Hadamard Matrices Related to Projective Planes

Hadi Kharaghani*

Department of Mathematics and
Computer Science
University of Lethbridge
Lethbridge, Alberta, T1K 3M4, Canada

kharaghani@uleth.ca

Sho Suda†

Department of Mathematics
National Defense Academy of Japan
Yokosuka, Kanagawa 239-8686, Japan

ssuda@nda.ac.jp

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Abstract

Let n be the order of a quaternary Hadamard matrix. It is shown that the existence of a projective plane of order n is equivalent to the existence of a balancedly multi-splittable quaternary Hadamard matrix of order n^2.

Mathematics Subject Classifications: 05B20, 05B25, 05B15

1 Introduction

K. A. Bush [2] was the first to establish a link between projective planes of even order and specific Hadamard matrices, that was later labeled as *Bush-type* in 1971. H. J. Ryser [11] found the same connection as an application of factors of design matrix in 1977. Eric Verheiden [12] provided a direct construction for the matrices using the incidence matrices of the corresponding projective planes.

Frans C. Bussemaker, Willem Haemers and Ted Spence [3] used an exhaustive search and found no strongly regular graph with parameters (36,15,6,6) and chromatic number six or, equivalently, there is no symmetric Bush-type Hadamard matrix of order 36. Many Bush-type Hadamard matrices of order 100 are constructed, but none is known to be symmetric. The proof of the nonexistence of a symmetric Bush-type Hadamard matrix of order 100 would be exciting and is an alternative to the proof of the nonexistence of projective plane of order 10, however, there has been no attempt at showing it so far. The nonexistence of the projective plane of order 10 was finally established by a long

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computational method by C. W. H. Lam et al. in [9, 10], and an alternate approach is still highly desirable.

The connection between projective planes and Hadamard matrices shown in [2, 11, 12] are all one-sided results in which from a projective plane of even order symmetric Bush-type Hadamard matrices are constructed.

Balancedly splittable Hadamard matrices were introduced by the authors in 2018 in [8], and the results were widely expanded in a recent paper by Jonathan Jedwab et al. in [5]. It is known [7] that the existence of a Hadamard matrix of order $4n$ would lead to a balancedly splittable Hadamard matrix of order $64n^2$. However, there is no balancedly splittable Hadamard matrix of order $4n^2$, n odd, see [8]. The case of Hadamard matrices of order $16n^2$, $n > 1$ odd, remains open, and no balancedly splittable Hadamard matrix of order 144 is known.

Concentrating on the order 144, the authors were led to some exotic classes of balancedly splittable Hadamard matrices, which is dubbed as *balancedly multi-splittable Hadamard matrices*. There is a balancedly multi-splittable Hadamard matrix of order 4^m for every positive integer m, and it seems that these are probably the only Hadamard matrices with this property.

It will be shown in this paper that the existence of a projective plane of order $4n$ is equivalent to the existence of a balancedly multi-splittable Hadamard matrix of order $16n^2$ provided that $4n$ is the order of a Hadamard matrix. In doing so we use the fact that the existence of projective planes are equivalent to the existence of orthogonal arrays, see [1, Theorems 3.18 and 3.20], and the latter is equivalent to the balancedly multi-splittable Hadamard matrices.

There is also a similar equivalence between the projective plane of order $2n$, n odd, and balancedly multi-splittable quaternary Hadamard matrices will be presented too.

The establishment of the nonexistence of a balancedly multi-splittable (quaternary) Hadamard matrix of order 144 (100) would be significant.

2 Preliminaries

2.1 Codes

Let n, q be positive integers $n, q \geqslant 2$, and let $Q = \{0, 1, \ldots, q-1\}$. A subset C of Q^n is called to be a *q-ary code of length n*. For $x, y \in Q^n$ with $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$, the *Hamming distance* between codewords x and y is given by $\operatorname{dist}(x, y) = |\{i : x_i \neq y_i\}|$. A code C is said to be an *equi-distance code* or a *1-distance set* if the Hamming distance $d(x, y)$ does not depend on $x, y \in C$ with $x \neq y$.

2.2 Hadamard matrices

An $n \times n$ matrix H is a *Hadamard matrix of order n* if its entries are $1, -1$ and it satisfies $HH^\top = I_n$, where I_n denotes the identity matrix of order n. A Hadamard matrix H of order n is said to be *balancedly splittable* if there is an $\ell \times n$ submatrix H_1 of H such that inner products for any two distinct column vectors of H_1 take at most two values.

More precisely, there exist integers a, b and the adjacency matrix A of a graph such that $H_1^\top H_1 = \ell I_n + aA + b(J_n - A - I_n)$, where J_n denotes the all-ones matrix of order n. We say the quadruple (v, ℓ, a, b) the parameter. In this case we say that H is balancedly splittable with respect H_1. Only the special case of $(v, \ell, a, b) = (4n^2, 2n^2, n, -n)$ will be used in this note.

The same concept can be extended to orthogonal designs [7]. Here, we adopt the following definition for quaternary Hadamard matrices. An $n \times n$ matrix H is a *quaternary Hadamard matrix of order* n if its entries are $\pm 1, \pm i$ and it satisfies $HH^* = nI_n$. A quaternary Hadamard matrix H of order n is said to be *balancedly splittable* if there is an $\ell \times n$ submatrix H_1 of H such that the off-diagonal entries of $H_1^* H_1$ are in the set

$$\{\varepsilon\alpha, \varepsilon\alpha^*, \varepsilon\beta, \varepsilon\beta^* \mid \varepsilon \in \{\pm 1, \pm i\}\},$$

where α, β are some complex numbers. In this paper, we restrict to the case $\alpha = \beta$ and we say that a quaternary Hadamard matrix H of order n is balancedly splittable if $H_1^* H_1 = \ell I + \alpha S$ where α is some positive real number and S is a $(0, \pm 1, \pm i)$-matrix with zero diagonal entries and nonzero off-diagonal entries.

2.3 Orthogonal arrays

An *orthogonal array* of strength t and index λ is an $N \times k$ matrix over the set $\{1, \ldots, q\}$ such that in every $N \times t$ subarray, each t-tuple in $\{1, \ldots, q\}^t$ appears λ times. We denote this property as $\mathrm{OA}_\lambda(N, k, q, t)$. Note that $N = \lambda q^t$ and (N, k, q, t) is the parameter of the orthogonal array. For $t = 2e$, the following lower bound on N was shown by Rao (see [6, Theorem 2.1]), namely, $N \geqslant \sum_{i=0}^{e} \binom{k}{i}(q-1)^i$. An orthogonal array with parameters $(N, k, q, 2e)$ is said to be complete if the equality holds in above.

When $t = 2$ and $\lambda = 1$, the complete orthogonal array has the parameters $\mathrm{OA}_1(q^2, q+1, q, 2)$, and it is known that its existence is equivalent to that of a projective plane of order q. For the orthogonal version of a projective plane is used in the next section.

The following lemmas will be used later.

Lemma 1. *Let A be an $N \times k$ matrix over $\{1, \ldots, q\}$. Write $A = \sum_{i=1}^{q} iA_i$, where A_i ($i \in \{1, \ldots, q\}$) are disjoint $N \times k$ $(0,1)$-matrices. Let D be the distance matrix, ie., D is an $N \times N$ matrix whose rows and columns indexed by the rows of A with (i,j)-entry defined by the Hamming distance between the i-th row and the j-th row of A. Then $\sum_{i=1}^{q} A_i A_i^\top = kJ_N - D$ holds.*

Proof. See the proof of [6, Lemma 2.5 (i)]. □

Lemma 2. *Assume that there exists an orthogonal array A with parameters $(q^2, q+1, q, 2)$. Write $A = \sum_{i=1}^{q} iA_i$, where A_i ($i \in \{1, \ldots, q\}$) are disjoint $q^2 \times (q+1)$ $(0,1)$-matrices. Then the matrices A_i satisfy*

(i) $\sum_{i=1}^{q} A_i A_i^\top = J_{q^2} + qI_{q^2}$,

(ii) $\sum_{i,j=1, i \neq j}^{q} A_i A_j^\top = q(J_{q^2} - I_{q^2})$.

(iii) Consider the code C obtained from the rows of A. Let $\{i_1,\dots,i_s\}$ be any s-element subset of $\{1,\dots,q+1\}$. The code C' obtained from C by restricting the coordinates on the set $\{i_1,\dots,i_s\}$ have the Hamming distances s or $s-1$ between the codewords in C'.

Proof. The proof for (i) and (ii) are exactly the same as [6, Lemma 2.5].

The assumed orthogonal array is a 2-design and 1-distance set with Hamming distance q in the Hamming association scheme. The case (iii) follows from the fact that C is a 1-distance set with Hamming distance q. □

Lemma 3. [4, Theorem 5.14] *Let C be an equidistance code of length $q+1$ over the symbol set $\{1,\dots,q\}$. Then*

$$|C| \leqslant q^2$$

holds. Equality holds if and only if the matrix whose rows consists of the codewords of C is an orthogonal array $OA_1(q^2,q+1,q,2)$.

3 Balancedly multi-splittable Hadamard matrices

We consider the following property of a Hadamard matrix. Let H be a Hadamard matrix of order $4n^2$. Assume that H is normalized so that the first column of H is the all-ones vector. A Hadamard matrix H is said to be *balancedly multi-splittable* if there is a block form of $H = \begin{bmatrix}\mathbf{1} & H_1 & \cdots & H_{2n+1}\end{bmatrix}$, where each H_i is of order $4n^2 \times (2n-1)$ such that H is balancedly splittable with respect to a submatrix $\begin{bmatrix}H_{i_1} & \cdots & H_{i_n}\end{bmatrix}$ for any n-element subset $\{i_1,\dots,i_n\}$ of $\{1,2,\dots,2n+1\}$, that is, the inner product of any distinct rows of $\begin{bmatrix}H_{i_1} & \cdots & H_{i_n}\end{bmatrix}$ is $\pm n$.

The main results of this paper are as follows:

Theorem 4. *Let n be a positive integer. The following are equivalent.*

(i) There exists a balancedly multi-splittable Hadamard matrix of order $16n^2$.

(ii) There exist an $OA_1(16n^2,4n+1,4n,2)$ and a Hadamard matrix of order $4n$.

Theorem 5. *Let n be a positive integer. The following are equivalent.*

(i) There exists a balancedly multi-splittable quaternary Hadamard matrix of order $4n^2$

(ii) There exist an $OA_1(4n^2,2n+1,2n,2)$ and a quaternary Hadamard matrix of order $2n$.

3.1 Proof of Theorem 4

The proof of (ii) ⇒ (i). Assume that there exists a Hadamard matrix H of order $4n$. Write H as

$$H = \begin{bmatrix} 1 & r_1 \\ 1 & r_2 \\ \vdots & \vdots \\ 1 & r_{4n} \end{bmatrix},$$

where r_i is a $1 \times (4n-1)$ matrix for any i.

Lemma 6. *(i) For any i, $r_i r_i^\top = 4n-1$.*

(ii) For any distinct i, j, $r_i r_j^\top = -1$.

Assume that there exists an $\mathrm{OA}(16n^2, 4n+1, 4n, 2)$, say A, of index 1 over $\{1, \ldots, 4n\}$. Write $A = \sum_{i=1}^{4n} iA_i$, where the A_i's are disjoint $16n^2 \times (4n+1)$ $(0,1)$-matrices. We then define the $16n^2 \times (16n^2-1)$ matrix D by $D = \sum_{i=1}^{4n} A_i \otimes r_i$ and $\tilde{D} = \begin{bmatrix} \mathbf{1} & D \end{bmatrix}$.

Lemma 7. *(i) $DD^\top = 16n^2 I_{16n^2} - J_{16n^2}$.*

(ii) $\tilde{D}$ is a Hadamard matrix of order $16n^2$.

Proof. (i): By Lemma 2 and Lemma 6,

$$\begin{aligned} DD^\top &= \sum_{i,j=1}^{4n} A_i A_j^\top \otimes r_i r_j^\top \\ &= \sum_{i=1}^{4n} A_i A_i^\top \otimes r_i r_i^\top + \sum_{i \neq j} A_i A_j^\top \otimes r_i r_j^\top \\ &= (4n-1) \sum_{i=1}^{4n} A_i A_i^\top - \sum_{i \neq j} A_i A_j^\top \\ &= (4n-1) J_{16n^2} + (4n-1) \cdot 4n I_{16n^2} - 4n(J_{16n^2} - I_{16n^2}) \\ &= 16n^2 I_{16n^2} - J_{16n^2}. \end{aligned}$$

(ii) immediately follows from (i). □

Let A' be a submatrix of A obtained by restricting the columns to a $2n$ element set. Write $A' = \sum_{i=1}^{4n} iA'_i$, where A'_i ($i \in \{1, \ldots, 4n\}$) are disjoint $16n^2 \times 2n$ $(0,1)$-matrices.

Lemma 8. *There exists a symmetric $(0,1)$-matrix B with diagonal entries 0 such that*

(i) $\sum_{i=1}^{4n} {A'}_i {A'}_i^\top = 2nJ_{16n^2} - \big(2nB + (2n-1)(J_{16n^2} - I_{16n^2} - B)\big)$, and

(ii) $\sum_{i,j=1, i\neq j}^{4n} {A'}_i {A'}_j^\top = 2nB + (2n-1)(J_{16n^2} - I_{16n^2} - B)$.

Proof. The rows of the matrix A is a 1-distance set with Hamming distance $4n$ and A' is obtained from A by restricting some $2n$ coordinates. Therefore by Lemma 1(iii), the Hamming distances between the rows of A' are $2n$ or $2n-1$. Thus, the distance matrix of the code of rows of A' is $2nB + (2n-1)(J_{16n^2} - I_{16n^2} - B)$ for some symmetric matrix $(0,1)$ B with zero diagonals.

Since $\sum_{i=1}^{4n} {A'}_i = J_{16n^2,2n}$, we have

$$\sum_{i,j=1}^{4n} {A'}_i {A'}_j^\top = (\sum_{i=1}^{4n} {A'}_i)(\sum_{j=1}^{4n} {A'}_j^\top) = J_{16n^2,2n} J_{2n,16n^2} = 2nJ_{16n^2}.$$

This with (i) shows (ii). □

Now we consider $D' = \sum_{i=1}^{4n} A'_i \otimes r_i$. Then, by Lemma 8,

$$\begin{aligned}
D'D'^\top &= \sum_{i,j=1}^{4n} A'_i A'^\top_j \otimes r_i r_j^\top \\
&= \sum_{i=1}^{4n} A'_i A'^\top_i \otimes r_i r_i^\top + \sum_{i \neq j} A'_i A'^\top_j \otimes r_i r_j^\top \\
&= (4n-1) \sum_{i=1}^{4n} A'_i A'^\top_i - \sum_{i \neq j} A'_i A'^\top_j \\
&= (4n-1)(2nJ_{16n^2} - (2nB + (2n-1)(J_{16n^2} - I_{16n^2} - B)) \\
&\quad - (2nB + (2n-1)(J_{16n^2} - I_{16n^2} - B)) \\
&= (8n^2 - 2n) I_{16n^2} + 2n(J_{16n^2} - I_{16n^2} - 2B).
\end{aligned}$$

Therefore the Hadamard matrix D' is balancedly multi-splittable. □

The proof of (i) ⇒ (ii). Assume that H is a balancedly multi-splittable Hadamard matrix of order $16n^2$ with respect to the following block form:

$$H = \begin{bmatrix} \mathbf{1} & H_1 & \cdots & H_{4n+1} \end{bmatrix},$$

where each H_i is a $16n^2 \times (4n-1)$ matrix.

Lemma 9. *For any i, $H_i H_i^\top$ is a $(4n-1, -1)$-matrix.*

Proof. We show the case $i = 1$. Since H is a Hadamard matrix of order $16n^2$, $HH^\top = 16n^2 I_{16n^2}$, that is,

$$J_{16n^2} + \sum_{i=1}^{4n+1} H_i H_i^\top = 16n^2 I_{16n^2}.$$

By the assumption of balanced multi-splittability, we have that all inner products of distinct rows in both $\begin{bmatrix} H_2 & \cdots & H_{2n+1} \end{bmatrix}$ and $\begin{bmatrix} H_{2n+2} & \cdots & H_{4n+1} \end{bmatrix}$ are $\pm 2n$. Thus,

$$\sum_{i=2}^{2n+1} H_i H_i^\top = (8n^2 - 2n) I_{16n^2} + 2nS, \qquad \sum_{i=2n+2}^{4n+1} H_i H_i^\top = (8n^2 - 2n) I_{16n^2} + 2nS',$$

where S and S' are $(0, 1, -1)$-matrices with diagonal entries 0 and off-diagonal entries ± 1. Then

$$\begin{aligned}
H_1 H_1^\top &= 16n^2 I_{16n^2} - J_{16n^2} - ((16n^2 - 4n) I_{16n^2} + 2nS + 2nS') \\
&= 4n I_{16n^2} - J_{16n^2} - 2n(S + S').
\end{aligned}$$

Since both S and S' are $(0, \pm 1)$-matrix, by inspecting the equation involving $H_1 H_1^\top$ it can be seen that $S + S'$ is a $(0, \pm 2)$-matrix with diagonal entries 0. However, the off-diagonal entries of $H_1 H_1^\top$ cannot be $-4n-1$, $S + S'$ is $(0, -2)$-matrix. Therefore, $H_1 H_1^\top$ is a $(4n-1, -1)$-matrix. □

For each i, consider the matrix $\tilde{H}_i = \begin{bmatrix} \mathbf{1} & H_i \end{bmatrix}$. Then, by Lemma 9, $\tilde{H}_i\tilde{H}_i^\top$ is a $(4n, 0)$-matrix. Thus some of rows of $\tilde{H}_i$ coincide. Since $\tilde{H}_i^\top\tilde{H}_i = 16n^2 I_{4n}$, the rank of $\tilde{H}_i$ is $4n$. Therefore there exist exactly $4n$ distinct rows of $\tilde{H}_i$ that correspond to the rows of a Hadamard matrix, say $\tilde{K}_i$, of order $4n$.

Write $\tilde{K}_i = \begin{bmatrix} \mathbf{1} & K_i \end{bmatrix}$ and fix i. Some rows of H_i also coincide and any row of H_i coincides with some row of K_i. In the matrix $\begin{bmatrix} \mathbf{1} & H_1 & \cdots & H_{4n+1} \end{bmatrix}$, we then assign a symbol j to any row in H_i, which equals the j-th row of K_i. Let A be the resulting $16n^2 \times (4n+1)$ matrix over the symbol set $\{1, \ldots, 4n\}$.

Lemma 10. *The code C with codewords consisting of the rows of A is an equidistance code with the number of codewords $16n^2$, equidistance $4n$, of length $4n+1$.*

Proof. It is enough to see the case for the first row and second row. Let the first and second rows of H be the following forms:

$$\begin{bmatrix} 1 & r_{1,1} & \cdots & r_{1,4n+1} \end{bmatrix},$$
$$\begin{bmatrix} 1 & r_{2,1} & \cdots & r_{2,4n+1} \end{bmatrix}.$$

Consider the inner product between them:

$$1 + \sum_{i=1}^{4n+1} r_{1,i} r_{2,i}^\top = 0.$$

By Lemma 9, $r_{1,i} r_{2,i}^\top \in \{4n-1, -1\}$ for any i. Then there exists i_0 such that $r_{1,i_0} r_{2,i_0}^\top = 4n-1$ and $r_{1,i} r_{2,i}^\top = -1$ for any $i \neq i_0$. Therefore the distance between the first row and second row is $4n$. □

Since the code C attains the upper bound in Lemma 3, A is an orthogonal array $\mathrm{OA}_1(16n^2, 4n+1, 4n, 2)$. □

3.2 Proof of Theorem 5

The proof of (ii) ⇒ (i). Assume that there exists a quaternary Hadamard matrix H of order $2n$. Write H as

$$H = \begin{bmatrix} 1 & r_1 \\ 1 & r_2 \\ \vdots & \vdots \\ 1 & r_{2n} \end{bmatrix},$$

where r_i is a $1 \times (2n-1)$ matrix for any i.

Lemma 11. *(i) For any i, $r_i r_i^* = 2n-1$.*

(ii) For any distinct i, j, $r_i r_j^* = -1$.

Assume that there exists an $\mathrm{OA}(4n^2, 2n+1, 2n, 2)$, say A, of index 1 over $\{1, \ldots, 2n\}$. Write $A = \sum_{i=1}^{2n} iA_i$, where the A_i's are disjoint $4n^2 \times (2n+1)$ $(0,1)$-matrices. We then define the $4n^2 \times (4n^2-1)$ matrix D by $D = \sum_{i=1}^{2n} A_i \otimes r_i$ and $\tilde{D} = \begin{bmatrix} \mathbf{1} & D \end{bmatrix}$.

Lemma 12. *(i)* $DD^\top = 4n^2 I_{4n^2} - J_{4n^2}$.

(ii) $\tilde{D}$ *is a quaternary Hadamard matrix of order* $4n^2$.

Proof. (i): By Lemma 2 and Lemma 11,

$$\begin{aligned}
DD^* &= \sum_{i,j=1}^{2n} A_i A_j^\top \otimes r_i r_j^* \\
&= \sum_{i=1}^{2n} A_i A_i^\top \otimes r_i r_i^* + \sum_{i\neq j} A_i A_j^\top \otimes r_i r_j^* \\
&= (2n-1)\sum_{i=1}^{2n} A_i A_i^\top - \sum_{i\neq j} A_i A_j^\top \\
&= (2n-1)J_{4n^2} + (2n-1)\cdot 2nI_{4n^2} - 2n(J_{4n^2} - I_{4n^2}) \\
&= 4n^2 I_{4n^2} - J_{4n^2}.
\end{aligned}$$

(ii) immediately follows from (i). □

Let A' be a submatrix of A obtained by restricting the columns to an n element set. Write $A' = \sum_{i=1}^{2n} iA'_i$, where A'_i ($i \in \{1,\dots,2n\}$) are disjoint $4n^2 \times n$ $(0,1)$-matrices.

Lemma 13. *There exists a symmetric* $(0,1)$*-matrix* B *with diagonal entries* 0 *such that*

(i) $\sum_{i=1}^{2n} {A'}_i {A'}_i^\top = nJ_{4n^2} - \big(nB + (n-1)(J_{4n^2} - I_{4n^2} - B)\big)$*, and*

(ii) $\sum_{i,j=1, i\neq j}^{2n} {A'}_i {A'}_j^\top = nB + (n-1)(J_{4n^2} - I_{4n^2} - B)$.

Proof. The rows of the matrix A is a 1-distance set with Hamming distance $2n$ and A' is obtained from A by restricting some n coordinates. Therefore by Lemma 1(iii), the Hamming distances between the rows of A' are n or $n-1$. Thus, the distance matrix of the code of rows of A' is $nB + (n-1)(J_{4n^2} - I_{4n^2} - B)$ for some symmetric matrix $(0,1)$ B with zero diagonals.

Since $\sum_{i=1}^{2n} {A'}_i = J_{4n^2,n}$, we have

$$\sum_{i,j=1}^{2n} {A'}_i {A'}_j^\top = (\sum_{i=1}^{2n} {A'}_i)(\sum_{j=1}^{2n} {A'}_j^\top) = J_{4n^2,n} J_{n,4n^2} = nJ_{4n^2}.$$

This with (i) shows (ii). □

Now we consider $D' = \sum_{i=1}^{2n} {A'}_i \otimes r_i$. Then, by Lemma 13,

$$\begin{aligned}
D'D'^* &= \sum_{i,j=1}^{2n} {A'}_i {A'}_j^\top \otimes r_i r_j^* \\
&= \sum_{i=1}^{2n} {A'}_i {A'}_i^\top \otimes r_i r_i^* + \sum_{i\neq j} {A'}_i {A'}_j^\top \otimes r_i r_j^*
\end{aligned}$$

$$
\begin{aligned}
&= (2n-1)\sum_{i=1}^{2n} A'_i A'^{\top}_i - \sum_{i\neq j} A'_i A'^{\top}_j \\
&= (2n-1)(nJ_{4n^2} - (nB + (n-1)(J_{4n^2} - I_{4n^2} - B)) \\
&\quad - (nB + (n-1)(J_{4n^2} - I_{4n^2} - B)) \\
&= (2n^2 - n)I_{4n^2} + n(J_{4n^2} - I_{4n^2} - 2B).
\end{aligned}
$$

Therefore the quaternary Hadamard matrix D' is balancedly multi-splittable. □

The proof of (i) ⇒ (ii). Assume that H is a balancedly multi-splittable quaternary Hadamard matrix of order $4n^2$ with respect to the following block form:

$$
H = \begin{bmatrix} \mathbf{1} & H_1 & \cdots & H_{2n+1} \end{bmatrix},
$$

where each H_i is a $4n^2 \times (2n-1)$ matrix.

Lemma 14. *For any i, $H_i H_i^*$ is a $(2n-1,-1)$-matrix.*

Proof. We show the case $i = 1$. Since H is a quaternary Hadamard matrix of order $4n^2$, $HH^* = 4n^2 I_{4n^2}$, that is,

$$
J_{4n^2} + \sum_{i=1}^{2n+1} H_i H_i^* = 4n^2 I_{4n^2}.
$$

By the assumption of balanced multi-splittability, we have that the inner product of distinct rows of matrices $\begin{bmatrix} H_2 & \cdots & H_{n+1} \end{bmatrix}$ or $\begin{bmatrix} H_{n+2} & \cdots & H_{2n+1} \end{bmatrix}$ are $\pm 2n, \pm 2i$. Thus,

$$
\sum_{i=2}^{n+1} H_i H_i^* = (2n^2 - n)I_{4n^2} + nS, \quad \sum_{i=n+2}^{2n+1} H_i H_i^* = (2n^2 - n)I_{4n^2} + nS',
$$

where S and S' are $(0, \pm 1, \pm i)$-matrix with diagonal entries 0 and off-diagonal entries $\pm 1, \pm i$. Then

$$
\begin{aligned}
H_1 H_1^* &= 4n^2 I_{4n^2} - J_{4n^2} - ((4n^2 - 2n)I_{100} + nS + nS') \\
&= 2nI_{4n^2} - J_{4n^2} - n(S + S').
\end{aligned}
$$

Since both S and S' are $(0, \pm 1, \pm i)$-matrix, by inspecting the equation involving $H_1 H_1^*$ it can be seen that $S + S'$ is a $(0, \pm 2, \pm 2i)$-matrix with diagonal entries 0. However, the absolute values of off-diagonal entries of $H_1 H_1^*$ cannot exceed $2n-1$, $S + S'$ is $(0,-2)$-matrix. Therefore, $H_1 H_1^*$ is a $(2n-1,-1)$-matrix. □

For each i, consider the matrix $\tilde{H}_i = \begin{bmatrix} \mathbf{1} & H_i \end{bmatrix}$. Then, by Lemma 14, $\tilde{H}_i \tilde{H}_i^*$ is a $(2n, 0)$-matrix. Thus some of rows of $\tilde{H}_i$ coincide. Since $\tilde{H}_i^* \tilde{H}_i = 4n^2 I_{2n}$, the rank of $\tilde{H}_i$ is $2n$. Therefore there exist exactly $2n$ distinct rows of $\tilde{H}_i$ that correspond to the rows of a Hadamard matrix, say $\tilde{K}_i$, of order $2n$.

Write $\tilde{K}_i = \begin{bmatrix} \mathbf{1} & K_i \end{bmatrix}$ and fix i. Some rows of H_i also coincide and any row of H_i coincides with some row of K_i. In the matrix $\begin{bmatrix} H_1 & \cdots & H_{2n+1} \end{bmatrix}$, we then assign a symbol j to any row in H_i, which equals the j-th row of K_i. Let A be the resulting $4n^2 \times (2n+1)$ matrix over the symbol set $\{1, \ldots, 2n\}$.

Lemma 15. *The code C with codewords consisting of the rows of A is an equidistance code with the number of codewords $4n^2$, equidistance $2n$, of length $2n+1$.*

Proof. It is enough to see the case for the first row and second row. Let the first and second rows of H be the following forms:

$$\begin{bmatrix} 1 & r_{1,1} & \cdots & r_{1,2n+1} \end{bmatrix},$$
$$\begin{bmatrix} 1 & r_{2,1} & \cdots & r_{2,2n+1} \end{bmatrix}.$$

Consider the inner product between them:

$$1 + \sum_{i=1}^{2n+1} r_{1,i} r_{2,i}^* = 0.$$

By Lemma 14, $r_{1,i} r_{2,i}^* \in \{2n-1, -1\}$ for any i. Then there exists i_0 such that $r_{1,i_0} r_{2,i_0}^* = 2n-1$ and $r_{1,i} r_{2,i}^* = -1$ for any $i \neq i_0$. Therefore the distance between the first row and second row is $2n$. □

Since the code C attains the upper bound in Lemma 3, A is an orthogonal array $\mathrm{OA}_1(4n^2, 2n+1, 2n, 2)$. □

4 Example

In this section, we present an example of balancedly multi-splittable Hadamard matrices following the construction in Theorem 4.

Example 16. Take an $\mathrm{OA}_1(16, 5, 4, 2)$ A and a Hadamard matrix H of order 4 as:

$$A^\top = \sum_{i=1}^{4} A_i^\top = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 & 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & r_1 \\ 1 & r_2 \\ 1 & r_3 \\ 1 & r_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Then the matrix D constructed in Theorem 4 is a balancedly multi-splittable Hadamard matrix of order 16:

$$
\begin{aligned}
D &= \sum_{i=1}^{4} iA_i \otimes r_i \\
&= \begin{bmatrix} \mathbf{1} & H_1 & H_2 & H_3 & H_4 & H_5 \end{bmatrix} \\
&= \left[\begin{array}{r|rrr|rrr|rrr|rrr|rrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1
\end{array}\right].
\end{aligned}
$$

Conversely, we demonstrate how an orthogonal array and a Hadamard matrix can be constructed from a balancedly multi-splittable Hadamard matrix D. Consider the 16×4 submatrix $\tilde{H}_1 = \begin{bmatrix} \mathbf{1} & H_1 \end{bmatrix}$ of D. Then, there are exactly four distinct rows

$$(1,1,1,1),\ (1,-1,1,-1),\ (1,1,-1,-1),\ (1,-1,-1,1)$$

in $\tilde{H}_1$. These form a Hadamard matrix of order 4 and set

$$\tilde{K}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & r_{1,1} \\ 1 & r_{1,2} \\ 1 & r_{1,3} \\ 1 & r_{1,4} \end{bmatrix}.$$

Similarly we define

$$\tilde{K}_i = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & r_{i,1} \\ 1 & r_{i,2} \\ 1 & r_{i,3} \\ 1 & r_{i,4} \end{bmatrix}.$$

for $i \in \{2,3,4,5\}$. In the matrix $\begin{bmatrix} H_1 & \cdots & H_5 \end{bmatrix}$, we assign a symbol $j \in \{1,2,3,4\}$ to any row in H_i, which equals the j-th row of K_i. The resulting matrix A is reconstructed as aforementioned.

Remark 17. There exist no balancedly multi-splittable quaternary Hadamard matrices of orders 36 and 100.

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