

How Many Cliques Can a Clique Cover Cover?

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Abstract

This work examines the problem of clique enumeration on a graph by exploiting its clique covers. The principle of inclusion/exclusion is applied to determine the number of cliques of size r in the graph union of a set $\mathcal{C} = \{c_1, \dots, c_m\}$ of m maximal cliques. This leads to a deeper examination of the sets involved and to an orbit partition, Γ , of the power set $\mathcal{P}(\mathcal{N}_m)$ of $\mathcal{N}_m = \{1, \dots, m\}$. Applied to the cliques, this partition gives insight into clique enumeration and yields new results on cliques within a clique cover, including expressions for the number of cliques of size r as well as generating functions for the cliques on these graphs. The quotient graph modulo this partition provides a succinct representation to determine cliques and maximal cliques in the graph union. The partition also provides a natural and powerful framework for related problems, such as the enumeration of induced connected components, by drawing upon a connection to extremal set theory through intersecting sets.

Mathematics Subject Classifications: 05A15, 05C30, 05C69

1 Introduction

For any graph $G = (V, E)$ and node subset $H \subseteq V$, the induced subgraph $G[H]$ has nodes H and those edges in E whose endpoints lie in H . A clique of size r is induced whenever $G[H]$ is a complete graph on r nodes. Allowing trivial cliques (i.e., $r = 1$ or $r = 2$), a

collection of cliques $\mathcal{C} = \{c_1, \dots, c_m\}$ can always be found (for some m) which *covers* the graph G – in the sense that the graph union, $G[c_1] \cup G[c_2] \cup \dots \cup G[c_m]$, of the induced subgraphs has the same vertex set as G .

Such a collection is called a *vertex clique cover* of G . A collection of cliques whose graph union contains all edges in G is called an *edge clique cover*.

This work sets to enumerate the number of r -cliques formed by the graph union of an edge clique cover, for any edge clique cover.

Section 2 outlines a method for determining the number of cliques in a graph G when the clique collection consists of all maximal cliques. Section 3 introduces the Γ partition, which can be applied to any union of sets, and a motivating example which we use for illustration throughout the remainder of the paper.

Section 3.1 shows that Γ partitions are orbit partitions and motivates the notion of compressing graph information via quotient graphs. In Section 3.2, the concept of signatures is introduced. Signatures provide a notion of graph isomorphism that takes into account the additional structure provided by the partition. Section 3.2.1 provides counting results regarding the signatures. Section 3.2.2 counts the number of connected induced subgraphs using signatures. Section 3.2.3 investigates the necessary and sufficient conditions to construct a clique using the quotient graph derived from the Γ -partition. These results are extended in Section 3.2.4 where we examine maximal cliques and the clique number.

Section 4 obtains clique count expressions for cliques that contain a particular subgraph H , establishes the clique count generating function of a clique collection and derives several expressions for the clique counts. Finally, Section 5 summarizes the results of this work and discusses potential research directions.

2 Counting by inclusion/exclusion

We begin by enumerating the r -cliques belonging to at least one of the members of the clique collection $\mathcal{C} = \{c_1, \dots, c_m\}$. Clearly, each of these cliques will appear in the union of the clique collection.

Proposition 1. *Let $\mathcal{C} = \{c_1, \dots, c_m\}$ be a collection of cliques. The number of r -cliques that are contained in at least one c_j for some $j \in \{1, \dots, m\}$ is*

$$\sum_{J: \emptyset \neq J \subseteq \{1, \dots, m\}} (-1)^{|J|+1} \binom{I_J}{r},$$

where $I_J := |\bigcap_{j \in J} c_j|$.

Proof. Let $\binom{c_j}{r} := \{\{v_1, \dots, v_r\} \subseteq c_j : v_1 \neq \dots \neq v_r\}$ denote the set of r -cliques contained in the clique c_j . We will prove that for any nonempty $J \subseteq \{1, \dots, m\}$,

$$\left| \bigcap_{j \in J} \binom{c_j}{r} \right| = \binom{I_J}{r},$$

by showing that

$$\bigcap_{j \in J} \binom{c_j}{r} = \binom{\bigcap_{j \in J} c_j}{r}.$$

If $\{v_1, \dots, v_r\} \in \bigcap_{j \in J} \binom{c_j}{r}$, then $\{v_1, \dots, v_r\} \subset c_j$ for all $j \in J$ and so

$$\{v_1, \dots, v_r\} \in \binom{\bigcap_{j \in J} c_j}{r}.$$

Conversely, if $\{v_1, \dots, v_r\} \in \binom{\bigcap_{j \in J} c_j}{r}$ then $\{v_1, \dots, v_r\} \subset c_j$ for all $j \in J$. Therefore,

$$\{v_1, \dots, v_r\} \in c_j,$$

for all $j \in J$ and the claim follows.

Therefore, the total number of r -cliques which are contained in at least one clique in $\mathcal{C}$ is $\left| \bigcup_{j \in \{1, \dots, m\}} \binom{c_j}{r} \right|$. By the principle of inclusion/exclusion (Wilf, 2005, p. 112),

$$\left| \bigcup_{j \in J} \binom{c_j}{r} \right| = \sum_{\emptyset \neq J \subseteq \{1, \dots, m\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} \binom{c_j}{r} \right| = \sum_{\emptyset \neq J \subseteq \{1, \dots, m\}} (-1)^{|J|+1} \binom{I_J}{r},$$

as needed to be shown. $\square$

We note that this is a lower bound for the number of r -cliques contained in the graph union of $\mathcal{C}$. This is because there could be cliques that arise from the graph union which do not properly belong to any member of the collection. For example, consider the collection $\mathcal{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. While this collection of cliques induces the triangle $\{1, 2, 3\}$, clearly the triangle is not contained within any individual clique.

When the collection $\mathcal{C}$ includes all the maximal cliques in $G = \bigcup_{C \in \mathcal{C}} C$, the lower bound is an equality.

Proposition 2. *Let $\mathcal{C} = \{c_1, \dots, c_m\}$ be the collection of all maximal cliques in G . The number of r -cliques induced by $\mathcal{C}$ is*

$$\sum_{J: \emptyset \neq J \subseteq \{1, \dots, m\}} (-1)^{|J|+1} \binom{I_J}{r},$$

where $I_J := \left| \bigcap_{j \in J} c_j \right|$.

Proof. Proposition 1 implies that the provided expression serves as a lower bound, as every clique present in a member c_j of $\mathcal{C}$ is necessarily a clique in G . Therefore, it remains to establish that all cliques in G belong to some $c_j \in \mathcal{C}$. Consider a clique H in G , which is a subset of some maximal clique H' belonging to $\mathcal{C}$. As such, H is contained in a member of $\mathcal{C}$. $\square$

Note that it is not sufficient that $\mathcal{C}$ consist of maximal cliques in the graph union G – all of the maximal cliques of G must be in $\mathcal{C}$. Otherwise, Proposition 2 can only provide a lower bound on the number of r –cliques in G . The bound is raised as the size of each clique in $\mathcal{C}$ is increased, reaching the actual value only if $\mathcal{C}$ consists of all maximal cliques in its graph union.

To get the exact number of r –cliques from any clique cover $\mathcal{C}$, a different approach must be taken, one based on a special graph partition, we call a Γ –partition. This more general theory is developed in the next section.

3 Γ –partitions

This section presents an approach to solving the problem of determining the induced number of cliques in the graph union of a given collection of cliques that cover a graph. A special partition of the node set is introduced that captures the membership of nodes in different cliques. Figure 1 provides a concrete example that illustrates the concepts presented in the rest of the paper.

Proposition 3 details the process of constructing the partition. In this work, $\overline{J}$ denotes the complement of a set $J \subseteq \mathcal{N}_m$ with respect to the set $\mathcal{N}_m$, and is formally defined as $\overline{J} := \mathcal{N}_m \setminus J$.

Proposition 3. *For any $m \geq 1$, given a sequence $(A_i)_{i=1}^m$ of subsets of $\mathcal{N}_n = \cup_{i=1}^m A_i$, the family of sets given by*

$$\Gamma := \left\{ \bigcap_{i \in J} A_i \setminus \left(\bigcup_{i \in \overline{J}} A_i \right) : J \subseteq \mathcal{N}_m \right\} := \{\Gamma_J : J \subseteq \mathcal{N}_m\}$$

is a partition of $\mathcal{N}_n$. Moreover, for any $i \in \mathcal{N}_m$,

$$A_i = \bigcup_{J \subseteq \mathcal{N}_m : i \in J} \Gamma_J.$$

Proof. First, we show that

$$\bigcup_{J \subseteq \mathcal{N}_m} \Gamma_J = \bigcup_{J \subseteq \mathcal{N}_m} \left[\bigcap_{i \in J} A_i \setminus \left(\bigcup_{i \in \overline{J}} A_i \right) \right] = \mathcal{N}_n.$$

For every $J \subseteq \mathcal{N}_m$,

$$\Gamma_J = \left[\bigcap_{i \in J} A_i \setminus \left(\bigcup_{i \in \overline{J}} A_i \right) \right] \subseteq \mathcal{N}_n,$$

as each $A_i \subseteq \mathcal{N}_n$. To see the reverse inclusion, fix *any* choice $x \in \cup_{i=1}^m A_i = \mathcal{N}_n$ and let $J_x := \{i : x \in A_i\} \subseteq \mathcal{N}_m$ denote the set of all indices i with $x \in A_i$, and its complement

in $\mathcal{N}_m$ as $\overline{J_x} = (\mathcal{N}_m \setminus J_x)$. Now $x \in \mathcal{N}_n$ appears in at least one A_i , since $\cup_{i=1}^m A_i = \mathcal{N}_n$, so it follows that $x \in \cap_{i \in J_x} A_i$ and $x \notin \cup_{i \in \overline{J_x}} A_i$. Thus,

$$x \in \left[\bigcap_{i \in J_x} A_i \setminus \left(\bigcup_{i \in \overline{J_x}} A_i \right) \right] = \Gamma_{J_x}$$

for any $x \in \mathcal{N}_n$, and hence

$$\mathcal{N}_n = \bigcup_{x \in \mathcal{N}_n} \Gamma_{J_x} = \bigcup_{J \subseteq \mathcal{N}_m} \Gamma_J.$$

It remains only to show that the intersection of any two distinct non-null members of Γ is empty – the proof is by contradiction. Let $J, H \subseteq \mathcal{N}_m$ be distinct, respectively producing

$$\Gamma_J = \left[\bigcap_{i \in J} A_i \setminus \left(\bigcup_{i \notin J} A_i \right) \right] \quad \text{and} \quad \Gamma_H = \left[\bigcap_{i \in H} A_i \setminus \left(\bigcup_{i \notin H} A_i \right) \right]$$

as members in Γ . Suppose $x \in \Gamma_J \cap \Gamma_H \neq \emptyset$, then $x \in \Gamma_J \implies x \in A_i \forall i \in J$ and $x \in \Gamma_H \implies x \in A_i \forall i \in H$. Since J and H are distinct, there exists some $k \in J \setminus H$ for which $x \in \Gamma_J$ appears in A_k . Now $k \notin H$ means $k \in \overline{H}$ and hence A_k appears in the union $\cup_{i \notin H} A_i$ being removed from $\cap_{i \in H} A_i$ in the definition of Γ_H . Therefore $x \notin \Gamma_H$ and, so, $x \notin \Gamma_J \cap \Gamma_H$, a contradiction. It follows that Γ_J and Γ_H are disjoint, whenever $J \neq H$ and hence that the sets of Γ form a partition of their union, $\mathcal{N}_n$.

Finally, for any $i \in \mathcal{N}_m$, it remains only to show that the original sets A_i are the union of those Γ -sets, Γ_J , whose index set J contains i . That is,

$$A_i = \bigcup_{J \subseteq \mathcal{N}_m: i \in J} \Gamma_J.$$

If $i \in J$, then $\Gamma_J = \left[\bigcap_{j \in J} A_j \setminus \bigcup_{j \in \overline{J}} A_j \right]$ intersects A_i , and hence $\Gamma_J \subseteq A_i$ whenever $i \in J$. It follows, then, that

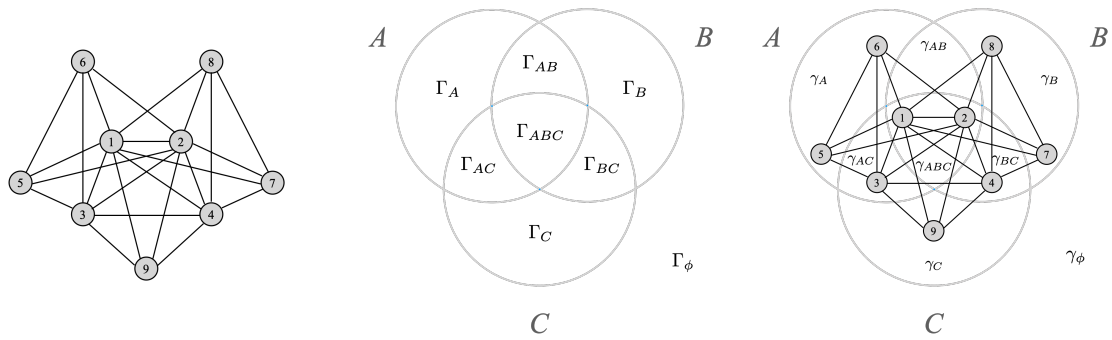
$$\bigcup_{J \subseteq \mathcal{N}_m: i \in J} \Gamma_J \subseteq A_i.$$

Conversely, for every $x \in A_i$, then $i \in J_x$ and

$$x \in \left[\bigcap_{j \in J_x} A_j \setminus \bigcup_{j \in \overline{J_x}} A_j \right] = \Gamma_{J_x} \subseteq \bigcup_{J \subseteq \mathcal{N}_m: i \in J} \Gamma_J.$$

So $A_i \subseteq \bigcup_{J \subseteq \mathcal{N}_m: i \in J} \Gamma_J \subseteq A_i$, and it follows that $A_i = \bigcup_{J \subseteq \mathcal{N}_m: i \in J} \Gamma_J$. $\square$

Note that Proposition 3 remains valid for any countable collection of sets $(A_i)_{i \in \mathbb{N}}$ and the set $\mathcal{N}_n$ replaced by any set Ω with $\Omega = \bigcup_{i \in \mathbb{N}} A_i$. However, for counting cliques over a collection of cliques, a weaker result suffices.



(a) The graph union of $\mathcal{C}$

(b) The Γ sets

(c) The Γ partition

Figure 1: $\mathcal{C} = \{A, B, C\}$ with $A = \{1, 2, 3, 5, 6\}$, $B = \{1, 2, 4, 7, 8\}$ and $C = \{1, 2, 3, 4, 9\}$.

Figure 1(a) depicts a graph covered by a collection of three 5-cliques. Corresponding partition indexing and membership classes of the nodes are depicted in Figure 1(b) and Figure 1(c), respectively.

We will call a partition produced as in Proposition 3, a Γ -partition and note that it will be peculiar to the sets A_i from which it is constructed.

For a collection of cliques $\mathcal{C} = \{c_1, \dots, c_m\}$, defined by index sets $c_j \subset \mathcal{N}_n$, with graph union $\bigcup_{j=1}^m c_j = \mathcal{N}_n$, Proposition 3 provides a general means to find Γ -sets, namely as $(\Gamma_J)_{J \subseteq \mathcal{N}_m}$ with

$$\Gamma_J = \left(\bigcap_{j \in J} c_j \right) \cap \left(\bigcap_{j \notin J} \overline{c_j} \right)$$

where complement is with respect to $\mathcal{N}_n$. That is, each cell Γ_J is the set of vertices common to all c_j for all $j \in J$ and absent from every c_j for which $j \notin J$. Again, the cardinality of Γ_J is denoted as $\gamma_J = |\Gamma_J|$.

The Γ -partition provides an equivalence relation on nodes $u, v \in \mathcal{N}_n$ via the indices of those cliques which contain u or v – namely, $J_u = \{j \in \mathcal{N}_m : u \in c_j\}$ and $J_v = \{j \in \mathcal{N}_m : v \in c_j\}$. The nodes u and v are equivalent, $u \equiv v$, if, and only if, $J_u = J_v$; that is, u and v are in the same Γ -set. For example, nodes 1 and 2 from Figure 1(c) are in the same class while nodes 1 and 4 are in different classes.

The Γ -partition can also be used directly to infer some properties of the graph union. For example, the adjacency of nodes in the graph union is related to the intersection of those Γ -sets which contain them:

Proposition 4. *Let u and v be two nodes in the graph union, $\bigcup_{j=1}^m c_j$, of the clique collection $\mathcal{C} = \{c_1, c_2, \dots, c_m\}$. If $u \in \Gamma_{J_u}$ and $v \in \Gamma_{J_v}$, then $u \sim v$ if, and only if, $J_u \cap J_v \neq \emptyset$.*

Proof. We note that $u \sim v$ if, and only if, for some $j \in \mathcal{N}_m$, $u \in c_j$ and $v \in c_j$, which is equivalent to $J_u \cap J_v \neq \emptyset$. $\square$

It follows, for example, that $u \sim v$ for every pair of nodes $u, v \in \Gamma_J$ (for any $J \subseteq \mathcal{N}_m$). Moreover, the cardinalities, γ_J , determine the degree of every vertex in Γ_J . Proposition 5 establishes that all nodes in a cell of Γ have the same degree.

Proposition 5. *For a non-null set $J \subseteq \mathcal{N}_m$, every vertex in Γ_J has degree d_J where*

$$d_J = \sum_{I \subseteq \mathcal{N}_m : I \cap J \neq \emptyset} \gamma_I - 1.$$

Proof. If $u \in \Gamma_J$, then $u \sim v$ if, and only if,

$$v \in \bigcup_{I \subseteq \mathcal{N}_m : I \cap J \neq \emptyset} \Gamma_I = \bigcup_{j \in J} c_j,$$

with $v \neq u$. Therefore, the degree of u is

$$\begin{aligned} \deg(u) &= \left| \bigcup_{j \in J} c_j \right| - 1 \\ &= \left| \bigcup_{j \in J} \left(\bigcup_{I: j \in I} \Gamma_I \right) \right| - 1 \\ &= \sum_{I: j \in I, \text{ for some } j \in J} \gamma_I - 1. \end{aligned}$$

$\square$

For instance, nodes 1 and 2 in Figure 1(a) both have degree 8.

Note that different clique collections having the same graph-union produce different Γ -partitions, these being peculiar to the particular cliques in the collection. The cliques of the collection Figure 1, for example, were all of size 5; had they all been of size 3 the same graph union of (now many more) cliques in the collection would be the same but the resulting Γ -sets would be different.

Having established the Γ -partitions, we investigate some of their traits. Notably, the Γ -partitions are found to be orbit partitions, a property hinted by Proposition 4.

3.1 An orbit partition

This section focuses on exploring the graphs that can be constructed from the indexing sets of the partition. Specifically, we analyze the quotient graphs of the graph union of the clique collection using the intersection property of the underlying indices.

Proposition 4 demonstrates that Γ is an *equitable partition* (Godsil & Royle, 2001; Lerner, 2005), that is, the number of neighbours in Γ_H of vertex $u \in \Gamma_J$ depends only on the choice of H and J . We will prove that they exhibit a stronger property, specifically,

that Γ is an *orbit partition* (defined in [Lerner, 2005](#), Definition 9.3.4 and Proposition 9.3.5).

That is, let G be a graph, $\text{Aut}(G)$ be the group of automorphisms of G , and $H \subseteq G$ be a subgroup of automorphisms. If two vertices u and v are equivalent under H , then there exists an automorphism in H that maps u to v . The orbits of H define the equivalence classes resulting from this equivalence, and the partition of G that contains the set of orbits by H is known as an orbit partition of G .

Proposition 6. *The partition $(\Gamma)_{\emptyset \neq J \subseteq \mathcal{N}_m}$ is an orbit partition for the group of automorphisms $\text{Aut}(G)$.*

Proof. Let V denote the set of nodes of G and for a nonempty $J \subseteq \mathcal{N}_m$, let π_J be any permutation of the elements of Γ_J . Let $\pi : V \rightarrow V$ be the extension of the π_J to V . It immediately follows that the orbits of π are the cells of Γ and, by Proposition 4, that π is an automorphism of V . $\square$

For example, we note that for any Γ -set depicted in Figure 1(b), the node numbers in Figure 1(c) can be permuted without changing the graph's structure.

It will be demonstrated later that the reason for this property is that the intersectionality of indexing sets provides a complete representation of the adjacency structure of the graph union. For example, we will see that collections of pairwise nonempty intersections indexing sets from the powerset of $\mathcal{N}_m$, which are known as *intersecting families* ([Meyerowitz, 1995](#)), corresponds to cliques in the graph union. As an illustration, consider the three cliques A , B , and C from Figure 1 are captured by the intersecting families $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$, where:

- (i) $\mathcal{F}_1 = \{\{A\}, \{A, C\}, \{A, B, C\}\},$
- (ii) $\mathcal{F}_2 = \{\{B\}, \{B, C\}, \{A, B, C\}\},$
- (iii) $\mathcal{F}_3 = \{\{C\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}.$

For any equitable partition, such as an orbit partition, $\Gamma = \{\Gamma_1, \dots, \Gamma_m\}$, of the vertex set of a graph G , a directed multi- (or weighted) *quotient* graph can be defined having nodes Γ_i and b_{ij} edges (or edge weights) from Γ_i to Γ_j where b_{ij} is the number of neighbours in Γ_j of every vertex in Γ_i – called the *quotient* of G modulo Γ and denoted G/Γ (e.g., [Lerner, 2005](#), Definition 9.3.2).

For the graph union of Figure 1(c), the partition $\Gamma = \{\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_{AC}, \Gamma_{BC}, \Gamma_{ABC}\}$ produces the quotient graph and matrix $\mathbf{B} = [b_{ij}]$ shown in Figure 2. This graph can be thought of as a compression of the original graph union. As such, some information will be lost, but much remains. Its (weighted) adjacency matrix and graph are enough to determine several properties of the graph union (for instance, see [Godsil, 1993](#)), including the path distances between nodes, the graph diameter, and a partial spectral decomposition – the characteristic roots of $\mathbf{B}$ are a subset of those of the adjacency matrix $\mathbf{A}$ of the graph union.

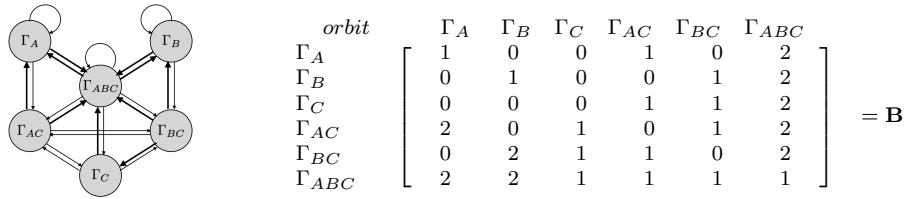


Figure 2: The quotient graph of the graph union of $\mathcal{C}$ modulo Γ and its edge weight matrix $\mathbf{B} = [b_{ij}]$. Edges are shown with width proportional to their weight in $\mathbf{B}$.

The compression of the original graph hints at a means of counting cliques using the quotient graph. If we choose an ordering of the orbits, say

$$(\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_{AB}, \Gamma_{AC}, \Gamma_{BC}, \Gamma_{ABC})$$

from Figure 1(c), then a unique tuple of the counts of nodes from each orbit identifies a set of subgraphs which are isomorphic to one another (under node permutation within each orbit). Thus, both $\{1, 3, 6\}$ and $\{2, 3, 6\}$ share the tuple $(1, 0, 0, 0, 1, 0, 1)$, but $\{1, 2, 3\}$ with tuple $(0, 0, 0, 0, 1, 0, 2)$ is a unique subgraph (under permutation within orbits). Each of these forms a 3-clique.

To prevent overcounting of cliques, it is crucial to ensure that the cliques identified through this partition framework are uniquely constructed. This requires analyzing cliques with respect to the notion of ‘type’, a generalization of graph isomorphism that plays a critical role in this context.

3.2 Type equivalent graphs

Given a collection of cliques $\mathcal{C} = \{c_1, \dots, c_m\}$, a subgraph H of $G = \cup_{i=1}^m c_i$, we define the *signature* of H with respect to the Γ -partition of $\mathcal{C}$ to be the *function* $f_H : \mathcal{P}(\mathcal{N}_m) \rightarrow \mathbb{N}_0$ defined by $f_H(J) := |H \cap \Gamma_J|$ for all $J \subseteq \mathcal{N}_m$. Note that this is defined for any subgraph H , not necessarily only cliques H . Two subgraphs H_1 and H_2 are said to be of the same *type*, or to be *type-isomorphic*, if, and only if, they have identical signatures (i.e., $f_{H_1} = f_{H_2}$). Finally, the *support* of H (or of f_H) is the set of all subsets J of $\mathcal{N}_m$ for which $f_H(J) > 0$; we write the support as $Supp(H) = \{J : J \subseteq \mathcal{N}_m \text{ and } f_H(J) > 0\}$, or as $Supp(f_H)$ when emphasizing the signature. Note also that all of these are predicated on the particular clique collection $\mathcal{C}$ and its associated Γ -partition.

For example, consider the clique collection of Figure 1(c) and the subgraphs $H_1 = \{1, 2, 3, 4\}$, $H_2 = \{1, 2, 3, 5\}$, $H_3 = \{1, 2, 3, 6\}$, and $H_4 = \{1, 2, 3, 5, 6\}$. The first three are graph isomorphic to each other and the complete graph, K_4 while H_4 is isomorphic to K_5 . In contrast only H_2 and H_3 are type isomorphic; H_1 has a different signature (and support), while H_4 shares the same support as H_2 and H_3 but is of a different type.

Because it differs from the usual graph equivalence, the notion of type could be of interest whenever the node labels, or the cliques defining the collection, carry additional meaning.

3.2.1 Γ -signatures

This section develops a number of counting results obtained types of subgraphs (as defined by signature) from any specific clique collection.

The number of different types of induced subgraphs is easily captured by the cell sizes of the partition:

Proposition 7. *The number of distinct signatures for the Γ -partition of a collection of m cliques is*

$$\prod_{J \in \mathcal{P}(\mathcal{N}_m)} (\gamma_J + 1).$$

Proof. A function $f : \mathcal{P}(\mathcal{N}_m) \rightarrow \mathbb{N}_0$ is a signature if, and only if, $|f(J)| \leq \gamma_J$. Thus, there are $\gamma_J + 1$ choices for every $J \in \mathcal{P}(\mathcal{N}_m)$. $\square$

Proposition 8. *For any signature f_H , the number of signatures having the same support, $\text{Supp}(H)$, is*

$$\prod_{J \in \text{Supp}(H)} \gamma_J.$$

Proof. For signatures f_{H_1} and f_{H_2} to have the same support, they must have the same Γ -cells, Γ_J for $J \in \text{Supp}(H_1) = \text{Supp}(H_2)$, and each signature can have values $1, \dots, \gamma_J$ for the J th cell. The total possible is therefore $\prod_{J \in \text{Supp}(H)} \gamma_J$. $\square$

Proposition 9. *Let $f : \mathcal{P}(\mathcal{N}_m) \rightarrow \mathbb{N}_0$. The number of induced subgraphs having signature f in the graph union of the clique collection $\{c_1, \dots, c_m\}$ is*

$$\prod_{J \in \text{Supp}(f)} \binom{\gamma_J}{f(J)}.$$

Proof. The signature is invariant to the choice of nodes within each Γ -cell – provided the same number of nodes from each cell is chosen, the signature is the same. Each cell has γ_j nodes giving

$$\prod_{J \in \text{Supp}(f)} \binom{\gamma_J}{f(J)}$$

choices for type-isomorphic induced subgraphs. $\square$

3.2.2 Connected subgraphs

The Γ -signature of an induced graph also indicates whether it is *connected*. This is captured by the notion of a *path-intersecting* collection of sets. We call a collection of sets $\mathcal{F}$ *path intersecting* if, for any $A, B \in \mathcal{F}$ there exists a sequence sets $J_1, J_2, \dots, J_\ell$ in $\mathcal{F}$ from $A = J_1$ to $B = J_\ell$ having that $J_j \cap J_{j+1} \neq \emptyset$ for all $j = 1, 2, \dots, \ell - 1$. For example, the collection $\{\{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}\}$ from Figure 1(b) is path intersecting, but is not an intersecting family.

Proposition 10. *A subgraph H of the graph union over a clique collection is connected if, and only if, its support is path-intersecting.*

Proof. Since, f_H is defined by the Γ -partition of $\mathcal{C}$, every node must appear in exactly one set J of $\text{Supp}(H)$. Moreover, any pair of nodes $u, v \in H$ appearing in the same set $J \in \text{Supp}(H)$ are connected by construction of the partition. So, we need only consider nodes u and v which lie in different sets of the support.

Suppose $\text{Supp}(H)$ is path-intersecting. Then for any pair of nodes $u, v \in H$, which appear in different subsets $J_u, J_v \in \text{Supp}(H)$, a sequence of sets $J_{w_1}, J_{w_2}, \dots, J_{w_\ell}$ can be found in $\text{Supp}(H)$ such that $J_u = J_{w_1}$, $J_v = J_{w_\ell}$, and $J_{w_i} \cap J_{w_{i+1}} \neq \emptyset$ for all $i = 1, \dots, (\ell - 1)$. From Proposition 4 $w_i \sim w_{i+1}$ for all $i = 1, \dots, (\ell - 1)$, $u = w_1 \rightarrow w_2 \rightarrow \dots \rightarrow w_\ell = v$, is a path from u to v in H , and so the subgraph H is connected.

Conversely, suppose H is connected. Every pair of nodes u, v appearing in separate sets J_u and J_v of $\text{Supp}(H)$ have a path connecting them in H . By the construction of Γ , this path can be chosen to be $u = w_1 \rightarrow w_2 \rightarrow \dots \rightarrow w_\ell = v$ such that each w_i comes from a different J_i in $\text{Supp}(H)$. Again, by Proposition 4, $w_i \sim w_{i+1}$ implies $J_i \cap J_{i+1} \neq \emptyset$, and hence that $\{J_1, \dots, J_\ell\}$ is path-intersecting. This holds for any $u, v \in H$ and hence any $J_u, J_v \in \text{Supp}(H)$, implying that it holds for the whole of $\text{Supp}(H)$. It follows that $\text{Supp}(H)$ is path-intersecting. $\square$

Proposition 11. *Let $\mathcal{I}_P$ be the set of all path-intersecting collections of non-empty cells from the Γ -partition of a clique collection $\mathcal{C}$. The number of distinct signatures that induce a connected subgraph in the graph union over $\mathcal{C}$ is*

$$\sum_{\mathcal{F} \in \mathcal{I}_P} \prod_{J \in \mathcal{F}} \gamma_J.$$

Proof. Proposition 10 states that for a subgraph H to be connected, its support must be path-intersecting; Proposition 10 determines the number of distinct signatures having the same support. Together they give the result. $\square$

It follows that the number of induced *disconnected* subgraphs is

$$\prod_{J \in \mathcal{P}(\mathcal{N}_m)} (\gamma_J + 1) - \sum_{\mathcal{F} \in \mathcal{I}_P} \prod_{J \in \mathcal{F}} \gamma_J$$

where $\mathcal{I}_P$ denotes the set of all path-intersecting collections of non-empty cells from Γ .

Proposition 12. Let $\mathcal{I}_P$ be the set of all path-intersecting collections of non-empty cells from the Γ -partition of a clique collection $\mathcal{C}$. The number of induced connected subgraphs of size k in the graph union over $\mathcal{C}$ is the k -th coefficient of the generating series

$$\sum_{\mathcal{F} \in \mathcal{I}_P} \prod_{J \in \mathcal{F}} [(1+x)^{\gamma_J} - 1].$$

Proof. By Proposition 10, every induced connected subgraph H is contained in some path-intersecting family. In fact, there exists a unique smallest path-intersecting family $\mathcal{F}_H := \text{Supp}(H)$ containing it. Clearly, the contribution of H to the generating function

$$\sum_{H'} x^{|V(H')|}$$

is x^k , where $|V(H)| = k$, and the sum is over all H' induced connected subgraphs whose support is $\mathcal{F}_H$.

Conversely, given a path-intersecting family $\mathcal{F}$, the induced connected subgraphs whose support is $\mathcal{F}$ are constructed uniquely by choosing $\alpha_J \geq 1$ nodes from Γ_J for every $J \in \mathcal{F}$. The generating series corresponding to this is

$$\prod_{J \in \mathcal{F}} [(1+x)^{\gamma_J} - 1]. \quad \square$$

3.2.3 Γ -support and cliques

The support of a subgraph H provides information on whether H is a clique and whether it is maximal.

Proposition 13. For any clique collection $\mathcal{C} = \{c_1, \dots, c_m\}$, the subgraph induced by H on the graph union $\bigcup_{j=1}^m c_m$, is a clique, if, and only if, its support, $\text{Supp}(H) = \{J : J \subseteq \mathcal{N}_m \text{ and } \Gamma_J \cap H \neq \emptyset\}$ is an intersecting family.

Proof. Suppose the induced graph on H is a clique. Fix two distinct sets $J_1, J_2 \in \text{Supp}(H)$. Let $u_1 \in \Gamma_{J_1} \cap H$ and $u_2 \in \Gamma_{J_2} \cap H$. Since $u_1 \sim u_2$, it must be that $u_1, u_2 \in c_j$ for some $j \in \mathcal{N}_m$. Therefore, it follows that $j \in J_1$ and $j \in J_2$, by the definition of the partition $(\Gamma_J)_{J \subseteq \mathcal{N}_m}$. Thus, $|J_1 \cap J_2| \geq 1$ and $\text{Supp}(H)$ is an intersecting family.

On the other hand, suppose that $\text{Supp}(H)$ is an intersecting family. Fix $u, v \in H$ and suppose that $u \in \Gamma_{J_u}$ and $v \in \Gamma_{J_v}$. Since $\text{Supp}(H)$ is an intersecting family, $|J_u \cap J_v| \geq 1$ and there exists some $j \in \mathcal{N}_m$ with $j \in J_u \cap J_v$. Thus, we have that $u, v \in c_j$ and since c_j is a clique, $u \sim v$. $\square$

So a subgraph H is connected if, and only if, its support is path-intersecting (Proposition 10) and is a clique if, and only if, its support is an intersecting family (Proposition 13).

When an intersecting family $\mathcal{F}$ is not a proper subset of any other intersecting family, it is called a *maximally intersecting family* (Meyerowitz, 1995). For example, the

family $\mathcal{F}_3 = \{C, AC, BC, ABC\}$ in Figure 1(b) is a maximally intersecting family and corresponds to the maximal clique C .

We note that maximal intersecting families are not necessarily isomorphic to maximal cliques. For example, the only other maximal intersecting family in Figure 1 (b) is $\mathcal{F}_4 = \{AB, AC, BC, ABC\}$ corresponds to the 4-clique $\{1, 2, 3, 4\}$ which is not maximal.

Theorem 14 provides the necessary and sufficient conditions for a clique to be maximal.

Theorem 14. *For any clique collection $\mathcal{C} = \{c_1, \dots, c_m\}$, a clique induced by H on the graph union $\bigcup_{j=1}^m c_m$, is maximal, if, and only if, for any $J \subseteq \mathcal{N}_m$,*

1. $J \in \text{Supp}(H) \implies |\Gamma_J \cap H| = \gamma_J$, and
2. $J \notin \text{Supp}(H) \implies$ either $\Gamma_J = \emptyset$ or $\Gamma_J \neq \emptyset$ and $\{J\} \cup \text{Supp}(H)$ is not an intersecting family.

Proof. First, to prove necessity, assume H is a maximal clique. For any $J \in \text{Supp}(H)$, at least one node in Γ_J is in H , and, so, connected to all other nodes in H . It follows from Proposition 4 that every node of Γ_J is also in H and hence $|\Gamma_J \cap H| = \gamma_J$ for all $J \in \text{Supp}(H)$. To show statement 2 holds, suppose now that $J \notin \text{Supp}(H)$. Further, suppose that $\{J\} \cup \text{Supp}(H)$ is an intersecting family and so, by Proposition 13, that $\Gamma_J \cup J$ is a clique. Since $J \notin \text{Supp}(H)$, $\Gamma_J \cap H = \emptyset$ and, since H is maximal, it follows that $\Gamma_J = \emptyset$.

To prove sufficiency, assume H is a clique and that both statements 1 and 2 hold. By statement 1, all nodes in Γ_J for $J \in \text{Supp}(H)$ are in H and no nodes remain in Γ_J to increase H . Statement 2 ensures that no nodes exist in any Γ_J with $J \notin \text{Supp}(H)$ that could enlarge H and still be a clique. Hence, H is maximal. $\square$

Statement 2 of Theorem 14 shows that, not only does a maximal clique have an intersecting family as its support (like all cliques), but that its intersecting family can only be expanded by sets $J \notin \text{Supp}(H)$ having no nodes in Γ_J .

3.2.4 The Γ -quotient graph and maximal cliques

Theorem 14 suggests that instead of considering intersecting families that are subsets of the entire power set, $\mathcal{P}(\mathcal{N}_m)$, we need only those that are subsets of the support of the graph union $G = \bigcup_{j=1}^m c_j$, namely, $\text{Supp}(G) = \{J : J \subseteq \mathcal{N}_m \text{ and } \Gamma_J \neq \emptyset\} \subseteq \mathcal{P}(\mathcal{N}_m)$.

This effectively ignores empty cells of the Γ partition to focus on intersecting families formed from the index sets that define the nodes of the quotient graph G/Γ . The relevant families are intrinsic to the quotient graph. For example,

- any path on G/Γ corresponds to a path-intersecting set (Proposition 10),
- any clique on G/Γ determines an intersecting family and hence a clique on G , and
- any maximal clique on G/Γ gives a maximal intersecting family and, so, a maximal clique on G .

The last two points are proved below in Proposition 15.

Proposition 15. *If $\mathcal{F}$ is a nonempty intersecting family on $\text{Supp}(G)$, then the graph $H_{\mathcal{F}}$ induced by $\{\Gamma_J : J \in \mathcal{F}\}$ is a clique. Furthermore, $\mathcal{F}$ is a maximal intersecting family on $\text{Supp}(G)$ if, and only if, $H_{\mathcal{F}}$ is a maximal clique.*

Proof. The fact that is a clique follows immediately from Proposition 4.

Suppose $\mathcal{F}$ is a maximal intersecting family on $\text{Supp}(G)$ and $H_{\mathcal{F}}$ is not a maximal clique. Then there exists some $u \in V(G)$ with u adjacent to all nodes in $H_{\mathcal{F}}$. Suppose $u \in \Gamma_{J_u}$, then Γ_{J_u} is nonempty and by Proposition 4, $\Gamma_{J_u} \cap J \neq \emptyset$ for all $J \in \mathcal{F}$. Therefore, either $\mathcal{F}$ is not a maximal intersecting family or $H_{\mathcal{F}}$ was not the subgraph induced by $\mathcal{F}$ – a contradiction.

The proof of the converse is almost identical. $\square$

Corollary 16. *If $\Gamma_J \neq \emptyset$ for all $\emptyset \neq J \subseteq \mathcal{N}_m$, then every maximal intersecting family on $\mathcal{P}(\mathcal{N}_m)$ induces a unique maximal clique in G .*

Proof. Suppose $\Gamma_J \neq \emptyset$ for all $\emptyset \neq J \subseteq \mathcal{N}_m$. Then $\text{Supp}(G)$ is the set of all nonempty subsets of $\mathcal{P}(\mathcal{N}_m)$. Therefore, by Proposition 15, each maximal intersecting family gives to a unique maximal clique. $\square$

This means that the number of maximal cliques, $M(\mathcal{C})$, in G is equal to the number of maximal intersecting families on $\text{Supp}(G)$ which in turn is bounded above by the number of maximal intersecting families on $\mathcal{N}_m$.

Corollary 17. *The number, $M(\mathcal{C})$, of maximal cliques in the graph union of $\mathcal{C} = \{c_1, \dots, c_m\}$ is bounded above by $\lambda(m)$, the number of maximal intersecting families on $\mathcal{N}_m$.*

Proof. By Theorem 14, each maximal intersecting family would correspond to at most one maximal clique in the graph union of the collection $\{c_1, \dots, c_m\}$. Thus, $\lambda(m)$ is an upperbound for $M(\mathcal{C})$. $\square$

Corollary 18. *The clique number of the graph union of the collection of cliques $\{c_1, \dots, c_m\}$ is*

$$\max_{\mathcal{F} \in \mathcal{M}} \sum_{J \in \mathcal{F}} \gamma_J,$$

where $\mathcal{M}$ is the set of all maximal intersecting families on $\mathcal{N}_m$.

Proof. By Theorem 14, a clique H is maximal if, and only if, its corresponding intersecting family $\mathcal{F}_H$ is only extendible by trivial elements and H uses all of the vertices in the cells Γ_J that contain members from H . Therefore, for every maximal intersecting family $\mathcal{F}$, there is a corresponding unique maximal clique H contained within the union of the cells $\{\Gamma_J : J \in \mathcal{F}\}$.

Since the clique number is the maximum of the size of all maximal cliques in a graph, and each maximal clique has the form $\sum_{J \in \mathcal{F}} \gamma_J$ for some maximal intersecting family $\mathcal{F}$, the proof follows. $\square$

To summarize, an intersecting family on $\text{Supp}(G)$ identifies a clique (Prop 13) and that clique is maximal if, and only if, its corresponding intersecting family is also maximal (Proposition 15). Whether an intersecting family, $\mathcal{F}$, is maximally intersecting can be determined from its cardinality, namely an intersecting $\mathcal{F} \subset \mathcal{N}_m$ is a maximal intersecting family if, and only if, $|\mathcal{F}| = 2^{m-1}$ (e.g., see Lemma 2.1 Meyerowitz, 1995); note that the intersecting family corresponding to an identified clique might have to be extended by adding subsets $J \in \mathcal{N}_m$ having $\Gamma_J = \emptyset$ to achieve this cardinality (Theorem 14). Every such maximal intersecting family produces a unique maximal clique (Corollary 16). The number of such maximal cliques is bounded above by $\lambda(m)$, the number of maximally intersecting families on $\mathcal{N}_m$ (Corollary 17). Unfortunately, $\lambda(m)$ is typically computationally intractable (e.g., see Brouwer, Mills, Mills, & Verbeek, 2013) though is presently feasible on today's laptops for $m \leq 10$, for example. In the special case where $\gamma_J > 0$ for all $J \subseteq \mathcal{N}_m$, every maximal intersecting family induces precisely one maximal clique so that the upper bound (Corollary 17) is achieved and $M(\mathcal{C}) = \lambda(m)$.

4 Counting cliques

For a family of sets $\mathcal{F}$, let $N(\mathcal{F}) := \sum_{J \in \mathcal{F}} \gamma_J$ denote the number of nodes in the sets contained in the family.

Given the collection of all maximal intersecting families on the support of G , we can apply the principle of inclusion-exclusion in the following manner.

Proposition 19. *Let H be a clique in the graph union of $\{c_1, \dots, c_m\}$ and let $\mathcal{F}_H$ denote its support. Let $\mathcal{M}_H$ be the set of all maximal intersecting families $\mathcal{F}$ on $\text{Supp}(G)$ that extend $\mathcal{F}_H$. The number of cliques that contain H in the graph union of $\{c_1, \dots, c_m\}$ is*

$$1 + \sum_{\mathcal{J} \subseteq \mathcal{M}_H} (-1)^{|\mathcal{J}|+1} \left(2^{N(\cap_{\mathcal{F} \in \mathcal{J}} \mathcal{F}) - |H|} - 1 \right).$$

Proof. Any clique that contains H would be a subclique of one of the maximal cliques that contain H . Therefore, by Theorem 14, it suffices to examine the collection $\mathcal{M}_H$ of maximal intersecting families that generate a unique maximal clique in the graph union of $\{c_1, \dots, c_m\}$. If $\mathcal{F} \in \mathcal{M}_H$ corresponds to a maximal clique with $N(\mathcal{F})$ total nodes, then the selection of a nonempty subset from $(\cup_{J \in \mathcal{F}} \Gamma_J) \setminus H$ corresponds to a clique that properly contains H . This can be done in

$$(2^{N(\mathcal{F}) - |H|} - 1)$$

ways.

Since some cliques are subgraphs of several different maximal cliques, we use the principle of inclusion/exclusion and obtain

$$\sum_{\mathcal{J} \subseteq \mathcal{M}_H} (-1)^{|\mathcal{J}|+1} \left(2^{N(\cap_{\mathcal{F} \in \mathcal{J}} \mathcal{F}) - |H|} - 1 \right)$$

cliques. However, this count does not include the clique H on its own and hence we add a 1. $\square$

The proof of Proposition 19 relies on the fact that every clique is contained in some maximal clique.

Then notion of signatures can also be used for obtaining clique count expressions.

Theorem 20. *The generating function for clique counts induced by a collection $\{c_1, \dots, c_m\}$ is*

$$\Phi(\mathbf{x}) = \sum_{\mathcal{F} \in \mathcal{I}_m} \prod_{J \in \mathcal{F}} [(1 + x_J)^{\gamma_J} - 1],$$

where $\mathcal{I}$ is the set of all intersecting families on $\mathcal{P}(\mathcal{N}_m)$, and $\mathbf{x}$ is the vector $(x_J : J \in \mathcal{P}(\mathcal{N}_m))$.

Proof. A clique H is determined uniquely by its signature and the node labels. By Proposition 13, the support must be an intersecting family on $\text{Supp}(G)$, and hence it is also an intersecting family on $\mathcal{P}(\mathcal{N}_m)$.

For a cell J to contribute $\alpha_J \geq 1$ nodes to H is accomplished in $\binom{\gamma_J}{\alpha_J}$ ways, which corresponds to the coefficient of $x_J^{\alpha_J}$ in the generating series

$$[(1 + x_J)^{\gamma_J} - 1],$$

and the result follows. $\square$

Extracting the coefficient of x^r in the generating function $\Phi(x_J \rightarrow x)$ in Theorem 20 yields the number of r -cliques as given in Corollary 21:

Corollary 21. *The number of r -cliques in the graph union of the clique collection $\{c_1, \dots, c_m\}$ is*

$$\sum_{\ell=1}^r \sum_{(\alpha_1, \dots, \alpha_\ell)} \sum_{(J_1, \dots, J_\ell)} \prod_{i=1}^{\ell} \binom{\gamma_{J_i}}{\alpha_i}$$

where $(J_1, \dots, J_\ell)$ is an intersecting family on $\text{Supp}(G)$ of size ℓ with signature $(\alpha_1, \dots, \alpha_\ell)$ being an integer composition of r having $1 \leq \alpha_i \leq \gamma_i$.

A third expression for the total number of cliques of any size, induced by the collection, can also be had by substituting $x_J = 1$ in the generating series in Theorem 20. The expression is given as Corollary 22:

Corollary 22. *The total number of cliques of size at least 1 induced by a collection $\{c_1, \dots, c_m\}$ is*

$$\sum_{\mathcal{F} \in \mathcal{I}_m} \prod_{J \in \mathcal{F}} [2^{\gamma_J} - 1],$$

where $\mathcal{I}_m$ is the set of all intersecting families on $\mathcal{P}(\mathcal{N}_m)$.

When $r = 2$, the interesting special case of the edge count is obtained (e.g., essential to edge count distributions for many random graph models, such as the Erdős-Rényi model):

Corollary 23. *The number of edges induced by the collection of r -cliques $\{c_1, \dots, c_m\}$ is*

$$\sum_{J \subseteq \mathcal{N}_m} \binom{\gamma_J}{2} + \frac{1}{2} \sum_{J \subseteq \mathcal{N}_m} \gamma_J \sum_{I \neq J: |I \cap J| \geq 1} \gamma_I.$$

Alternatively, edges can also be enumerated via the degree sequences of the vertices in the various cells Γ_J . For every $J \subseteq \mathcal{N}_m$, any two nodes within Γ_J have the same degree. For instance, if $u \in \Gamma_{\{k\}}$ for some $k \in \mathcal{N}_m$, then it must be that $\deg(u) = r - 1$ because $u \in c_k$ and $u \notin c_j$ for all $j \neq k$ by the definition of $\Gamma_{\{k\}}$. On the other extreme, if $u \in \Gamma_{\mathcal{N}_m}$, then $u \in c_j$ for all $j \in \mathcal{N}_m$ and hence u must be adjacent to all other nodes in G which are in at least one of the $\{c_1, \dots, c_m\}$. Therefore,

$$\deg(u) = n - \gamma_\emptyset - 1 = n - 1,$$

The “handshaking lemma” immediately gives the number of edges induced by the collection as below:

Proposition 24. *The number of edges induced by the collection of cliques $\{c_1, \dots, c_m\}$ is*

$$\frac{1}{2} \sum_{J: \emptyset \neq J \subseteq \mathcal{N}_m} \gamma_J \left(\sum_{I: |I \cap J| \geq 1} \gamma_I - 1 \right).$$

Proof. Follows immediately from Proposition 5 and the fact that number of edges in the graph is half the sum of the degrees in the graph. $\square$

5 Discussion

In this work, connections were established and exploited between several graph-theoretic properties of clique covers, and notions of intersecting families on a special partition, the Γ -partition, of a graph $G = \cup_{i=1}^m c_i$ formed from a collection of $\mathcal{C} = \{c_1, \dots, c_m\}$ of m cliques c_i . The Γ -partition frames the unique contributions to G from the various cliques of $\{c_1, \dots, c_m\}$ via sets from the power set of $\mathcal{N}_m$.

The Γ -partition induces the quotient graph, G/Γ , which efficiently summarizes the information present in the clique collection. This creates a mapping between graph-theoretic concepts, such as cliques, maximal cliques, and induced subgraphs, and their corresponding set-theoretic counterparts, such as intersecting families, maximal intersecting families, and path-intersecting families. Further investigation is needed to explore the potential of this framework in studying other graphic structures.

Of course, the Γ -partition and quotient graph are determined by the particular cliques given as elements of the collection. Coarser partitions (those which produce fewer Γ_J cells) are preferred – ideally, the collection would consist of a minimal number of unique maximal cliques. Investigating the notion of optimal representations of graphs by means of clique collections is an interesting avenue for research.

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