Hadamard Matrices Related to Projective Planes

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Abstract

Let $n$ be the order of a quaternary Hadamard matrix. It is shown that the existence of a projective plane of order $n$ is equivalent to the existence of a balancedly multi-splittable quaternary Hadamard matrix of order $n^2$.

Mathematics Subject Classifications: 05B20, 05B25, 05B15

1 Introduction

K. A. Bush [2] was the first to establish a link between projective planes of even order and specific Hadamard matrices, that was later labeled as Bush-type in 1971. H. J. Ryser [11] found the same connection as an application of factors of design matrix in 1977. Eric Verheiden [12] provided a direct construction for the matrices using the incidence matrices of the corresponding projective planes.

Frans C. Bussemaker, Willem Haemers and Ted Spence [3] used an exhaustive search and found no strongly regular graph with parameters $(36,15,6,6)$ and chromatic number six or, equivalently, there is no symmetric Bush-type Hadamard matrix of order 36. Many Bush-type Hadamard matrices of order 100 are constructed, but none is known to be symmetric. The proof of the nonexistence of a symmetric Bush-type Hadamard matrix of order 100 would be exciting and is an alternative to the proof of the nonexistence of projective plane of order 10, however, there has been no attempt at showing it so far. The nonexistence of the projective plane of order 10 was finally established by a long

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computational method by C. W. H. Lam et al. in [9, 10], and an alternate approach is still highly desirable.

The connection between projective planes and Hadamard matrices shown in [2, 11, 12] are all one-sided results in which from a projective plane of even order symmetric Bush-type Hadamard matrices are constructed.

Balancedly splittable Hadamard matrices were introduced by the authors in 2018 in [8], and the results were widely expanded in a recent paper by Jonathan Jedwab et al. in [5]. It is known [7] that the existence of a Hadamard matrix of order $4n$ would lead to a balancedly splittable Hadamard matrix of order $64n^2$. However, there is no balancedly splittable Hadamard matrix of order $4n^2$, $n$ odd, see [8]. The case of Hadamard matrices of order $16n^2$, $n > 1$ odd, remains open, and no balancedly splittable Hadamard matrix of order 144 is known.

Concentrating on the order 144, the authors were led to some exotic classes of balancedly splittable Hadamard matrices, which is dubbed as balancedly multi-splittable Hadamard matrices. There is a balancedly multi-splittable Hadamard matrix of order $4^m$ for every positive integer $m$, and it seems that these are probably the only Hadamard matrices with this property.

It will be shown in this paper that the existence of a projective plane of order $4n$ is equivalent to the existence of a balancedly multi-splittable Hadamard matrix of order $16n^2$ provided that $4n$ is the order of a Hadamard matrix. In doing so we use the fact that the existence of projective planes are equivalent to the existence of orthogonal arrays, see [1, Theorems 3.18 and 3.20], and the latter is equivalent to the balancedly multi-splittable Hadamard matrices.

There is also a similar equivalence between the projective plane of order $2n$, $n$ odd, and balancedly multi-splittable quaternary Hadamard matrices will be presented too.

The establishment of the nonexistence of a balancedly multi-splittable (quaternary) Hadamard matrix of order 144 (100) would be significant.

2 Preliminaries

2.1 Codes

Let $n, q$ be positive integers $n, q \geq 2$, and let $Q = \{0, 1, \ldots, q - 1\}$. A subset $C$ of $Q^n$ is called to be a $q$-ary code of length $n$. For $x, y \in Q^n$ with $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$, the Hamming distance between codewords $x$ and $y$ is given by $\text{dist}(x, y) = |\{i : x_i \neq y_i\}|$. A code $C$ is said to be an equi-distance code or a 1-distance set if the Hamming distance $d(x, y)$ does not depend on $x, y \in C$ with $x \neq y$.

2.2 Hadamard matrices

An $n \times n$ matrix $H$ is a Hadamard matrix of order $n$ if its entries are 1, $-1$ and it satisfies $HH^T = I_n$, where $I_n$ denotes the identity matrix of order $n$. A Hadamard matrix $H$ of order $n$ is said to be balancedly splittable if there is an $\ell \times n$ submatrix $H_1$ of $H$ such that inner products for any two distinct column vectors of $H_1$ take at most two values.
More precisely, there exist integers $a, b$ and the adjacency matrix $A$ of a graph such that $H_1^\top H_1 = tI_n + aA + b(J_n - A - I_n)$, where $J_n$ denotes the all-ones matrix of order $n$. We say the quadruple $(v, \ell, a, b)$ the parameter. In this case we say that $H$ is balancedly splittable with respect $H_1$. Only the special case of $(v, \ell, a, b) = (4n^2, 2n^2, n, -n)$ will be used in this note.

The same concept can be extended to orthogonal designs [7]. Here, we adopt the following definition for quaternary Hadamard matrices. An $n$-quaternary Hadamard matrix $H$ of order $n$ is a quaternary Hadamard matrix such that the off-diagonal entries of $H^\top H_1$ are in the set

$$\{\varepsilon\alpha, \varepsilon\alpha^*, \varepsilon\beta, \varepsilon\beta^* \mid \varepsilon \in \{\pm 1, \pm i\}\},$$

where $\alpha, \beta$ are some complex numbers. In this paper, we restrict to the case $\alpha = \beta$ and we say that a quaternary Hadamard matrix $H$ of order $n$ is balancedly splittable if $H_1^\top H_1 = tI + \alpha S$ where $\alpha$ is some positive real number and $S$ is a $(0, \pm 1, \pm i)$-matrix with zero diagonal entries and nonzero off-diagonal entries.

### 2.3 Orthogonal arrays

An orthogonal array of strength $t$ and index $\lambda$ is an $N \times k$ matrix over the set $\{1, \ldots, q\}$ such that in every $N \times t$ subarray, each $t$-tuple in $\{1, \ldots, q\}^t$ appears $\lambda$ times. We denote this property as OA$_\lambda(N, k, q, t)$. Note that $N = \lambda q^t$ and $(N, k, q, t)$ is the parameter of the orthogonal array. For $t = 2e$, the following lower bound on $N$ was shown by Rao (see [6, Theorem 2.1]), namely, $N \geq \sum_{i=0}^{e} \binom{t}{i} (q - 1)^i$. An orthogonal array with parameters $(N, k, q, 2e)$ is said to be complete if the equality holds in above.

When $t = 2$ and $\lambda = 1$, the complete orthogonal array has the parameters OA$_1(q^2, q + 1, q, 2)$, and it is known that its existence is equivalent to that of a projective plane of order $q$. For the orthogonal version of a projective plane is used in the next section.

The following lemmas will be used later.

**Lemma 1.** Let $A$ be an $N \times k$ matrix over $\{1, \ldots, q\}$. Write $A = \sum_{i=1}^{q} iA_i$, where $A_i$ ($i \in \{1, \ldots, q\}$) are disjoint $N \times k$ $(0, 1)$-matrices. Let $D$ be the distance matrix, ie., $D$ is an $N \times N$ matrix whose rows and columns indexed by the rows of $A$ with $(i, j)$-entry defined by the Hamming distance between the $i$-th row and the $j$-th row of $A$. Then

$$\sum_{i=1}^{q} A_iA_i^\top = kJ_N - D$$

holds.

**Proof.** See the proof of [6, Lemma 2.5 (i)].

**Lemma 2.** Assume that there exists an orthogonal array $A$ with parameters $(q^2, q+1, q, 2)$. Write $A = \sum_{i=1}^{q} iA_i$, where $A_i$ ($i \in \{1, \ldots, q\}$) are disjoint $q^2 \times (q + 1)$ $(0, 1)$-matrices. Then the matrices $A_i$ satisfy

(i) $\sum_{i=1}^{q} A_iA_i^\top = J_{q^2} + qI_{q^2}$,

(ii) $\sum_{i,j=1, i\neq j}^{q} A_iA_j^\top = q(J_{q^2} - I_{q^2})$. 

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(iii) Consider the code $C$ obtained from the rows of $A$. Let $\{i_1, \ldots, i_s\}$ be any $s$-element subset of $\{1, \ldots, q + 1\}$. The code $C'$ obtained from $C$ by restricting the coordinates on the set $\{i_1, \ldots, i_s\}$ have the Hamming distances $s$ or $s - 1$ between the codewords in $C'$.

**Proof.** The proof for (i) and (ii) are exactly the same as [6, Lemma 2.5].

The assumed orthogonal array is a 2-design and 1-distance set with Hamming distance $q$ in the Hamming association scheme. The case (iii) follows from the fact that $C$ is a 1-distance set with Hamming distance $q$. \(\square\)

**Lemma 3.** [4, Theorem 5.14] Let $C$ be an equidistance code of length $q + 1$ over the symbol set $\{1, \ldots, q\}$. Then

$$|C| \leq q^2$$

holds. Equality holds if and only if the matrix whose rows consists of the codewords of $C$ is an orthogonal array $OA_1(q^2, q + 1, q, 2)$.

\section{Balancedly multi-splittable Hadamard matrices}

We consider the following property of a Hadamard matrix. Let $H$ be a Hadamard matrix of order $4n^2$. Assume that $H$ is normalized so that the first column of $H$ is the all-ones vector. A Hadamard matrix $H$ is said to be balancedly multi-splittable if there is a block form of $H = \left[ \begin{array}{c|c} 1 & H_1 \\ \vdots & \vdots \\ 1 & H_{4n} \end{array} \right]$, where each $H_i$ is of order $4n^2 \times (2n - 1)$ such that $H$ is balancedly splittable with respect to a submatrix $[H_{i_1}, \ldots, H_{i_n}]$ for any $n$-element subset $\{i_1, \ldots, i_n\}$ of $\{1, 2, \ldots, 2n + 1\}$, that is, the inner product of any distinct rows of $[H_{i_1}, \ldots, H_{i_n}]$ is $\pm n$.

The main results of this paper are as follows:

**Theorem 4.** Let $n$ be a positive integer. The following are equivalent.

(i) There exists a balancedly multi-splittable Hadamard matrix of order $16n^2$.

(ii) There exist an $OA_1(16n^2, 4n + 1, 4n, 2)$ and a Hadamard matrix of order $4n$.

**Theorem 5.** Let $n$ be a positive integer. The following are equivalent.

(i) There exists a balancedly multi-splittable quaternary Hadamard matrix of order $4n^2$.

(ii) There exist an $OA_1(4n^2, 2n + 1, 2n, 2)$ and a quaternary Hadamard matrix of order $2n$.

\subsection{Proof of Theorem 4}

The proof of (ii) $\Rightarrow$ (i). Assume that there exists a Hadamard matrix $H$ of order $4n$. Write $H$ as

$$H = \begin{bmatrix}
1 & r_1 \\
1 & r_2 \\
\vdots & \vdots \\
1 & r_{4n}
\end{bmatrix},$$

where $r_1, r_2, \ldots, r_{4n}$ are elements of $\{1, 2, \ldots, q + 1\}$. This table represents a balancedly multi-splittable Hadamard matrix.
where \( r_i \) is a \( 1 \times (4n - 1) \) matrix for any \( i \).

**Lemma 6.** (i) For any \( i \), \( r_i r_i^\top = 4n - 1 \).

(ii) For any distinct \( i, j \), \( r_i r_j^\top = -1 \).

Assume that there exists an OA(\( 16n^2, 4n + 1, 4n, 2 \)), say \( A \), of index 1 over \( \{1, \ldots, 4n\} \). Write \( A = \sum_{i=1}^{4n} iA_i \), where the \( A_i \)'s are disjoint \( 16n^2 \times (4n + 1) \) (0,1)-matrices. We then define the \( 16n^2 \times (16n^2 - 1) \) matrix \( D \) by \( D = \sum_{i=1}^{4n} A_i \otimes r_i \) and \( \tilde{D} = [1 \quad D] \).

**Lemma 7.** (i) \( DD^\top = 16n^3 I_{16n^2} - J_{16n^2} \).

(ii) \( \tilde{D} \) is a Hadamard matrix of order \( 16n^2 \).

**Proof.** (i): By Lemma 2 and Lemma 6,

\[
DD^\top = \sum_{i,j=1}^{4n} A_i A_j^\top \otimes r_i r_j^\top \\
= \sum_{i=1}^{4n} A_i A_i^\top \otimes r_i r_i^\top + \sum_{i \neq j} A_i A_j^\top \otimes r_i r_j^\top \\
= (4n - 1) \sum_{i=1}^{4n} A_i A_i^\top - \sum_{i \neq j} A_i A_j^\top \\
= (4n - 1)J_{16n^2} + (4n - 1) \cdot 4nI_{16n^2} - 4n(J_{16n^2} - I_{16n^2}) \\
= 16n^3 I_{16n^2} - J_{16n^2}.
\]

(ii) immediately follows from (i). \( \square \)

Let \( A' \) be a submatrix of \( A \) obtained by restricting the columns to a \( 2n \) element set. Write \( A' = \sum_{i=1}^{4n} iA'_i \), where \( A'_i (i \in \{1, \ldots, 4n\}) \) are disjoint \( 16n^2 \times 2n \) (0,1)-matrices.

**Lemma 8.** There exists a symmetric (0,1)-matrix \( B \) with diagonal entries 0 such that

(i) \( \sum_{i=1}^{4n} A'_i A'_i^\top = 2nJ_{16n^2} - (2nB + (2n - 1)(J_{16n^2} - I_{16n^2} - B)) \), and

(ii) \( \sum_{i,j=1, i \neq j} A'_i A'_j^\top = 2nB + (2n - 1)(J_{16n^2} - I_{16n^2} - B) \).

**Proof.** The rows of the matrix \( A \) is a 1-distance set with Hamming distance \( 4n \) and \( A' \) is obtained from \( A \) by restricting some \( 2n \) coordinates. Therefore by Lemma 1(iii), the Hamming distances between the rows of \( A' \) are 2\( n \) or 2\( n - 1 \). Thus, the distance matrix of the code of rows of \( A' \) is \( 2nB + (2n - 1)(J_{16n^2} - I_{16n^2} - B) \) for some symmetric matrix \( (0,1) \) \( B \) with zero diagonals.

Since \( \sum_{i=1}^{4n} A'_i = J_{16n^2, 2n} \), we have

\[
\sum_{i,j=1}^{4n} A'_i A'_j^\top = \left( \sum_{i=1}^{4n} A'_i \right) \left( \sum_{j=1}^{4n} A'_j^\top \right) = J_{16n^2, 2n} J_{2n, 16n^2} = 2nJ_{16n^2}.
\]

This with (i) shows (ii). \( \square \)
Now we consider $D' = \sum_{i=1}^{4n} A'_i \otimes r_i$. Then, by Lemma 8,

$$D'D'^\top = \sum_{i,j=1}^{4n} A'_i A'_j^\top \otimes r_i r_j^\top$$

$$= \sum_{i=1}^{4n} A'_i A'_i^\top \otimes r_i r_i^\top + \sum_{i \neq j} A'_i A'_j^\top \otimes r_i r_j^\top$$

$$= (4n - 1) \sum_{i=1}^{4n} A'_i A'_i^\top - \sum_{i \neq j} A'_i A'_j^\top$$

$$= (4n - 1)(2nJ_{16n^2} - (2nB + (2n - 1)(J_{16n^2} - I_{16n^2} - B))$$

$$- (2nB + (2n - 1)(J_{16n^2} - I_{16n^2} - B))$$

$$= (8n^2 - 2n)I_{16n^2} + 2n(J_{16n^2} - I_{16n^2} - 2B).$$

Therefore the Hadamard matrix $D'$ is balancedly multi-splitable. □

The proof of $(i) \Rightarrow (ii)$. Assume that $H$ is a balancedly multi-splitable Hadamard matrix of order $16n^2$ with respect to the following block form:

$$H = \begin{bmatrix} 1 & H_1 & \cdots & H_{4n+1} \end{bmatrix},$$

where each $H_i$ is a $16n^2 \times (4n - 1)$ matrix.

**Lemma 9.** For any $i$, $H_i H_i^\top$ is a $(4n - 1, -1)$-matrix.

**Proof.** We show the case $i = 1$. Since $H$ is a Hadamard matrix of order $16n^2$, $HH^\top = 16n^2I_{16n^2}$, that is,

$$J_{16n^2} + \sum_{i=1}^{4n+1} H_i H_i^\top = 16n^2I_{16n^2}.$$

By the assumption of balanced multi-splitability, we have that all inner products of distinct rows in both $[H_2 \cdots H_{2n+1}]$ and $[H_{2n+2} \cdots H_{4n+1}]$ are $\pm 2n$. Thus,

$$\sum_{i=2}^{2n+1} H_i H_i^\top = (8n^2 - 2n)I_{16n^2} + 2nS,$$

$$\sum_{i=2n+2}^{4n+1} H_i H_i^\top = (8n^2 - 2n)I_{16n^2} + 2nS',$$

where $S$ and $S'$ are $(0, 1, -1)$-matrices with diagonal entries 0 and off-diagonal entries $\pm 1$. Thus

$$H_1 H_1^\top = 16n^2I_{16n^2} - J_{16n^2} - ((16n^2 - 4n)I_{16n^2} + 2nS + 2nS')$$

$$= 4nI_{16n^2} - J_{16n^2} - 2n(S + S').$$

Since both $S$ and $S'$ are $(0, \pm 1)$-matrix, by inspecting the equation involving $H_1 H_1^\top$ it can be seen that $S + S'$ is a $(0, \pm 2)$-matrix with diagonal entries 0. However, the off-diagonal entries of $H_1 H_1^\top$ cannot be $-4n - 1$, $S + S'$ is $(0, -2)$-matrix. Therefore, $H_1 H_1^\top$ is a $(4n - 1, -1)$-matrix. □

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For each $i$, consider the matrix $\tilde{H}_i = [1 \ H_i]$. Then, by Lemma 9, $\tilde{H}_i\tilde{H}_i^\top$ is a $(4n,0)$-matrix. Thus some rows of $H_i$ coincide. Since $\tilde{H}_i^\top\tilde{H}_i = 16n^2 I_{4n}$, the rank of $\tilde{H}_i$ is $4n$. Therefore there exist exactly $4n$ distinct rows of $\tilde{H}_i$ that correspond to the rows of a Hadamard matrix, say $\tilde{K}_i$, of order $4n$.

Write $\tilde{K}_i = [1 \ K_i]$ and fix $i$. Some rows of $H_i$ also coincide and any row of $H_i$ coincides with some row of $K_i$. In the matrix $[1 \ H_i \cdots \ H_{4n+1}]$, we then assign a symbol $j$ to any row in $H_i$, which equals the $j$-th row of $K_i$. Let $A$ be the resulting $16n^2 \times (4n+1)$ matrix over the symbol set $\{1, \ldots, 4n\}$.

**Lemma 10.** The code $C$ with codewords consisting of the rows of $A$ is an equidistance code with the number of codewords $16n^2$, equidistance $4n$, of length $4n+1$.

**Proof.** It is enough to see the case for the first row and second row. Let the first and second rows of $H_i$ be the following forms:

$$
\begin{bmatrix}
1 & r_{1,1} & \cdots & r_{1,4n+1} \\
1 & r_{2,1} & \cdots & r_{2,4n+1}
\end{bmatrix}.
$$

Consider the inner product between them:

$$1 + \sum_{i=1}^{4n+1} r_{1,i}r_{2,i}^\top = 0.
$$

By Lemma 9, $r_{1,i}r_{2,i}^\top \in \{4n-1, -1\}$ for any $i$. Then there exists $i_0$ such that $r_{1,i_0}r_{2,i_0}^\top = 4n-1$ and $r_{1,i}r_{2,i}^\top = -1$ for any $i \neq i_0$. Therefore the distance between the first row and second row is $4n$.

Since the code $C$ attains the upper bound in Lemma 3, $A$ is an orthogonal array $OA_1(16n^2, 4n+1, 4n, 2)$.

### 3.2 Proof of Theorem 5

The proof of (ii) $\Rightarrow$ (i). Assume that there exists a quaternary Hadamard matrix $H$ of order $2n$. Write $H$ as

$$H = \begin{bmatrix}
1 & r_1 \\
1 & r_2 \\
\vdots & \vdots \\
1 & r_{2n}
\end{bmatrix},$$

where $r_i$ is a $1 \times (2n-1)$ matrix for any $i$.

**Lemma 11.** (i) For any $i$, $r_ir_i^* = 2n - 1$.

(ii) For any distinct $i, j$, $r_ir_j^* = -1$.

Assume that there exists an $OA(4n^2, 2n+1, 2n, 2)$, say $A$, of index 1 over $\{1, \ldots, 2n\}$. Write $A = \sum_{i=1}^{2n} i A_i$, where the $A_i$’s are disjoint $4n^2 \times (2n+1) (0,1)$-matrices. We then define the $4n^2 \times (4n^2 - 1)$ matrix $D$ by $D = \sum_{i=1}^{2n} A_i \otimes r_i$, and $\tilde{D} = [1 \ D]$. 

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Lemma 12.  

(i) $DD^\top = 4n^2I_{4n^2} - J_{4n^2}$.

(ii) $\tilde{D}$ is a quaternary Hadamard matrix of order $4n^2$.

Proof. (i): By Lemma 2 and Lemma 11,

\begin{align*}
DD^* &= \sum_{i,j=1}^{2n} A_iA_j^\top \otimes r_ir_j^* \\
&= \sum_{i=1}^{2n} A_iA_i^\top \otimes r_ir_i^* + \sum_{i \neq j} A_iA_j^\top \otimes r_ir_j^* \\
&= (2n - 1) \sum_{i=1}^{2n} A_iA_i^\top - \sum_{i \neq j} A_iA_j^\top \\
&= (2n - 1)J_{4n^2} + (2n - 1) \cdot 2nI_{4n^2} - 2n(J_{4n^2} - I_{4n^2}) \\
&= 4n^2I_{4n^2} - J_{4n^2}.
\end{align*}

(ii) immediately follows from (i). \hfill \Box

Let $A'$ be a submatrix of $A$ obtained by restricting the columns to an $n$ element set. Write $A' = \sum_{i=1}^{2n} iA_i'$, where $A_i'$ ($i \in \{1, \ldots, 2n\}$) are disjoint $4n^2 \times n (0,1)$-matrices.

Lemma 13. There exists a symmetric $(0,1)$-matrix $B$ with diagonal entries 0 such that

(i) $\sum_{i=1}^{2n} A_i'A_i'^\top = nJ_{4n^2} - (nB + (n - 1)(J_{4n^2} - I_{4n^2} - B))$, and

(ii) $\sum_{i,j=1, i \neq j} A_i'A_j'^\top = nB + (n - 1)(J_{4n^2} - I_{4n^2} - B)$.

Proof. The rows of the matrix $A$ is a 1-distance set with Hamming distance $2n$ and $A'$ is obtained from $A$ by restricting some $n$ coordinates. Therefore by Lemma 1(iii), the Hamming distances between the rows of $A'$ are $n$ or $n - 1$. Thus, the distance matrix of the code of rows of $A'$ is $nB + (n - 1)(J_{4n^2} - I_{4n^2} - B)$ for some symmetric matrix $(0,1)$ $B$ with zero diagonals.

Since $\sum_{i=1}^{2n} A_i' = J_{4n^2,n}$, we have

$$\sum_{i,j=1}^{2n} A_i'A_j'^\top = \sum_{i=1}^{2n} A_i'\left(\sum_{j=1}^{2n} A_j'^\top\right) = J_{4n^2,n}J_{n,4n^2} = nJ_{4n^2}.$$ 

This with (i) shows (ii). \hfill \Box

Now we consider $D' = \sum_{i=1}^{2n} A_i' \otimes r_i$. Then, by Lemma 13,

\begin{align*}
D'D^* &= \sum_{i,j=1}^{2n} A_i'A_j'^\top \otimes r_ir_j^* \\
&= \sum_{i=1}^{2n} A_i'A_i'^\top \otimes r_ir_i^* + \sum_{i \neq j} A_i'A_j'^\top \otimes r_ir_j^*
\end{align*}
where $S ± K$ coincides with some row of $H$. By the assumption of balanced multi-splittability, we have that the inner product of $HH^*$ is $4n^2$, that is, $H^*H = 4n^2I_{4n^2}$. Therefore the quaternary Hadamard matrix $D'$ is balancedly multi-splittable.  

The proof of (i) $\implies$ (ii). Assume that $H$ is a balancedly multi-splittable quaternary Hadamard matrix of order $4n^2$ with respect to the following block form:

$$H = \begin{bmatrix} 1 & H_1 & \cdots & H_{2n+1} \end{bmatrix},$$

where each $H_i$ is a $4n^2 \times (2n-1)$ matrix.

**Lemma 14.** For any $i$, $H_iH_i^*$ is a $(2n-1, -1)$-matrix.

**Proof.** We show the case $i = 1$. Since $H$ is a quaternary Hadamard matrix of order $4n^2$, $HH^* = 4n^2I_{4n^2}$, that is,

$$J_{4n^2} + \sum_{i=1}^{2n+1} H_iH_i^* = 4n^2I_{4n^2}.
$$

By the assumption of balanced multi-splittability, we have that the inner product of distinct rows of matrices $[H_2 \cdots H_{n+1}]$ or $[H_{n+2} \cdots H_{2n+1}]$ are $\pm 2n, \pm 2i$. Thus,

$$\sum_{i=2}^{n+1} H_iH_i^* = (2n^2 - n)I_{4n^2} + nS, \quad \sum_{i=n+2}^{2n+1} H_iH_i^* = (2n^2 - n)I_{4n^2} + nS',$

where $S$ and $S'$ are $(0, \pm 1, \pm i)$-matrix with diagonal entries 0 and off-diagonal entries $\pm 1, \pm i$. Then

$$H_1H_1^* = 4n^2I_{4n^2} - J_{4n^2} - ((4n^2 - 2n)I_{2n} + nS + nS')$$

$$= 2nI_{4n^2} - J_{4n^2} - n(S + S').$$

Since both $S$ and $S'$ are $(0, \pm 1, \pm i)$-matrix, by inspecting the equation involving $H_1H_1^*$ it can be seen that $S + S'$ is a $(0, \pm 2, \pm 2i)$-matrix with diagonal entries 0. However, the absolute values of off-diagonal entries of $H_1H_1^*$ cannot exceed $2n - 1$, $S + S'$ is $(0, -2)$-matrix. Therefore, $H_1H_1^*$ is a $(2n-1, -1)$-matrix.  

For each $i$, consider the matrix $\tilde{H}_i = [1 \ H_i]$. Then, by Lemma 14, $\tilde{H}_i\tilde{H}_i^*$ is a $(2n, 0)$-matrix. Thus some of rows of $\tilde{H}_i$ coincide. Since $\tilde{H}_i^*\tilde{H}_i = 4n^2I_{2n}$, the rank of $\tilde{H}_i$ is $2n$. Therefore there exist exactly $2n$ distinct rows of $\tilde{H}_i$ that correspond to the rows of a Hadamard matrix, say $K_i$, of order $2n$.

Write $\tilde{K}_i = [1 \ K_i]$ and fix $i$. Some rows of $H_i$ also coincide and any row of $H_i$ coincides with some row of $K_i$. In the matrix $[H_1 \cdots H_{2n+1}]$, we then assign a symbol $j$ to any row in $H_i$, which equals the $j$-th row of $K_i$. Let $A$ be the resulting $4n^2 \times (2n + 1)$ matrix over the symbol set $\{1, \ldots, 2n\}$. 


Lemma 15. The code \( C \) with codewords consisting of the rows of \( A \) is an equidistance code with the number of codewords \( 4n^2 \), equidistance \( 2n \), of length \( 2n + 1 \).

Proof. It is enough to see the case for the first row and second row. Let the first and second rows of \( H \) be the following forms:

\[
\begin{bmatrix}
1 & r_{1,1} & \cdots & r_{1,2n+1} \\
1 & r_{2,1} & \cdots & r_{2,2n+1}
\end{bmatrix}.
\]

Consider the inner product between them:

\[
1 + \sum_{i=1}^{2n+1} r_{1,i}r_{2,i}^* = 0.
\]

By Lemma 14, \( r_{1,i}r_{2,i}^* \in \{2n - 1, -1\} \) for any \( i \). Then there exists \( i_0 \) such that \( r_{1,i_0}r_{2,i_0}^* = 2n - 1 \) and \( r_{1,i}r_{2,i}^* = -1 \) for any \( i \neq i_0 \). Therefore the distance between the first row and second row is \( 2n \).

Since the code \( C \) attains the upper bound in Lemma 3, \( A \) is an orthogonal array \( OA_{1}(4n^2, 2n + 1, 2n, 2) \).

4 Example

In this section, we present an example of balancedly multi-splittable Hadamard matrices following the construction in Theorem 4.

Example 16. Take an \( OA_{1}(16, 5, 4, 2) \) \( A \) and a Hadamard matrix \( H \) of order 4 as:

\[
A^\top = \sum_{i=1}^{4} A_i^\top = \begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 \\
1 & 2 & 3 & 4 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 & 2 & 1 & 4 & 3 \\
1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2
\end{bmatrix},
\]

\[
H = \begin{bmatrix}
1 & r_1 \\
1 & r_2 \\
1 & r_3 \\
1 & r_4
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}.
\]

Then the matrix \( D \) constructed in Theorem 4 is a balancedly multi-splittable Hadamard
matrix of order 16:

\[ D = \sum_{i=1}^{4} iA_i \otimes r_i \]

\[ = \begin{bmatrix} 1 & H_1 & H_2 & H_3 & H_4 & H_5 \end{bmatrix} \]

\[ = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\end{bmatrix} \]

Conversely, we demonstrate how an orthogonal array and a Hadamard matrix can be constructed from a balancedly multi-splittable Hadamard matrix \( D \). Consider the 16 \times 4 submatrix \( \tilde{H}_1 = \begin{bmatrix} 1 & H_1 \end{bmatrix} \) of \( D \). Then, there are exactly four distinct rows

\((1, 1, 1, 1), (1, -1, 1, -1), (1, 1, -1, -1), (1, -1, -1, 1)\)

in \( \tilde{H}_1 \). These form a Hadamard matrix of order 4 and set

\[ \tilde{K}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 \\
\end{bmatrix} = \begin{bmatrix} 1 & r_{1,1} \\
1 & r_{1,2} \\
1 & r_{1,3} \\
1 & r_{1,4} \\
\end{bmatrix}. \]

Similarly we define

\[ \tilde{K}_i = \begin{bmatrix} 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 \\
\end{bmatrix} = \begin{bmatrix} 1 & r_{i,1} \\
1 & r_{i,2} \\
1 & r_{i,3} \\
1 & r_{i,4} \\
\end{bmatrix}. \]

for \( i \in \{2, 3, 4, 5\} \). In the matrix \( [H_1 \cdots H_5] \), we assign a symbol \( j \in \{1, 2, 3, 4\} \) to any row in \( H_i \), which equals the \( j \)-th row of \( K_i \). The resulting matrix \( A \) is reconstructed as aforementioned.

**Remark 17.** There exist no balancedly multi-splittable quaternary Hadamard matrices of orders 36 and 100.
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References


