. Combining this with Lemma 19, we see that ${B'_0}^n$ satisfies condition 12 of Definition 12 with respect to e'_0. Note that the only hyperedges that are present in ${B'_0}^n \setminus \{e'_0\}$ and absent from $B^n_0 \setminus \{e_0\}$ are e_0 and $\{u_1, v_1, v_n\}$. Thus, since B^n_0 is K_4^3-civilized with respect to e_0, if there is a copy of K_4^3 that is susceptible to ${B'_0}^n \setminus \{e'_0\}$, then it must contain e_0 or $\{u_1, v_1, v_n\}$. However, every such copy has at least two hyperedges that are missing from ${B'_0}^n \setminus \{e'_0\}$, and so condition 12 of Definition 12 is satisfied. ☐

Next, we show that, by carefully chaining together augmented beachball hypergraphs, one obtains a K_4^3-civilized hypergraph with quadratic running time. This implies Theorem 4 via Corollary 17.

Theorem 21. *For any $n \geqslant 4$ and $m \geqslant 2$ there exists a K_4^3-civilized hypergraph H_0 with $n+m$ vertices such that*

$$M_4^3(H_0) = (m-1)\,n.$$

Proof. Let $v_0, \ldots, v_{n-1}$ be distinct vertices whose indices are viewed "cyclically" modulo n. In particular, for any i, we have $v_{-i} = v_{n-i} = v_{2n-i}$, and so on. Let $u_0, \ldots, u_{m-1}$ be m additional vertices which are distinct from one another, and from $v_0, \ldots, v_{n-1}$. For $0 \leqslant i \leqslant m-2$, define $S_i := \{v_0, \ldots, v_{n-1}\} \cup \{u_i, u_{i+1}\}$. We let H_0 be the hypergraph

$$B'(v_0, v_1, \ldots, v_{n-1}, u_0, u_1) \cup \left(\bigcup_{i=1}^{m-2} B'(v_{-i}, v_{1-i}, \ldots, v_{n-1-i}, u_i, u_{i+1}) \setminus \{\{v_{-i}, v_{1-i}, u_i\}\} \right).$$

The construction can be thought of intuitively as follows. Start with the hypergraph $B'(v_0, \ldots, v_{n-1}, u_0, u_1)$. Then, we take the same beachball hypergraph again, except that we delete one specific hyperedge, rotate the middle vertices by one, move the bottom vertex to the top, and insert a new vertex on the bottom. Repeat until there are no remaining vertices to add. See Figure 3 for an illustration.

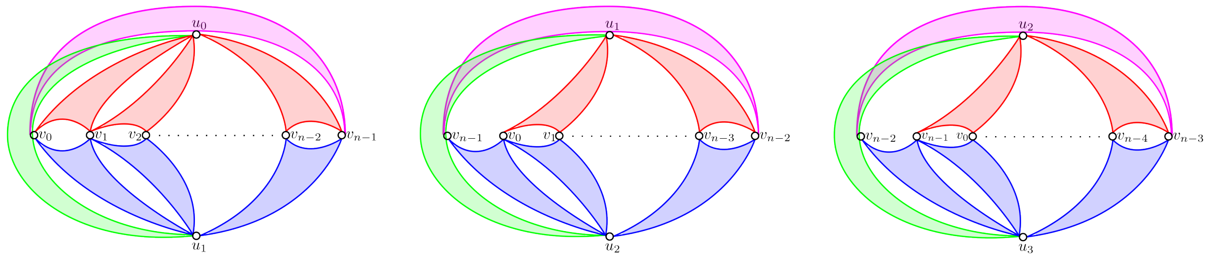

Figure 3: The hypergraph H_0 constructed in the proof of Theorem 21. We have drawn the only hyperedges of H_0 that are on S_0, S_1 and S_2 for illustration purposes.

First, we claim that the hyperedges of H_0 within S_0 are precisely the hyperedges of the augmented beachball hypergraph on its vertices. Every hyperedge of the augmented beachball hypergraph on S_0 was added, by definition of H_0, and so it suffices to show that no additional hyperedges are present. For $i > 1$, all of the hyperedges of $B'(v_{-i}, v_{1-i}, \ldots, v_{n-1-i}, u_i, u_{i+1})$ contain u_i or u_{i+1} and thus are not contained in S_0. Moreover, the only hyperedge of $B'(v_{n-1}, v_0, \ldots, v_{n-2}, u_1, u_2)$ contained in S_0 that is not a hyperedge of the augmented beachball on its vertices is $\{v_{n-1}, v_0, u_1\}$ which was deleted during the construction of H_0. This proves the claim. The same argument also proves that, for all $i \geqslant 1$, the hyperedges within the vertices S_i are precisely those of the augmented beachball hypergraph on these vertices, except that the hyperedge $\{v_{-i}, v_{1-i}, u_i\}$ is missing.

Now, we also observe that, for any $t \geqslant 0$, the hypergraph H_t does not contain any hyperedge which includes u_i and $u_{i'}$ with $|i - i'| > 1$. To see this, consider the first time that it fails and consider the susceptible K_4^3 which caused it; as H_0 contains no such hyperedges containing u_i and $u_{i'}$, we get a contradiction.

By Lemma 20, every augmented beachball in the chain is K_4^3-civilised, and the corresponding trajectory implies that the hyperedge $\{v_{-i}, v_{1-i}, u_i\}$ is contained in the unique K_4^3 that is susceptible to $B'(v_{-i}, v_{1-i}, \ldots, v_{n-1-i}, u_i, u_{i+1})$.

So we see that, for any $i \geqslant 1$, the first hyperedge in S_i that becomes infected does so due to a susceptible copy of K_4^3 with precisely one vertex outside of S_i (since it cannot be contained entirely inside S_i); until such a copy appears, no additional hyperedges in S_i become infected. Note that this statement is not true for $i = 0$, as the first hyperedge in S_0 becomes infected due to a susceptible copy of K_4^3 contained within S_0.

Putting this together, we see that the only hyperedges that become infected in the first n steps are exactly those which become infected in the K_4^3-bootstrap process in $B'(v_0, v_1, \ldots, v_{n-1}, u_0, u_1)$. By Lemma 20, all of these hyperedges are disjoint from S_i for all $i \geqslant 1$, except for the last one, namely $\{u_1, v_0, v_{n-1}\}$, which happens to be the unique hyperedge of $B'(v_{n-1}, v_0, \ldots, v_{n-2}, u_1, u_2)$ that is missing from H_0. From this point, the infection follows the K_4^3-bootstrap process in $B'(v_{n-1}, v_0, \ldots, v_{n-2}, u_1, u_2)$ for n steps. This pattern repeats itself $m - 1$ times. Thus, H_0 is K_4^3-tame. The proof that the additional conditions of Definition 12 hold with respect to $e_0' = \{u_0, v_0, v_1\}$ is analogous to the arguments given in the proofs of the lemmas in this section, and so we omit it.

This completes the proof. $\square$

Proof of Theorem 4. For each $n \geqslant 8$, Theorem 21 gives us a K_4^3-civilized hypergraph H_0 on n vertices such that
$$M_4^3(H_0) \geqslant \left(\left\lceil \frac{n}{2} \right\rceil - 1\right) \left\lfloor \frac{n}{2} \right\rfloor.$$
The result now follows by Corollary 17. $\square$

4 Cubic Time Construction for K_5^3

Our final task is to use the beachball hypergraph again to obtain a construction for the K_5^3-bootstrap process with cubic running time.

Theorem 22. *For any $n \geqslant 2$ and $m \geqslant 2$, there exists a K_5^3-civilized hypergraph H_0 with $2n + 4m + 4$ vertices such that*

$$M_5^3(H_0) = (2n + 4)\, m^2 - 1.$$

Proof. The vertices of the construction are naturally divided into two "halves" which share exactly two special vertices, w_1 and w_2. For $i \in \{1, 2\}$, the vertices that are "exclusive" to the ith half are $v_1^i, \ldots, v_n^i, u_1^i, \ldots, u_m^i, z_1^i, \ldots, z_m^i$ and x^i. Each half of the construction has $n + 2m + 1$ exclusive vertices. Together with w_1 and w_2, this makes $2n + 4m + 4$ vertices in total.

For $1 \leqslant j \leqslant m^2$, let (a_j, b_j) be the jth element of $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$ in lexicographic order. For $1 \leqslant j \leqslant m^2$, let C_j^1 be the copy of K_5^3 with vertex set $\left\{u_{a_j}^1, z_{b_j}^1, u_{a_j}^2, z_{b_j}^2, w_2\right\}$ and, if $j \leqslant m^2 - 1$, then we additionally define C_j^2 to be the copy of K_5^3 with vertex set $\left\{u_{a_j}^2, z_{b_j}^2, u_{a_{j+1}}^1, z_{b_{j+1}}^1, w_1\right\}$. Let

$$B_j^1 := B\left(w_1, v_1^1, \ldots, v_n^1, w_2, u_{a_j}^1, z_{b_j}^1\right) \vee x^1$$

and

$$B_j^2 := B\left(w_2, v_1^2, \ldots, v_n^2, w_1, u_{a_j}^2, z_{b_j}^2\right) \vee x^2$$

for all $1 \leqslant j \leqslant m^2$. Define H_0 to be the union of the following five hypergraphs

$$B_1^1,$$

$$\bigcup_{j=2}^{m^2} B_j^1 \setminus \left\{\left\{u_{a_j}^1, z_{b_j}^1, w_1\right\}\right\},$$

$$\bigcup_{j=1}^{m^2} B_j^2 \setminus \left\{\left\{u_{a_j}^2, z_{b_j}^2, w_2\right\}\right\},$$

$$\bigcup_{j=1}^{m^2} C_j^1 \setminus \left\{ \left\{ u_{a_j}^1, z_{b_j}^1, w_2 \right\}, \left\{ u_{a_j}^2, z_{b_j}^2, w_2 \right\} \right\}, \text{ and}$$

$$\bigcup_{j=1}^{m^2-1} C_j^2 \setminus \left\{ \left\{ u_{a_j}^2, z_{b_j}^2, w_1 \right\}, \left\{ u_{a_{j+1}}^1, z_{b_{j+1}}^1, w_1 \right\} \right\}.$$

Intuitively, the way that the infection evolves can be described as follows. It starts by propagating through B_1^1, the first beachball hypergraph on the first half, in the way described by Lemma 19 and Proposition 11. This takes $n+1$ steps. The hyperedge that gets infected in the last step is $\left\{u_{a_1}^1, z_{b_1}^1, w_2\right\}$. Consequently, now all of the hyperedges within $V\left(C_1^1\right)$ except for $\left\{u_{a_1}^2, z_{b_1}^2, w_2\right\}$ are infected, and so this hyperedge gets infected in the next step; at this point, we have done $n+2$ steps in total. This was the only hyperedge of B_1^2 that is not in H_0, and so its infection triggers the beachball hypergraph B_1^2 on the right half of the construction. The process inside B_1^2 ends with the infection of the hyperedge $\left\{u_{a_1}^2, z_{b_1}^2, w_1\right\}$, which is in $V\left(C_1^2\right)$. So now all the hyperedges of $V\left(C_1^2\right)$ except $\left\{u_{a_2}^1, z_{b_2}^1, w_1\right\}$ are infected, and hence this hyperedge gets infected in the next step. This transfers the infection back over to the left half of the construction by triggering B_2^1, and so on. If the process does indeed progress in this manner, then, in total, it will take $n+2$ steps for every $1 \leqslant j \leqslant m^2$ and every $i \in \{1,2\}$, with the exception of $j = m^2$ and $i = 2$, which only contributes $n+1$ steps (since $C_{m^2}^2$ has not been defined). Thus, there are $(2n+4)\, m^2 - 1$ steps.

Let us now make this rigorous. Our goal is to show that H_0 is K_5^3-civilized with respect to $e_0 = \left\{u_{a_1}^1, z_{b_1}^1, w_1\right\}$; on the way to that goal, we will need to establish several claims. First, we observe that any pair of vertices which are not contained together in a hyperedge of H_0 are not contained together in any hyperedge of H_t for any $t \geqslant 0$; to see this, consider the first time that such a hyperedge becomes infected, look at the susceptible copy of K_5^3 which caused it and get a contradiction. This argument immediately yields the following claim. Throughout the statement of the claim, keep in mind that w_1 and w_2 are regarded as being on both halves of the construction, but all other vertices are exclusive to one half or the other.

Claim 23. *For any $t \geqslant 0$, H_t does not contain any hyperedge e which satisfies any of the following conditions:*

(i) e contains x^i but is not contained in the ith half of the construction for some $i \in \{1,2\}$,

(ii) e contains v_j^i but is not contained in the ith half of the construction for some $1 \leqslant j \leqslant n$ and $i \in \{1,2\}$,

(iii) e contains two of $u_1^i, \ldots, u_m^i$ or two of $z_1^i, \ldots, z_m^i$ for some $i \in \{1,2\}$.

The purpose of the next claim is similar to the previous one; i.e. to rule out certain types of hyperedges from becoming infected.

Claim 24. *For any $t \geqslant 0$, every hyperedge of H_t containing both w_1 and w_2 must also contain one of x^1 or x^2.*

Proof of Claim 24. Consider the first time t such that H_t contains a hyperedge e that contains both of w_1 and w_2 but neither of x^1 nor x^2. Note that $t \geqslant 1$ by construction of H_0. Let F be the copy of K_5^3 containing e which is susceptible to H_{t-1}. By minimality of t, the only two hyperedges of H_{t-1} containing w_1 and w_2 are $\{w_1, w_2, x^1\}$ and $\{w_1, w_2, x^2\}$. So, the vertices of F must consist of w_1, w_2, x^1, x^2 and a fifth vertex, say y. However, y is exclusive to one of the two halves of the construction, and so, regardless of which half it is, H_{t-1} contains several hyperedges which satisfy condition 23 of Claim 23, which is a contradiction. ☐

Let us now show that H_0 is K_5^3-tame with the trajectory that was described earlier in the proof. Suppose that this is not the case, let $t \geqslant 1$ be the minimum time in which there is an unexpected infected hyperedge, say e. Let F be a copy of K_5^3 containing e which is susceptible to H_{t-1}. By Lemma 19 and minimality of t, we cannot have $V(F) \subseteq B_j^i$ for any $1 \leqslant j \leqslant m^2$ and $i \in \{1, 2\}$. Now, any five vertices on the ith half of the construction are either contained in B_j^i for some j or contain two of $u_1^i, \ldots, u_m^i$ or two of $z_1^i, \ldots, z_m^i$. So, as there are no hyperedges satisfying condition 23 of Claim 23, $V(F)$ must contain at least one vertex that is exclusive to each half of the construction. But now, since no hyperedges satisfy Claim 23 23 or 23, we get that $V(F)$ cannot contain any of the vertices x^i or v_j^i for $i \in \{1, 2\}$ and $1 \leqslant j \leqslant n$. This also means that it cannot contain both of w_1 and w_2 by Claim 24. On the other hand, it must contain at least one of w_1 or w_2; if not, then three vertices of F are exclusive to one of the sides of the partition, and we get a contradiction from Claim 23 23. Using Claim 23 23 one more time, we see that $V(F)$ has the form $\{u_a^1, z_b^1, w_\ell, u_c^2, z_d^2\}$ for some $1 \leqslant a, b, c, d \leqslant m$ and $\ell \in \{1, 2\}$. We now divide the proof into cases.

Case 1. e does not contain w_ℓ.

We assume that $e = \{u_a^1, z_b^1, u_c^2\}$ and note that the other three cases follow from similar arguments. Let $1 \leqslant j \leqslant m^2$ be chosen so that $a = a_j$ and $b = b_j$; such a j exists by construction of H_0. The hypergraph H_{t-1} contains the hyperedge $e' = e \setminus \{u_c^2\} \cup \{z_d^2\}$ and so, by minimality of t, we get that e' is contained in either $V\left(C_j^1\right)$ or $V\left(C_{j-1}^2\right)$. Indeed, under the trajectory described at the beginning of the proof, every hyperedge containing two vertices exclusive to one side and one exclusive to the other had this property. The fact that e' is contained in $V\left(C_j^1\right)$ or $V\left(C_{j-1}^2\right)$ implies that e is as well. However, every hyperedge of C_j^1 or C_{j-1}^2 not containing w_1 or w_2 was added to H_0 originally, which contradicts our choice of e.

Case 2. e contains w_ℓ.

Again, let $1 \leqslant j \leqslant m^2$ be chosen so that $a = a_j$ and $b = b_j$. The hyperedge $\left\{u_{a_j}^1, z_{b_j}^1, u_c^2\right\}$ is contained in H_{t-1} and so, by minimality of t, it must be contained in either $V\left(C_j^1\right)$ or $V\left(C_{j-1}^2\right)$. By definition of $V\left(C_j^1\right)$ and $V\left(C_{j-1}^2\right)$, and our specific choice of lexicographic order in the construction of H_0, this implies that either $c = a_j$ or $c = a_j - 1$ and $b_j = 1$.

Suppose first that $c = a_j$. Then we get that H_{t-1} contains $\left\{z^1_{b_j}, u^2_{a_j}, z^2_d\right\}$ which, by minimality of t and construction of H_0, implies that either $d = b_j$ or $d = b_j - 1$. If $d = b_j$, then we must have that $\ell = 2$ by minimality of t since $\left\{z^1_{b_j}, w_1, z^2_{b_j}\right\}$ is not contained in any of the hypergraphs $C^1_1, \ldots, C^1_{m^2}$ or $C^2_1, \ldots, C^2_{m^2-1}$. So, in the case that $c = a_j$ and $d = b_j$, we get that $V(F) = V\left(C^1_j\right)$. This implies that e is one of the only two hyperedges of C_j that are missing from H_0. If $e = \left\{u^1_{a_j}, z^1_{b_j}, w_2\right\}$, then the hyperedge $\left\{w_2, u^2_{a_j}, z^2_{b_j}\right\}$ is present before e, which contradicts minimality of t. If $e = \left\{w_2, u^2_{a_j}, z^2_{b_j}\right\}$, then e is being infected due to C^1_j being susceptible, which fits the description of the trajectory from earlier in the proof, and so this contradicts the definition of e. Now, suppose that $c = a_j$ and $d = b_j - 1$. In this case, we must have $\ell = 1$ by minimality of t since $\left\{z^1_{b_j}, w_2, z^2_{b_j-1}\right\}$ is not contained in any of the sets $C^1_1, \ldots, C^1_{m^2}$ or $C^2_1, \ldots, C^2_{m^2-1}$. So, what we end up with is $V(F) = V\left(C^2_{j-1}\right)$ and we get a contradiction similar to the case that we just analyzed.

Now, suppose that $c = a_j - 1$ and $b_j = 1$. Since H_{t-1} contains $\left\{z^1_1, u^2_{a_j-1}, z^2_d\right\}$, we must have either $d = 1$ or $d = m$. If $d = 1$, we get a contradiction, since $\left\{u^1_{a_j}, u^2_{a_j-1}, z^2_1\right\}$ is not contained in H_{t-1} by minimality of t and the fact that $m \geqslant 2$ and so, under lexicographic order, $(a_j - 1, 1)$ does not immediately precede or follow any pair involving a_j in the first coordinate. So, $d = m$. Now, we get that $\ell = 1$ by minimality of t since $\{z^1_1, w_2, z^2_m\}$ is not contained in any of the sets $C^1_1, \ldots, C^1_{m^2}$ or $C^2_1, \ldots, C^2_{m^2-1}$. So, we get that $V(F) = C^2_{j-1}$ which leads us to a contradiction, as in the previous paragraph.

Similar arguments also show that, for $e_0 = \left\{u^1_{a_1}, z^1_{a_1}, w_1\right\}$, the hypergraph $H_0 \setminus \{e_0\}$ is K^3_5-stable; thus, condition 12 of Defintion 12 holds. Clearly each of the copies of K^3_5 in the trajectory of H_0, as described above, has precisely two hyperedges missing from $H_0 \setminus \{e_0\}$ and so 12 holds, too. This completes the proof. $\square$

Proof of Theorem 5. For n sufficiently large and $n \equiv 4 \bmod 6$, Theorem 22 provides a K^3_5-civilized hypergraph H'_0 with n vertices such that

$$M^3_5(H'_0) \geqslant \left(\frac{n+8}{3}\right)\left(\frac{n-4}{6}\right)^2 - 1.$$

The result follows by Corollary 17. $\square$

5 Open Problems

While Theorem 5 determines $M^r_k(n)$ up to a constant factor for all $r \geqslant 3$ and $k \geqslant r + 2$, it would be interesting to have a better understanding of the growth rate in the case $k = r + 1$. The most interesting question here seems to be whether there is a non-trivial upper bound on the running time of the K^3_4-bootstrap process; we conjecture that the quadratic lower bound from Theorem 4 is tight up to a constant factor.

Conjecture 25. $M^3_4(n) = O(n^2)$.

In contrast, we do not believe that Theorem 4 is tight for all $r \geqslant 3$. It is even conceivable that, for large enough r, the maximum running time of the K_{r+1}^r-process on n vertices eventually becomes $\Theta(n^r)$. We ask whether this is, indeed, the case.

Question 26. Does there exist an integer r_0 such that, if $r \geqslant r_0$, then $M_{r+1}^r(n) = \Theta(n^r)$?

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Remarks

After completing this work, Hartarsky and Lichev [32] and Espuny Díaz, Janzer, Kronenberg and Lada [23] independently disproved 25 by showing that $M_4^3(n) = \Theta(n^3)$, and consequently answered Question 26 in the affirmative in a rather strong sense with $r_0 = 3$. Hartarsky and Lichev [32] also determine the leading asymptotics of the prefactor when $r \to \infty$. Additionally, Espuny Díaz et al. [23] provide the first nontrivial exact result about the maximum running times of hypergraph bootstrap percolation by showing that the maximum running time for the H-bootstrap process on n vertices when H is K_4^3 minus an edge is exactly $2n - \lfloor \log_2(n-2) \rfloor - 6$.

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