Solutions to Seven and a Half Problems on Tilings

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Abstract

Four problems about tilings, related to the so-called: Heesch number, isohedral number, m-morphic figures, and σ-morphic figures, can be asked in four variations of the notion of tiling: protosets with more elements, disconnected tiles, colored tiles and tessellations in larger-dimensional spaces. That makes 16 combinations in total. Five among them have been previously solved in the literature, and one has been partially solved. We here solve seven of the remaining combinations, and additionally complete that partial solution.

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1 Introduction

Once upon a time, the topic of tilings was on the verge between mathematics and recreation. Today, there is an extensive theory behind this topic, and there are numerous articles witnessing the constantly growing popularity of this research area. When speaking about some comprehensive references on tilings, it is impossible not to recommend the monumental work by Grünbaum and Shephard [11], which is, although somewhat aged, still taken as the "bible" for the subject matter. And we would also like to mention two very fresh monographs: by Fathauer [7] (written in a more popular style) and by Adams [1] (written in a more serious style).

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Let T be a closed topological disk in the Euclidean plane $\mathbb{E}^2$. We say that T *tiles* (or *tessellates*) the plane if there exists a collection $\mathscr{T}$ of congruent copies of T such that every two elements of $\mathscr{T}$ have disjoint interiors and $\bigcup \mathscr{T} = \mathbb{E}^2$. The collection $\mathscr{T}$ is called a *tiling* or *tessellation* (sometimes even *paving, mosaic,* or *parquetting,* but those terms are much rarer), the elements of $\mathscr{T}$ are called *tiles,* while T is said to be the *prototile* of that tiling.

Given a figure T, the first question that comes to mind in this context is whether T tiles the plane or not. But even if that question is answered, there are other questions we may ask about T related to its "tiling behavior". Here are some parameters that can be assigned to a planar figure T, relevant to the present work. Descriptions given here will be quite informal, while formal definitions will follow later.

1) The **Heesch number** of T is either a nonnegative integer or ∞ that says how many times T can be completely surrounded by its congruent copies. The Heesch number of T is ∞ if and only if T tiles the plane. The notion of the Heesch number is more interesting for figures that do not tile the plane; in that case, the Heesch number can be perceived as a kind of measure how "far" toward a tiling we can advance with congruent copies of the given figure (the larger it is, the figure "behaves more nicely").

   It is an open question whether for each positive integer m there exists a figure whose Heesch number is m. (In particular, the answer to the following weaker question is also unknown: is the set of nonnegative integers that can be the Heesch number of some figure bounded from above? This is usually referred to as *Heesch's problem,* after Heesch considered it [14] in 1968.)

2) A somewhat "complementary" parameter is the so-called **isohedral number.** To say it in short, we first define the isohedral number of a particular tiling to be the number of equivalence classes of the tiles in that tiling, where two tiles are equivalent if and only if they are identified by a symmetry of the tiling. It, in a way, measures how "chaotic" a given tiling is, and then the isohedral number of a figure T (here assumed that it tiles the plane) is defined as the smallest possible isohederal number among all the possible tilings by tiles congruent to T. (The larger isohedral number is, the figure is deemed "harder to work with".) If the isohedral number of a figure T equals m, we also say that T is an m-*anisohedral* figure.

   It is an open question, asked by Berglund [5] in 1993, whether for each positive integer m there exists an m-anisohedral figure.

3) The third parameter that we introduce has a simple definition: a figure is said to be ***m*-morphic** if it tiles the plane in exactly m noncongruent ways.

   It is an open question, asked by Grünbaum and Shephard [9] in 1977, whether for each positive integer m there exists an m-morphic figure.

4) Finally, in relation to the previous point, we say that a figure is **$\boldsymbol{\sigma}$-morphic** if it tiles the plane in infinitely many ways, but only countably many.

   It is an open question, asked by Grünbaum and Shephard [10] in 1981, whether a σ-morphic figure exists.

Apart from the presented, "plain" notion of tiling, some variations (or generalizations) of it have also been considered in the literature. And although the four mentioned problems are unsolved (and thus, presumably, quite hard) in the presented formulations, some of them turned out to be more approachable in some of those modified settings. Let us first introduce these variations of the notion of tiling.

A) **Protosets with more elements:**

   We may consider tilings with more different types of tiles. Namely, a collection $\mathscr{T}$ of closed topological disks in the plane (not necessarily congruent) is called a *tiling* if the elements of $\mathscr{T}$ have pairwise disjoint interiors and $\bigcup \mathscr{T} = \mathbb{E}^2$. Let $\mathscr{P}$ be a set obtained by choosing one representative from each class of mutually congruent tiles from $\mathscr{T}$. Then we say that $\mathscr{P}$ is the *protoset* of the tiling $\mathscr{T}$, the elements of $\mathscr{P}$ are called *prototiles,* and we say that the protoset $\mathscr{P}$ *admits* the tiling $\mathscr{T}$. Also, more generally, any set of pairwise noncongruent topological disks in the plane will be called a *protoset,* and then it could be asked whether such a protoset admits any tiling at all.

   If $|\mathscr{P}| = 1$, we say that the tiling is *monohedral,* if $|\mathscr{P}| = 2$, we say that the tiling is *dihedral* and so on.

B) **Disconnected tiles:**

   We may allow a prototile to be disconnected. In particular, instead of requiring that a prototile T is a closed topological disk, in this variation we loosen this requirement by letting T be any finite union of closed topological disks.

C) **Colored tiles:**

   In this variation, each edge of a prototile T is colored by some color, and there is a set of matching rules that specify which colors can be matched together. To be more precise, for any tiling $\mathscr{T}$ admitted by T, there is the following requirement: if T' and T'' are two tiles from $\mathscr{T}$ that have a common segment, and if c_1 and c_2 are colors by which this segment is colored in T', respectively T'' (ignoring the vertices), then $\{c_1, c_2\}$ has to be among the pairs allowed to be matched together. (Of course, in the event that T' and T'' have multiple common segments, then each of them should obey this requirement.)

D) **Tessellations in $\mathbb{E}^d$ for $d > 2$:**

   Tessellations in $\mathbb{E}^d$ are more-or-less straightforward generalization of the situation in $\mathbb{E}^2$, but there are a few details that we have to pay attention to.

The first is just a terminological issue: the word *tiling* usually associates to a two-dimensional space, and thus for $d > 2$ we shall use the word *tessellation.* Elements of a tessellation will be called *cells,* and one representative of cells will be called the *protocell.*

Second, in $\mathbb{E}^2$, the constraint that a prototile is "without holes" (that is, that its genus is 0, which is implicitly contained in the requirement that the prototile is a closed topological disk) is not a real restriction, since a prototile with holes clearly cannot tile the plane. However, requiring that a protocell in $\mathbb{E}^d$ for $d > 2$ must be a closed topological d-ball would be a too severe restriction (and is almost never assumed in the literature on this topic). Instead of that, it is more natural to allow that a protocell can be any region that is bounded, has connected interior, and (to eliminate pathological cases) is regular closed (in other words, equals the closure of its interior).

Note: the terminology around tessellations in more-dimensional spaces is anything but firmly established. Grünbaum and Shephard [11] treat only two-dimensional spaces, but in Section 10.7 they briefly mention some references to more-dimensional spaces, and here they also use the term *tiling.* Mann [16] also touches on $\mathbb{E}^3$ only briefly, and in [17] he also mentions $\mathbb{E}^d$; in those cases he uses the terms *tiling* and *tessellation* interchangeably, and their elements are called *tiles* and *cells,* also interchangeably. Adams [1] sticks with *tilings* and *tiles* (in both $\mathbb{E}^2$ and $\mathbb{E}^3$, without mentioning larger dimensions). Coxeter [6] uses the term *honeycomb* in n-dimensional spaces for any n, while two-dimensional honeycombs are called *plane tessellations* by him, and three-dimensional honeycombs are called *solid tessellations*; elements of a honeycomb are called *cells.* In a recent book, Johnson [15] uses the term *n-honeycomb* (in n dimensions), where 2-honeycombs are called *tessellations,* and 3-honeycombs are called *cellulations*; elements of an n-honeycomb are called *cellules.* And for the end of this paragraph we mention Olshevsky, who had put up a website [20] that provided a quite comprehensive glossary of many terms used in more-dimensional spaces (the site ceased to exist a long time ago, but can still be accessed via WayBack Machine). By it, the term *tessellation* is appropriate for any number of dimensions, and two-dimensional tessellations are called *tilings,* while tessellations in more than two dimensions are called *honeycombs* (and the same usage is also adopted by Fathauer [7]); in any case, elements of a tessellation are called *cellets.* In the present article we use the terms from two paragraphs ago.

Each of the open questions marked by ⚠ under the description of the notions 1)–4) can be asked in each of the settings A)–D). That makes a total of 16 possible combinations, some of which have already been solved in the literature. In Table 1 we give an overview of what is known so far (fields with a reference in square brackets), as well as what are new results from the present article (fields referring to a theorem from some later section). For questions asked for protosets with more elements (column A), our results give answers for protosets with k elements for every k, $k \geqslant 2$. For questions asked in more dimensions (column D), our results give answers for spaces $\mathbb{E}^d$ for every d, $d \geqslant 3$.

	A	B	C	D
1	[3] & T6	T7	?	?
2	T9	[28]	[28]	[28]
3	[12]	T11	T13	T16
4	[22]	?	T17	T20

Table 1: Overview of the state-of-the-art knowledge.

In the following section we shall formally define the notions 1)–4), as well as provide some more background on the problems researched here.

2 Preliminaries

In the four subsections within this section we give a more detailed elaboration of the necessary notions.

2.1 The Heesch number

Let us first give a formal definition of the Heesch number.

Definition 1. We say that a figure T in the plane can be *surrounded n times* if and only if there exist finite collections of figures $\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_n$ in the plane such that:

- for each i, $1 \leqslant i \leqslant n$, each figure from $\mathscr{C}_i$ is congruent to T;
- every two different figures from $\{T\} \cup \bigcup_{i=1}^{n} \mathscr{C}_i$ have disjoint interiors;
- for each i, $1 \leqslant i \leqslant n$, each figure from $\mathscr{C}_i$ has a common boundary point with some figure from $\mathscr{C}_{i-1}$ (where, by convention, we let $\mathscr{C}_0 = \{T\}$);
- for each i, $1 \leqslant i \leqslant n$, $\bigcup \left(\bigcup_{j=0}^{i} \mathscr{C}_j \right)$ contains $\bigcup \left(\bigcup_{j=0}^{i-1} \mathscr{C}_j \right)$ completely in its interior.

The collection $\mathscr{C}_i$ is called the i^{th} *corona*.

Definition 2. The *Heesch number* of a given figure T is the maximal nonnegative integer n such that T can be surrounded n times. If such a maximum does not exist, then we define the Heesch number to be infinite.

One would be tempting to say that the Heesch number of T is infinite if and only if T tiles the plane. It turns out that this conclusion is not really trivial as it maybe seems, since theoretically it is possible that a figure can be surrounded n times for any positive integer n, but that it cannot be surrounded infinitely many times (that is, that it does not tile the plane). However, after some work it can be proved that the statement from the first sentence is indeed correct, as given by the so-called Extension Theorem [11, Theorem 3.8.1].

Note. The definition of the Heesch number is not fully consistent across the literature. In some works, it is defined as given here. In other works (including some works by some of the present authors), the last bullet point in Definition 1 additionally demands that $\bigcup \left(\bigcup_{j=0}^{i} \mathscr{C}_j \right)$ should be a closed topological disk. However, as we shall see in Subsubsection 3.1.1, this more restrictive definition is badly flawed, at least in the case of protosets with more elements, in the sense that it would allow the existence of a protoset that tiles the plane while its Heesch number is finite! And furthermore, there seems to be no immediate reason why this flaw could not manifest itself even in the case of tiling with a single prototile. Because of that, it seems safer to ally with the school of thought that does not care about the form of $\bigcup \left(\bigcup_{j=0}^{i} \mathscr{C}_j \right)$. In any case, our Theorem 6 does not depend on this nuance (and the same is true for the results that will be summarized in this section).

The generalization of the definition of the Heesch number to disconnected tiles, colored tiles, and spaces $\mathbb{E}^d$ for $d > 2$ is clear. As far as the current knowledge reaches, the largest known finite Heesch number equals 6 (demonstrated by Bašić [2], after the previous record, which was 5, set by Mann [16], stood for almost twenty years). Introducing colors does not change the picture substantially, since even with a somewhat more general matching rules (which take into account also orientations of the sides when specifying which sides can be matched together), everything one gets is that the value of the largest known finite Heesch number is pushed a little bit further in that case, up to 11 (by DeWeese and Coronaldi from 2010, unpublished, but presented in, e.g., [18]). In more-dimensional spaces, we have the result by Bašić and Slivková [4] that the Heesch number in $\mathbb{E}^d$ is asymptotically unbounded for $d \to \infty$ (in other words, given any nonnegative integer n, there exists a dimension d in which there is a protocell whose Heesch number is finite and greater than n). As a curiosity, we also mention that Heesch's problem in the hyperbolic plane has been solved by Tarasov [29]. And if we allow disconnected tiles (in the Euclidean plane), Theorem 7 settles the matter in that case.

This leaves the question of the Heesch number of protosets with more elements. Let us first define the Heesch number in that case.

Definition 3. The *Heesch number* of a given protoset $\mathscr{P}$ is the maximal nonnegative integer n such that each prototile from $\mathscr{P}$ can be surrounded n times. (Here, *surrounding* a prototile T from $\mathscr{P}$ is defined as in Definition 1, where in the first bullet point the condition "congruent to T" is replaced by "congruent to some prototile from $\mathscr{P}$.") If such a maximum does not exist, then we define the Heesch number to be infinite.

Grünbaum and Shephard conjectured that, given a positive integer k, there is an upper bound on all finite Heesch numbers of protosets with k prototiles [11, Section 3.8]. This conjecture was refuted in [3], where it was shown that, given any positive integer n, there exists a protoset with 3 prototiles whose Heesch number is n; it is then an easy corollary that the same holds with 3 replaced by k for any k, $k > 3$. This almost fills the field 1-A in Table 1, but the case $k = 2$ falls apart (of course, excluding the case $k = 1$, which is the "plain" Heesch's problem). With Theorem 6, the picture is completed.

Note. The following point needs a clarification. Note that the definition of a tiling by a protoset allows an event that a protoset $\mathscr{P}$ does not tile the plane, altough some of its (strict) subsets possibly do. (Indeed, note that, if $\mathscr{P}$ admits a tiling $\mathscr{T}$, then the definition dictates that $\mathscr{T}$ must contain a congruent copy of *each* prototile from $\mathscr{P}$.) This is the definition from (among other places) [11], and the conjecture from the previous paragraph is posed having that definition in mind. The Heesch number of $\mathscr{P}$, as introduced in Definition 3, is thus defined from the perspective of the "worst" prototile from $\mathscr{P}$. Perhaps one would say that it would be more appropriate to define that a protoset $\mathscr{P}$ tiles the plane if the plane can be tiled by tiles congruent to prototiles from $\mathscr{P}$, not necessarily using them all. In accordance with that definition, the Heesch number of a protoset would then be naturally defined by replacing the phrase "*each* prototile" in Definition 3 by "*at least one* prototile." This approach has also been considered in the literature, and the corresponding question for that version of the Heesch number has been asked in [8, Question 4.1(a)]. We here follow the definitions from [11], and unfortunately, our Theorem 6 does not work (and, as it seems, cannot be easily adapted) for the alternative approach.

2.2 The isohedral number

Let $\mathscr{T}$ be a monohedral tiling. Let $\mathbf{S}(\mathscr{T})$ be the group of all isometries of the plane that leave $\mathscr{T}$ invariant ($\mathbf{S}(\mathscr{T})$ is also called the *symmetry group* of $\mathscr{T}$). For each tile T of $\mathscr{T}$, the *orbit* (or the *transitivity class*) of T is the set of all tiles into which T can be mapped by an isometry from $\mathbf{S}(\mathscr{T})$. This leads us to the following definition.

Definition 4. The *isohedral number* of a given tiling $\mathscr{T}$ is the total number of orbits into which $\mathscr{T}$ is divided under the action of the group $\mathbf{S}(\mathscr{T})$. The isohedral number of a given figure T is the smallest possible isohedral number among all the tilings admitted by T. If the isohedral number of T is m, we say that the figure T is *m-anisohedral.*

The question whether for each positive integer m there exists an m-anisohedral figure is attributed to Berglund [5] in the previous section, but we also want to mention that a closely related question was posed by Grünbaum and Shephard [11, Exercise 9.3.2] a few years earlier. A figure with the largest known (finite) isohedral number is discovered by Myers [19], and its isohedral number equals 10. What is also interesting to mention is that the question of existence of a polyhedron with isohedral number greater than 1 was posed as the second part of Hilbert's 18^{th} problem, and there are some beliefs that Hilbert posed the question in $\mathbb{E}^3$ because he was convinced that in $\mathbb{E}^2$ it is easy to show that the answer is negative. Hilbert's question was solved in the affirmative by Reinhardt [21] in 1928, and then its two-dimensional version was solved, also in the affirmative, by Heesch [13] in 1935.

Note. In the previous paragraph, we snuck in the word "finite" in parenthesis, but the reader could, after giving it one more thought, ask himself what was the purpose of that, is there a figure with infinite isohedral number? Clearly, the isohedral number of every periodic tiling is finite. Therefore, if a figure with infinite isohedral number existed, that would imply the existence of a figure that admits only aperiodic tilings. The existence of such a figure had been a long-standing open question, and was open at the moment of submission of the present article. However, as fate would have it, this fiendish problem finally met its end practically an eyeblink afterwards: the answer is affirmative. See the (already famed) article [27] for the rest of the story (and you can also check the very freshly minted [26], which makes a further step: here the authors present such a figure with an additional property that no reflections are used in any of the tilings).

For disconnected tiles, colored tiles and more-dimensional spaces, the same definition works without changes. The existence of m-anisohedral figures for any positive integer m was answered affirmatively by Socolar [28] for all those three settings.

For tilings by protosets with more elements, we can also use the same definition without changes (this follows the approach from, e.g., [11, Section 1.3]); note that then the isohedral number of a protoset is at least as large as the number of elements in the protoset. This is the approach adopted in the present article, but a somewhat different definition has also been seen in the literature: in [8], the isohedral number of a tiling $\mathscr{T}$ has been defined by first calculating, for a given prototile P from the protoset $\mathscr{P}$ of $\mathscr{T}$, how many different orbits exist whose elements are tiles congruent to P, and then the isohedral number of $\mathscr{T}$ is defined as the maximum among all these numbers as P ranges through $\mathscr{P}$. And furthermore, another place for inconsistencies is what it exactly means that a protoset admits a tiling (see the note at the end of the previous subsection). The good news is that our Theorem 9 solves the problem regardless of these deviations (in particular, the point from the last sentence does not influence Theorem 9 in any way, while under the alternative definition of the isohedral number of $\mathscr{T}$ we may replace "$m \geqslant k \geqslant 2$" by "$m \geqslant 1$ and $k \geqslant 2$" in the statement of Theorem 9).

2.3 Polymorphic figures

The definition of an m-morphic figure given in the introduction is clear enough (it is a figure that admits exactly m noncongruent tilings). This definition directly transfers to each of the four generalized settings discussed in the present article.

In [9], Grünbaum and Shephard posed the question whether there exists an m-morphic figure for each positive integer m (currently, the largest known m for which there exists an m-morphic figure is $m = 11$, discovered by Myers [19]), and in the same article, they posed the same question for protosets with more elements. This latter question has been answered (in the affirmative) by Harborth [12]. However, we here again point to the note at the end of Subsection 2.1, and make a remark that Harborth's solution does not work for the alternative definition from that note (namely, one prototile from his two-element protoset is a rhomb, which tiles the plane in infinitely many ways). The same Grünbaum and Shephard later [11] posed the same question but under this stricter definition, and this version of the problem has been solved by Schmitt (in particular, first one solution appeared in [22], and then another solution was given in [24], which differs from the first one as in the new solution no reflection is used in any of the obtained tilings, and all the obtained tilings are periodic; one more approach was later published in [23], which is, for the first time, based on hexagonal/triangular tilings, with the author commenting that "the construction is simpler, and the corresponding tilings posses a higher degree of symmetry", but adding that there is a single tiling that uses only one of the two prototiles).

In Theorems 11, 13 and 16 we solve the problem for disconnected tiles, colored tiles, and tessellations in $\mathbb{E}^d$ for any d, $d \geqslant 3$.

2.4 $\boldsymbol{\sigma}$-morphic figures

Finally, the question of existence of a figure that tiles the plane in infinitely many ways but only countably many was posed by Grünbaum and Shephard [10]. Its version for protosets with more elements has been solved by Schmitt in [22], and then he gave another solution in [25]. There is an essential difference between these two solutions, as in the first one, among all the obtained tilings there are only finitely many periodic ones, while in the second solution all the obtained tilings but finitely many of them are periodic.

In Theorems 17 and 20 we solve the problem for colored tiles and tessellations in $\mathbb{E}^d$ for any d, $d \geqslant 3$.

3 The Heesch number

We show that, for each positive integer m, there exists a two-element protoset, as well as a disconnected prototile, whose Heesch number is m.

3.1 Protosets

Given a positive integer m, we shall now describe a protoset consisting of two prototiles whose Heesch number is m.

- The first prototile in our protoset, which we shall call S (where S stands for "square"), is obtained by adding a symmetric bump to one side of a unit square, and the matching nick to the opposite side, where the only way to fill the nick is to place the bump in it, and vice versa (see Figure 1).

- The other prototile, denoted by C_m, is obtained in the following manner. We start from a rectangle of dimensions $(2m + 5) \times (m + 2)$. We first remove from it a rectangle of dimensions $(2m + 3) \times m$, centered within the initial rectangle. From the lower side of the remaining "frame", starting two units from the right side, we remove a rectangle of dimensions $(2m + \frac{1}{2}) \times 1$. We now add some bumps and nicks in the manner shown in Figure 1.

We are now ready to prove the following proposition.

Proposition 5. *The protoset consisting of the prototiles S and C_m has Heesch number m.*

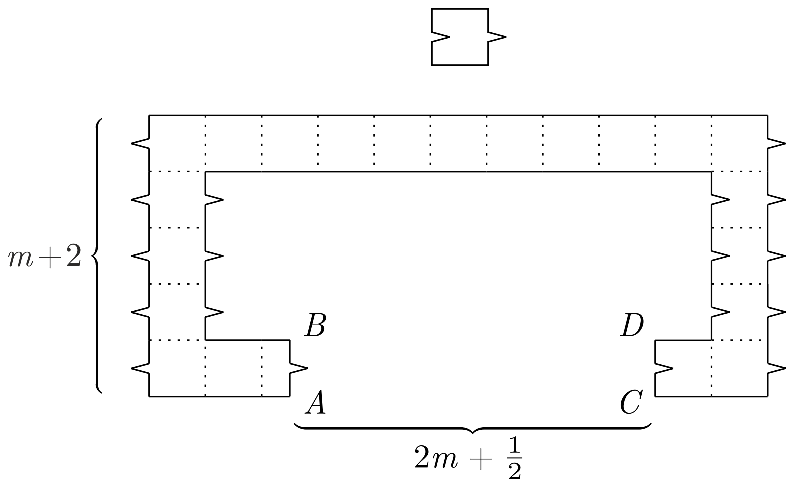

Figure 1: Prototiles S and C_m for $m = 3$.

Proof. We have to prove that each prototile can be surrounded m times, and that at least one prototile cannot be surrounded $m+1$ times. As it is obvious that the prototile S tiles the plane, let us prove that the prototile C_m cannot be surrounded more than m times. Note that the bump on the edge denoted by AB in the figure has to be accommodated by a nick on some tile from the first corona. That tile can be either S, or even another copy of C_m, but in any case, that tile has a bump positioned at least 1 unit to the right of the edge AB. This bump has to be accommodated by a nick on some tile from either the second corona or from the first corona again, which implies the existence of another bump positioned at least 2 units to the right of the edge AB. Repeating the argument i times, we conclude that there is a bump at least i units to the right of the edge AB, and this bump belongs to some tile from at most the i^{th} corona. And we have the analogous conclusion for the edge CD. But since the distance between the edges AB and CD is only $2m + \frac{1}{2}$, we infer that no more than m coronas can be formed. And finally, that m coronas can indeed be formed is shown in Figure 2 (the example for $m = 3$, and the generalization is obvious), which completes the proof. □

Together with the results obtained in [3], or by simply cutting one prototile into a number of pieces distinctive enough so that they can be fitted only in such a way to assemble the initial prototile, we obtain the following theorem.

Theorem 6. *For every positive integers m and k, where $k \geqslant 2$, there exists a protoset consisting of k prototiles whose Heesch number is m.*

3.1.1 A reflection on the definition of the Heesch number

We here return to Definition 1. Recall that we remarked that in some works this definition includes an additional requirement that $\bigcup \left(\bigcup_{j=0}^{i} \mathscr{C}_j\right)$ is a closed topological disk for each i. We now construct a protoset that, with this additional requirement, has finite Heesch number but nevertheless tiles the plane. Because of the existence of such a protoset, we

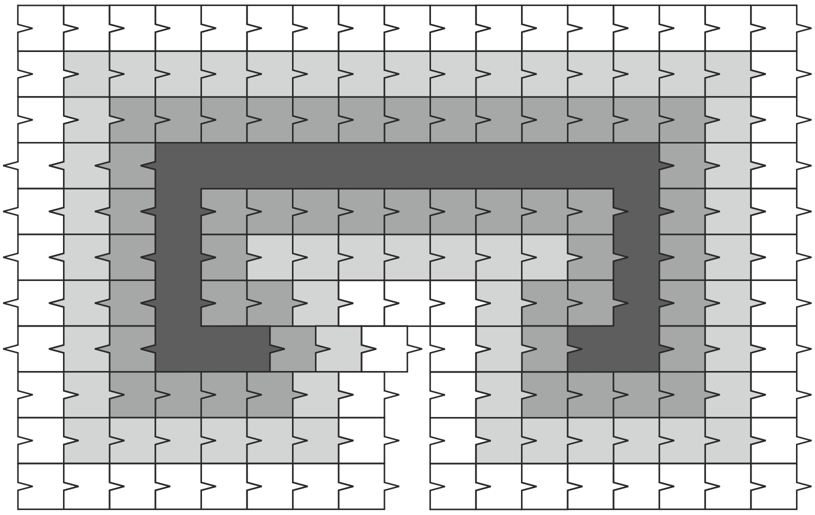

Figure 2: Three coronas formed around the prototile C_3.

conclude that this alternative definition is not appropriate (at least in the setting with protosets with more elements, but we also cannot discard the possibility that even a single prototile exists that unveils a similar issue).

The two-element protoset that we construct bears a resemblance to the protoset from Figure 1. The first prototile, call it S', is again a unit square, but now *asymmetric* bump and nick are added to it (instead of symmetric ones). For the second prototile, call it U_m, we now start from a rectangle of dimensions $(2m + 10) \times (2m + 5)$, remove from it a concentric rectangle of dimensions $(2m + 8) \times (2m + 3)$, and remove a rectangle of dimensions $(2m + 2) \times 1$ from the middle of the lower side of the remaining "frame". In the end, we add some bumps and nicks as shown in Figure 3, where the tile of the darkest shade of gray is U_1. (Let us just mention, there is no real need to make the bumps and nicks asymmetric, the claim would also be correct with symmetric ones. But then the argument from the following paragraph when we deduce forced positions of some S'-tiles would not be that immediate, as it would be possible, at least theoretically, that another copy of U_m is used to accommodate some bump/nick. Therefore, asymmetric bumps and nicks make the exposition a little bit tidier.)

Let us show that the Heesch number of the protoset $\{S', U_m\}$ equals m. The prototile S' clearly can be surrounded infinitely many times, so we prove that the prototile U_m can be surrounded m times but cannot be surrounded $m + 1$ times. We present the proof on the example $m = 1$, the generalization for any m is obvious (and, in fact, even the example $m = 1$ alone is enough to establish the existence of a counter-intuitive protoset that we are seeking). Let the prototile U_1 be positioned in the plane. Suppose that two coronas can be formed around it. In Figure 3, positions of all the S'-tiles with the exception of the first two and the last two rows are forced as shown (while we have some degrees of freedom within the first two and the last two rows, which will not affect the proof). Note that all among those tiles that are of the median shade of gray belong to the first corona, while the tiles of the lightest shade of gray belong to the second corona (in general, if the initial tile were U_m for $m > 1$, in the continuation of this argument we would have tiles that belong to the third corona *at most,* the fourth corona *at most* etc., that is, we would not be able to say *exactly* which corona; but this bound from one side is enough for the proof to work). But now note that the white part in the middle represents a hole in $\bigcup\left(\bigcup_{j=0}^{2}\mathscr{C}_j\right)$, that is, the second corona cannot be formed if we require that $\bigcup\left(\bigcup_{j=0}^{2}\mathscr{C}_j\right)$ must be a topological disk. On the other hand, that hole can be filled with S'-tiles as shown by the dashed lines, and it is clear that this configuration can be extended to a tiling of the whole plane. This shows that the protoset $\{S', U_m\}$ has the "wild" properties that we were discussing.

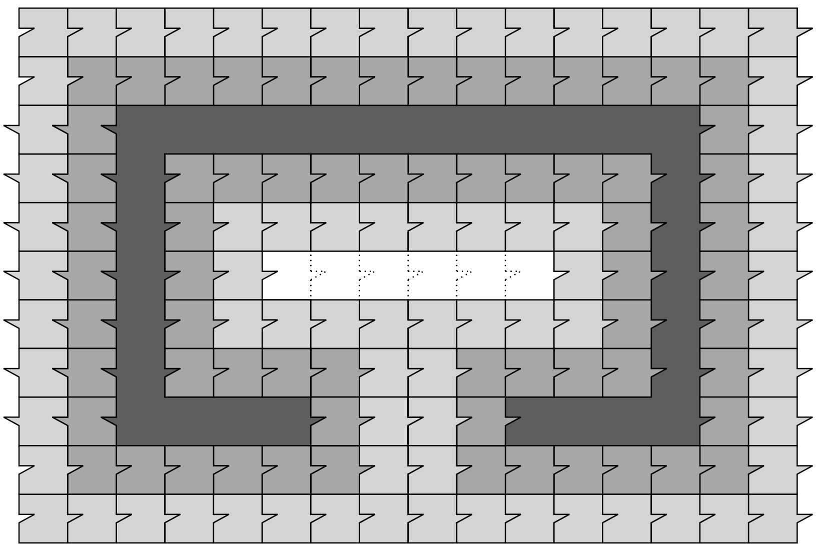

Figure 3: A try to surround the prototile U_1 twice.

3.2 Disconnected tiles

We now describe, for a given positive integer m, a two-part disconnected prototile whose Heesch number is m. It is composed of two copies of the prototile S from Section 3.1, translated by $2m + \frac{1}{2}$ units along the horizontal direction (see Figure 4). The following theorem can now be proved by a similar idea as in Proposition 5.

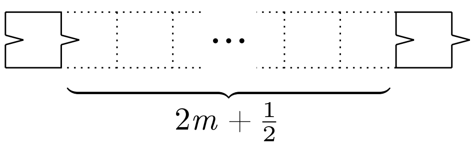

Figure 4: The described two-part tile with Heesch number m.

Theorem 7. *For every positive integer m, there exists a two-part prototile with Heesch number m.*

Proof. Of course, we claim that the Heesch number of the two-part prototile that has just been described equals m. Note that the two initial squares, distanced $2m + \frac{1}{2}$ units from each other, force the placement of another two squares between them, distanced $2m - 2 + \frac{1}{2}$ units from each other, which are parts of some tiles from the first corona. Those two new squares force the placement of further two squares, distanced $2m - 4 + \frac{1}{2}$ units from each other, which are parts of some tiles from at most the second corona. Repeating the argument m times in total, we arrive to two squares distanced $\frac{1}{2}$ units from each other, which are parts of some tiles from at most the m^{th} corona. Therefore, no more than m coronas can be formed, and it is easy to see that m coronas indeed can be formed, which completes the proof. $\square$

4 The isohedral number

We show that, for each positive integer m, there exists a two-element protoset whose isohedral number is m.

4.1 Protosets

We shall now describe, for a given positive integer n, a protoset consisting of two prototiles whose isohedral number is $n + 1$. The first prototile is a heptagon shown in Figure 5 left (basically, a rectangle with a nick). The second prototile is a rectangle with n cavities of the form of the first prototile; this prototile should not have any axes of symmetry (e.g., assume that the n cavities are all at the same side and before the midpoint of that side, as seen in Figure 5 right).

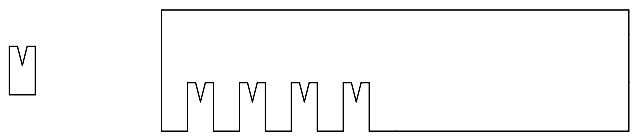

Figure 5: The described protoset for $n = 4$.

Let us call these two prototiles H and R_n, respectively. We are ready to prove the following proposition.

Proposition 8. *The protoset consisting of the prototiles H and R_n has isohedral number $n + 1$.*

Proof. Note that the n cavities in the prototile R_n can be uniquely filled by n copies of the prototile H. This forms a rectangle that can be then used to tile the plane in many ways. We claim that each such tiling has isohedral number at least $n + 1$. Let $\mathscr{T}$ be one such tiling. Any congruence mapping $\mathscr{T}$ to itself maps each R_n-tile to another copy of R_n. We further see that all the H-tiles (a total of n of them) filling the cavities in an R_n-tile are in different orbits; together with at least one orbit containing R_n-tiles, we conclude that there are at least $n + 1$ orbits in total. And Figure 6 shows that there indeed exists a tiling with $n + 1$ orbits, which completes the proof. $\square$

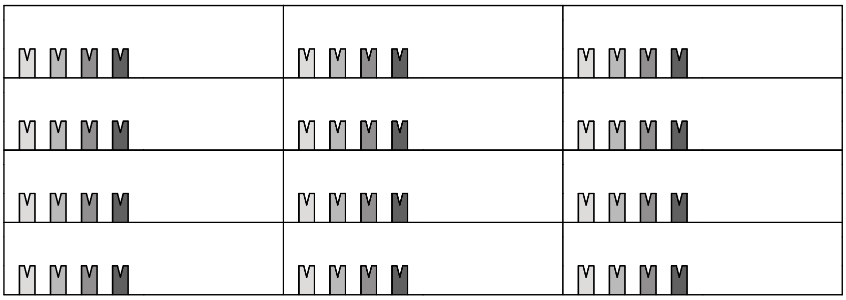

Figure 6: A tiling with $n+1$ orbits.

As cardinality of a protoset can be increased by the same trick that was already mentioned when we were discussing Theorem 6, we reach the following theorem.

Theorem 9. *For every positive integers m and k, where $m \geqslant k \geqslant 2$, there exists a protoset consisting of k prototiles whose isohedral number is m.*

Note. One could ask whether a simpler shape can be taken for the prototile H (e.g., just a rectangle). The point behind this choice of H is to ensure that it does not tile the plane by itself, in order to accommodate for some inconsistencies in the relevant definitions across the literature, as commented at the end of Subsetion 2.2.

5 Polymorphic figures

We show that, for each positive integer m, there exists an m-morphic disconnected prototile, an m-morphic colored prototile, and an m-morphic protocell in $\mathbb{E}^d$ for each d, $d \geqslant 3$.

5.1 Disconnected tiles

We now describe a two-part disconnected prototile admitting a prescribed number of noncongruent tilings. The prototile will depend upon a given positive integer n, and with that it mind, we shall say that it has n *segments.*

Look at Figure 7 left.

- The bottom part, shaped like the letter C in the picture, will be called the *plug.* Of course, the shape itself is not really important, almost any shape is suitable; to be on the safe side, we only impose the restriction that it should not have any axes of symmetry, because such axes could lead to some unwanted tilings.
- The upper (and larger) part is obtained by starting from a rectangle and then modifying it as follows. There is a hole cut in it in the shape of the plug, and the plug is positioned on the same vertical as the hole and exactly two heights of the rectangle below the hole. There is also a bump on the right-hand edge of the rectangle (shaped like the letter A in the picture), and the matching nick in

the corresponding position on the left-hand side. Finally, there are n uniformly distributed bumps on the top edge of the rectangle (shaped like the letter B in the picture) and n matching nicks in the corresponding positions on the bottom edge.

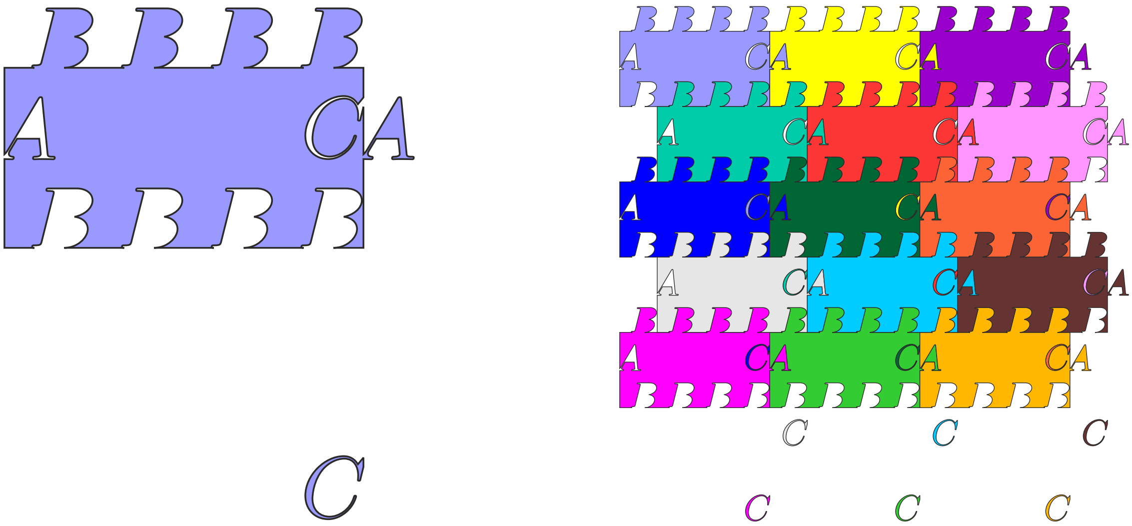

Figure 7: The two-part prototile with 4 segments (left), and one tiling admitted by it (right).

The following proposition is the key result of this subsection.

Proposition 10. *The described two-part prototile with* n *segments tiles the plane in* $\lfloor \frac{n}{2} \rfloor + 1$ *noncongruent ways.*

Proof. Consider one such described tile in the plane. By construction, the A-nick on the left-hand edge can be filled only by the corresponding A-bump, by translating the tile horizontally. By repeating this argument, we conclude that the considered tile uniquely forces the formation of a doubly-infinite strip whose height is the height of the rectangle (ignoring bumps, nicks and plugs).

Now pay attention to the C-holes. By construction, they can be filled only by the plugs, which forces the existence of another such strip that is exactly the translate of the observed strip along the vertical direction and for the distance of two rectangle heights. By repeating this argument, we conclude that the formation of infinitely many such strips is uniquely forced. These are the strips in odd rows in Figure 7 right.

Finally, note that such a conglomerate of strips still leaves an uncovered part of the plane that is exactly of the same form. Let us consider how to fill this "interspace." Note that any tile placed in this interspace uniquely forces (by the same arguments) another conglomerate of strips (and thus completes the tiling). Further, note that the first B-bump (enumerated from left to right) from a tile placed in this interspace can fill either the 1^{st} B-nick, or the 2^{nd} one, ..., or the n^{th} one of a tile immediately above. This gives a total of n tilings of the plane by the considered prototile, which differ only by the amount by which the "even" strips are shifted with respect to the "odd" strips; Figure 7 right shows the case when the even strips are shifted by 1 unit. However, some of these tilings are congruent since “even” and “odd” strips can swap roles; in particular, shifting by i units and shifting by $n-i$ units produce congruent tilings (since, if the even strips are shifted by i units with respect to the odd strips, then the odd strips are shifted by $n-i$ units with respect to the even strips). Therefore, noncongruent tilings are obtained by shifting by $0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ units, and thus there is a total of $\lfloor \frac{n}{2} \rfloor + 1$ noncongruent tilings. $\square$

The following theorem is now immediate.

Theorem 11. *For every positive integer m, there exists an m-morphic two-part prototile.*

5.2 Colored tiles

In this section we describe a prototile with colored edges admitting a prescribed number of noncongruent tilings.

As our construction will rely on a construction of a 2-element m-morphic protoset by Schmitt [24], we first describe his construction.

- His “main” prototile is a rectangle with some so-called *keys* and *holes* added to its sides. A key is formed in the following way. Consider two concentric half-circles. The so-called *empty key*, denoted by $k(0)$, will be the smaller half-disk. Apart from this one, there exist six more types of keys. Each of them can be obtained by dividing the half-ring between the two circles into six congruent parts, and choosing three of those six parts; a key will be the union of the smaller half-disk and those three parts. A number of keys can be obtained this way; we shall need only six of them, so we take six, denote them by $k(1), k(2), \ldots, k(6)$, and ignore the rest. Those six that we take can be almost any six, there is only one restriction: no two among them can be mirror images of each other (it is easily seen that this is possible: in fact, such a set of keys can be chosen with even up to 10 elements).

  There are also 7 holes, denoted by $h(0), h(1), \ldots, h(6)$, where the hole $h(i)$ is simply the complement of the key $k(i)$ (they perfectly match each other).

  See Figure 8 for a rectangle with the keys $k(0), k(1), \ldots, k(6)$ added to the upper side, and the holes to the bottom side in the corresponding order. (Note: the rect-

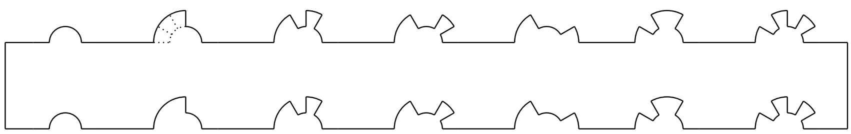

Figure 8: An illustration of the 7 keys and 7 holes.

  angle shown here is **not** the rectangle from Schmitt’s construction; for his rectangle a much more complicated pattern of keys and holes is needed, and furthermore, there are also some keys and holes added to the lateral sides, not only to the top and the bottom one. All of this will be described soon. For now, we just want to illustrate how individual keys and holes look like.) In one of the keys, the dashed lines show how the key is composed of the mentioned half-disk and the mentioned parts of the half-ring.

- The second prototile, called *lid,* is simply one of the six congruent parts into which the half-ring considered in the previous bullet point is divided (that is, one of those parts three of which are needed for the construction of a key).

Note that, by construction, each hole can be filled either by its matching key, or by the empty key together with three lids; any other combination of a key and a hole is impossible.

We now describe how exactly a rectangle should be "decorated" by keys and holes. Let m be a given positive integer. Take a rectangle $ABCD$, where the shorter sides, AD and BC, are 10 units long, while the longer sides, AB and DC, are $10(\frac{m(m+1)}{2}-1)$ units long. Divide each side into unit segments, and place the keys and holes along the sides as follows (each key/hole is placed in the center of one such unit segment; for the sake of brevity, the keys are denoted by $0, 1, \dots, 6$ instead of $k(0), k(1), \dots$, and the corresponding holes are denoted by $\overline{0}, \overline{1}, \dots, \overline{6}$):

- AD: $0\overline{5}4\overline{0}0\overline{0}0\overline{0}0\overline{6}$;
- BC: $\overline{1}0\overline{0}0\overline{0}0\overline{0}3\overline{2}0$;
- DC: $Q_1 + Q_2 + Q_3$;
- AB: $P_1 + P_2 + P_3$,

where:

$$\begin{aligned}
Q_1 &= 6\overline{2}0\overline{0}0\overline{0}\cdots 0\overline{0}5\overline{1};\\
P_1 &= \overline{6}2\overline{0}0\overline{0}0\cdots \overline{0}0\overline{5}1;\\
Q_2 &= \tfrac{m(m+1)}{2}-1 \text{ copies of } c^-(4), \text{ where } c^-(4) = 0\overline{0}0\overline{0}0\overline{0}0\overline{4}0\overline{0};\\
P_2 &= \tfrac{m(m+1)}{2}-1 \text{ copies of } c^-(3), \text{ where } c^-(3) = \overline{0}0\overline{3}0\overline{0}0\overline{0}0\overline{0}0;\\
Q_3 &= H(1)H(2)\cdots H(m-1);\\
P_3 &= H(m)H(m+1)\cdots H(2m-2),
\end{aligned}$$

and, for $1 \leqslant i \leqslant m-1$,

$$H(i) = 0\overline{0}0\overline{1}\,\underbrace{c(3)\cdots c(3)}_{i \text{ copies}}\,0\overline{2}0\overline{0}0\overline{0}, \text{ where } c(3) = 0\overline{0}0\overline{0}0\overline{0}0\overline{3}0\overline{0},$$

and

$$H(2m-1-i) = \overline{0}0\overline{0}0\overline{5}0\,\underbrace{c(4)\cdots c(4)}_{i \text{ copies}}\,\overline{6}0\overline{0}0, \text{ where } c(4) = \overline{0}0\overline{4}0\overline{0}0\overline{0}0\overline{0}0.$$

The addition that appears in the description of AB and DC is simply the position-wise addition. Note that, by construction, the numbers in a fixed position in P_1, P_2 and P_3 (and also Q_1, Q_2 and Q_3) either all three have the overline or none of them does; therefore, the addition is well-defined (the result in each position will have/not have the overline depending on whether the summands have it or not). Actually, it can be checked that even more is true: by construction, there will be at most one nonzero number in each position, and thus the sum in that position will in fact be equal to that number (if such one exists, and be $0/\overline{0}$ otherwise).

By [24, Section 7], we have the following theorem.

Theorem 12. *The described two-element protoset tiles the plane in m noncongruent ways. In each of these tilings, all the tiles are rotated and translated images of each other (without reflections).*

We are now ready to deduce the main theorem of this subsection.

Theorem 13. *For every positive integer m, there exists an m-morphic colored prototile.*

Proof. Let us introduce 14 colors, named $0, 1, \ldots, 6, \overline{0}, \overline{1}, \ldots, \overline{6}$. Take the same rectangle $ABCD$, again divide its side into unit segments, but now, instead of adding keys and holes to these segments, color them by the considered 14 colors, following the same pattern that was used for keys and holes. Assume for a moment that we impose the restriction that any overlapping sides of two neighboring tiles must overlap exactly for some integer number of unit segments.

The matching rules are as follows: pairs of colors that can be placed against each other are only $\{i, \overline{\imath}\}$ for any i, as well as $\{0, \overline{\imath}\}$ for any i.

The idea is to make matching rules such that they allow exactly the same possibilities (in comparison to Schmitt's construction) of placements of two neighboring tiles relative to each other; then appeal to Theorem 12 and conclude that, since we achieve by coloring exactly the same options as Schmitt achieved by keys and holes (with the help of the lid—which, of course, is unnecessary in our case), our colored rectangle tiles the plane in the same number of ways. It seems that the matching rules from the previous paragraph serve exactly that purpose, but there are two issues that we have to take care of. First, in each tiling allowed by Schmitt's construction only direct copies (no mirror images) of the tiles are used; in our case, nothing (at least so far) excludes the possibility that some rectangle in a tiling has a common boundary part with its mirrored copy. Second, recall the assumption introduced at the end of the first paragraph; we have to find a way to geometrically enforce that assumption.

Let us show a way to handle both these issues simultaneously. Replace each colored unit segment by a 5-segment broken line; all these broken lines should be translations/rotations of each other, each of them should be centrally symmetric, but apart from that, they should exhibit no other regularities. Now color the middle three segments of each broken line by the original color of the corresponding unit segment, and color the remaining two segments by a new color (this will be the 15^{th} color in total; for matching rules concerning it, we can say, e.g., that it can be placed only against itself, though this is not of a key importance). These broken lines clearly can be positioned only against each other, which resolves the second issue. And as these broken lines exhibit no mirror symmetry, no two neighboring rectangles can be mirrored copies of each other, which implies that all the rectangles in any tiling are direct copies of each other; this resolves the first issue, and completes the proof. □

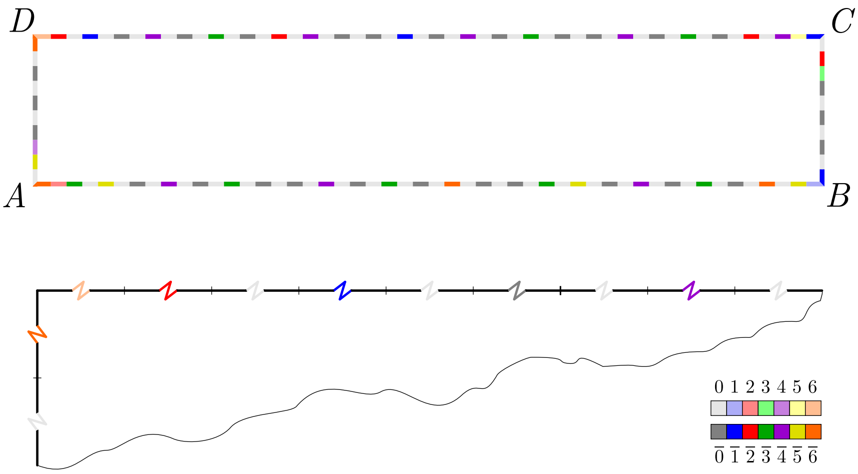

Figure 9: The colored prototile for $m = 3$.

In Figure 9 we show the example for $m = 3$. The upper part presents our rectangle with colored unit segments, while in the lower part we see how the introduced broken lines look like in the magnified top left corner (the short thin lines are not an integral part of the prototile in any sense, they only suggest where the old unit segments were before the replacement by the broken lines was performed).

5.3 More dimensions

We shall now use a modification of the idea from Subsection 5.1 in order to construct a solid in $\mathbb{E}^3$ admitting a prescribed number of noncongruent tessellations. The result will then be generalized to $\mathbb{E}^d$ for any d, $d \geqslant 3$.

Let us first describe the announced solid. The solid will depend upon a given positive integer n, and with that it mind, we shall say that it has n *pillars*. We describe the construction step by step.

- We start from three rectangular boxes of dimensions $3n \times 4 \times 3$ (width $\times$ depth $\times$ height); the exact dimensions are, of course, not crucial, but we specify them for the reader's convenience, who can follow the description by referring to Figure 10. These three boxes are arranged in a "staircase" form, connecting along one edge (to give a more formal description, if two opposite corners of one of them have coordinates $(0, 0, 0)$ and $(3n, 4, 3)$, then the corresponding two corners of the next box have coordinates $(0, 4, 3)$ and $(3n, 8, 6)$, and similarly for the third one).

- Additionally, there is a “rod” (a $3n \times 1 \times 1$ box) placed at the joint of the upper face of the first box and the front face of the second box, and analogously for the second and the third box. On the other hand, such a rod has been cut out from the lower-back edge of the second box, and also of the third box. (Altogether, in other words, two arrays of unit cubes, of length $3n$ each, have been translated in the direction to the front for the distance of 4 units.) These rods are not essential to the construction, their only role is to make the interior of the constructed solid connected.

- There is an asymmetric “hook” protruding from the right face of the top box, and the matching hole in the corresponding place in its left face. The only way to fill that hole is to put the hook in it, and vice versa (the only way to accomodate the hook is to put it in the hole). Also, the top side of the top box has an attached asymmetric bump to it, and there is a matching nick in the corresponding position in the bottom side of the bottom box (for them also holds that they can be matched only with each other).

- The bottom box has n pillars attached to its top side. The distance between every two consecutive pillars is 2 units, and the leftmost and the rightmost pillars are at the distance of 1 unit from the left, respectively right end of the box. All the pillars with the exception of the rightmost one are 3 units tall (that is, their top is at the same height as the top of the second box), while the rightmost one is 6 units tall (that is, its top is at the same height as the top of the third box, ignoring the bump). The second box has n holes cut through it (matching the pillars), and the third box has 1 hole cut through it, at the position corresponding to the rightmost pillar. The bases of these pillars should be an *asymmetric* shape (in Figure 10 they are shown in a simplified form, with square bases), ensuring that the pillars can be matched with holes only by translations (and the pillars themselves are translations of each other). It can be demonstrated that the shape described so far satisfies the statement of Proposition 14. However, the proof can be a little bit simplified if we introduce one further modification: we distort the top portion of the rightmost pillar, and distort the hole in the top box in the corresponding way, so that the pillar and the hole can be matched only with each other (by a translation along the front-back direction). With this final touch, the description of our protocell is complete.

We now show the following proposition.

Proposition 14. *The described protocell with n pillars tessellates the space in $\lfloor \frac{n}{2} \rfloor + 1$ noncongruent ways.*

Proof. Consider one such described cell in the space. We first note that the “hook” and the corresponding hole force the placement of infinitely many cells along the horizontal axis, that is, we reach a “staircase” that is infinitely wide. Further, the bump on the top and the matching hole at the bottom force the unique prolongation of the “staircase”

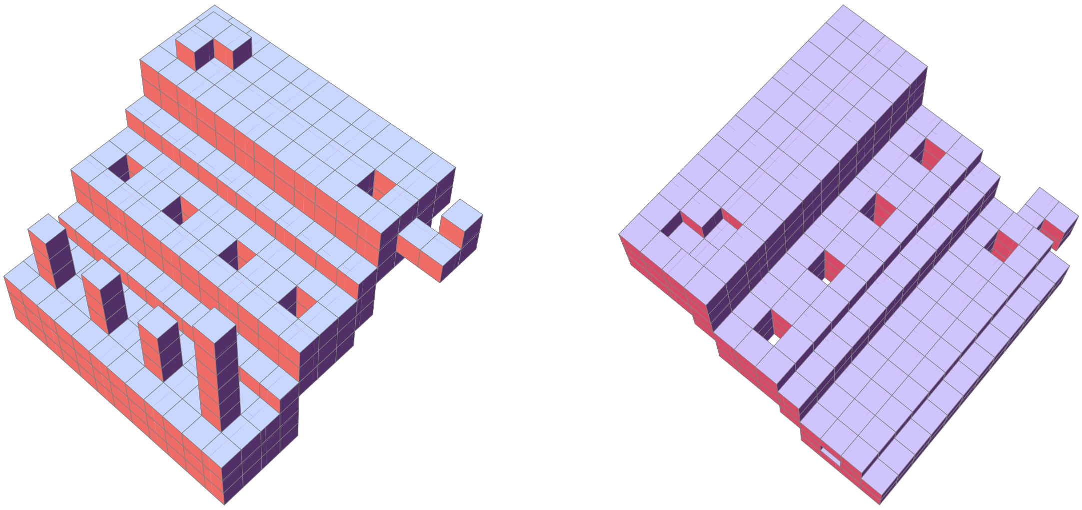

Figure 10: The protocell with 4 pillars (top and bottom view).

to infinitely many "stairs". Let us call one such (infinite and infinitely wide) staircase a *stairway*.

Note that the taller pillars on this stairway, due to the distortions introduced in the last step of the construction, force the placement of a translated copy of such a stairway, where the translation is along the front-back direction and for the distance of 8 units. By repeating this argument, we conclude that the considered stairway uniquely forces placements of infinitely many such stairways, where every two consecutive ones are translated by 8 units with respect to each other.

We now observe that such a conglomerate of stairways leaves an unfilled part of the space that is exactly of the same form. Having in mind a similar argument as in the proof of Proposition 10, we see that a conglomerate of stairways that fills this unfilled part can be shifted with respect to the first conglomerate in a total of n ways; as then, some of them produce congruent tessellations, and after taking this into account, there remain $\lfloor \frac{n}{2} \rfloor + 1$ noncongruent tessellations. One of them is shown in Figure 11 (note, in particular, how the top part resembles the tiling from Subsection 5.1). $\square$

In order to generalize this result to $\mathbb{E}^d$, we first show the following lemma.

Lemma 15. *Let $d \geqslant 2$. Assume that an m-morphic protocell C in $\mathbb{E}^d$ exists, where in each of the m tessellations admitted by C all the cells are translatory copies of each other. Further assume that C can be formed by joining some d-dimensional unit hypercubes facet to facet. Then there exists an m-morphic protocell in $\mathbb{E}^{d+1}$ such that, in each of the m tessellations admitted by it, all the cells are translatory copies of each other, and furthermore, this new protocell also can be formed from unit hypercubes in $\mathbb{E}^{d+1}$.*

Proof. Let us first define an auxiliary hypersolid Q in $\mathbb{E}^{d+1}$. It is constructed as follows. We let Γ be a fixed asymmetric d-dimensional shape inside the hypercube $[0,1]^d$. We now start from the hypercube $[0,1]^{d+1}$ and, for each of the first d coordinate axes, we

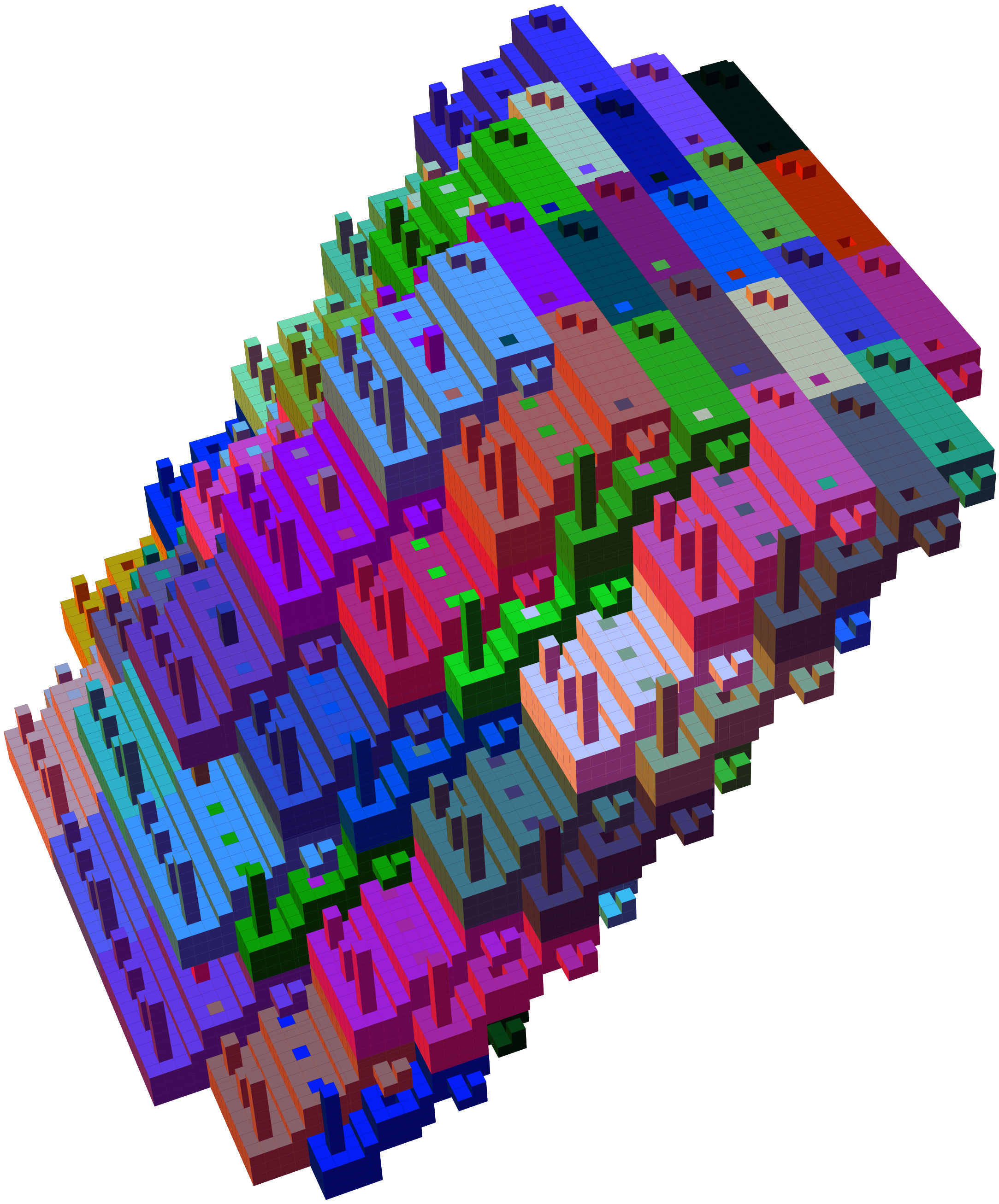

Figure 11: One of the tessellations admitted by the protocell with 4 pillars.

drill a hole of shape Γ, with some fixed depth l, through the first facet orthogonal to the considered axis, and we attach the (hyper)prism with base Γ and height l to the second facet orthogonal to the considered axis. To make this more formal, this description can be summarized as:

$$Q = \Bigg([0,1]^{d+1} \cup \bigcup_{i=1}^{d} \{(x_1, \ldots, x_{i-1}, p, x_i, \ldots, x_d) : (x_1, x_2, \ldots, x_d) \in \Gamma,\ 1 < p \leqslant 1 + l\} \Bigg) \setminus \bigcup_{i=1}^{d} \{(x_1, \ldots, x_{i-1}, p, x_i, \ldots, x_d) : (x_1, x_2, \ldots, x_d) \in \operatorname{int}\Gamma,\ 0 \leqslant p < l\}.$$

Now, assume that a protocell C is given in $\mathbb{E}^d$ (satisfying the conditions from the statement). We can describe C as $C = \bigcup_{i=1}^{s}([0,1]^d + t_i)$ for some vectors $t_1, t_2, \ldots, t_s$ from $\mathbb{Z}^d$. Let t'_i be the vector from $\mathbb{Z}^{d+1}$ obtained by attaching 0 to the end of t_i, and let ❟ be an asymmetric shape inside C. We define the protocell C' in $\mathbb{E}^{d+1}$ by:

$$C' = \left(\bigcup_{i=1}^{s} (Q + t'_i) \cup (\text{❟} \times (1, \tfrac{6}{5}]) \right) \setminus (\operatorname{int} \text{❟} \times [0, \tfrac{1}{5})).$$

When explained this way, all this looks much more complicated than it really is. Basically, given a protocell C in $\mathbb{E}^d$, we add 1 unit of “thickness” to it along the $(d+1)^{\text{st}}$ axis, we add some Γ-shaped “hooks” and matching holes to the lateral sides of the obtained structure (assuming that the $(d+1)^{\text{st}}$ axis is “vertical”), and add an asymmetric bump to the top side and the matching hole to the bottom side (its height, respectively depth, is here taken to be $\frac{1}{5}$, though this value does not really matter). In Figure 12 we show an example for $d = 2$, where C is the L-tetromino (for the sake of this illustration, we ignore the fact that the L-tetromino actually tiles the plane in infinitely many ways). The shape C' obtained in the described way is shown from the top and from the bottom. Note that, strictly speaking, due to the bump in the form of a comma, the presented structure cannot be built from cubes; however, that form of a bump is chosen only for a prettier visual effect, any other asymmetric (and distinctive enough) form would work equally as well, and thus it could also be chosen to be composed of cubes (and, of course, if we want it to be composed of *unit* cubes, in the end we may simply scale everything by the appropriate factor).

Now, assume that the space $\mathbb{E}^{d+1}$ is tessellated by cells congruent to C'. Consider one of the cells. Its Γ-hooks and corresponding holes imply that all the cells that have a common part (of nonzero measure) with a lateral side of the considered cell have their top and bottom side in level with the top and the bottom side of the considered cell, and all these cells (including the initial one) have the bump on the same side. By applying the same argument to those neighboring cells and then iterating it again and again, we reach the conclusion that all the cells listed this way constitute a layer of the form $\mathbb{E}^d \times [0,1]$ (of course, ignoring the bump). The arrangement of cells forming this layer mimics a tessellation of $\mathbb{E}^d$ by cells congruent to C. Now note that the bumps on these cells force the position of an identical layer immediately above this one, and similarly the holes force

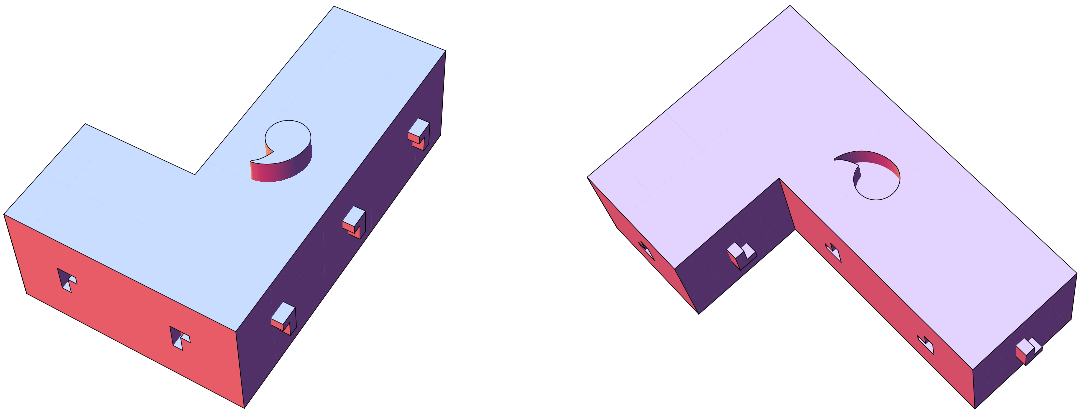

Figure 12: The protocell C' obtained from the L-tetromino (top and bottom view).

the position of an identical layer immediately below this one. By iterating this argument we conclude that the considered layer uniquely forces positions of all the cells in the tessellation of $\mathbb{E}^{d+1}$.

Therefore, we only have to count how many such layers can be made from cells congruent to C'. As we have already mentioned, each such layer mimics a tessellation of $\mathbb{E}^d$ by cells congruent to C. Recall that (by the conditions of the lemma) there are m such noncongruent tessellations, and that each of them consists of translatory copies of C. Therefore, by adding 1 unit of thickness along the $(d+1)^{\text{st}}$ axis, and adding Γ-hooks and the corresponding holes, as well as the bumps and their corresponding holes, we see that each such tessellation can be extended to a described layer (the fact that the tessellation is obtained only by translations ensures that the Γ-hooks and holes will be placed consistently with each other, that is, all the obtained $(d+1)$-dimensional cells will indeed be congruent to C'). This shows that C' is m-morphic, and it is also clear that, in each of the m tessellations admitted by it, all the cells are translatory copies of each other. The proof is thus finished. $\square$

The following theorem is now a direct corollary.

Theorem 16. *Given a positive integer d, $d \geqslant 3$, in the space $\mathbb{E}^d$ there exists an m-morphic protocell for every positive integer m.*

Proof. If $d = 3$, the protocell from Proposition 14 with $2m - 2$ (or $2m - 1$) pillars is m-morphic. Also note that each of these m tessellations is by translations only. Therefore, the conclusion for $d > 3$ can be obtained by iteratively applying Lemma 15 the necessary number of times. $\square$

6 σ-morphic figures

We show that there exists a σ-morphic colored prototile, and a σ-morphic protocell in $\mathbb{E}^d$ for each d, $d \geqslant 3$.

6.1 Colored tiles

We shall now describe a prototile with colored edges admitting countably many noncongruent tessellations. We start from a square $ABCD$, and replace each of its sides by a 5-segment broken line. These four broken lines should all be translations/rotations of each other, each of them should be centrally symmetric, but apart from that, they should exhibit no other regularities. We now color each segment of each of these lines either black or red as follows (where "0" stands for a black segment, and "r" stands for a red segment):

- AB: $rr000$;
- BC: $rr00r$;
- CD: $rrr00$;
- DA: $0rr00$.

See Figure 13 for the final result. The matching rule is that a red segment cannot be placed against another red segment (the other combinations are permitted).

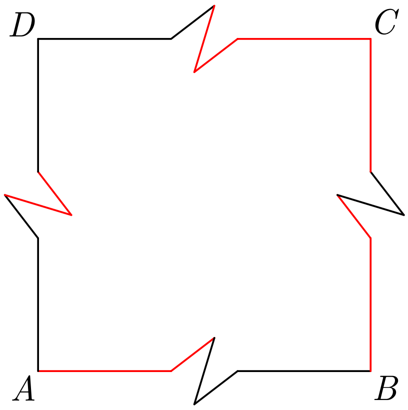

Figure 13: A σ-morphic colored prototile.

We now show the following theorem.

Theorem 17. *There is a σ-morphic prototile with colored edges.*

Proof. Of course, we are proving that the prototile just described is σ-morphic.

It is easy to see that, in any tiling admitted by this prototile, the tiles have to be arranged in the square lattice. We can also check that the colors together with the matching rule leave only these possibilities for the pairs of broken lines that can be placed against each other:

- AB against BA;
- AB against DC;

- AB against AD;
- BC against AD.

Consider a tile in the plane oriented as in Figure 13 (for the rest of the proof, we shall refer to this as the *natural orientation*). Since BC can only be placed against AD, this forces a translated copy of the considered tile immediately to the right of it. Also, since DC can only be placed against AB, a translated copy of the considered tile is forced immediately above it. Iterating this argument leads to the conclusion that the considered tile forces the whole quadrant to the right of and above it of its translated copies.

Let T be a tile in the natural orientation, and assume that the tile immediately to the left of it is oriented differently. Note that then all the tiles to the right of T are in the natural orientation, and none of the tiles to the left of T can be in the natural orientation. Let us now consider the row immediately below this one. We claim that, if there exists a tile in the natural orientation in that row, then all such ones are the tile below T and all the tiles to the right of it, and no other tile. Indeed, we first note that no tile from that row to the left of T can be in the natural orientation (because then the tile immediately above it would also have to be in the natural orientation, which we know that it is not). Therefore, in the considered row there exists the leftmost tile in the natural orientation, say T'. Suppose the contrary: it is positioned somewhere to the right of T. Then, the tile immediately to the left of T' is not in the natural orientation, and it is placed against the line AD of T', which leaves only the possibility that it is oriented 90 degrees counterclockwise with respect to the natural orientation. But then its top part, which is CB, is placed against AB from the tile immediately above (since that tile is in the natural orientation), which is a contradiction.

Therefore, if there is a tile in the natural orientation that is the leftmost such tile in some row, then in each row that contains such tiles there is the leftmost such tile, and all those leftmost tiles are along the same vertical (and additionally, all the rows with such configurations are consecutive). In a similar way we prove the same claim for bottommost tiles in the natural orientation: if there is such a one in some column, then in each column that contains tiles of the natural orientation there is the bottommost one, always at the constant height.

We now conclude: if there is no leftmost nor bottommost tile in the natural orientation, then (assuming that at least one such tile exists, which we can do without loss of generality) we have the periodic tiling by translations of such a tile, as shown in Figure 14 top left. Otherwise, all the tiles in the natural orientation form either a half-plane or a quadrant. Applying the same logic to tiles in other orientations, we conclude that each group of tiles oriented in the same way forms either a half-plane or a quadrant, and we are left only to see in how many ways they can be combined for the completion of the tiling.

Note that the half-planes obtained this way have the border composed either of AB-lines or of AD-lines, while the quadrants have one border composed of AB-lines and the other border of AD-lines. Therefore, if two half-planes are combined for a tiling, there are two possibilities: in one of them AB-lines are placed against BA-lines, while in the

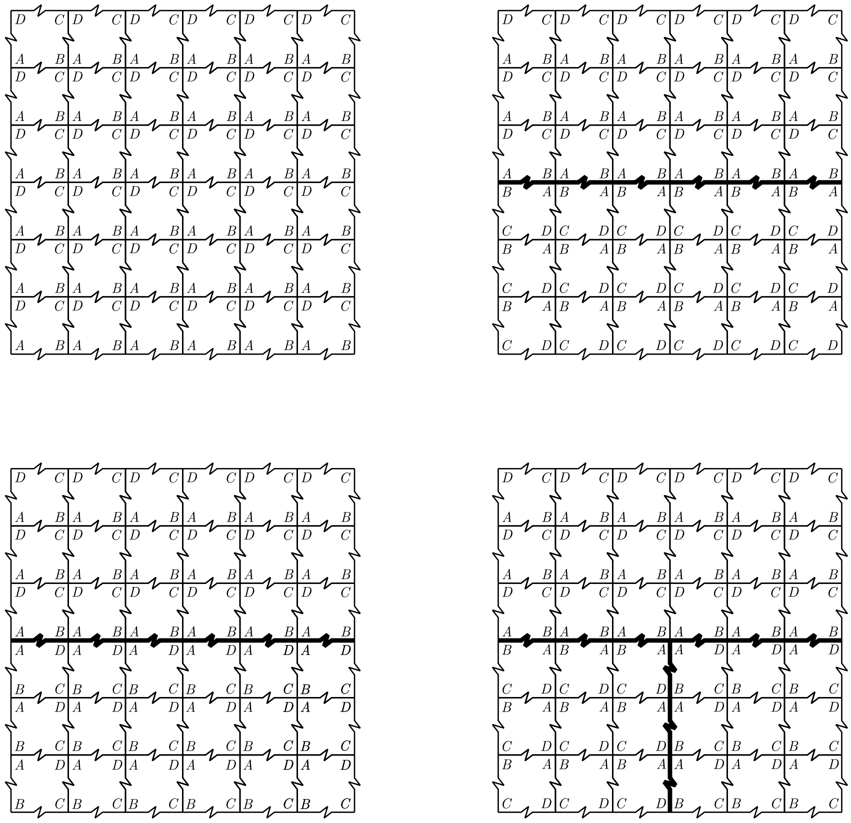

Figure 14: The four disparate tilings.

other one AB-lines are placed against AD-lines (and recall that it is not permitted to place AD-lines against DA-lines); see Figure 14, top right and bottom left. If one half-plane is combined with two quadrants, there is the unique way to do it, shown in Figure 14 bottom right. And finally, if we are combining four quadrants, we get a countable family of noncongruent tilings, following the pattern shown in Figure 15 (the two vertical borders can be separated by the distance of any nonnegative integer); this completes the proof. □

6.2 More dimensions

Finally, we reach what maybe is the main result of this article: we describe a solid in $\mathbb{E}^3$ admitting infinitely but only countably many noncongruent tessellations. The result will then be generalized to $\mathbb{E}^d$.

Our starting point is the 12-omino shown in Figure 16; due to its shape, we shall refer to it as the *faucet*. As discovered by Myers [19], it is dimorphic, that is, tiles the plane in exactly two noncongruent ways, which are shown in Figure 17. Note that the tiling shown on the left is periodic (a patch that periodically repeats is outlined by a thick line), but the one on the right is not: the shaded (doubly-infinite) region appears nowhere else in the tiling. We call the shaded region the *frontier*; it will play a key role in our construction.

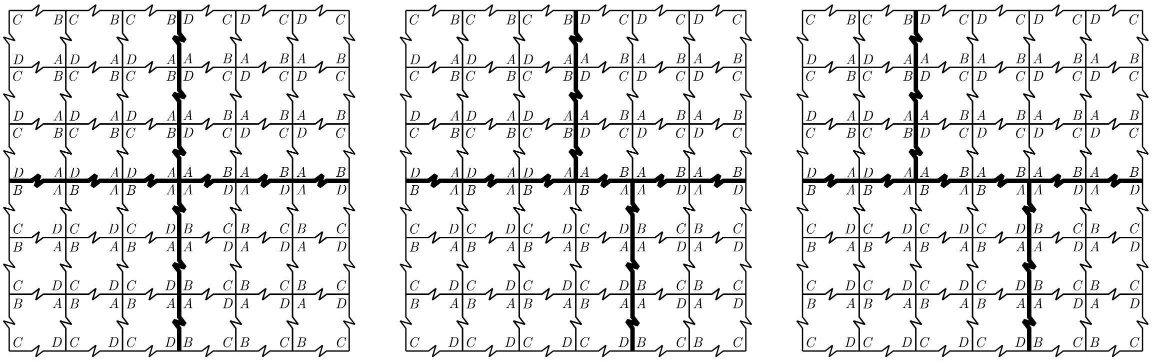

Figure 15: Countably many tilings composed of 4 quadrants.

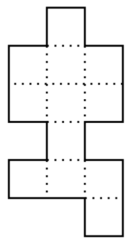

Figure 16: The faucet.

We now describe our solid. It is formed as follows.

- The basic structure that we start from is a right prism of height 1 whose base is the faucet. Basically, we arrange 12 unit cubes to obtain the form of the faucet.
- Each unit cube has four pillars on its top face, four holes in the bottom face, and four tunnels all the way from bottom to top. We describe them in more details.
  - The base of each pillar is a small square, and its height is (e.g.) 1.3. Additionally, each pillar has an attached "bead" at height around 0.3; each bead is shaped like a small cube (but larger than the base of the pillars), and the pillar is centered through it.
  - Each hole is of depth 0.3, and its base is the same as for the pillars.
  - The base of each tunnel is again the same, and they have an additional cavity shaped like the beads on the pillars, at the same height as the beads on the pillars are.

  All pillars, holes and tunnels are placed on the diagonals of the top/bottom face of the unit cube. The four pillars are placed in the vertices of a square of side length 0.7, the four holes in the same positions directly below the pillars, and the four tunnels are placed in the vertices of a square of side length 0.3. The exact values 0.7 and 0.3 are not so important, but the point is that four pillars grouped around

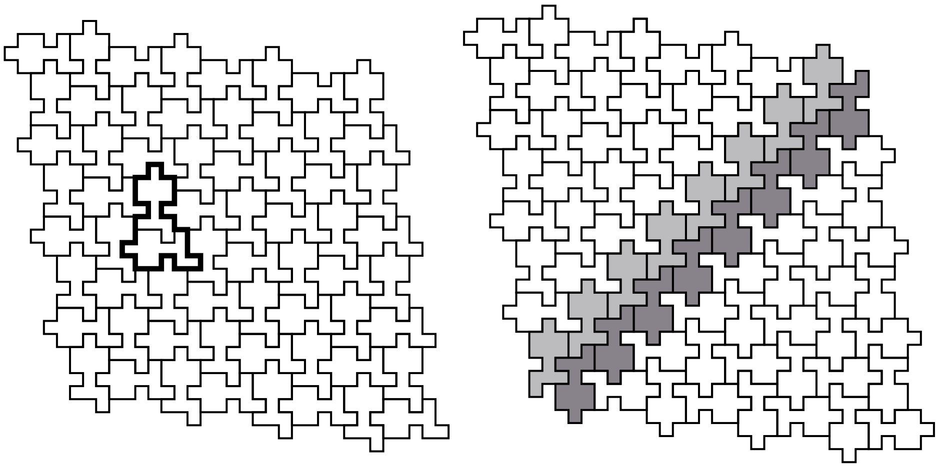

Figure 17: The two tilings by the faucet.

a common vertex of four *neighboring* cubes form a structure that matches the four tunnels of a single cube.

- Among all the $12 \cdot 4$ pillars, one is different from the others, in the following way. It has a kind of “pompom” attached near the top, which is of an asymmetric shape, and attached to only one side of the pillar (that is, the symmetry is broken as much as possible). In the same unit cube, the hole directly below this pillar has an additional “pocket” that matches this pompom (by shape and by position).

In Figure 18 we see a blueprint for the described protocell: the left-hand part shows the side and the top view of a single building cube, while the right-hand part shows the top view of the whole protocell. The pillar with the pompom is the rightmost bottommost one (and it is the unit cube with this pillar that is the one shown on the left). In Figure 19 we see the described protocell in 3D.

We are now ready for the following proposition.

Proposition 18. *The described protocell tessellates the space in countably many ways.*

Proof. We shall start from the following claim.

Claim 19. *Assume that the space is tessellated by the considered cells. Let C be one of the cells, and let α be the plane through centers of all the building unit cubes of C. Then all the lateral neighbors of C have centers of their building unit cubes in α, and all of their pillars point in the same direction as the pillars of C do. By lateral neighbors of C we mean those cells that have a common part (of nonzero area) with one of the lateral sides of the building unit cubes of C (that is, with one of those sides that are not parallel with α).*

Proof of the claim. For the sake of simplicity, let α be a horizontal plane, say the one where the z-coordinate is 0. Furthermore, assume that all the building unit cubes of C have centers at integer coordinates (therefore, in particular, from $\mathbb{Z}^2 \times \{0\}$).

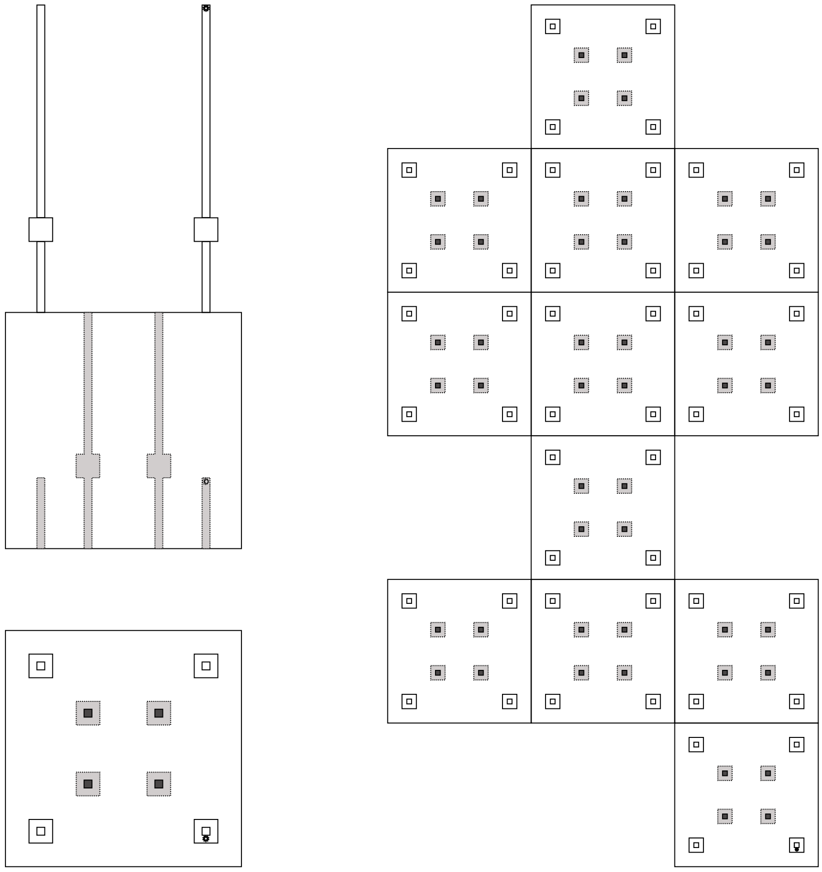

Figure 18: Blueprint for the described protocell.

Note that the pompom of C forces the position of a translated copy of C directly above C and 1 unit apart. (In fact, by iterating this argument, we see that positions of infinitely many translated copies of C are forced, all along the same vertical and where every two consecutive ones are 1 unit apart. This will be relevant later in the proof, but is of no relevance for the current claim.) There are some pillars between these two cells. Note that beads on these pillars can be accommodated only by another copy of C whose pillars also point upwards, and whose centers of the building unit cubes are shifted along the both horizontal axes by $\frac{1}{2}$ with respect to the integer lattice (in other words, the coordinates of the mentioned centers belong to $(\mathbb{Z}+\frac{1}{2})^2 \times \{1\}$); see Figure 20. Note: a cell that is sandwiched between the other two (one such is shown in a paler tone in Figure 20) does not necessarily have to be a translated copy of those two, it could be rotated and/or mirrored; but it has to be "horizontal", with additional details about its placement as described.

Now, note that, wherever such an intermediate cell is "leaning over" an edge of the initial cell C, we conclude that a lateral neighbor of C immediately below this cell indeed has all centers of the building unit cubes in α, and its pillars point upwards, which was to be proved. Since for each top edge of C we can find such a cell that is leaning over

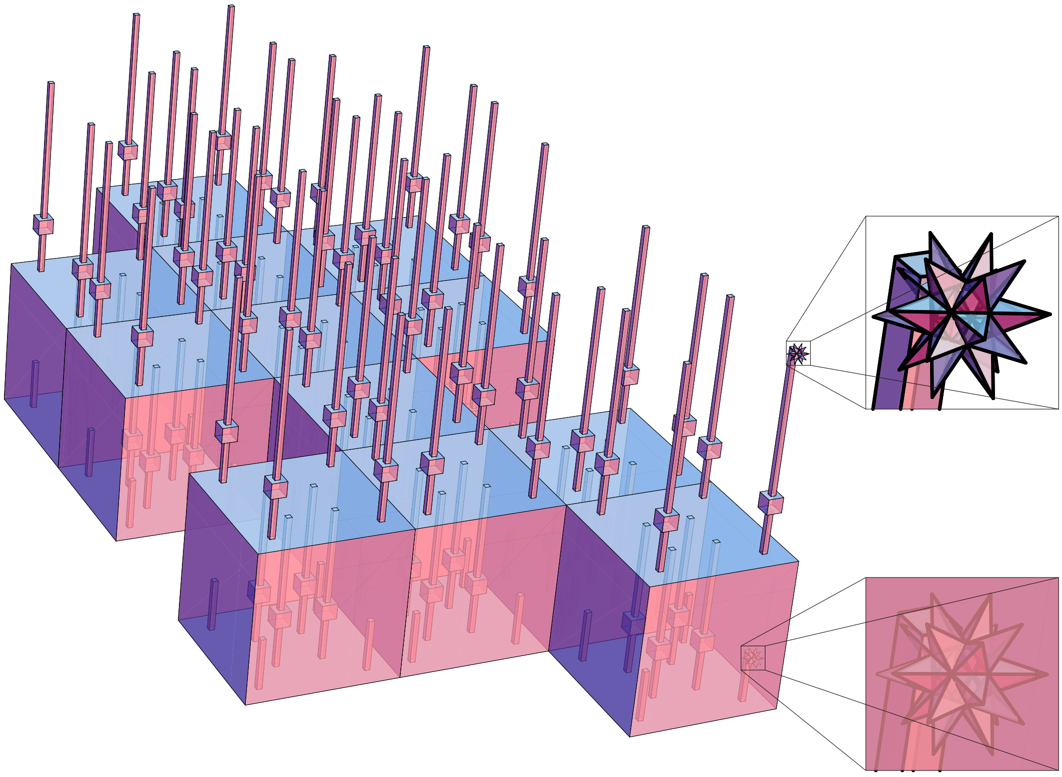

Figure 19: The described protocell in 3D.

that edge (the outermost pillars with their beads guarantee this), the proof of the claim is finished. □

Given a cell C, by applying this claim to it, then to its lateral neighbors, then to their lateral neighbors and so on, we conclude that all the cells listed this way, when pillars and holes are ignored, form a flat layer 1 unit thick. Clearly, the arrangement of cells forming this layer mimics a tiling of the plane by the faucet. Further, the argument from the beginning of the proof of the claim shows that the cells from this layer force infinitely many exactly the same layers (where "exactly the same" means that they differ only by a translation along the direction of the pillars), where every two consecutive layers are 1 unit apart. The (so far) untessellated part of the space is of completely the same shape as the filled part, and we similarly conclude that it is also composed of layers that all mimic mutually the same tiling of the plane by the faucet (however, that tiling is not necessarily the same as the one from the layers from the previous sentence).

Therefore, we only have to count in how many noncongruent ways the two tilings from those two conglomerates of layers can be combined. Note that, if both those tilings are the one from Figure 17 left (including its rotated/mirrored versions), then, because of the periodicity, they can be positioned with respect to each other in only finitely many ways. Similarly, if one of the two tilings is the one from Figure 17 left and the other one from the right, then the first one can be positioned with respect to the second one in only

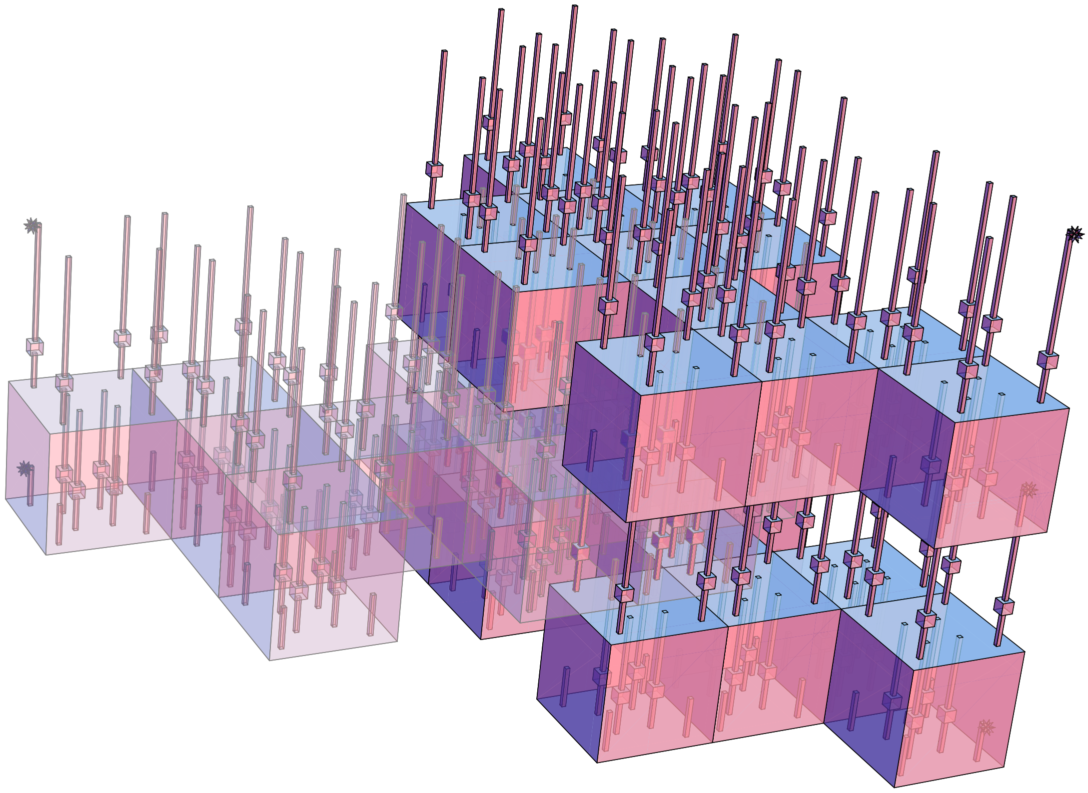

Figure 20: Two cells 1 unit apart, and a cell between them.

finitely many ways. Now assume that both the tilings are those from Figure 17 right. Then, if their frontiers do not have the same direction (actually, the only possibility for that is that they are orthogonal), we again have only finitely many their noncongruent mutual positions. Finally, assume that their frontiers have the same direction. Then note that the two frontiers can be arbitrarily far from each other, which means that in this case there are countably many ways the two tilings can be positioned with respect to each other. (To be more precise, we actually have two countable families of noncongruent tessellations: in one of them the two considered tilings both look exactly as on Figure 17 right, while in the other family one of the two tilings is as on Figure 17 right while the other one is mirrored and rotated for $\frac{\pi}{2}$. But this does not matter, since all this together, plus the finitely many tessellations we have previously described, is still countable.)

The proof is thus completed. $\square$

And we conclude by appropriately generalizing this result to more-dimensional spaces.

Theorem 20. *Given a positive integer d, $d \geqslant 3$, a σ-morphic protocell exists in the space $\mathbb{E}^d$.*

Proof. Given a σ-morphic protocell in $\mathbb{E}^d$ that is formed by joining some d-dimensional unit hypercubes facet to facet, we shall show how to obtain a σ-morphic protocell in $\mathbb{E}^{d+1}$. This is enough since the protocell from the previous proposition can be indeed formed from unit cubes (note that the only part in Figure 19 that is not drawn as a cuboidal structure is the pompom, but its shape is almost arbitrary, and thus can also be formed from small cubes); therefore, by iterating the construction that is going to be described, we can obtain a σ-morphic protocell in the Euclidean space of an arbitrary dimension.

The idea is actually completely the same as the "transition" that was used between the (two-dimensional) faucet and the (three-dimensional) protocell figuring in Proposition 18. Let us extract the core of that construction. For each unit square, a thickness was added along the new coordinate axis (thus obtaining a 3D cube), and pillars, tunnels and holes were constructed along that axis arranged in the described pattern. If we want to build a protocell in $\mathbb{E}^{d+1}$ from a protocell in $\mathbb{E}^d$, we perform the same steps on each of the building d-dimensional unit hypercubes (not forgetting, after all that, to add a pompom, together with the corresponding pocket, on exactly one pillar of all the pillars attached to the whole protocell). In Figure 21 we see how the 4-dimensional unit hypercube obtained this way looks like. (Note: a little bit different idea is also possible. Instead of first chopping everything up into small cubes and then adding pillars etc. to each of them, we can immediately add pillars and the rest only to the main 12 cubes. In this case it is not hard to see that, with each new dimension, the structures added for that dimension can indeed be matched only with the corresponding structures added for the same dimension, not with any structures added for some previous dimension. But we felt that the approach described first was a little bit smoother to follow, though this is probably a matter of taste.)

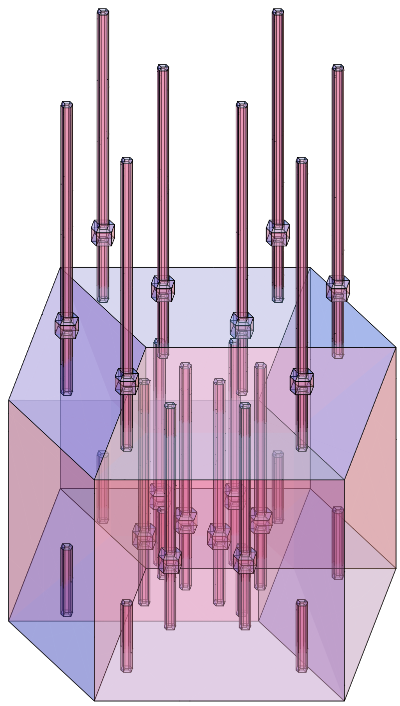

Figure 21: Four-dimensional unit hypercube with "decorations".

In the same way as in the proof of Proposition 18, we conclude that we only have to count in how many noncongruent ways two tessellations of the space $\mathbb{E}^d$ by the original σ-morphic protocell can be positioned with respect to each other. For each of the two tessellations we can choose one of countably many possibilities. If we fix a point of origin in each of the two of them, we see that there are countably many different mutual positions (to be more precise, *at most* countably many, since many of them could be congruent). Taken all together (since $\aleph_0^3 = \aleph_0$), there are no more than countably many noncongruent tessellations of $\mathbb{E}^{d+1}$. But clearly, the number of noncongruent tessellations is infinite, since the protocell in $\mathbb{E}^d$ that we started from is σ-morphic (and thus its obtained derivative tessellates the space $\mathbb{E}^{d+1}$ in at least countably many ways). This completes the proof. $\square$

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