On counting double centralizers of symmetric groups

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Abstract

Let S_{2m} be symmetric group, $h_0 = (1\ 2)\cdots(2m-1\ 2m)$ and $H = C(h_0)$. We clarify the structure of $gHg^{-1} \cap H, g \in S_{2m}$, and using tools from analytic combinatorics we prove that the permutations g such that $|gHg^{-1} \cap H|$ bounded by $m^{O(1)}$ have density zero.

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1 Introduction

For any positive integer n, let S_n be the symmetric group on the symbols $\{1, 2, \cdots, n\}$ and $h \in S_n$ be any permutation. We want to know how common elements $g \in S_n$ can make the centralizer $C(\langle h, ghg^{-1}\rangle)$ small. Here by "small" we mean of size in polynomial of n. The interest of such problems is originated from growth in groups as follows. Consider a permutation subgroup $G \subset S_n$ acting on S_n by inner automorphisms. For any subset $A \subset G$, define the orbit of h under A to be $O_A(h) = \{a^{-1}ha \mid a \in A\}$. Clearly, if there are $a_1, a_2 \in A$ such that $a_1^{-1}ha_1 = a_2^{-1}ha_2$, then $a_2a_1^{-1} \in C(h)$ and we can bound the size of $O_A(h)$ from below by

$$|O_A(h)| = |A/(AA^{-1} \cap C(h))| \geqslant |A|/|AA^{-1} \cap C(h)|,$$

where $A/\sim$ is modulo by the equivalence relation with $a_1 \sim a_2$ if $a_2a_1^{-1} \in C(h)$. This is a form of the orbit-stabilizer principle. The above lower bound is effective only if $C(h)$ is small. If $C(h)$ is large and we still want a profitable lower bound, we can study the action of G on $S_n \times S_n$ via

$$g \cdot (h_0, h_1) := (g^{-1}h_0g, g^{-1}h_1g), \forall g \in G, h_0, h_1 \in S_n.$$

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Then at least one of the orbits $O_A(h_i), i = 0, 1$, has size at least square root of $|O_A((h_0, h_1))|$, which has lower bound

$$|O_A((h_0, h_1))| \geqslant |A|/|AA^{-1} \cap C(h_0) \cap C(h_1)| = |A|/|AA^{-1} \cap C(\langle h_0, h_1 \rangle)|.$$

Especially when h_0 and h_1 are conjugate, $|O_A(h_0)| = |O_A(h_1)|$ and the above gives a lower bound for its square. This arithmetic sets up the first step for Helfgott-Seress [8] to bound the diameter of permutation groups, for which they needed to control the growth of chains of stabilizers more carefully, see section 1.5 therein.

Hence it makes sense to ask, given $h \in S_n$, how easily we can find $g \in S_n$ such that $C(\langle h, ghg^{-1} \rangle)$ is small. If h has k fixed points, then $C(h)$ is at least of size $k!$. Assuming h has no fixed points, $C(h)$ can still be large if the block partition of the support of h is fixed in many ways. Such an extreme example occurs for $n = 2m$ even and $h_0 = (1\ 2)(3\ 4) \cdots (2m-1\ 2m)$, written in left-to-right convention for composition of cycles. Let H be the subgroup of S_{2m} consisting of permutations preserving the partition $\{1, 2\}, \{3, 4\}, \cdots, \{2m-1, 2m\}$, then it is not hard to see that $H = C(h_0)$. To make the notion of being "small" more precise, we introduce the following notations. In an asymptotic convention, call $g \in S_{2m}$ *good* if $|H \cap gHg^{-1}| = m^{O(1)}$; call it *bad* otherwise. Harald Helfgott wonders in [1] about the structure of good elements and postulated that the good permutations asymptotically have density 1 in S_{2m}. There seems to be a fair share of good permutations in S_{2m}, for example if the cycle decomposition of g does not contain "too many" cycles of the same length, g may be checked good. The paper contributes to studying the structure of good elements, and however shows that the good permutations asymptotically have density zero as a negative answer to Helfgott's postulation.

More precisely we prove

Theorem 1. *For any $c > 0$,*

$$\mathrm{Prob}(|H \cap gHg^{-1}| < m^c) \to 0, \ \ \text{as } m \to \infty.$$

Consequently, good elements of S_{2m} have density zero.

The right tail is also estimated to show that

Theorem 2. *For some constant $C > 0$,*

$$\mathrm{Prob}\left(|H \cap gHg^{-1}| > Cm^{\log m}\right) \to 0, \ \ \text{as } m \to \infty.$$

In particular, the bad elements $g \in S_{2m}$ with $|H \cap gHg^{-1}| \gg m^{\log m}$ have zero density.

The above results are based on classifying the structure of $H \cap gHg^{-1}$ for arbitrary $g \in S_{2m}$. It turns out that the isomorphism class of $H \cap gHg^{-1}$ depends on the double coset HgH and moreover

Theorem 3. *Each HgH has a representative $x \in \mathrm{Sym}\{2, 4, \cdots, 2m\} \leqslant S_{2m}$ determined by a partition of m and there is a 1-1 correspondence $H \backslash S_{2m}/H \leftrightarrow \{\text{paritition of } m\}$.*

Furthermore, for any $g \in HxH$ *with* $x \in \mathrm{Sym}\{2,4,\cdots,2m\}$ *whose cycle decomposition has* r_i *cycles of length* i, $i = 1,\cdots,k$,

$$H \cap gHg^{-1} \simeq \bigoplus_{i=1}^{k} D_i \wr S_{r_i},$$

where D_i *is the dihedral group with* $2i$ *elements and* $\wr$ *denotes wreath product of groups. (For convenience we write* D_1 *for* C_2 *or* S_2*.)*

Thus $|H \cap gHg^{-1}|$ can be seen as a random variable on partitions of m with probability distribution of counting measure $P(\lambda) = \frac{|HxH|}{|S_{2m}|}$, if $g \in HxH$ for $x \in \mathrm{Sym}\{2,4,\cdots,2m\}$ with cycle type λ. Details are examined in section 2.

Outline of paper.

The 1-1 correspondence $H\backslash S_{2m}/H \leftrightarrow \{\text{paritition of } m\}$ of Theorem 3 is established by studying the left and right action of H on S_{2m} in details in section 2.3. It can also be verified by a character formula in section 2.4. Then combined with an idea of bipartite graph automorphism construction introduced by J. P. James [14], we prove the structural result in Theorem 3 in section 2.5. As a byproduct we prove that they are all rational groups in aspect of representation theory.

Explicitly shown in section 3.1, the distribution of $|H \cap gHg^{-1}|$ happens to be $P = \mathrm{ESF}(\frac{1}{2})$, where $\mathrm{ESF}(\frac{1}{2})$ is the Ewens' distribution with bias $\frac{1}{2}$. Then we estimate the left tail $P(\leqslant m^c)$ by the moment bound. The expectations for each m involved in the moment bound are brought together into a special generating function. Then asymptotics of the expectations can be extracted from coefficients of singular expressions of the generating function around its singularities which are of logarithmic type, see section 3.2.2. We use techniques from analytic combinatorics, especially the hybrid method introduced by Flajolet-Fusy-Gourdon-Panario-Pouyanne [4], to find the correct asymptotics and prove Theorem 1 in section 3.3.

In the same probabilistic setting, the expectations involved in the moment bound of the right tail are brought together into generating functions with singularities of exponential type. Then to prove Theorem 2, we use asymptotics of coefficients of generating functions of exponential type which was given by E. M. Wright [15], in section 4.

2 Structures of double cosets and $H \cap gHg^{-1}$

2.1 Preliminaries on H and $H \cap gHg^{-1}$

This section includes some necessary basic group theoretic results on $H = C(h_0) \leqslant S_{2m}$ for $h_0 = (1\ 2)(3\ 4)\cdots(2m-1\ 2m)$ and $H \cap gHg^{-1}$ for general $g \in S_{2m}$.

Firstly, viewed as preserving the block partition $\{1,2\},\{3,4\},\cdots,\{2m-1,2m\}$ of $1,2\cdots,2m$, the structure of H is as simple as follows:

Proposition 4. *H has the wreath product structure* $H = C(h_0) \simeq C_2 \wr S_m$.

This is also an easy corollary of 4.1.19 of James-Kerber [13] which describes centralizers of arbitrary permutations in a symmetric group as wreath products of cyclic groups with smaller symmetric groups.

One immediately notices that $H \cap gHg^{-1}$ is identical for any g in a common left coset of H. Moreover, for any $h_1, h_2 \in H$ and $g \in S_{2m}$,

$$H \cap h_1 g h_2 H (h_1 g h_2)^{-1} = H \cap h_1 g H g^{-1} h_1^{-1} = h_1 \left(H \cap gHg^{-1}\right) h_1^{-1},$$

hence the structure of $H \cap gHg^{-1}$ depends only on the double coset HgH.

Example 5. For $m = 2$, $H = D_1 \wr S_2 \simeq (C_2)^2 \rtimes S_2$ and $S_4/H = \{\bar{1}, (1\ \bar{}3), (1\ \bar{}4)\}$. Computing by hand we get

$$H \cap (1\ 3)H(1\ 3) = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} = K_4,$$

where K_4 is the Klein four group. Again by hand

$$H \cap (1\ 4)H(1\ 4) = H \cap (1\ 3)H(1\ 3) = K_4.$$

There is no wonder because there are only 2 double cosets in $H \backslash S_4 / H$ with representatives 1 (which may be seen as supported on any single symbol for convenience) and (1 3), and clearly (3 4)(1 3)(3 4) = (1 4) (note that $(3\ 4) \in H$).

2.2 Double coset decomposition of S_{2m}

Counting the left cosets contained in HgH gives

$$|HgH| = |H|[H : H \cap gHg^{-1}] = \frac{|H|^2}{|H \cap gHg^{-1}|}.$$

Thus if each double coset determines a distinct structure (or size) of $H \cap gHg^{-1}$, the density of those g is assigned by

$$\frac{|HgH|}{|S_{2m}|} = \frac{|H|^2}{|S_{2m}||H \cap gHg^{-1}|} = \frac{(2^m m!)^2}{(2m)!|H \cap gHg^{-1}|}.$$

In addition, the double coset decomposition of S_{2m} by H gives

$$|S_{2m}| = \sum_{g \in H \backslash S_{2m}/H} |HgH| = \sum_{g \in H \backslash S_{2m}/H} \frac{|H|^2}{|H \cap gHg^{-1}|},$$

and consequently

$$\sum_{g \in H \backslash S_{2m}/H} \frac{1}{|H \cap gHg^{-1}|} = \frac{|S_{2m}|}{|H|^2} = \frac{(2m)!}{(2^m m!)^2} \sim \frac{1}{\sqrt{\pi m}},$$

by Stirling's formula. These formulae become the starting point of studying distribution of $|H \cap gHg^{-1}|$ in section 3.

2.3 Counting double cosets by partition number

To describe the structure of $H\backslash S_{2m}/H$, we first prove the following lemma on double coset representatives.

Lemma 6. *Each double coset of $H\backslash S_{2m}/H$ has a representative with support contained in the odd integers $M = \{1, 3, \cdots, 2m-1\}$ or the even integers $M' = \{2, 4, \cdots, 2m\}$.*

Proof. For any $x \in S_{2m}$, we first split its cycles containing some 2-blocks $\{2k-1, 2k\}$ as follows. A proper cycle (of length $\geqslant 2$) containing both $2k-1$ and $2k$ can be written as (in left-to-right convention for cycle composition):

$$(2k-1\ l_1 \cdots\ l_s\ 2k\ l'_1\ \cdots\ l'_t)$$

$$= (2k-1\ l_1) \cdots (2k-1\ l_s)(2k-1\ 2k)(2k-1\ l'_1) \cdots (2k-1\ l'_t),$$

with all numbers distinct. Multiplying $(2k-1\ 2k)$ ($\in H$) on the left of both sides above we get

$$(2k-1\ 2k)x = (2k\ l_1\ \cdots\ l_s)(2k-1\ l'_1\ \cdots\ l'_t) \cdots,$$

i.e. we can decompose the cycle into two cycles which split $\{2k-1, 2k\}$. Repeat the procedure using suitable $(2k_i - 1\ 2k_i)$ ($\in H$) $i = 1, \cdots, r$, until $(2k_1 - 1\ 2k_1) \cdots (2k_r - 1\ 2k_r)x$ has no cycles containing any $\{2k-1, 2k\}$. (This is doable since $(2k_i - 1\ 2k_i)$ commutes with the cycles not intersecting $\{2k_i - 1, 2k_i\}$.)

For a representative with such cycle type, by multiplying $(2k-1\ 2k)$'s simultaneously on left and right, we get a product of cycles which either contains only odd numbers or even numbers. Then move all cycles of even numbers to the left. Now we can replace them by corresponding cycles of the complementary odd numbers, by multiplying the unique element in H supported on the corresponding 2-blocks on the left. (For example, $(2\ 6\ 4)(8\ 10)$ can be replaced by $(1\ 5\ 3)(7\ 9)$ since $(2\ 6\ 4)(8\ 10)(1\ 5\ 3)(7\ 9) = (2\ 6\ 4)(1\ 5\ 3)(8\ 10)(7\ 9) \in H$.) Thus we get a representative of HxH supported on odd numbers. Replacing the cycles of odd number by complementary even numbers we get a representatives supported on even integers. $\square$

In addition, the following explicit expression of Proposition 4 is crucial to proving the main result of this section.

Lemma 7. *Let $M = \{1, 3, \cdots, 2m-1\}, M' = \{2, 4, \cdots, 2m\}$, $C = \prod_{i=1}^{m} \mathrm{Sym}\{2i-1, 2i\} \leqslant S_{2m}$, and $T = \mathrm{Sym}\{(1, 2), \cdots, (2m-1, 2m)\} \leqslant S_{2m}$ (the symmetric group of the ordered pairs $(2k-1, 2k)$'s). Then $H = TC$ and explicitly for any $h \in H$, there is a unique decomposition*

$$h = \bar{h}\tilde{h} = \bar{h}_M \bar{h}_{M'} \tilde{h} = \bar{h}_{M'} \bar{h}_M \tilde{h},$$

in which $\tilde{h} \in C$, $\bar{h} \in T$, $\bar{h}_M$ and $\bar{h}_{M'}$, commuting with each other, are the complementary permutation actions of $\bar{h}$ restricted onto M and M' respectively. We call it the TC-decomposition of H.

Proof. For any $h \in H$ and $k \leqslant m$, let $\bar{h}$ be the permutation action defined as

$$\bar{h} \cdot (2k) = \begin{cases} h(2k), & \text{if } h(2k) \text{ is even,} \\ h(2k-1), & \text{if } h(2k) \text{ is odd,} \end{cases}$$

and

$$\bar{h} \cdot (2k-1) = \begin{cases} h(2k-1), & \text{if } h(2k) \text{ is odd,} \\ h(2k), & \text{if } h(2k) \text{ is even,} \end{cases}$$

where $h(i)$ denotes the number that h moves i to.

The definition guarantees that $\bar{h}$ sends even numbers to even numbers and odd to odd while still preserving the partition $\{1, 2\}, \cdots, \{2m-1, 2m\}$, hence belongs to H. The case separation in the definition where $2k-1$ and $2k$ are switched by h gives a product of transpositions $(2k-1, 2k)$'s, denoted by $\tilde{h}$. This amounts to the decomposition $h = \bar{h}\tilde{h}$ which is unique simply because $C \cap T = \{1\}$. Restriction of $\bar{h}$ onto M and M' gives the 3-term decomposition

$$h = \bar{h}_M \bar{h}_{M'} \tilde{h} = \bar{h}_{M'} \bar{h}_M \tilde{h},$$

whose uniqueness is due to the decomposition $T = \mathrm{Sym}(M) \times \mathrm{Sym}(M')$. □

Remark 8. Note that alternatively we have the CT-decomposition $H = CT$, i.e. $h = \tilde{h}'\bar{h}$ for some $\tilde{h}' \in C$ which switches $h(2k)$ and $h(2k-1)$ when necessary.

Now we can prove the main result on the structure of $H \backslash S_{2m} / H$.

Proposition 9. *Keep the notations from last lemma. Each conjugacy class of* $\mathrm{Sym}(M)$ *(*$\mathrm{Sym}(M')$*) is contained in a distinct double coset of* $H \backslash S_{2m}/H$*, and each double coset intersects with* $\mathrm{Sym}(M)$ *(*$\mathrm{Sym}(M')$*) at a conjugacy class of* $\mathrm{Sym}(M)$ *(*$\mathrm{Sym}(M')$*). Consequently,* $|H \backslash S_{2m}/H| = p(m)$*, the partition number of* m.

Proof. For any two conjugates $x_1, x_2 \in \mathrm{Sym}(M)$, say conjugated by $x = (2k_1 - 1\ 2k_2 - 1\ \cdots\ 2k_s - 1) \cdots (2k'_1 - 1\ 2k'_2 - 1\ \cdots\ 2k'_t - 1)$, they are conjugate in S_{2m} by $x' = x(2k_1\ 2k_2\ \cdots\ 2k_s) \cdots (2k'_1\ 2k'_2\ \cdots\ 2k'_t) \in H$.

Hence $x_2 \in Hx_1H$.

On the other hand, if $x_2 \in Hx_1H$, then there exists $h \in H$ such that $x_1 h x_2^{-1} \in H$. By Lemma 7 we get

$$x_1 h x_2^{-1} = x_1 \bar{h}_M \bar{h}_{M'} \tilde{h} x_2^{-1} = x_1 \bar{h}_M \bar{h}_{M'} (\tilde{h} x_2^{-1} \tilde{h}) \tilde{h}. \tag{1}$$

It is easy to check that $chc^{-1} = chc \in \mathrm{Sym}(M)$ $(\mathrm{Sym}(M'))$ for any $h \in \mathrm{Sym}(M)$ $(\mathrm{Sym}(M'))$ and any $c \in C$ such that h preserves the support of c, denoted by $\mathrm{supp}(c)$.

We claim that x_2^{-1} preserves $\mathrm{supp}(\tilde{h})$. For any $k \leqslant m$, if $2k-1 \notin \mathrm{supp}(\tilde{h})$, then $x_1 h x_2^{-1}(2k) = x_1 h(2k) = h(2k)$ is even. Since $x_1 h x_2^{-1} \in H$, $x_1 h x_2^{-1}(2k-1)$ must be odd, which indicates $x_2^{-1}(2k-1) \notin \mathrm{supp}(\tilde{h})$. If $2k-1 \in \mathrm{supp}(\tilde{h})$, then $x_1 h x_2^{-1}(2k) = x_1 h(2k)$ is odd. Hence $x_1 h x_2^{-1}(2k-1) = h x_2^{-1}(2k-1)$ is even, and $x_2^{-1}(2k-1) \in \mathrm{supp}(\tilde{h})$. This shows

$$x_2^{-1}(M \smallsetminus \mathrm{supp}(\tilde{h})) = M \smallsetminus \mathrm{supp}(\tilde{h}),\ x_2^{-1}(M \cap \mathrm{supp}(\tilde{h})) = M \cap \mathrm{supp}(\tilde{h}),$$

and consequently $\tilde{h}x_2^{-1}\tilde{h} \in \mathrm{Sym}(M)$.

Therefore in (1), we can switch $\bar{h}_{M'}$ and $(\tilde{h}x_2^{-1}\tilde{h})$ to get

$$x_1hx_2^{-1} = \left(x_1\bar{h}_M(\tilde{h}x_2^{-1}\tilde{h})\right)\bar{h}_{M'}\tilde{h},$$

which must be the 3-term TC-decomposition of h. Hence $x_1hx_2^{-1} = h$ and

$$x_1\bar{h}_M(\tilde{h}x_2^{-1}\tilde{h}) = \bar{h}_M,$$

i.e. x_1 is conjugate to $\tilde{h}x_2\tilde{h}$ by $\bar{h}_M \in \mathrm{Sym}(M)$.

Furthermore we can choose $h \in H$ with $\tilde{h} = 1$, so that x_1 is conjugate to x_2 by $\bar{h}_M \in \mathrm{Sym}(M)$. Actually since x_2^{-1} preserves $\mathrm{supp}(\tilde{h})$, it is easy to verify that $x_2\tilde{h}x_2^{-1} \in H$. Then the TC-decomposition $x_1hx_2^{-1} = x_1\bar{h}x_2^{-1}x_2\tilde{h}x_2^{-1}$ $(\in H)$ implies $x_1\bar{h}x_2^{-1} \in H$. Thus we can choose $h \in T$ in the beginning.

Finally, since $\mathrm{Sym}(M) \simeq S_m$ in an obvious way, the conjugacy classes of $\mathrm{Sym}(M)$ hence the double cosets $H\backslash S_{2m}/H$ are in one-to-one correspondence with the partitions of m. □

Remark 10. Note that if x preserves the support of $c \in C$, then cxc is the *truncation* of x from $\mathrm{supp}(c)$, i.e. $cxc\,|_{\mathrm{supp}(c)}$ is the trivial permutation and cxc is the same permutation as x outside of $\mathrm{supp}(c)$.

Remark 11. Now we can show by Stirling's formula that

$$\text{Average of } |H \cap gHg^{-1}| = \sum_{g \in H\backslash S_{2m}/H} \frac{|HgH||H \cap gHg^{-1}|}{|S_{2m}|}$$

$$= |H\backslash S_{2m}/H|\frac{|H|^2}{|S_{2m}|} = p(m)\frac{2^{2m}(m!)^2}{(2m)!}$$

$$\sim p(m)\frac{2^{2m}\cdot 2\pi m(m/e)^{2m}}{\sqrt{2\pi\cdot 2m}(2m/e)^{2m}} = p(m)\sqrt{\pi m}.$$

Since $p(m) \sim \dfrac{1}{4\sqrt{3}m}e^{\pi\sqrt{2m/3}}$ by Hardy-Ramanujan [9], the average is of super-polynomial growth, which is a sign that the density of good elements should be low.

2.4 Counting double cosets by character formula

Apart from the combinatorics in section 2.3, there is a representation theoretic way of counting (self-inverse) double cosets by character formula following Frame [6].

Proposition 12. *The number of self-inverse double cosets of a finite group G with respect to a subgroup $H \leqslant G$ equals*

$$\sum_{\chi \in \mathrm{Irr}G, \langle \chi, \mathrm{Ind}_H^G 1_H \rangle \neq 0} \mathrm{FS}(\chi),$$

where the sum is over Frobenius-Schur indicators of irreducible characters occurring in the induced character of G from the trivial character of H. Here for any character χ of G,

$$\mathrm{FS}(\chi) := \frac{1}{|G|}\sum_{x\in G}\chi(x^2).$$

Note that $\mathrm{Ind}_H^G 1_H$ is afforded by the permutation representation of G through its action on the right cosets $H\backslash G$.

Proof. We follow the ideas of [6].

First, we show that the number of self-inverse double cosets of G with respect to H is

$$\frac{\#\{g_i x^2 = hg_i \mid g_i \in H\backslash G, x \in G, h \in H\}}{|G|}.$$

(See Theorem 3.1 of [6].) It suffices to show that each self-inverse double coset corresponds to $|G|$ solutions to the equation

$$h(g_i x g_i^{-1}) = (g_i x g_i^{-1})^{-1}, \tag{2}$$

which says that the inverse of $t = g_i x g_i^{-1}$ belongs to its own right coset. Each double coset HgH decomposes into right cosets as

$$HgH = \coprod_{y\in H/(g^{-1}Hg\cap H)} Hgy,$$

hence each left coset $h'gH \subset HgH$ intersects with each right coset Hgy at $h'g(g^{-1}Hg \cap H)y$, all of which have $d = |g^{-1}Hg \cap H|$ elements. In particular, the inverse of each right coset is a left coset, so it intersects with its own right coset at d elements, which amount to d values of t. Summing over all right cosets in HgH, we get $[H : (g^{-1}Hg \cap H)]d = |H|$ solutions to (2) in HgH if it is an self-inverse double coset. Varying the right cosets $g_i \in H\backslash G$, for each solution $(x_0, h_0) \in HgH \times H$ $hx = x^{-1}$, we get solutions $(g_i^{-1}xg_i, h)$ to $hg_i x g_i^{-1} = (g_i x g_i^{-1})^{-1}$, which amount to $[G : H]|H| = |G|$ solutions.

Now let G act on $H\backslash G$ by right multiplication and consider the corresponding permutation representation of G, which affords $\mathrm{Ind}_H^G 1_H$ by definition. Since the character value of a permutation representation on every element is the number of its fixed points, we get

$$\begin{aligned}
&\frac{\#\{g_i x^2 = hg_i \mid g_i \in H\backslash G, x \in G, h \in H\}}{|G|}\\
=&\frac{1}{|G|}\sum_{x\in G}\mathrm{Ind}_H^G 1_H(x^2)\\
=&\mathrm{FS}(\mathrm{Ind}_H^G 1_H(x^2))\\
=&\sum_{\chi\in\mathrm{Irr}G,\langle\chi,\mathrm{Ind}_H^G 1_H\rangle\neq 0}\mathrm{FS}(\chi).
\end{aligned}$$

□

Next, we resort to an interesting result of Inglis-Richardson-Saxl [11] on multiplicity free decomposition of the permutation representation $\mathrm{Ind}_H^{S_{2m}} 1_H$.

Proposition 13. *Let $H = C(h_0), h_0 = (1\ 2)(3\ 4)\cdots(2m-1\ 2m)$, then*

$$\mathrm{Ind}_H^{S_{2m}} 1_H = \bigoplus_{|\lambda|=m} S^{2\lambda},$$

where S^ν for any partition ν denotes the Specht module (over $\mathbb{Q}$).

By Proposition 9, the double cosets of S_{2m} with respect to H are all self-inverse for x conjugate to x^{-1} in $\mathrm{Sym}\{2, 4, \cdots, 2m\}$. Also note that all irreducible representations of symmetric groups are of real type, i.e. $\mathrm{FS}(\chi) = 1$ for any $\chi \in \mathrm{Irr}S_{2m}$. Then Proposition 13 and Proposition 12 show that the number of double cosets $H\backslash S_{2m}/H$ equals

$$\sum_{\chi \in \mathrm{Irr}G, \langle \chi, \mathrm{Ind}_H^G 1_H \rangle \neq 0} \mathrm{FS}(\chi) = \sum_{|\lambda|=m} \mathrm{FS}(S^{2\lambda}) = \sum_{|\lambda|=m} 1 = p(m),$$

the partition number of m.

2.5 Structure of $H \cap gHg^{-1}$ and proof of Theorem 3

With the structure description of double cosets $H\backslash S_{2m}/H$, this section proves Theorem 3 using an idea of constructing bipartite graph automorphisms introduced by J.P. James [14].

Let $\mathcal{G} = (V, E)$ be a *bipartite* graph (non-directed), i.e. its vertex set $V = V_1 \coprod V_2$ is a disjoint union of two *parties* $V_i, i = 1, 2$ and the edge set E is a collection of (unordered) pairs $\{v_1, v_2\}, v_i \in V_1, i = 1, 2$. We allow one edge to be duplicated. A graph automorphism is a permutation of vertices that sends edges to edges. Denote $\mathrm{Aut}_b(\mathcal{G})$ the set of automorphisms preserving $V_i, i = 1, 2$. Suppose $\mathcal{G}$ is k-regular, i.e. each vertex belongs to k edges, then $|E| = kl$ for some positive integer l. Label the edges by integers between 1 and kl. Define two k-partitions of $\{1, \cdots, kl\}$ as

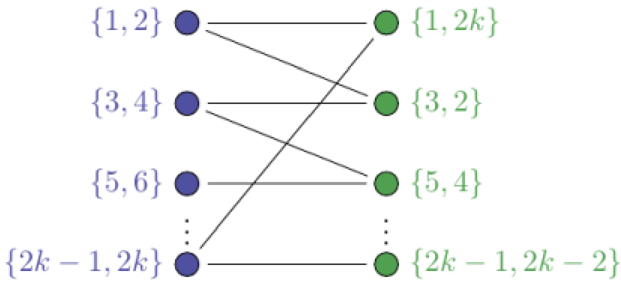
$$\alpha_i = \{U_v, v \in V_i\}, i = 1, 2,$$

in which $U_v = \{1 \leqslant i \leqslant kl, v \text{ belongs to } i\}$, the set of all edges containing v. Then any automorphism of $\mathrm{Aut}_b(\mathcal{G})$ is a permutation of $\{1, \cdots, kl\}$ that preserves the two k-partitions α_1, α_2. Denote the group of such permutations $(S_{kl})_{\alpha_1,\alpha_2}$, then by definition $\mathrm{Aut}_b(\mathcal{G}) \leqslant (S_{kl})_{\alpha_1,\alpha_2}$.

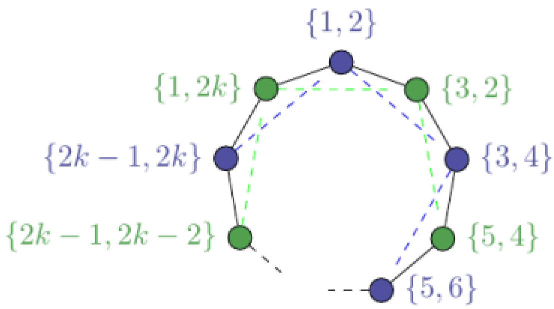
On the other hand, each permutation of $(S_{kl})_{\alpha_1,\alpha_2}$ is an automorphism of $\mathrm{Aut}_b(\mathcal{G})$. This is simply because each part of α_i (a k-subset of $\{1, \cdots .kl\}$) corresponds to a vertex in V_i, hence a permutation preserving α_i sends a vertex to a vertex, which also sends edges to edges by definition. We summarize Lemma 2.2 and 2.3 of [14] as follows

Proposition 14. $(S_{kl})_{\alpha_1,\alpha_2} \simeq \mathrm{Aut}_b(\mathcal{G})$.

Proof of Theorem 3. The one-to-one correspondence $H\backslash S_{2m}/H$ was already established in Proposition 9. In application of Proposition 14 to our case, let $k = 2, l = m$, the edges be $1, 2, \cdots, 2m$, and the two parties of vertices be $\alpha_1 = \{\{1, 2\}, \cdots, \{2m-1, 2m\}\}$ and $\alpha_2 = \{\{g(1), g(2)\}, \cdots, \{g(2m-1), g(2m)\}\}$ for any $g \in S_{2m}$. The edge i connects two vertices (blocks $\{2k-1, 2k\}$'s) that contain i. Then by definition, $(S_{2m})_{\alpha_1,\alpha_2} = H \cap gHg^{-1}$. Recall that $H = C(h_0), h_0 = (1\ 2) \cdots (2m-1\ 2m)$. By Proposition 9, the structure of $H \cap gHg^{-1}$ depends only on those g supported on even (or odd) numbers and their cycle type determined by partitions of $M' = \{2, 4, \cdots, 2m\}$. Hereinafter we denote a partition by $\lambda = \{1^{r_1} \cdots k^{r_k}\}$ which means λ has r_i parts equal to i and by $N_\lambda = \sum_{i=1}^{k} r_i$ the number of parts of λ. If $g \in HxH$ for x in the conjugacy class of $\mathrm{Sym}(M')$ with cycle type λ, then the constructed bipartite graph $\mathcal{G}$ has N_λ connected components corresponding to parts of λ, i.e. cycles of x. For instance, the component corresponding to a part k of λ, which may be expressed as the standard cycle $(2\ 4 \cdots\ 2k) \in \mathrm{Sym}(M')$, looks like



When unfolded, it becomes a $2k$-gon



Denote such a bipartite graph by $\mathcal{G}_k$. Clearly as a proper subgroup of the automorphism group of the above $2k$-gon, i.e. D_{2k}, $\mathrm{Aut}_b(\mathcal{G}_k)$ contains the automorphism group of the k-polygon with blue nodes (or equivalently the k-gon with green nodes) and dashed edges, i.e. D_k. Hence $\mathrm{Aut}_b(\mathcal{G}_k) \simeq D_k$, the dihedral group with $2k$ elements. Any automorphism in $\mathrm{Aut}_b(\mathcal{G})$ can also permute components of the same size, i.e. those corresponding to cycles of the same length. Thus the above construction using bipartite graphs replicates the definition of wreath product with symmetric groups. Hence for any permutation $x \in \mathrm{Sym}(M')$ with cycle type $\{i^r\}$, by Proposition 14 we have the wreath product presentation

$$H \cap xHx^{-1} \simeq \mathrm{Aut}_b(\mathcal{G}) \simeq \mathrm{Aut}_b\mathcal{G}_i \wr S_r = D_i \wr S_r,$$

In general for any $g \in HxH$ and $x \in \mathrm{Sym}(M')$ of cycle type $\lambda = \{1^{r_1}2^{r_2} \cdots k^{r_k}\}$, we get

$$H \cap gHg^{-1} \simeq \bigoplus_{i=1}^{k} D_i \wr S_{r_i},$$

and in particular,

$$|H \cap gHg^{-1}| = \prod_{i=1}^{k} (2i)^{r_i} r_i!,$$

simply by $|D_i \wr S_{r_i}| = |D_i|^{r_i}|S_{r_i}|$. This completes the proof. □

Using Theorem 3 we can measure the double cosets as follows.

Corollary 15. *For any $g \in HxH$ with $x \in \mathrm{Sym}\{2, 4, \cdots, 2m\}$ with x of cycle type $\lambda = \{1^{r_1} \cdots k^{r_k}\}$,*

$$|HgH| = |H|[H : H \cap gHg^{-1}] = (2^m m!)^2/(\prod_{i=1}^{k} (2i)^{r_i} r_i!).$$

By Theorem 4.4.8 of James-Kerber [13], the wreath product of a rational finite group with any symmetric group is also rational, hence Theorem 3 implies

Corollary 16. *All irreducible representations of $H \cap gHg^{-1}$ are realizable over $\mathbb{Q}$.*

2.6 Some computational verification of Theorem 3

For convenience, we denote $g \sim \lambda$ for any $g \in S_{2m}$ and λ a partition of m, if $g \in HxH$ with $x \in \mathrm{Sym}\{2, 4, \cdots, 2m\}$ of cycle type λ.

For the simplest example, if $x \sim \{1^m\}$, then Theorem 3 gives

$$H \cap xHx^{-1} \simeq D_1 \wr S_m = C_2 \wr S_m,$$

which coincides with Proposition 4 because $HxH = H$.

For $m = 2$, S_4 has $p(2) = 2$ double cosets, the nontrivial of which has a representative $x \sim \{2^1\}$, then Theorem 3 gives

$$H \cap xHx^{-1} \simeq D_2 \simeq K_4,$$

which coincides with our computation by hand in Example 5.

For $m = 3$, there are $p(3) = 3$ double cosets in $H\backslash S_6/H$ with representatives $1, (4\ 5)$, $(2\ 3)(4\ 5)$. Computed by `GAP` (the *StructureDescription* function), we get

$$H \cap (4\ 5)H(4\ 5) \simeq C_2 \times C_2 \times C_2 \simeq D_1 \times D_2,$$

and

$$H \cap (2\ 3)(4\ 5)H(2\ 3)(4\ 5) \simeq S_3 \simeq D_3,$$

where D_i denotes the dihedral group with $2i$ elements and for convenience, we write C_2 as D_1. Note that $(4\ 5) \sim \{1^1 2^1\}$ and $(2\ 3)(4\ 5) \sim \{3^1\}$, the structure results by Theorem 3 coincide with computation by `GAP`.

For $m = 4$, computed by `GAP` (the *DoubleCosetRepsAndSizes* function), there are $p(4) = 5$ double cosets in $H \backslash S_8 / H$ with representatives

$$1, (6\ 7) \sim \{1^2 2^1\}, (4\ 5)(6\ 7) \sim \{1^1 3^1\}, (2\ 3)(6\ 7) \sim \{2^2\}, (2\ 3)(4\ 5)(6\ 7) \sim \{4^1\}.$$

`GAP` gives the following structure description in coincidence with Theorem 3

$$\begin{aligned}
&H \cap (6\ 7)H(6\ 7) \simeq C_2 \times C_2 \times D_4 \simeq (D_1 \wr S_2) \times D_2, \\
&H \cap (2\ 3)(6\ 7)H(2\ 3)(6\ 7) \simeq C_2^4 \rtimes C_2 \simeq D_2 \wr S_2, \\
&H \cap (4\ 5)(6\ 7)H(4\ 5)(6\ 7) \simeq D_6 \simeq D_1 \times D_3, \\
&H \cap (2\ 3)(4\ 5)(6\ 7)H(2\ 3)(4\ 5)(6\ 7) \simeq D_4.
\end{aligned}$$

For $m = 5$, by `GAP`, there are $p(5) = 7$ double cosets in $H \backslash S_{10} / H$ with representatives

$$1, (8\ 9) \sim \{1^3 2^1\}, (6\ 7)(8\ 9) \sim \{1^2 3^1\}, (4\ 5)(8\ 9) \sim \{1^1 2^2\},$$

$$(4\ 5)(6\ 7)(8\ 9) \sim \{1^1 4^1\}, (2\ 3)(6\ 7)(8\ 9) \sim \{1^1 2^1 3^1\}, (2\ 3)(4\ 5)(6\ 7)(8\ 9) \sim \{5^1\}.$$

`GAP` gives the following structure description in coincidence with Theorem 3

$$\begin{aligned}
&H \cap (8\ 9)H(8\ 9) \simeq C_2 \times C_2 \times C_2 \times S_4 \simeq (D_1 \wr S_3) \times D_2, \\
&H \cap (6\ 7)(8\ 9)H(6\ 7)(8\ 9) \simeq D_4 \times S_3 \simeq (D_1 \wr S_2) \times D_3, \\
&H \cap (4\ 5)(8\ 9)H(4\ 5)(8\ 9) \simeq C_2 \times (C_2^4 \rtimes C_2) \simeq D_1 \times (D_2 \wr S_2), \\
&H \cap (4\ 5)(6\ 7)(8\ 9)H(4\ 5)(6\ 7)(8\ 9) \simeq C_2 \times D_4 = D_1 \times D_4, \\
&H \cap (2\ 3)(6\ 7)(8\ 9)H(2\ 3)(6\ 7)(8\ 9) \simeq C_2 \times C_2 \times S_3 \simeq D_1 \times D_2 \times D_3, \\
&H \cap (2\ 3)(4\ 5)(6\ 7)(8\ 9)H(2\ 3)(4\ 5)(6\ 7)(8\ 9) \simeq D_5.
\end{aligned}$$

More computational verification by `GAP` for $m \geqslant 6$ can also be checked.

3 Counting good elements

With the structural results on $H \cap gHg^{-1}$, we are prepared to count good elements in S_{2m}. Recall that $g \in S_{2m}$ is *good* if $|H \cap gHg^{-1}| = O(m^c)$ for some universal constant $c > 0$.

3.1 Counting with random permutation statistics

We show that the distribution of $|H\cap gHg^{-1}|$ happens to be the Ewens' distribution with bias $\theta=\frac{1}{2}$. By definition (see Example 2.19 of Arratia-Barbour-Tavaré [2]), the Ewens' distribution $\mathrm{ESF}(\theta)$ is the distribution equipped with the following probability density on partitions $\lambda=\{1^{r_1}\cdots k^{r_k}\}$ of m

$$P_\theta(\lambda)=\frac{m!}{\theta(\theta+1)\cdots(\theta+m-1)}\prod_{i=1}^{k}\left(\frac{\theta}{i}\right)^{r_i}\frac{1}{r_i!}. \tag{3}$$

By Theorem 3 and Corollary 15, the distribution of $|H\cap gHg^{-1}|$ over $g\in S_{2m}$ is equivalent to the following probability density on partitions of m, i.e. for any $x\in \mathrm{Sym}\{2,4,\cdots,2m\}$ of cycle type λ,

$$P(\lambda)=\frac{|HxH|}{|S_{2m}|}=\frac{2^{2m}(m!)^2}{(2m)!\prod_{i=1}^{k}(2i)^{r_i}r_i!}=\frac{m!}{\prod_{j=1}^{m}(j-\frac{1}{2})}\prod_{i=1}^{k}\left(\frac{\frac{1}{2}}{i}\right)^{r_i}\frac{1}{r_i!}, \tag{4}$$

which is exactly $P_{\frac{1}{2}}(\lambda)$ as in (3).

This turns the study of distribution of $|H\cap gHg^{-1}|$ into study of Ewens' distribution $\mathrm{ESF}(\frac{1}{2})$. By Theorem 5.1 of [2], as $m\to\infty$, $\mathrm{ESF}(\theta)$ point-wise converges to the joint distribution of independent Poisson distributions $(Z_1,Z_2,\cdots)$ on $\mathbb{N}^\infty$, where $Z_i\sim Po(\theta/i)$ for any $i\geqslant 1$ with $Prob(Z_i=j)=e^{-\theta/i}\frac{(\theta/i)^j}{j!}$. However, the unmanageable errors appearing in [2] between Ewens' distributions and joint Poisson distribution make it inaccessible to calculate the tail distribution of $\mathrm{ESF}(\theta)$. In the next section, we use methods of analytic combinatorics to estimate the left tail $P(|H\cap gHg^{-1}|\leqslant m^c)$, i.e. the probability of good elements.

3.2 Left tail of Ewen's distribution

First we define $|H\cap gHg^{-1}|$ as a random variable on partitions of m, i.e. let $f:\{\text{partition of } m\}\to\mathbb{R}$ be $f(\lambda)=|H\cap gHg^{-1}|=\prod_{i=1}^{k}(2i)^{r_i}r_i!$ for any partition $\lambda=(1^{r_1}\cdots k^{r_k})$ of m such that $g\in HxH$ for any $x\in\mathrm{Sym}\{2,4,\cdots,2m\}$ of cycle type λ, by Theorem 3.

For any $a\in\mathbb{R}$, define $W_{a,m}:=\sum_{|\lambda|=m}f(\lambda)^{-a}$. Especially for $a=0$ we get the partition number $W_{0,m}=p(m)\sim\frac{1}{4\sqrt{3}m}e^{\pi\sqrt{2m/3}}$ and for $a=1$, $W_{1,m}=\frac{(2m)!}{2^{2m}(m!)^2}\sim\frac{1}{\sqrt{\pi m}}$ by section 2.2. Also note that $W_{a,m}$ strictly decreases as a increases. In this notation we can write the distribution P defined in (4) as $P(\lambda)=W_{1,m}^{-1}f(\lambda)^{-1}$.

To estimate $P(f(\lambda)\leqslant m^c)$, i.e. the probability of good elements, we introduce the moment bound. For any nonnegative random variable X from a sample space Ω to $\mathbb{R}_{\geqslant 0}$ with probability distribution F, define the α-th moment for any $\alpha>0$ by

$$M_X^\alpha:=\mathbb{E}(X^\alpha)=\int_\Omega X^\alpha(\omega)dF(\omega).$$

Then by Markov's inequality, we have for any $C > 0$,

$$F(X > C) = F(X^\alpha > C^\alpha) \leqslant \frac{M_X^\alpha}{C^\alpha}.$$

Since α is arbitrary, we get

Proposition 17 (Moment bound). *For any $\alpha > 0$ and nonnegative random variable X with distribution F,*

$$F(X \geqslant C) \leqslant \inf_{\alpha>0} \frac{M_X^\alpha}{C^\alpha},\ \forall C > 0.$$

Now for the distribution P defined in (4), the moment bound applied to $X = f^{-1}$ gives for any $c > 0$,

$$P(f \leqslant m^c) = P(f^{-1} \geqslant m^{-c}) \leqslant \inf_{\alpha>0} m^{c\alpha} M_{f^{-1}}^\alpha = \inf_{\alpha>0} m^{c\alpha} W_{1,m}^{-1} W_{\alpha+1,m}, \tag{5}$$

since we have the expectation

$$\mathbb{E} f^{-\alpha} = W_{1,m}^{-1} \sum_{|\lambda|=m} f(\lambda)^{-\alpha} f^{-1}(\lambda) = W_{1,m}^{-1} \sum_{|\lambda|=m} f(\lambda)^{-(\alpha+1)} = W_{1,m}^{-1} W_{\alpha+1,m}.$$

Hence the task is to find an appropriate estimate of $W_{\beta,m}$ for $\beta > 1$. This is accessible through a hybrid method introduced by Flajolet et al [4] which we present in section 3.3.

3.2.1 Generating function of $W_{\beta,m}$

Before applying the hybrid method, it is necessary to introduce the following generating function for any $\beta \in \mathbb{R}$,

$$W_\beta(z) = \sum_{m\geqslant 0} W_{\beta,m} z^m = \sum_{m\geqslant 0} \sum_{|\lambda|=m} \frac{z^{r_1+2r_2+\cdots+kr_k}}{\prod_{i=1}^k (2i)^{r_i\beta} (r_i!)^\beta} = \prod_{i\geqslant 1} I_\beta(z^i/(2i)^\beta), \tag{6}$$

where $I_\beta(z) = \sum_{j\geqslant 0} \dfrac{z^j}{(j!)^\beta}$ defines an entire function (called Le Roy function, see [10]). For $\beta > 0$, W_β is an analytic function in the open unit disk of convergence radius $\geqslant 1$ at the origin, since

$$(W_{\beta,m})^{1/m} \leqslant W_{0,m}^{1/m} = p(m)^{1/m} \sim e^{\sqrt{m}/m} \to 1,\ \text{as } m \to \infty. \tag{7}$$

To further determine the convergence radius of $W_\beta(z), \beta > 0$, we need a lower bound for $W_{\beta,m}$. For any $\alpha \in \mathbb{R}$, let μ_α be the distribution on $\{\text{partition of } m\}$ with $\mu_\alpha(\lambda) = W_{\alpha,m}^{-1} f(\lambda)^{-\alpha}$ for any partition λ of m. For example, μ_0 is the uniform distribution and μ_1 is the distribution $P = P_{\frac{1}{2}}$ in the notation of Ewen's distribution defined in (3). For $0 < \gamma < 1$, $x^{1/\gamma}$ is a convex function, hence by Jensen's inequality (with expectation $\mathbb{E}_{\mu_\beta}$ over μ_β), for any $\alpha, \beta \in \mathbb{R}$,

$$\left(\mathbb{E}_{\mu_\beta} f^{-\alpha}\right)^{1/\gamma} \leqslant \mathbb{E}_{\mu_\beta} \left((f^{-\alpha})^{1/\gamma}\right),$$

i.e.

$$W_{\beta,m}^{-1}\sum_{|\lambda|=m} f^{-\alpha}(\lambda)f^{-\beta}(\lambda) = W_{\beta,m}^{-1}W_{\alpha+\beta,m}$$

$$\leqslant \left(W_{\beta,m}^{-1}\sum_{|\lambda|=m} f^{-\alpha/\gamma}f^{-\beta}\right)^{\gamma} = W_{\beta,m}^{-\gamma}W_{\alpha/\gamma+\beta,m}^{\gamma}.$$

Thus we get

Proposition 18. *For any* $\alpha, \beta \in \mathbb{R}$, $0 < \gamma < 1$, *and* $m \in \mathbb{Z}_+$,

$$W_{\alpha+\beta,m} \leqslant W_{\beta,m}^{1-\gamma}W_{\alpha/\gamma+\beta,m}^{\gamma}.$$

Remark 19. For $\beta = 0$ and $0 < \gamma = \alpha < 1$, we get

$$((\mathbb{E}_{\mu_0} f^{-\alpha}))^{1/\alpha} \leqslant \mathbb{E}_{\mu_0}(f^{-\alpha})^{1/\alpha} = \frac{W_{1,m}}{p(m)},$$

i.e.

$$\sum_{|\lambda|=m} f(\lambda)^{-\alpha} = W_{\alpha,m} \leqslant W_{1,m}^{\alpha}p(m)^{1-\alpha}.$$

Let $\alpha = 1 - \frac{1}{\sqrt{m}}\frac{\sqrt{3}}{\pi\sqrt{2}}t\ln m$ for any $0 < t < \frac{1}{2}$. By the asymptotics of $W_{1,m}$, $p(m)$ and $m^{\frac{\ln m}{\sqrt{m}}} = O(1)$, we get $W_{1,m}^{\alpha} = O(m^{-\frac{1}{2}})$ and $p(m)^{1-\alpha} = O(m^t)$, hence

$$W_{\alpha,m} \leqslant O(m^{-\frac{1}{2}+t}).$$

However, this bound is not sufficient for estimating the left tail in (5).

Remark 20. Proposition 18 is a log-convex constraint on $W_{\alpha,m}$, since

$$(1-\gamma)\beta + \gamma(\alpha/\gamma + \beta) = \alpha + \beta.$$

Especially for $\gamma = \frac{1}{2}$ we get

$$W_{\alpha+\beta} \leqslant W_{\beta,m}^{\frac{1}{2}}W_{2\alpha+\beta,m}^{\frac{1}{2}},$$

or

$$W_{2\alpha+\beta,m} \geqslant W_{\alpha+\beta}^{2}W_{\beta,m}^{-1}.$$

By the above remark, we can prove

Corollary 21. *For any* $\beta \geqslant 0$, $W_{\beta,m}^{1/m} \to 1$. *Consequently, the convergence radius of* $W_\beta(z)$ *equals* 1.

Proof. Acknowledging the upper bound (7), we need only to prove the lower bound. For any $\beta \in [0,1]$, $W_{\beta,m}^{1/m} \to 1$, due to $W_{0,m} = p(m)$, $W_{1,m} \sim \frac{1}{\sqrt{\pi m}}$ and the monotonicity of $W_{\beta,m}$ on β. Since $\alpha/\gamma + \beta$ with $\alpha, \beta, \gamma \in (0,1)$ ranges over $(0, +\infty)$, the case of $\beta > 1$ easily follows from Proposition 18. $\square$

3.2.2 Exp-log schema for $W_\beta(z)$

Let $H_\beta(z) = \log(I_\beta(z)) = \sum_{l\geqslant 1} h_{\beta,l} z^l$ ($h_{\beta,0} = 0$ since $I_\beta(0) = 1$). Then (6) becomes

$$W_\beta(z) = \exp\left(\sum_{i\geqslant 1} H_\beta\left(\frac{z^i}{2i^\beta}\right)\right) = \exp\left(\sum_{l\geqslant 1}\sum_{i\geqslant 1} h_{\beta,l}\frac{z^{il}}{(2i)^{\beta l}}\right) \tag{8}$$

$$= \exp\left(\sum_{l\geqslant 1}\frac{h_{\beta,l}}{2^{\beta l}}\mathrm{Li}_{\beta l}(z^l)\right),$$

where $\mathrm{Li}_\gamma(z) = \sum_{k\geqslant 1}\dfrac{z^k}{k^\gamma}$ is the polylogarithm for any $\gamma \in \mathbb{C}$.

Directly by definition, for $\gamma > 1$, $\mathrm{Li}_\gamma(1) < \infty$ and $\mathrm{Li}_\gamma(1)$ monotonically decrease to 1 as $\gamma \to \infty$. Also note that $\sum_{l\geqslant 1}\frac{h_{\beta,l}}{2^{\beta l}} = H_\beta(\frac{1}{2^\beta}) < \infty$ since the Le Roy function $I_\beta(z)$ is entire and positive for $z > 0$. Hence by Dirichlet's criterion, $\sum_{l>\lfloor\frac{1}{\beta}\rfloor}\dfrac{h_{\beta,l}}{2^{\beta l}}(\mathrm{Li}_{\beta l}(1) - 1)$ converges, and

$$\sum_{l>\lfloor\frac{1}{\beta}\rfloor}\frac{h_{\beta,l}}{2^{\beta l}}\mathrm{Li}_{\beta l}(1) = \sum_{l>\lfloor\frac{1}{\beta}\rfloor}\frac{h_{\beta,l}}{2^{\beta l}} + O(1) \tag{9}$$

$$= H_\beta(\frac{1}{2^\beta}) + O(1) = O(1).$$

Hence if $\beta > 1$, $W_\beta(z)$ is bounded in the unit disc $|z| < 1$ (the convergence region), or of *global order* 0 in notation of the next subsection where we introduce the hybrid method in details. The boundedness also prevents us from directly using (Hardy-Littlewood-Karamata) Tauberian theorem to derive asymptotics for $W_{\beta,m}, \beta > 1$.

Fortunately, the following result on singularities of polylogarithms is particularly helpful in this perspective.

Lemma 22 (Lemma 5 of [4]). *For any $\gamma \in \mathbb{C}$, the polylogarithm $\mathrm{Li}_\gamma(z)$ is analytically continuable to the slit plane $\mathbb{C}\setminus\mathbb{R}_{\geqslant 1}$. Moreover, the singular expansion of $\mathrm{Li}_\gamma(z)$ near the singularity $z = 1$ for non-integer γ is*

$$\mathrm{Li}_\gamma(z) \sim \Gamma(1-\gamma)\tau^{\gamma-1} + \sum_{j\geqslant 0}\frac{(-1)^j}{j!}\zeta(\gamma - j)\tau^j, \tag{10}$$

where $\tau := -\log z = \sum_{l\geqslant 1}\dfrac{(1-z)^l}{l}$, $\Gamma(z)$ is the gamma function and $\zeta(z)$ is the Riemann zeta function. For $m \in \mathbb{Z}_+$,

$$\mathrm{Li}_m(z) = \frac{(-1)^m}{(m-1)!}\tau^{m-1}(\log\tau - H_{m-1}) + \sum_{j\geqslant 0, j\neq m-1}\frac{(-1)^j}{j!}\zeta(m-j)\tau^j, \tag{11}$$

where H_k is the harmonic number $1 + 1/2 + \cdots + 1/k$.

In (10) (similar to (11)), the first term is the singular part for γ with real part $\mathrm{Re}\gamma \leqslant 1$ and the regular remainder tends to $\zeta(\gamma) = \mathrm{Li}_\gamma(1)$ if $\mathrm{Re}\gamma > 1$, as $\tau \to 0$ (or $z \to 1$). The lemma indicates that for $0 < \beta < 1$, Tauberian theorem is also not directly applicable to $W_\beta(z)$, since $e^{a(-\log z)^{\beta-1}} \gg (1-|z|)^{-a}$ for any $a > 0$, i.e. is of infinite global order. In section 4, we will deduce asymptotics of coefficients of this type through application of a saddle point method following E. M. Wright [15].

Note that $\mathrm{Li}_\gamma(z^k)$ only has singularities at k-th roots $\xi_1, \dots, \xi_k$ of unity, the above lemma gives the corresponding singular expansion

$$\mathrm{Li}_m(z^k) = \frac{(-1)^m}{(m-1)!k^{m-1}}(k\tau)^{m-1}(\log(k\tau) - H_{m-1}) \qquad (12)$$

$$+ \sum_{j\geqslant 0, j\neq m-1} \frac{(-1)^j}{j!k^j}\zeta(m-j)(k\tau)^j,$$

which becomes a series of $(1 - z/\xi_i)$ by substitution

$$k\tau = -k\log(z/\xi_i) = \sum_{l\geqslant 1} \frac{k}{l}(1 - z/\xi_i)^l.$$

3.3 Proof of Theorem 1 by hybrid method asymptotics for $W_{\beta,m}$

We first introduce some necessary notions following Flajolet et al [4].

Definition 23. The *global order* of an analytic function $f(z)$ in the open unit disc, is a number $a \leqslant 0$ such that $|f(z)| = O((1-|z|)^a), \forall |z| < 1$, that is, there exists $M > 0$ such that $|f(z)| < M(1-|z|)^a$ for all z with $|z| < 1$.

Since for any $\beta > 1$, $W_\beta(z)$ is bounded in the unit disc, its global order is zero. It can be shown by Cauchy's integral formula that a function $f(z)$ of global order $a \leqslant 0$ has coefficients satisfying $[z^n]f(z) = O(n^{-a})$, see section 1.1 of [4].

Definition 24. A *log-power function* at 1 is a finite sum of the form

$$\sigma(z) = \sum_{k=1}^{r} c_k(\log(\frac{1}{1-z})(1-z)^{\alpha_k},$$

where $\alpha_1 < \cdots < \alpha_k$ and each c_k is a polynomial. A log-power function at a finite set of points $Z = \{\zeta_1, \cdots, \zeta_m\}$, is a finite sum

$$\Sigma(z) = \sum_{j=1}^{m} \sigma_j\left(\frac{z}{\zeta_j}\right),$$

where σ_j is a log-power function at 1.

Since $\mathrm{Li}_0(z) = \frac{z}{1-z}, \mathrm{Li}_1(z) = \log(\frac{1}{1-z})$, a log-power function can be seen as approximation by combinations of these two polylogarithms. Asymptotics of coefficient of log-power functions are known, see Lemma 1 of [4].

Definition 25. Let $h(z)$ be analytic in $|z| < 1$ and s be a nonnegative integer. $h(z)$ is said to be $\mathcal{C}^s$*-smooth* on the unit disc, or of class $\mathcal{C}^s$, if for all $k = 0, \cdots, s$, its k-th derivative $h^{(k)}(z)$ defined for $|z| < 1$ admits a continuous extension to $|z| = 1$.

The smoothness condition relates to the coefficients of a function in an obvious way: if $h(z) = \sum_{n \geqslant 0} h_n z^n$ with $h_n = O(n^{-s-1-\delta})$ for some $\delta > 0$ and $s \in \mathbb{Z}_{\geqslant 0}$, then it is $\mathcal{C}^s$-smooth. Conversely, we have the Darboux's transfer (Lemma 2 of [4]): if $h(z)$ is $\mathcal{C}^s$-smooth, then $h_n = o(n^{-s})$. By (9) and the easy differentiation formula $\mathrm{Li}'_\gamma(z) = \mathrm{Li}_{\gamma-1}(z)/z$, we can see that for any $\beta \geqslant 2$, $W_\beta(z)$ is at least $\mathcal{C}^{\lfloor \beta \rfloor - 2}$-smooth on the unit disc.

Definition 26. An analytic function $Q(z)$ in the open unit disc is said to admit a *log-power expansion* of class $\mathcal{C}^t$ if there exist a finite set of points $Z = \{\zeta_1, \cdots, \zeta_m\}$ on the unit circle $|z| = 1$ and a log-power function $\Sigma(z)$ at the points of Z such that $Q(z) - \Sigma(z)$ is $\mathcal{C}^t$-smooth on the unit circle.

By (9) and Lemma 22, $W_\beta(z)$ has a non-trivial log-power expansion only for $\beta = 1$ and for $0 < \beta < 1$ there exists no such expansion.

Definition 27. Let $f(z)$ be analytic in the open unit disc. For ζ a point on the unit circle, we define the *radial expansion* of f at ζ with order $t \in \mathbb{R}$ as the smallest (in terms of numbers of monomials) log-power function $\sigma(z)$ at ζ, provided it exists, such that

$$f(z) = \sigma(z) + O((z - \zeta)^t),$$

when $z = (1 - x)\zeta$ and x tends to 0^+. The quantity $\sigma(z)$ is written

$$\mathrm{asymp}(f(z), \zeta, t).$$

Now we are prepared to introduce the main theorem of the hybrid method.

Proposition 28 (Theorem 2 of [4])**.** *Let $f(z)$ be analytic in the open unit disc D, of finite global order $a \leqslant 0$, and such that it admits a factorization $f = P \cdot Q$, with P, Q analytic in D. Assume the following conditions on P and Q, relative to a finite set of points $Z = \{\zeta_1, \ldots, \zeta_m\}$ on the unit circle ∂D:*
D1*: The "Darboux factor" $Q(z)$ is $\mathcal{C}^s$- smooth on ∂D ($s \in \mathbb{Z}_{\geqslant 0}$).*
D2*: The "singular factor" $P(z)$ is analytically continuable to an indented domain of the form $\mathfrak{D} = \cap_{j=1}^m (\zeta_j \cdot \Delta)$, where a Δ-domain is $\Delta(R, \phi) := \{z \in \mathbb{C} \mid |z| < R, \phi < arg(z-1) < 2\pi - \phi, z \neq 1\}$ for some radius $R > 1$ and angle $\phi \in (0, \frac{\pi}{2})$. For some non-negative real number t_0, it admits, at any $\zeta_j \in Z$, an asymptotic form (a log-power expansion of class $\mathcal{C}^{t_0}$)*

$$P(z) = \sigma_j(z/\zeta_j) + O((z - \zeta_j)^{t_0}) \ (z \to \zeta_j, z \in \mathfrak{D}),$$

where $\sigma_j(z)$ *is a log-power function at* 1.
D3*:* $t_0 > u_0 := \lfloor \frac{s+\lfloor a \rfloor}{2} \rfloor$.
Then f *admits radial expansions at every* $\zeta_j \in Z$ *with order* $u_0 = \lfloor \frac{s+\lfloor a \rfloor}{2} \rfloor$*. The coefficients of* z^n *of* $f(z)$ *satisfy:*

$$[z^n]f(z) = [z^n]A(z) + o(n^{-u_0}),$$

where $A(z) := \sum_{j=1}^m \operatorname{asymp}(f(z), \zeta_j, u_0)$.

Now we turn to approximating the coefficients of $W_\beta(z), \beta > 1$, to the order $o(n^{-u_0})$ for some $u_0 \in \mathbb{Z}_+$ which will be specified later as needed. We follow the hybrid method in close steps.

3.3.1 Darboux factor

By Proposition 28 we should choose a Darboux factor of $\mathcal{C}^s$-smooth for $s = 2u_0$, noting that the global order of $W_\beta(z)$ is zero. Provided the exp-log schema (8), we can factorize $W_\beta(z)$ into

$$\begin{aligned} W_\beta(z) &= \exp\left(\sum_{l<\lfloor \frac{2u_0+2}{\beta} \rfloor} \frac{h_{\beta,l}}{2^{\beta l}} \mathrm{Li}_{\beta l}(z^l) \right) \cdot \exp\left(\sum_{l \geqslant \lfloor \frac{2u_0+2}{\beta} \rfloor} \frac{h_{\beta,l}}{2^{\beta l}} \mathrm{Li}_{\beta l}(z^l) \right) \\ &=: e^{U(z)} \cdot e^{V(z)}. \end{aligned} \tag{13}$$

Note that for $l \geqslant \lfloor \frac{2u_0+2}{\beta} \rfloor$, i.e. $\beta l \geqslant 2u_0+2 = s+2$ and all $k = 0, \cdots, s$, the k-th derivative of $\mathrm{Li}_{\beta l}$ admits a continuous extension onto the unit circle. Hence by Dirichlet's criterion as (9), $V(z)$ as of (13) is $\mathcal{C}^s$-smooth and we can take the Darboux factor as $Q(z) = e^{V(z)}$.

3.3.2 Singular factor

Clearly we should take $P(z) = e^{U(z)}$ as the singular factor. Here as of (13), $U(z) = \sum_{l<\lfloor \frac{2u_0+2}{\beta} \rfloor} \frac{h_{\beta,l}}{2^{\beta l}} \mathrm{Li}_{\beta l}(z^l)$ as a truncation of the infinite sum, only has singularities at the l-th roots of unity for $l \leqslant \lfloor \frac{2u_0+2}{\beta} \rfloor - 1$, by Lemma 22. This is to say $P(z)$ is analytically continuable to the intersection of Δ-domains pointed at those roots, which form the set Z as in Proposition 28. Also the lemma readily shows that $P(z)$ admits the required asymptotic expansion to any order at each point of Z.

Hence by Proposition 28, for any $\beta > 1$, $W_\beta(z)$ admits a radial expansion at any point of Z with the chosen order u_0 and the hybrid method could give us the wanted asymptotics for $W_{\beta,m}$ once the radial expansions is calculated explicitly at each singularity. To simplify calculation, we set $u_0 = \lfloor \beta \rfloor$ so that we only need to consider the expansion at l-th roots of unity for $l \leqslant \left\lfloor \frac{2u_0+2}{\beta} \right\rfloor - 1$, which evaluates as follows

$$\left\lfloor \frac{2\lfloor \beta \rfloor + 2}{\beta} \right\rfloor - 1 = \begin{cases} 2 & \text{if } 1 < \beta \leqslant \frac{4}{3}, \\ 1 & \text{if } \frac{4}{3} < \beta < 2, \\ 2 & \text{if } \beta = 2, \\ 1 & \text{if } \beta > 2. \end{cases}$$

In application due to Lemma 22, we are mainly concerned with the cases where $\beta \in \mathbb{Z}_{\geqslant 2}$ and $\beta \to 2^-$.

3.3.3 The expansion at $z = 1, \beta \in \mathbb{Z}_{\geqslant 2}$

We first consider $\beta \in \mathbb{Z}_{\geqslant 2}$. Note that for any (real part) $\Re\gamma > 1$, $\zeta(\gamma) = \mathrm{Li}_\gamma(1)$ and

$$W_\beta(1) = \exp\left(\sum_{l\geqslant 1} \frac{h_{\beta,l}}{2^{\beta l}} \mathrm{Li}_{\beta l}(1)\right),$$

by taking out $W_\beta(1)$ and using Lemma 22 we get ($\tau = -\log z$)

$$W_\beta(z) = W_\beta(1) \exp\left(\sum_{l\geqslant 1} \frac{h_{\beta,l}}{2^{\beta l}} \frac{(-1)^{\beta l} l^{\beta l-1}}{(\beta l-1)!} \tau^{\beta l-1}(\log\tau + \log l - H_{\beta l-1})\right) \tag{14}$$

$$\cdot \exp\left(\sum_{l\geqslant 1} \frac{h_{\beta,l}}{2^{\beta l}} \sum_{j\geqslant 1, j\neq \beta l-1} \frac{(-1)^j}{j!} \zeta(\beta l - j) l^j \tau^j\right)$$

$$= W_\beta(1) \exp\left(A_\beta(\tau)\log\tau + B_\beta(\tau) + \delta_\beta(\tau)\right)$$

$$= W_\beta(1) + W_\beta(1) \sum_{n=1}^{\infty} \frac{1}{n!} \left(A_\beta(\tau)\log\tau + B_\beta(\tau) + \delta_\beta(\tau)\right)^n,$$

in which $A_\beta, B_\beta, \delta_\beta$ are series of τ correspondingly.

Noticing that $\tau = -\log z = \sum_{l=1}^{\infty} \frac{(1-z)^l}{l}$, to approximate $W_\beta(z)$ by log-power functions at $z = 1$ to the order $u_0 = \lfloor \beta \rfloor$ is to approximate it to the order $O(\tau^\beta)$. Simply we have

$$A_\beta(\tau) = \frac{(-1)^\beta h_{\beta,1}}{2^\beta(\beta-1)!} \tau^{\beta-1} + O(\tau^{2\beta-1}),$$

$$B_\beta(\tau) = \frac{(-1)^\beta h_{\beta,1}}{2^\beta(\beta-1)!} (-H_{\beta-1}) \tau^{\beta-1} + O(\tau^{2\beta-1}),$$

$$\delta_\beta(\tau) = \sum_{j=1}^{\beta-1} \frac{(-1)^j}{j!} \tau^j \left(\sum_{l\geqslant 1, \beta l-1\neq j} \frac{h_{\beta,l}}{2^{\beta l}} \zeta(\beta l - j) l^j\right) + O(\tau^\beta)$$

$$= \sum_{j=1}^{\beta-1} \frac{(-1)^j H_{\beta,j}}{j!} \tau^j + O(\tau^\beta),$$

where $H_{\beta,j} = \sum_{l\geqslant 1, \beta l-1\neq j} \frac{h_{\beta,l}}{2^{\beta l}} \zeta(\beta l - j) l^j$ are convergent series.

Hence in (14), we only need to care about the following terms

$$A_\beta(\tau)\log\tau,\ B_\beta(\tau),\ \sum_{n=1}^{\beta-1} \frac{1}{n!} \delta_\beta^n(\tau).$$

We investigate the log-power expansion of these three terms separately.

First we write $\log\tau$ as

$$\log\tau = \log\left((1-z)\sum_{l=0}^{\infty}\frac{(1-z)^l}{l+1}\right) = \log(1-z) + \log\left(1+\sum_{l=1}^{\infty}\frac{(1-z)^l}{l+1}\right)$$
$$= \log(1-z) + O(1-z).$$

Then

$$A_\beta(\tau)\log\tau = \frac{(-1)^\beta h_{\beta,1}}{2^\beta(\beta-1)!}\tau^{\beta-1}\log(1-z) + O(\tau^\beta)$$
$$= \frac{(-1)^\beta h_{\beta,1}}{2^\beta(\beta-1)!}\left(\sum_{l=1}^{\infty}\frac{(1-z)^l}{l}\right)^{\beta-1}\log(1-z) + O(\tau^\beta)$$
$$= \frac{(-1)^\beta h_{\beta,1}}{2^\beta(\beta-1)!}\left((1-z)^{\beta-1} + \frac{\beta-1}{2}(1-z)^\beta\right)\log(1-z) + O(\tau^\beta).$$

The other two terms $B_\beta(\tau)$ and $\delta_\beta(\tau)$ do not involve $\log(1-z)$, hence for large enough n, do not contribute to $[z^n]W_\beta(z)$ by the following lemma

Lemma 29 (Lemma 1 of [4]). *The general shape of coefficients of a log-power function is computable by the two rules:*

$$[z^n](1-z)^\alpha \sim \frac{1}{\Gamma(-\alpha)}n^{-\alpha-1}, \alpha \notin \mathbb{Z}_{\geqslant 0},$$

$$[z^n](1-z)^r(-\log(1-z))^k \sim (-1)^r k(r!)n^{-r-1}(\log n)^{k-1}, r \in \mathbb{Z}_{\geqslant 0}, k \in \mathbb{Z}_+.$$

Note that $\Gamma(z)$ has poles at negative integers which makes the first formula in the lemma coincide with the obvious fact that $(1-z)^\alpha, \alpha \in \mathbb{Z}_{\geqslant 0}$ do not contribute to asymptotics of coefficients eventually. Combined with the above calculation, we get

$$\begin{aligned}
&[z^n]A_\beta(\tau)\log\tau \\
=&[z^n]\frac{(-1)^\beta h_{\beta,1}}{2^\beta(\beta-1)!}\left((1-z)^{\beta-1} + (\beta-1)(1-z)^\beta\right)\log(1-z) + o(n^{-\beta}) \\
=&\frac{h_{\beta,1}}{2^\beta n^\beta} + o(n^{-\beta}) = (2n)^{-\beta} + o(n^{-\beta}),
\end{aligned} \tag{15}$$

recalling that $\sum_{l\geqslant 1} h_{\beta,l}z^l = \log(I_\beta(z)) = \log\left(\sum_{j\geqslant 0} z^j/(j!)^\beta\right)$ and $h_{\beta,1} = 1$ for any $\beta \in \mathbb{R}$. In general, the coefficients $h_{\beta,l}$ can be computed by Faà di Bruno's formula. Hence we get the expansion for $W_\beta(z)$ at $z = 1$ in this shape.

3.3.4 The expansion at $z = 1, \beta > 1, \beta \notin \mathbb{Z}_{\geqslant 0}$

By Lemma 22 we get ($\tau = -\log z$)

$$W_\beta(z) = W_\beta(1) \exp\left(\sum_{l\geqslant 1} \frac{h_{\beta,l} l^{\beta l-1}}{2^{\beta l}} \Gamma(1-\beta l)(\tau)^{\beta l-1}\right) \tag{16}$$
$$\cdot \exp\left(\sum_{l\geqslant 1} \frac{h_{\beta,l}}{2^{\beta l}} \sum_{j\geqslant 1} \frac{(-1)^j}{j!} \zeta(\beta l - j) l^j \tau^j\right)$$
$$= W_\beta(1) \exp\left(A_\beta(\tau) + \delta_\beta(\tau)\right),$$

in which (recall that $h_{\beta,1} = 1$)

$$A_\beta(\tau) = \frac{h_{\beta,1}}{2^\beta} \Gamma(1-\beta)\tau^{\beta-1} + O(\tau^{2\beta-1})$$
$$= \frac{\Gamma(1-\beta)}{2^\beta}(1-z)^{\beta-1} + O((1-z)^\beta)$$

and $\delta_\beta(\tau)$ involves only integer powers of $(1-z)$. Hence by Lemma 29, we only need to concern about $A_\beta(\tau)$ and

$$[z^n]A_\beta(\tau) = \frac{\Gamma(1-\beta)}{2^\beta \Gamma(1-\beta)} n^{-\beta} + o(n^{-\beta}) = \frac{1}{2^\beta n^\beta} + o(n^{-\beta}).$$

3.3.5 The expansion at $z = -1$

By Lemma 22, only $\mathrm{Li}_{2\beta l}(z^{2l})$ in (8) contribute singularities at $z = -1$, hence contribute to the asymptotics of $W_{\beta,n}$ to the order $O(n^{-2\beta})$ by 3.3.3 and 3.3.4.

Thus combining 3.3.1-3.3.4 and 3.3.5, we conclude from the hybrid method Proposition 28 that

Proposition 30. *For any $\beta > 1$,*

$$W_{\beta,m} = \frac{W_\beta(1)}{2^\beta m^\beta} + o(m^{-\beta}).$$

Remark 31. We omit the calculation of $W_\beta(1)$ for now, but according to Proposition 4 of [4], it should be less than $4.26341/2^\beta$ for $\beta \geqslant 2$. Also note that by Stirling's formula $W_{1,m} \sim \frac{1}{\sqrt{\pi m}}$, an abrupt jump of order in n. This is caused by $W_\beta(1) \to \infty$ as $\beta \to 1$.

Proof of Theorem 1. Now by the moment bound from Proposition 17, for P defined as (4) and f the random variable on {patition of m} defined at the beginning of subsection 3.2, and for any $c > 0, \alpha > 0$, we have

$$P(f < m^c) \leqslant m^{c\alpha} W_{1,m}^{-1} W_{\alpha+1,m} = m^{c\alpha} \cdot O(m^{-1/2-\alpha}) = O(m^{-1/2+(c-1)\alpha}).$$

In particular since for any $c > 0$ there always exists α small enough such that $(c-1)\alpha < 1/2$, we have

$$P(f < m^c) \to 0, \text{ as } m \to \infty,$$

which proves Theorem 1. ☐

4 Proof of Theorem 2 by Wright's expansion

Again by Markov's inequality, for any $c > 0, 0 < \beta < 1$ and expectation $\mathbb{E}_P$ on the probability measure P defined in (4),

$$P\left(f(\lambda) > m^c\right) = P\left(f^{1-\beta}(\lambda) > m^{c(1-\beta)}\right) \leqslant \frac{1}{m^{c(1-\beta)}}\mathbb{E}_P(f^{1-\beta}) \tag{17}$$

$$= m^{-c(1-\beta)} \sum_{|\lambda|=m} \left(\prod_{i=1}^{k}(2i)^{r_i} r_i!\right)^{1-\beta} \frac{2^{2m}(m!)^2}{(2m)!\prod_{i=1}^{k}(2i)^{r_i}r_i!}$$

$$= m^{-c(1-\beta)} W_{1,m}^{-1} W_{\beta,m}.$$

Only when β tends 1 could the above inequality give an appropriate bound for $P(f(\lambda) > m^c)$. The upper bound of $W_{\beta,m}$ in remark 19 can only best possibly give

$$P(f(\lambda) > m^c) = O(1),$$

for $\beta = 1 - \frac{1}{\sqrt{m}}\frac{\sqrt{3}}{\pi\sqrt{2}} t \log m$ and any $0 < t < \frac{1}{2}$. Hence we need more precise asymptotics for $W_{\beta,m}, 0 < \beta < 1$.

Let $\frac{1}{2} < \beta < 1$, we can split $W_\beta(z)$ as of (8) into

$$W_\beta(z) = \exp\left(\sum_{l \geqslant 1} \frac{h_{\beta,l}}{2^{\beta l}} \mathrm{Li}_{\beta l}(z^l)\right) = \exp\left(2^{-\beta}\mathrm{Li}_\beta(z)\right) \cdot e^{V_\beta(z)}, \tag{18}$$

where $V_\beta(z) = \sum_{l \geqslant 2} \frac{h_{\beta,l}}{2^{\beta l}} \mathrm{Li}_{\beta l}(z^l)$ and also note that $h_{\beta,1} = 1$. By calculation using the hybrid method in subsection 3.2, it is clear that

$$[z^n]e^{V_\beta(z)} = O(n^{-2\beta}). \tag{19}$$

For the first factor, by Lemma 22, we get ($\tau = -\log z$)

$$\exp\left(2^{-\beta}\mathrm{Li}_\beta(z)\right) = \exp\left(2^{-\beta}\left(\Gamma(1-\beta)\tau^{\beta-1} + \zeta(\beta) + \delta_\beta(\tau)\right)\right), \tag{20}$$

where

$$\delta_\beta(\tau) = \sum_{j \geqslant 1} \frac{(-1)^j}{j!}\zeta(\beta - j)\tau^j.$$

Similar to 3.3.3, since $\delta_\beta(\tau)$ (or $e^{\delta_\beta(\tau)}$) only involves integer powers of $(1-z)$, by Lemma 29 it does not contribute to the asymptotics of $[z^n]\exp\left(2^{-\beta}\mathrm{Li}_\beta(z)\right)$ in order of n. Thus it is essential to approximate the coefficients of

$$U_\beta(z) = \exp\left(2^{-\beta}\Gamma(1-\beta)(-\log z)^{\beta-1}\right).$$

Together with the factorization (18) and asymptotics (19), this gives

$$W_{\beta,m} = [z^m]W_\beta(z) = Ce^{2^{-\beta}\zeta(\beta)} \sum_{k=0}^{m} [z^n]U_\beta(z)[z^{m-n}]e^{V_\beta(z)}. \tag{21}$$

Now we focus on the asymptotics of $[z^n]e^{2^{-\beta}\zeta(\beta)}U_\beta(z)$. We notice that functions of same type with U_β were already handled in 1930s by E. M. Wright [15].

Proposition 32 (Wright's expansions, Theorem 5,6,7 of [15]). *For any $a, b, c \in \mathbb{C}, a \neq 0$ and $\rho > 0$, let*

$$\chi(z) = \frac{z^c}{(-\log(z))^b} \exp\left(\frac{a}{(-\log(z))^\rho}\right),$$

and

$$F(z) = \sum_{n=\lceil \Re c \rceil+1}^{\infty} (n-c)^{b-1} \phi(a(n-c)^\rho) z^n,$$

in which $\Re c$ is the real part of c and

$$\phi(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(l+1)\Gamma(\rho l + b)}.$$

(It is called a generalized Fox-Wright function, see [5].) Then $F(z)$ forms the singular part of $\chi(z)$ and $G(z) = F(z) - \chi(z)$ is a regular function around $z = 1$ where it behaves uniformly in terms of a and ρ. Moreover, define the asymptotic expansion

$$H(z) \sim z^{1/2-b} e^{(1+1/\rho)z} \left(\sum_{j=0}^{r} \frac{(-1)^j a_j}{z^j} + O(\frac{1}{|z|^{r+1}})\right),$$

where the term $O(|z|^{r+1})$ and a_j are uniformly bounded for $\rho > -1$, for example,

$$a_0 = \{2\pi(\rho+1)\}^{-\frac{1}{2}}, \quad a_1 = \frac{12b^2 - 12b(\rho+1) + (\rho+2)(2\rho+1)}{24(\rho+1)\{2\pi(\rho+1)\}^{\frac{1}{2}}}.$$

For $arg(z) = \xi, |\xi| \leqslant \pi - \epsilon$, let

$$Z = (\rho|z|)^{1/(\rho+1)} e^{i\xi/(\rho+1)},$$

then $\phi(z)$ has the asymptotics (by a saddle point analysis which Wright did not perform in [15] but in [16])

$$\phi(z) = H(Z),$$

and the error term in H depends on ϵ.

Since $V_\beta(z)$ is regular of global order 0 at the singularity $z = 1$, it does not contribute to asymptotics of coefficients (by Cauchy's integral formula). Thus we conclude from Proposition 32 that

Corollary 33. *Let $b = c = 0, a = 2^{-\beta}\Gamma(1-\beta)$ and $\rho = 1 - \beta$, then*

$$[z^n]U_\beta(z) = n^{-1}\phi(2^{-\beta}\Gamma(1-\beta)n^{1-\beta}).$$

In particular $\xi = 0$ (keeping notations of the above proposition), hence

$$Z = \left((1-\beta)2^{-\beta}\Gamma(1-\beta)n^{1-\beta}\right)^{1/(2-\beta)} = \left(2^{-\beta}\Gamma(2-\beta)n^{1-\beta}\right)^{1/(2-\beta)}.$$

Then

$$
\begin{aligned}
[z^n]e^{2^{-\beta}\zeta(\beta)}U_\beta(z) &= e^{2^{-\beta}\zeta(\beta)} \cdot n^{-1} H\left(\left(2^{-\beta}\Gamma(2-\beta)n^{1-\beta}\right)^{1/(2-\beta)}\right) \\
&= n^{-1}e^{2^{-\beta}\zeta(\beta)}\left(2^{-\beta}\Gamma(2-\beta)n^{1-\beta}\right)^{1/(4-2\beta)} \\
&\cdot \exp\left(\frac{2-\beta}{1-\beta}\left(2^{-\beta}\Gamma(2-\beta)n^{1-\beta}\right)^{1/(2-\beta)}\right) \cdot C \\
&= C'n^{-1} \cdot n^{\frac{1-\beta}{4-2\beta}} \exp\left(\frac{g(1-\beta)}{1-\beta}\right),
\end{aligned}
$$

where

$$
g(1-\beta) = (2-\beta)\left(2^{-\beta}\Gamma(2-\beta)n^{1-\beta}\right)^{1/(2-\beta)} + 2^{-\beta}\zeta(\beta)(1-\beta),
$$

C is bounded independent of $1-\beta$ and n, and $C' = C \cdot 2^{-\beta}\Gamma(2-\beta) \sim C/2$ as $\beta \to 1^-$.

Let $\epsilon = 1-\beta \to 0^+$, then we can rewrite $g(1-\beta)$ as

$$
g(\epsilon) = (1+\epsilon)\left(2^{\epsilon-1}\Gamma(1+\epsilon)n^{\epsilon}\right)^{\frac{1}{1+\epsilon}} + 2^{\epsilon-1}\zeta(1-\epsilon)\epsilon.
$$

Hence to figure out the asymptotics of $W_{\beta,n}$ we need to compute the limit

$$
\begin{aligned}
\lim_{\epsilon\to 0^+}\frac{g(\epsilon)}{\epsilon} &= \frac{1}{2} + \lim_{\epsilon\to 0^+}\frac{\left(2^{\epsilon-1}\Gamma(1+\epsilon)n^{\epsilon}\right)^{\frac{1}{1+\epsilon}} + 2^{\epsilon-1}\zeta(1-\epsilon)\epsilon}{\epsilon} \\
&= \frac{1}{2} + \frac{1}{2}\lim_{\epsilon\to 0^+}\frac{2^{\frac{\epsilon(1-\epsilon)}{1+\epsilon}}n^{\frac{\epsilon}{1+\epsilon}}\Gamma(1+\epsilon)^{\frac{1}{1+\epsilon}} + \zeta(1-\epsilon)\epsilon}{\epsilon}.
\end{aligned}
$$

First, the limit exists since $\zeta(1-\epsilon)\epsilon \to -1$ and then

$$
g(\epsilon) \to 1 \cdot (2^{-1})^1 + 2^{-1}(-1) = 0, \text{ as } \epsilon \to 0.
$$

Moreover, we have the Laurent series of $\zeta(s)$

$$
\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n\gamma_n}{n!}(s-1)^n,
$$

where γ_n are the Stieltjes constants and especially γ_0 is the Euler-Mascheroni constant. Thus we get

$$
\zeta(1-\epsilon)\epsilon = -1 + \gamma_0\epsilon + \sum_{n=0}^{\infty}\frac{\gamma_n}{n!}\epsilon^{n+1} \sim -1 + \gamma_0\epsilon + O(\epsilon^2),
$$

and we can rewrite the limit as

$$
\lim_{\epsilon\to 0^+}\frac{g(\epsilon)}{\epsilon} = \frac{1}{2} + \frac{1}{2}\lim_{\epsilon\to 0^+}\frac{2^{\frac{\epsilon(1-\epsilon)}{1+\epsilon}}n^{\frac{\epsilon}{1+\epsilon}}\Gamma(1+\epsilon)^{\frac{1}{1+\epsilon}} - 1}{\epsilon} + \frac{\gamma_0}{2}
$$

$$= \frac{1+\gamma_0}{2} + \frac{1}{2}\lim_{\epsilon\to 0^+} \frac{n^{\frac{\epsilon}{1+\epsilon}}\Gamma(1+\epsilon)^{\frac{1}{1+\epsilon}} - 2^{-\frac{\epsilon(1-\epsilon)}{1+\epsilon}}}{\epsilon}$$

$$= \frac{1+\gamma_0}{2} + \frac{1}{2}\lim_{\epsilon\to 0^+} \frac{g_1(\epsilon)-g_2(\epsilon)}{\epsilon}.$$

Easily $g_1(0) = g_2(0) = 1$. Now we calculate their first derivatives at 0,

$$g_1'(\epsilon) = n^{\frac{\epsilon}{1+\epsilon}} \frac{1}{(1+\epsilon)^2} \log n \cdot \Gamma(1+\epsilon)^{\frac{1}{1+\epsilon}}$$

$$+ n^{\frac{\epsilon}{1+\epsilon}} \cdot \Gamma(1+\epsilon)^{\frac{1}{1+\epsilon}} \left(-\frac{1}{(1+\epsilon)^2} \log \Gamma(1+\epsilon) + \frac{1}{1+\epsilon} \frac{\Gamma'(1+\epsilon)}{\Gamma(1+\epsilon)} \right),$$

hence

$$g_1'(0) = \log n - \gamma_0,$$

note that $\Gamma(1) = 1, \Gamma'(1) = -\gamma_0$. (Moreover, inductively we have estimate that $g_1^{(k)}(0) \sim (\log n)^k$.)

$$g_2'(\epsilon) = 2^{-\frac{\epsilon(1-\epsilon)}{1+\epsilon}} \log 2 \cdot \left(1 - \frac{2}{(1+\epsilon)^2}\right),$$

hence

$$g_2'(0) = -\log 2.$$

We are plugged into the limit and get

$$\lim_{\epsilon\to 0^+} \frac{g(\epsilon)}{\epsilon} = \frac{1+\gamma_0}{2} + \frac{1}{2}(g_1'(0) - g_2'(0)) = \frac{1+\gamma_0}{2} + \frac{1}{2}(\log 2n - \gamma_0)$$

$$= \frac{1+\log 2n}{2}.$$

Moreover, we have (for $n \leqslant m$)

$$\frac{g(\epsilon)}{\epsilon} - \frac{1+\log 2n}{2} = O\left(\sum_{k\geqslant 1} \frac{(\log n)^{k+1}}{(k+1)!}\epsilon^k\right),$$

where the constant in $O(*)$ is independent of m and $1-\beta$.

Finally we get

$$[z^n]e^{2^{-\beta}\zeta(\beta)}U_\beta(z) = O\left(n^{-1} \cdot n^{\frac{1-\beta}{4-2\beta}} \cdot \exp\left(\frac{g(1-\beta)}{1-\beta}\right)\right)$$

$$= O\left(n^{-1+\frac{1-\beta}{4-2\beta}} \exp\left(\frac{\log 2n}{2} + \log n \cdot O\left(\sum_{k\geqslant 1} \frac{((1-\beta)\log n)^k}{(k+1)!}\right)\right)\right)$$

$$= O\left(n^{-\frac{1}{2}+\frac{1-\beta}{4-2\beta}+O\left(\sum_{k\geqslant 1}\frac{((1-\beta)\log n)^k}{(k+1)!}\right)}\right).$$

Returning to (21) we finally get

$$W_{\beta,m} = [z^m]W_\beta(z) = O\left(\sum_{n=0}^{m}[z^n]e^{2^{-\beta}\zeta(\beta)}U_\beta(z)[z^{m-n}]e^{V_\beta(z)}\right)$$

$$= O\left(m^{-\frac{1}{2}+\frac{1-\beta}{4-2\beta}+O\left(\sum_{k\geqslant 1}\frac{((1-\beta)\log m)^k}{(k+1)!}\right)}\right),$$

note that $[z^{m-n}]e^{V_\beta(z)} = O\left(m^{-2\beta}\right)$.

Returning to (17) and noting that $W_{1,m} \sim (\pi m)^{-1/2}$, for any $c > 0$ we get

$$P(f > m^c) \leqslant m^{-c(1-\beta)}W_{1,m}^{-1}W_{\beta,m}$$

$$= O\left(m^{\left(-c+\frac{1}{4-2\beta}\right)(1-\beta)+O\left(\sum_{k\geqslant 1}\frac{((1-\beta)\log m)^k}{(k+1)!}\right)}\right).$$

For $\beta = 1 - \frac{t}{(\log m)^2}$ (t constant) and $c > \frac{1}{2} + \log m$, the above term goes to zero as $m \to \infty$. This amounts to proving Theorem 2.

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