Monk's Rule for Demazure Characters of the General Linear Group

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Abstract

Key polynomials are characters of Demazure modules for the general linear group that generalize the Schur polynomials. We prove a nonsymmetric generalization of Monk's rule by giving a cancellation-free, multiplicity-free formula for the key polynomial expansion of the product of an arbitrary key polynomial with a degree one key polynomial.

Mathematics Subject Classifications: 05E05, 05E10

1 Introduction

Demazure generalized the Weyl character formula [4] to certain submodules of irreducible modules generated by extremal weight spaces under the action of a Borel subalgebra of a Lie algebra. These *Demazure characters* are naturally indexed by a highest weight and an element of the Weyl group, or, in the case of the general linear group, a partition and a permutation. When the permutation is taken to be the longest element, the Demazure character is a Schur polynomial.

Demazure characters of the general linear group form a basis of the polynomial ring often called the basis of *key polynomials*. As key polynomials contain the Schur polynomials, it is natural to consider the expansion of a product of key polynomials into the key basis. However the coefficients appearing are not, in general, nonnegative.

We prove, in Theorem 25, a combinatorial formula for these structure constants when one of the factors is degree one, parallel to Monk's case for Schur polynomials. This formula is cancellation-free, that is, the terms in the expansion into key polynomials are

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pairwise distinct. Moreover, it is multiplicity-free in the sense that the only nonzero coefficients appearing are 1 and -1.

Our proof of this new rule is combinatorial, utilizing the combinatorial model of *Kohnert diagrams* [7] that generates key polynomials. Briefly, given a weak composition $\mathbf{a}$, let $\mathrm{KD}(\mathbf{a})$ denote the set of Kohnert diagrams for $\mathbf{a}$. In Theorem 10, we prove for any positive integer $k \leqslant n$, there exists a weight-preserving bijection

$$\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_k) \xrightarrow{\sim} \bigcup_{\substack{\mathbf{b} \preceq \mathbf{a} \\ 1 \leqslant j \leqslant k}} \mathrm{KD}(\mathbf{b} + \mathbf{e}_j),$$

where $\mathbf{e}_k = (0^{k-1}, 1, 0^{n-k})$ is the composition with a single nonzero part of value 1 in position k. Our *nonsymmetric Monk's rule*, Theorem 25, follows from this via inclusion–exclusion since the union on the right is not disjoint.

This bijection generalizes the Robinson–Schensted–Knuth insertion algorithm [6, 12, 13] that can be used to prove Pieri's formula for Schur polynomials. We give simple, direct proofs of the bijection for two extreme cases in Section 4, and we develop new tools for studying the combinatorics of Kohnert diagrams to prove the general case in Section 5.

Haglund, Luoto, Mason and van Willigenburg [5] give a *nonnegative* formula for the key expansion of a product of a key polynomial and a Schur polynomial with a certain number of variables. Our formula is more general in that we do not restrict the number of variables for the Schur polynomial, though it is less general in that we consider only Schur polynomials indexed by one part. When both formulas apply, our proof simplifies and our formulas agree.

Subsequent to this paper, the second author [10] generalized the constructions above to prove a nonsymmetric generalization of the Pieri rule. That is, Quijada gives a multiplicity-free, signed formula for the key polynomial expansion of the product of a key polynomial with a single part key polynomial. More recently, the first author [2] has given a signed expansion for the product of a key polynomial with any Schur polynomial using an explicit insertion algorithm on Kohnert diagrams.

2 Key combinatorics

The Demazure characters for the general linear group, studied combinatorially as *standard bases* by Lascoux and Schützenberger [8] and Kohnert [7], then under the name *key polynomials* by Reiner and Shimozono [11], Mason [9], Assaf and Searles [1], and others, can be characterized in many equivalent ways. In Section 2.1, we review Kohnert's [7] elegant combinatorial algorithm for computing a key polynomial based on diagrams, which lies at the heart of our rule for key polynomials. In Section 2.2, we review a partial order on weak compositions studied by Assaf and Searles [1] that allows us to characterize when diagrams corresponding to one key polynomial also correspond to another.

2.1 Kohnert diagrams

We fix a positive integer n throughout, and we consider *weak compositions* to be sequences of nonnegative integers of length n. A *partition* is a weakly decreasing composition for which we often omit trailing 0s.

A *diagram* is any finite collection of unit cells in the first quadrant. To distinguish generic diagrams from Young diagrams, we draw cells of generic diagrams as unit circles.

The *key diagram* for a weak composition $\mathbf{a} = (a_1, \ldots, a_n)$, denoted by $\mathrm{key}_{\mathbf{a}}$ is the set of left justified cells with a_i in row i indexed in Cartesian coordinates.

Definition 1 ([7])**.** A *Kohnert move* on a diagram selects the rightmost cell of a given row and moves the cell down within its column to the first available position below, if it exists, jumping over other cells in its way as needed.

Denote the set of diagrams that can be obtained by Kohnert moves from the key diagram $\mathrm{key}_{\mathbf{a}}$ by $\mathrm{KD}(\mathbf{a})$. For example, Fig. 1 shows all diagrams that can be obtained via Kohnert moves from the key diagram for $(0, 3, 2)$, depicted at the top.

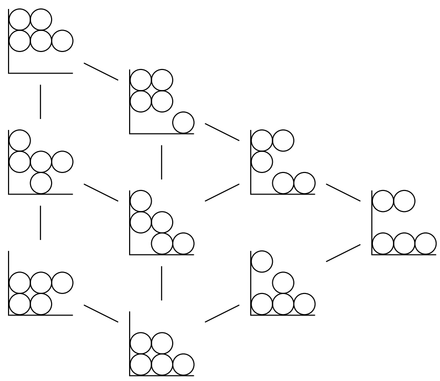

Figure 1: Kohnert diagrams for $(0, 3, 2)$, where southeast edges (resp. south or southwest edges)indicate Kohnert moves on the second row (resp. third row).

To each diagram T, we associate the weak composition $\mathbf{wt}(T)$ whose ith component is equal to the number of cells in row i of T. For example, the weights of diagrams in the leftmost column of Fig. 1 are $(0, 3, 2), (1, 3, 1), (2, 3, 0)$, from top to bottom.

We take as our definition the following result of Kohnert [7].

Definition 2. The *key polynomial* $\kappa_{\mathbf{a}}$ is given by

$$\kappa_{\mathbf{a}} = \sum_{T \in \mathrm{KD}(\mathbf{a})} x_1^{\mathbf{wt}(T)_1} \cdots x_n^{\mathbf{wt}(T)_n}. \tag{1}$$

For example, from Fig. 1 we compute the key polynomial

$$\kappa_{(0,3,2)} = x_2^3 x_3^2 + x_1 x_2^2 x_3^2 + x_1^2 x_2 x_3^2 + x_1^3 x_3^2 + x_1^2 x_2^2 x_3 + x_1^3 x_2 x_3 + x_1^3 x_2^2 + x_1 x_2^3 x_3 + x_1^2 x_2^3.$$

In particular, for $\mathbf{a}$ weakly increasing of length n, we have

$$\kappa_{\mathbf{a}} = s_{\mathrm{rev}(\mathbf{a})}(x_1, \dots, x_n), \tag{2}$$

where $\mathrm{rev}(\mathbf{a})$ is the partition $(a_n, a_{n-1}, \dots, a_1)$ and s_λ is the Schur polynomial.

Note $\kappa_{(0,3,2)}$ is not equal to $s_{(3,2)}(x_1, x_2, x_3)$. However, every Kohnert diagram for $(0, 3, 2)$ is also a Kohnert diagram for $(0, 2, 3)$, that is, $\mathrm{KD}(0, 3, 2) \subset \mathrm{KD}(0, 2, 3)$. Containment of Kohnert diagrams leads to a useful partial order on weak compositions, originally defined in [1], that we explore next.

2.2 Left swap order

Not all diagrams, for instance the diagram with two cells in positions $(1, 1)$ and $(2, 2)$, can result from Kohnert moves on key diagrams. We will often need to distinguish between diagrams that can arise from Kohnert moves on a key diagram with those that cannot.

Definition 3. A diagram T is a *generic Kohnert diagram* if there exists a weak composition $\mathbf{a}$ for which $T \in \mathrm{KD}(\mathbf{a})$.

Given a diagram T, the *column weight of* T, denoted by $\mathbf{cwt}(T)$, is the weak composition whose ith part is the number of cells in the ith column of T. Abusing notation, given a weak composition $\mathbf{a}$, the *column weight of* $\mathbf{a}$ is $\mathbf{cwt}(\mathbf{a}) = \mathbf{cwt}(\mathrm{key}_{\mathbf{a}})$.

Since Kohnert moves preserve the column weight, we have the following.

Corollary 4. *For T a generic Kohnert diagram, $\mathbf{cwt}(T)$ is a partition.*

Assaf and Searles [1, Lemma 2.2] give a useful criterion to determine if T is a generic Kohnert diagram.

Proposition 5 ([1])**.** *A diagram T is a Kohnert diagram for some weak composition if and only if for every position $(c, r) \in \mathbb{N} \times \mathbb{N}$ with $c > 1$, we have*

$$\#\{(c-1, s) \in T \mid s \geqslant r\} \geqslant \#\{(c, s) \in T \mid s \geqslant r\}. \tag{3}$$

We define a partial order on generic Kohnert diagrams with fixed column weight μ by $S \prec T$ whenever S can be obtained from T by a sequence of Kohnert moves. For example, see Fig. 1. Notice this partial order is neither ranked nor is it a lattice.

Related to this order, Assaf and Searles [1] considered a partial order on weak compositions that sort to a given partition defined as follows.

Definition 6 ([1])**.** A *left swap* on a weak composition $\mathbf{a} = (a_1, a_2, \cdots)$ exchanges two parts $a_i < a_j$ with $i < j$. The *left swap order* on weak compositions is the transitive closure of the relations $\mathbf{b} \preceq \mathbf{a}$ whenever $\mathbf{b}$ is a left swap of $\mathbf{a}$.

Given a weak composition $\mathbf{a}$, let lswap($\mathbf{a}$) denote the set of weak compositions $\mathbf{b}$ for which $\mathbf{b} \preceq \mathbf{a}$. For example, we have

$$\mathrm{lswap}(0,3,2) = \{(0,3,2),(3,0,2),(3,2,0),(2,3,0)\}.$$

Notice the remaining two weak compositions that sort to the partition $(3,2)$, namely $(2,0,3)$ and $(0,2,3)$, are not included in this set. Comparing with Fig. 1, the weak compositions in lswap$(0,3,2)$ are precisely those whose key diagrams are Kohnert diagrams for $(0,3,2)$, and this observation holds in general. To prove this, we require the following definition from [1, Definition 3.5].

Definition 7 ([1])**.** The *thread decomposition* of a generic Kohnert diagram partitions the cells into *threads* as follows. Beginning with the rightmost column, select the lowest available cell to begin the thread. After threading a cell in column $j+1$, thread the lowest available cell in column j that is weakly above the threaded cell in column $j+1$. Continue the thread until all columns are threaded or no choices remain. Continue threading until all cells are part of some thread.

As noted in [1], Proposition 5 ensures each thread of the thread decomposition ends in the first column. Following [1, Lemma 3.6], we may define the *thread weight* of a Kohnert diagram T to be the weak composition $\boldsymbol{\theta}(T)$ whose ith part is the number of cells in the thread occupying row i in the first column of T. For example, the diagram in Fig. 2 has $\boldsymbol{\theta}(T) = (4,1,5,0,4)$.

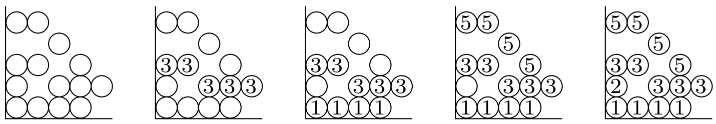

Figure 2: An example of thread decomposition of a generic Kohnert diagram, where the cells of a given thread are labeled the same.

Implicit in [1, Theorem 3.7], we have the following useful fact.

Lemma 8 ([1])**.** *For a generic Kohnert diagram T, we have $T \in \mathrm{KD}(\mathbf{a})$ if and only if $\boldsymbol{\theta}(T) \preceq \mathbf{a}$.*

An easy and useful consequence is the following.

Proposition 9. *Given weak compositions $\mathbf{a}$ and $\mathbf{b}$, we have $\mathbf{b} \preceq \mathbf{a}$ if and only if $\mathrm{key}_{\mathbf{b}} \in \mathrm{KD}(\mathbf{a})$.*

Proof. Suppose $\mathbf{b}$ is obtained from $\mathbf{a}$ by a left swap for $i < j$. Beginning with $\mathrm{key}_{\mathbf{a}}$, we may apply a Kohnert move to the last cell in row j until the cell lands in row i below it, jumping over cells in rows k if $a_k \geqslant a_j$. Repeating this for the rightmost $a_j - a_i$ cells in row j, we obtain $\mathrm{key}_{\mathbf{b}}$. Thus $\mathrm{key}_{\mathbf{b}}$ is a Kohnert diagram for $\mathbf{a}$.

Conversely, suppose $\mathrm{key}_{\mathbf{b}} \in \mathrm{KD}(\mathbf{a})$. By Definition 7, the threads of a key diagram necessarily consist of all cells in a given row, and so $\boldsymbol{\theta}(\mathrm{key}_{\mathbf{b}}) = \mathbf{b}$. Thus by Lemma 8, we have $\mathbf{b} \preceq \mathbf{a}$. ☐

3 Key results

In Section 3.1, we state our main result giving a bijection that takes a pair of Kohnert diagrams, the latter being a single cell, and maps it bijectively to another Kohnert diagram. Our expression for the union in the image has redundancy, and in Section 3.2 we reduce the indexing set to the maximal elements which can be described in terms of addable cells, parallel to the classical case. In Section 3.3, we state our second main result giving an inclusion-exclusion formula for the key expansion of the product of key polynomials from our bijection.

3.1 Main bijection

The key polynomial product requires an additional parameter k that governs the number of variables in the right term of the product. Given a positive integer $k \leqslant n$, let $\mathbf{e}_k$ denote the weak composition with a single nonzero part of value 1 in position k, i.e. $\mathbf{e}_k = (0^{k-1}, 1, 0^{n-k})$.

Our bijection, whose proof begins in Section 4, is stated succinctly as follows.

Theorem 10. *Given any weak composition* $\mathbf{a}$ *and any positive integer* $k \leqslant n$*, there exists a weight-preserving bijection*

$$\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_k) \xrightarrow{\sim} \bigcup_{\substack{\mathbf{b} \preceq \mathbf{a} \\ 1 \leqslant j \leqslant k}} \mathrm{KD}(\mathbf{b} + \mathbf{e}_j), \tag{4}$$

where the addition of weak compositions on the right is coordinate-wise.

Notice Theorem 10 generalizes Monk's rule as follows.

Corollary 11. *For* $\mathbf{a}$ *a weakly increasing weak composition, we have a bijection*

$$\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_n) \xrightarrow{\sim} \bigsqcup_{\substack{1 \leqslant j \leqslant n \\ a_j < a_{j+1}}} \mathrm{KD}(\mathbf{a} + \mathbf{e}_j). \tag{5}$$

Proof. We consider the right-hand side of Eq. (4) when $\mathbf{a}$ is weakly increasing and $k = n$. We claim

$$\bigcup_{\substack{\mathbf{b} \preceq \mathbf{a} \\ 1 \leqslant j \leqslant n}} \mathrm{KD}(\mathbf{b} + \mathbf{e}_j) = \bigcup_{\substack{1 \leqslant j \leqslant n \\ a_j < a_{j+1}}} \mathrm{KD}(\mathbf{a} + \mathbf{e}_j).$$

To see this, suppose $\mathbf{b} \prec \mathbf{a}$. Then for any $j \leqslant n$, we have $\mathbf{b} + \mathbf{e}_j \prec \mathbf{a} + \mathbf{e}_i$ where $i \geqslant j$ is the largest index for which $b_j = a_i$. Therefore by Proposition 9, $\mathrm{KD}(\mathbf{b} + \mathbf{e}_j) \subset \mathrm{KD}(\mathbf{a} + \mathbf{e}_i)$. Therefore the pair $(\mathbf{b}, j)$ can be removed from the indexing set for all j, proving the claimed equality.

Similarly, if $a_j = a_{j+1}$, then $\mathbf{a} + \mathbf{e}_j \prec \mathbf{a} + \mathbf{e}_{j+1}$, so again by Proposition 9, $\mathrm{KD}(\mathbf{a} + \mathbf{e}_j) \subset \mathrm{KD}(\mathbf{a} + \mathbf{e}_{j+1})$ and so the index j may be removed from the union on the right. Since $\mathbf{a}$ is weakly increasing, the indexing is as stated. Moreover, the set of pairs $(\mathbf{a}, j)$ for j such that $a_j < a_{j+1}$ result in weak compositions $\mathbf{a} + \mathbf{e}_j$ with different column weights, and so the union must be disjoint. $\square$

Notice, by Proposition 9, the union on the right-hand side of Eq. (4) has redundancy whenever $\mathbf{b}' + \mathbf{e}_{j'} \prec \mathbf{b} + \mathbf{e}_j$ for some $\mathbf{b}', \mathbf{b} \preceq \mathbf{a}$ and some $j', j \leqslant k$. Furthermore, the image of the bijection in Theorem 10 is not, in general, *disjoint*, even after accounting for this redundancy. Therefore when taking generating polynomials, we must use inclusion–exclusion for the nontrivial intersections.

3.2 Addable cells

As suggested by Corollary 11, the maximal, in the sense of Proposition 9, weak compositions appearing on the right-hand side of Eq. (4) can be described in terms of *addable cells* for key diagrams.

Definition 12. Given a weak composition $\mathbf{a}$, the cell in row r and column c is an *addable cell for* $\mathbf{a}$ if $a_r < c$ and there exists $s \geqslant r$ such that $a_s = c - 1$.

For partitions, we may add a cell in row r and column c only if row r has length $c-1$. For weak compositions, this condition is relaxed so that row r has length *at most* $c-1$ and some row s weakly above row r has length *exactly* $c-1$ since the excess cells of row s can be dropped down to row r via Kohnert moves.

When the cell in row r and column c is an addable cell for $\mathbf{a}$, we also require a way to construct the weak composition $\mathbf{b} \preceq \mathbf{a}$ such that $\mathbf{b} + \mathbf{e}_r$ appears as a term on the right-hand side of Eq. (4) corresponding to this addition.

Definition 13. Given a weak composition $\mathbf{a}$ and an addable position (c, r), the *maximal support composition for* $\mathbf{a}$ *at* (c, r) is the weak composition

$$\mathbf{supp}_{\mathbf{a}}^{(c,r)} = t_{r_0,r_1} \cdots t_{r_{q-1},r_q} \cdot \mathbf{a}, \tag{6}$$

where $t_{i,j}$ is the transposition interchanging parts in positions i and j, and where $r = r_0 < r_1 < \cdots < r_q$ is the unique increasing sequence of row indices such that $a_{r_{i-1}} < a_{r_i}$ with $a_{r_q} = c - 1$ and if $r_{i-1} < s < r_i$, then either $a_s \leqslant a_{r_{i-1}}$ or $a_s > a_{r_i}$.

For an example, look ahead to Fig. 3. We re-characterize the right-hand side of Eq. (4) in terms of addable cells.

Lemma 14. *Given a weak composition* $\mathbf{a}$ *and positive integer* $k \leqslant n$*, we have*

$$\bigcup_{\substack{\mathbf{b} \preceq \mathbf{a} \\ 1 \leqslant j \leqslant k}} \mathrm{KD}(\mathbf{b} + \mathbf{e}_j) = \bigcup_{\substack{1 \leqslant j \leqslant k \\ (c,j) \text{ addable for } \mathbf{a}}} \mathrm{KD}(\mathbf{supp}_{\mathbf{a}}^{(c,j)} + \mathbf{e}_j). \tag{7}$$

Proof. By Definition 13, we have $\mathbf{supp}_{\mathbf{a}}^{(c,j)} \preceq \mathbf{a}$, and so we have containment of the right-hand side of Eq. (7) in the left-hand side. For the other direction, suppose $\mathbf{b} \preceq \mathbf{a}$ and $j \leqslant k$ satisfies $b_j = c - 1$. Then by the choice of r_i's in Definition 13, $\mathbf{b} \preceq \mathbf{supp}_{\mathbf{a}}^{(c,j)}$, and so $\mathbf{b} + \mathbf{e}_j \preceq \mathbf{supp}_{\mathbf{a}}^{(c,j)} + \mathbf{e}_j$. Therefore, by Proposition 9, $\mathrm{KD}(\mathbf{b} + \mathbf{e}_j) \subseteq \mathrm{KD}(\mathbf{supp}_{\mathbf{a}}^{(c,j)} + \mathbf{e}_j)$, and so we may eliminate these terms from the union on the left, giving the desired equality. □

Lemma 14 reduces the index set for the union in Eq. (4) to addable cells for $\mathbf{a}$. In order to characterize the maximal terms for the union, we strengthen Definition 12 to the following notion of *k-addable cells*.

Definition 15. Given a weak composition $\mathbf{a}$ and positive integer k, the cell in row $r \leqslant k$ and column c is a *k-addable cell for* $\mathbf{a}$ if

1. $a_r < c$ and if $a_r < c-1$, then there exists some $l > k$ such that $a_l = c-1$;
2. for all $r < i \leqslant k$, either $a_i < a_r$ or $a_i \geqslant c$.

Example 16. To find addable cells for $\mathbf{a} = (4,6,4,3,0,1,1,2,5,4)$ in column $c = 5$, we look for a row index r for which $a_r < c$, giving $r = 1,3,4,5,6,7,8,10$. Rows $4,5,6,7,8$ have $a_r < c-1$, and so require the existence of a row index $l > r$ for which $a_l = c-1$. Since $a_{10} = c-1$, this condition is met.

Now let $k = 5$ and consider which of these is k-addable. We eliminate all $r > k$, leaving those positions depicted in Fig. 3. Considering the cell in row 5, the maximal support composition is constructed using the sequence $r_0 = 5 < 6 < 8 < 10$ so that the cells in each of these rows marked by $\otimes$ drop down to row 5 to create $\mathbf{supp}_{\mathbf{a}}^{(5,5)} = (4,6,4,3,4,0,1,1,5,2)$. The maximal support compositions for the other addable cells in column c are similarly marked.

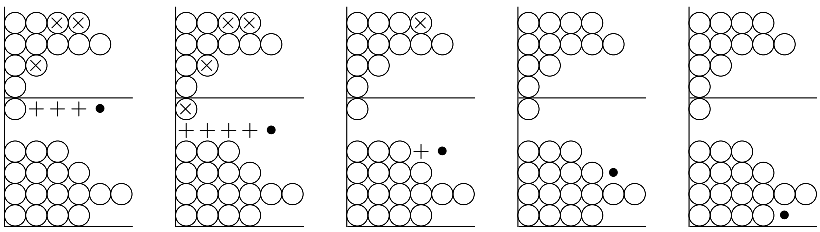

Figure 3: The five addable cells ($\bullet$) for $(4,6,4,3,0,1,1,2,5,4)$ in column 5. Here the marked cells ($\otimes$) drop to positions ($+$) in row r in creating the maximal support composition. The first, third and fourth are 6-addable but the second and fifth are not.

Notice Definition 15(1) is stronger than Definition 12 since in this case we require the supporting row to be above row k, not just above row r. Taken together, the conditions of Definition 15 imply a k-addable cell is an addable cell such that none of the row indices in Definition 13 except the first lies below row k.

Lemma 17. *Given a weak composition* $\mathbf{a}$ *and a positive integer* k*, if* c *is a column index for which there exists some row index* $r \leqslant k$ *such that* (c,r) *is an addable cell for* $\mathbf{a}$*, then there exists a row index* $r \leqslant s \leqslant k$ *such that* (c,s) *is* k*-addable for* $\mathbf{a}$ *and* $\mathbf{supp}_{\mathbf{a}}^{(c,r)} \preceq \mathbf{supp}_{\mathbf{a}}^{(c,s)}$.

Proof. If (c,r) is k-addable for $\mathbf{a}$, then we may take $s = r$. Otherwise, we consider three cases based on how (c,r) fails to be k-addable for $\mathbf{a}$.

Suppose Definition 15(1) fails for (c,r). Since (c,r) is addable, there exists a maximal row index s such that $r < s \leqslant k$ and $a_s = c-1$. Then (c,s) is k-addable, satisfying Definition 15(1) since $a_s = c-1$ and Definition 15(2) since, by choice of s, no row index $i > s$ has $a_i = c-1$. Moreover, the final index r_q appearing in Definition 13 for (c,r) satisfies $r_q \leqslant s$ and $a_{r_q} = a_s$, ensuring we have $\mathbf{supp}_{\mathbf{a}}^{(c,r)} \preceq \mathbf{supp}_{\mathbf{a}}^{(c,r_q)} \preceq \mathbf{supp}_{\mathbf{a}}^{(c,s)}$ as desired.

Suppose Definition 15(1) holds but Definition 15(2) fails for (c,r) with some index $r < i \leqslant k$ for which $a_r < a_i \leqslant c-1$. Then there exists a row index $r_j \leqslant k$ appearing in Definition 13 for (c,r), and we may take s' to be the maximum such index. Define s to be the largest row index such that $s \leqslant k$ and $a_s = a_{s'}$. Then (c,s) is k-addable, satisfying Definition 15(1) since (c,r) does and Definition 15(2) since by Definition 13 for (c,r), no row index $i > s$ with $i \leqslant k$ has $a_s < a_i \leqslant c-1$, and by choice of s, no row index $i > s$ with $i \leqslant k$ has $a_s = a_i$. Moreover, since $s' \leqslant s$ and $a_{s'} = a_s$, we have $\mathbf{supp}_{\mathbf{a}}^{(c,r)} \preceq \mathbf{supp}_{\mathbf{a}}^{(c,s')} \preceq \mathbf{supp}_{\mathbf{a}}^{(c,s)}$ as desired.

Suppose Definition 15(1) holds but Definition 15(2) fails for (c,r) only for some index $r < i \leqslant k$ for which $a_i = a_r$. We may take s to be the maximal index such that $r < s \leqslant k$ and $a_s = a_r$. By Definition 13, the sequences of row indices for the cells (c,r) and (c,s) differ only for the first index r_0, and no other index is weakly less than k. In particular, (c,s) is k-addable, satisfying Definition 15(1) since (c,r) does and Definition 15(2) since by Definition 13 for (c,r), no row index $i > s$ with $i \leqslant k$ has $a_s < a_i \leqslant c-1$ and by choice of s no row index $i > s$ with $i \leqslant k$ has $a_s = a_i$. Once again, since $r \leqslant s$ and $a_r = a_s$, we have $\mathbf{supp}_{\mathbf{a}}^{(c,r)} \preceq \mathbf{supp}_{\mathbf{a}}^{(c,s)}$ as desired. $\square$

Lemma 17 allows us to reduce the indexing set still further.

Lemma 18. *Given a weak composition* $\mathbf{a}$ *and a positive integer* k*, if both* (c,r) *and* (c,s) *are* k*-addable cells for* $\mathbf{a}$ *with* $r < s$*, then* $a_r > a_s$ *and* $\mathbf{supp}_{\mathbf{a}}^{(c,r)} + \mathbf{e}_r$ *and* $\mathbf{supp}_{\mathbf{a}}^{(c,s)} + \mathbf{e}_s$ *are incomparable in left swap order.*

Proof. Since both cells (c,r) and (c,s) are addable, we must have $a_r, a_s < c$. Since (c,r) is k-addable, by Definition 15(2) we must have $a_r > a_s$. The first $r-1$ parts of $\mathbf{supp}_{\mathbf{a}}^{(c,r)} + \mathbf{e}_r$ and $\mathbf{supp}_{\mathbf{a}}^{(c,s)} + \mathbf{e}_s$ must agree, and the rth part of the former is strictly larger, ensuring it cannot be above the latter in left swap order by Proposition 9. On the other hand, there are $a_r - a_s > 0$ fewer cells above row k in the latter than in the former, ensuring the latter cannot be above the former in left swap order. Thus the two are incomparable. $\square$

We may now state the minimal indexing set for the union in Eq. (4).

Theorem 19. *For a weak composition* $\mathbf{a}$ *and positive integer* k*, we have*

$$\bigcup_{\substack{\mathbf{b} \preceq \mathbf{a} \\ 1 \leqslant j \leqslant k}} \mathrm{KD}(\mathbf{b} + \mathbf{e}_j) = \bigcup_{\substack{1 \leqslant j \leqslant k \\ (c,j)\ k\text{-addable for } \mathbf{a}}} \mathrm{KD}(\mathbf{supp}_{\mathbf{a}}^{(c,j)} + \mathbf{e}_j), \tag{8}$$

where no term on the right is strictly contained in another.

Proof. By Lemma 17, if (c, r) is an addable cell for $\mathbf{a}$ that it not k-addable, then there exists a k-addable cell (c, s) such that $\mathbf{supp}_{\mathbf{a}}^{(c,r)} \preceq \mathbf{supp}_{\mathbf{a}}^{(c,s)}$. Since $r < s$, we also have $\mathbf{supp}_{\mathbf{a}}^{(c,r)} + \mathbf{e}_r \preceq \mathbf{supp}_{\mathbf{a}}^{(c,s)} + \mathbf{e}_s$. Therefore, by Proposition 9, the set of Kohnert diagrams of the former is contained in the set of Kohnert diagrams of the latter, so the addable cell (c, r) may be removed from the indexing set on the right side of Eq. (7). Thus the equality follows from Lemma 14.

To see that the terms on the right side of Eq. (8) are pairwise not contained in one another, note that for $c' \neq c$, the column weights of $\mathbf{supp}_{\mathbf{a}}^{(c,j')} + \mathbf{e}_{j'}$ and $\mathbf{supp}_{\mathbf{a}}^{(c,j)} + \mathbf{e}_j$ are different, regardless of the values for j', j, and so the sets of Kohnert diagrams in this case are disjoint. For cells added within the same column, Lemma 18 ensures there is no pairwise containment. $\square$

3.3 Drop sets

In contrast with the classical case, the image of the bijection in Theorem 10 is not, in general, disjoint. Therefore when taking generating polynomials to obtain our key analog of Monk's rule for multiplying key polynomials, we must use inclusion–exclusion to account for the nontrivial intersections.

As remarked in the proof of Theorem 19, if $c' \neq c$, then the column weights of $\mathbf{supp}_{\mathbf{a}}^{(c,j')} + \mathbf{e}_{j'}$ and $\mathbf{supp}_{\mathbf{a}}^{(c,j)} + \mathbf{e}_j$ are different, regardless of the values for j', j, and so the sets of Kohnert diagrams in this case are disjoint. Therefore we focus our attention on cells added within a given column.

Definition 20. Given a weak composition $\mathbf{a}$, a positive integer k, and a column index c, the *k-addable row set for* $\mathbf{a}$ *in column* c is given by

$$\mathrm{Row}_{\mathbf{a},k}^{c} = \{r \leqslant k \mid (c, r) \text{ is } k\text{-addable for } \mathbf{a}\}. \tag{9}$$

We say that c is a *k-addable column for* $\mathbf{a}$ whenever the set $\mathrm{Row}_{\mathbf{a},k}^{c}$ is nonempty.

By Lemma 18, if we take elements of $\mathrm{Row}_{\mathbf{a},k}^{c}$ as increasing, $r_1 < \cdots < r_p$, then the corresponding parts of $\mathbf{a}$ are decreasing, $a_{r_1} > \cdots > a_{r_p}$.

Similar to Definition 13, we can construct the weak compositions that index the intersections of the sets $\mathrm{KD}(\mathbf{supp}_{\mathbf{a}}^{(c,r)} + \mathbf{e}_r)$ using left swap order.

Definition 21. Let $\mathbf{a}$ be a weak composition, k a positive integer, and c a k-addable column for $\mathbf{a}$. Given a nonempty subset $R \subseteq \mathrm{Row}_{\mathbf{a},k}^{c}$, the *maximal drop composition for* $\mathbf{a}$ *in column* c *at rows* R is the weak composition

$$\mathbf{drop}_{\mathbf{a}}^{(c,R)} = t_{r_{-p},r_{-p+1}} \cdots t_{r_{-1},r_0} \cdot \mathbf{supp}_{\mathbf{a}}^{(c,r_0)}, \tag{10}$$

where $R = \{r_{-p} < \cdots < r_{-1} < r_0\}$.

In particular, for singleton sets we have the equivalence $\mathbf{drop}_{\mathbf{a}}^{(c,\{r\})} = \mathbf{supp}_{\mathbf{a}}^{(c,r)}$.

Example 22. Consider again the weak composition $\mathbf{a} = (4, 6, 4, 3, 0, 1, 1, 2, 5, 4)$ and $k = 6$. In addition to the three k-addable cells for $\mathbf{a}$ in column 5 giving $\mathrm{Row}^c_{\mathbf{a},k} = \{3, 4, 6\}$, there are three doubleton subsets as well as the entire set to consider for R, as indicated in Fig. 4. Notice the cells that fall from above row k are the same as for the maximal support composition of the highest row of R, but now not all cells fall to the same row.

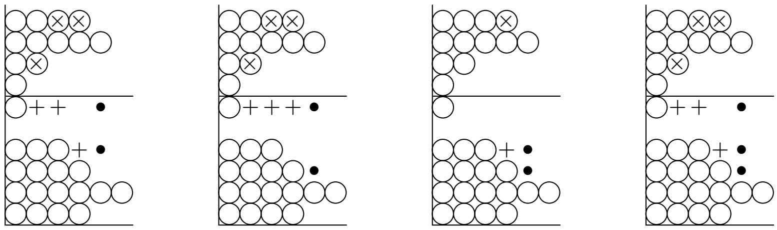

Figure 4: The four nonempty, non-singleton subsets of rows ($\bullet$) of the 6-addable cells for $(4, 6, 4, 3, 0, 1, 1, 2, 5, 4)$ in column 5. Here marked cells ($\otimes$) will drop down to the indicated position ($+$) below in creating the maximal drop composition.

Lemma 23. *Let* $\mathbf{a}$ *be a weak composition,* k *a positive integer, and* c *a* k*-addable column for* $\mathbf{a}$*. Given a nonempty subset* $R \subseteq \mathrm{Row}^c_{\mathbf{a},k}$ *and a row index* $s \in \mathrm{Row}^c_{\mathbf{a},k}$ *such that* $s > \max R$*, we have*

$$\mathrm{KD}\left(\mathbf{supp}_{\mathbf{a}}^{(c,s)} + \mathbf{e}_s\right) \cap \mathrm{KD}\left(\mathbf{drop}_{\mathbf{a}}^{(c,R)} + \mathbf{e}_{\min(R)}\right) = \mathrm{KD}\left(\mathbf{drop}_{\mathbf{a}}^{(c,R\cup\{s\})} + \mathbf{e}_{\min(R)}\right). \tag{11}$$

Proof. By Definition 21, $\mathbf{drop}_{\mathbf{a}}^{(c,R)} \preceq \mathbf{supp}_{\mathbf{a}}^{(c,\max(R))}$ for any nonempty $R \subseteq \mathrm{Row}^c_{\mathbf{a},k}$. Since $s = \max(R \cup \{s\}) > \min(R)$, it follows from Proposition 9 that

$$\mathrm{KD}\left(\mathbf{drop}_{\mathbf{a}}^{(c,R\cup\{s\})} + \mathbf{e}_{\min(R)}\right) \subseteq \mathrm{KD}\left(\mathbf{supp}_{\mathbf{a}}^{(c,s)} + \mathbf{e}_s\right).$$

Similarly, adding a new maximum row index to R expands the set of cells above row k that are dropped but does not affect the resulting positions of cells from $\max(R)$ down, so $\mathbf{drop}_{\mathbf{a}}^{(c,R\cup\{s\})} \preceq \mathbf{drop}_{\mathbf{a}}^{(c,R)}$. Since $\min(R \cup \{s\}) = \min(R)$, by Proposition 9 again we have

$$\mathrm{KD}\left(\mathbf{drop}_{\mathbf{a}}^{(c,R\cup\{s\})} + \mathbf{e}_{\min(R)}\right) \subseteq \mathrm{KD}\left(\mathbf{drop}_{\mathbf{a}}^{(c,R)} + \mathbf{e}_{\min(R)}\right).$$

Therefore the right-hand side of Eq. (11) is contained in the left-hand side.

For brevity, let $\mathbf{b} = \mathbf{drop}_{\mathbf{a}}^{(c,R\cup\{s\})} + \mathbf{e}_{\min(R)}$. Given a generic Kohnert diagram T such that $\mathbf{cwt}(T) = \mathbf{cwt}(\mathbf{b})$ but $T \notin \mathrm{KD}(\mathbf{b})$, we aim to show T is not contained in the left-hand side of Eq. (11). By Lemma 8 and Proposition 9, T is a Kohnert diagram for a weak composition if and only if the key diagram for its thread weight, $\mathrm{key}_{\boldsymbol{\theta}(T)}$, is a Kohnert diagram for the same weak composition. Since the left-hand and right-hand sides of Eq. (11) consist of a set or an intersection of sets of Kohnert diagrams of weak compositions, it thus suffices to assume $T = \mathrm{key}_{\boldsymbol{\theta}(T)}$. Let j denote the largest index such that no weak composition $\mathbf{c} \preceq \mathbf{b}$ satisfies $c_i = \mathbf{wt}(T)_i$ for all $i \geqslant j$. If $j > s$, then since $\mathbf{drop}_{\mathbf{a}}^{(c,R\cup\{s\})}$ and $\mathbf{supp}_{\mathbf{a}}^{(c,s)}$ have the same values beyond index s, we must have $\mathbf{wt}(T) \not\preceq \mathbf{supp}_{\mathbf{a}}^{(c,s)} + \mathbf{e}_s$, and so T does not appear in the set on the left-hand side of Eq. (11). If $j \leqslant s$, then since $\mathbf{drop}_{\mathbf{a}}^{(c,R\cup\{s\})}$ and $\mathbf{drop}_{\mathbf{a}}^{(c,R)}$ have the same values before index s, we must have $\mathbf{wt}(T) \not\preceq \mathbf{drop}_{\mathbf{a}}^{(c,R)} + \mathbf{e}_{\min R}$, and so again T does not appear in the set on the left-hand side of Eq. (11). $\square$

Example 24. Beginning with Theorem 19 our running example of the weak composition $\mathbf{a} = (4,6,4,3,0,1,1,2,5,4)$ with $k = 6$ in column $c = 5$ gives

$$\begin{aligned}\bigcup_{r\in\mathrm{Row}^c_{\mathbf{a},k}} \mathrm{KD}(\mathbf{supp}_{\mathbf{a}}^{(c,r)} + \mathbf{e}_r) \quad &= \quad \mathrm{KD}(4,6,4,3,0,5,1,1,5,2)\\ &\qquad \cup\mathrm{KD}(4,6,4,5,0,1,1,2,5,3)\\ &\qquad \cup\mathrm{KD}(4,6,5,3,0,1,1,2,5,4).\end{aligned}$$

Taking the generating polynomial by iteratively applying Lemma 23 gives

$$\begin{aligned}&\kappa_{(4,6,4,3,0,5,1,1,5,2)} + \kappa_{(4,6,4,5,0,1,1,2,5,3)} + \kappa_{(4,6,5,3,0,1,1,2,5,4)}\\ &\qquad - \kappa_{(4,6,4,5,0,3,1,1,5,2)} - \kappa_{(4,6,5,3,0,4,1,1,5,2)} - \kappa_{(4,6,5,4,0,1,1,2,5,3)}\\ &\qquad\qquad + \kappa_{(4,6,5,4,0,3,1,1,5,2)}\end{aligned}$$

We finally have all the ingredients needed to state the key analog of Monk's rule.

Theorem 25. *Given a weak composition* $\mathbf{a}$ *and positive integer* k*, we have*

$$\kappa_{\mathbf{a}} \cdot s_{(1)}(x_1,\ldots,x_k) = \sum_{\substack{c\ k\text{-addable for }\mathbf{a}\\ \varnothing\neq R\subseteq \mathrm{Row}^c_{\mathbf{a},k}}} (-1)^{\#R-1} \kappa_{\mathbf{drop}_{\mathbf{a}}^{(c,R)} + \mathbf{e}_{\min(R)}}. \tag{12}$$

Moreover, the terms on the right-hand side are pairwise distinct.

Proof. Combining Theorems 10 and 19, we have a weight-preserving bijection

$$\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_k) \xrightarrow{\sim} \bigcup_{\substack{1\leqslant j\leqslant k\\ (c,j)\ k\text{-addable for }\mathbf{a}}} \mathrm{KD}(\mathbf{supp}_{\mathbf{a}}^{(c,j)} + \mathbf{e}_j).$$

The generating polynomial on the left-hand side is $\kappa_{\mathbf{a}} \cdot s_{(1)}(x_1,\ldots,x_k)$. The generating polynomial on the right-hand side can be computed by first noting the sets are disjoint for different columns c, then using Lemma 23 iteratively to compute intersections. The equality now follows from the inclusion–exclusion. $\square$

In particular, notice the right hand side of Eq. (12) is nonnegative if and only if the k-addable row set for $\mathbf{a}$ for each k-addable column c is a singleton.

4 Extremal cases

We prove Theorem 10 for certain extremal cases via a reversible insertion of a single cell into a generic Kohnert diagram. To ease notation, given a weak composition $\mathbf{a}$ and positive integer k, we denote the *target space* of the bijection by

$$\mathcal{D}(\mathbf{a}, k) = \bigcup_{\substack{\mathbf{b} \preceq \mathbf{a} \\ 1 \leqslant j \leqslant k}} \mathrm{KD}(\mathbf{b} + \mathbf{e}_j). \tag{13}$$

In Section 4.1, we give a simple insertion algorithm for the case $k = 1$, developing along the way several tools for understanding Kohnert diagrams that are essential for all cases. In Section 4.2, we review *rectification*, a generalization of the classical RSK insertion algorithm on tableaux to an insertion algorithm on diagrams [3] and use it to insert for the case where $a_i = 0$ for all $i > k$.

4.1 Bottom insertion

The bijection of Theorem 10 for the case $k = 1$ is simple to state, though the proof requires several additional tools.

Definition 26. Let T be a generic Kohnert diagram, and let

$$c = \min\{i \mid (i, 1) \notin T\}$$

be the column of the left-most empty position of T in the first row. Then the *bottom insertion map* Δ_1 sends T to the diagram

$$\Delta_1(T) = T \sqcup \{(c, 1)\}. \tag{14}$$

In order to show $\Delta_1(T)$ is a generic Kohnert diagram, we reformulate the criterion given in Proposition 5 by generalizing the thread decomposition given in Definition 7 to *matching sequences* defined as follows.

Definition 27. Let C (respectively, D) be a diagram consisting of cells in some column i (respectively, $i+1$). A *matching* from D to C is a directed graph with vertex set $D \sqcup C$ such that for every $x \in D$ and every $y \in C$ we have

1. x has out-degree 1 and in-degree 0;

2. y has out-degree 0 and in-degree at most 1;

3. if $y \leftarrow x$, then the row index of y is weakly greater than that of x.

For a given matching M, we say a cell $x \in D$ *matches to* a cell $y \in C$, written $(y \leftarrow x)$ or $M(x) = y$, whenever M has a directed edge from x to y.

Definition 28. For T an arbitrary diagram, a *matching sequence on* T is a directed graph M with vertex set the cells of T such that for every pair of adjacent columns i and $i+1$ for which column $i+1$ is nonempty in T, the restriction of M to the cells of T in columns i and $i+1$ is a matching.

If T is a generic Kohnert diagram, then the thread decomposition of T induces the matching sequence $\mathcal{M}_{\boldsymbol{\theta}}(T)$ on T defined by x matching to y for cells $x \in T$ in column $i+1$ and $y \in T$ in column i if and only if x and y are in the same thread. For example, Fig. 5 shows the four possible matching sequences for the given diagram.

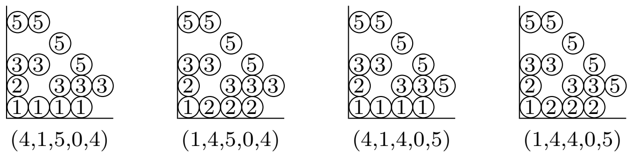

Figure 5: The four possible matching sequences of a generic Kohnert diagram along with their anchor weights (below), where the matched cells in adjacent columns are labeled the same.

By Hall's Marriage Theorem, the characterization of generic Kohnert diagrams T in Proposition 5 is equivalent to the existence of a matching sequence on T.

While matchings allow us to determine if an arbitrary diagram is a generic Kohnert diagram, in order to prove Theorem 10 we must be able to determine as well for which weak compositions $\mathbf{b}$ a generic Kohnert diagram lies in $\mathrm{KD}(\mathbf{b})$.

Definition 29. Given a matching M on a diagram T, the *anchor weight of* M is the weak composition $\mathbf{wt}(M)$ whose ith part is the number of cells along the path in M that terminates in column 1, row i.

Lemma 30. *Let T be a generic Kohnert diagram, and let M be a matching sequence on T. Then $T \in \mathrm{KD}(\mathbf{wt}(M))$.*

Proof. We proceed by induction on the number of columns c occupied by T. If $c = 1$, then T is a key diagram with $T = \mathrm{key}_{\mathbf{wt}(M)} \in \mathrm{KD}(\mathbf{wt}(M))$.

Suppose $c > 1$ and assume the result for any generic Kohnert diagram occupying $c-1$ columns. Let T' be the diagram obtained from T by removing all the cells in the first column and pushing each cell in columns 2 to c to the left from position (i,j) to position $(i-1,j)$. Let M' be the matching sequence on T' induced from M by preserving all the existing matchings between cells in T that were moved left to get T'. In particular, T' is a generic Kohnert diagram by Proposition 5, and by induction we have $T' \in \mathrm{KD}(\mathbf{wt}(M'))$.

Let T'' be the diagram with the same first column as T and the key diagram $\mathrm{key}_{\mathbf{wt}(M')}$ in columns 2 and beyond. Then $T \preceq T''$, so it suffices to show $T'' \in \mathrm{KD}(\mathbf{wt}(M))$. We do this by induction on the number of connected components of M. Observe the second column of T'' coincides with the first column of T', and so T and T'' coincide in the first two columns. Let M'' be the matching sequence on T'' defined by $M''(y)$ is the cell to the left of y for y strictly right of the second column, and $M''(y) = M(y)$ for y in the second column. Then $\mathbf{wt}(M'') = \mathbf{wt}(M)$.

If M'' has one component, then each column of T'' beyond the first has at most one cell. Letting y denote the cell in column 2, we have apply reverse Kohnert moves to columns $2, 3$, and so on until the cell lies in the same row as $M''(y)$. The corresponding matching is preserved, and so $T'' \in \mathrm{KD}(\mathbf{wt}(M))$ as desired. If M'' has more than one component, then let x denote the highest cell in the first column of T'', and we may similarly apply reverse Kohnert moves to columns $2, 3, \ldots$ to the cells on the component of x until they lie in the same row as x. Having done this, we may remove the top row from the result, correspondingly removing one component of the matching. By induction, the remainder lifts as well, and so once again $T'' \in \mathrm{KD}(\mathbf{wt}(M))$. $\square$

We may now strengthen Proposition 5 as follows.

Theorem 31. *For T an arbitrary diagram, T is a Kohnert diagram for $\mathbf{a}$ if and only if there exists a matching sequence M on T with $\mathbf{wt}(M) \preceq \mathbf{a}$.*

Proof. If a diagram T is not a generic Kohnert diagram, then by Proposition 5 both statements are indeed false for all weak compositions $\mathbf{a}$. Suppose, then, T is a generic Kohnert diagram. Statement (1) implies (2) using the thread decomposition by Lemma 8. Finally, to see (2) implies (1), we have $T \in \mathrm{KD}(\mathbf{wt}(M))$ by Lemma 30 and $\mathbf{wt}(M) \preceq \mathbf{a}$, so, by Proposition 9, $T \in \mathrm{KD}(\mathbf{a})$. $\square$

Theorem 31 yields the following useful characterization of the left swap order.

Corollary 32. *For weak compositions $\mathbf{a}$ and $\mathbf{b}$, we have $\mathbf{b} \preceq \mathbf{a}$ if and only if $\mathbf{wt}(M) \preceq \mathbf{a}$ for some matching sequence M on $\mathrm{key}_{\mathbf{b}}$.*

We now have enough tools to show $\Delta_1(T)$ is indeed a generic Kohnert diagram. Moreover, we can show Δ_1 sends T to the appropriate target space. That is, if $T \in \mathrm{KD}(\mathbf{a})$, then $\Delta_1(T) \in \mathcal{D}(\mathbf{a}, 1)$. We will want to show Δ_1 is in fact an injective map, and the following lemma will be instrumental in helping us recover T from $\Delta_1(T)$. More generally, we use the following lemma for constructing maps in the reverse direction from the target space in subsequent sections.

Lemma 33. *Let $\mathbf{a}$ be a weak composition. For every diagram $U \in \mathcal{D}(\mathbf{a}, n)$, there exists a unique column index c such that $\mathbf{cwt}(U) = \mathbf{cwt}(\boldsymbol{\theta}(U)) = \mathbf{cwt}(\mathbf{a}) + \mathbf{e}_c$. Moreover, for every weak composition $\mathbf{b} \in \mathrm{lswap}(\mathbf{a})$ and positive integer $k \leqslant n$ such that $U \in \mathrm{KD}(\mathbf{b} + \mathbf{e}_k)$, we have $c = b_k + 1$.*

Proof. Since $U \in \mathcal{D}(\mathbf{a}, n)$, there exist a weak composition $\mathbf{b} \in \mathrm{lswap}(\mathbf{a})$ and positive integer $k \leqslant n$ such that $U \in \mathrm{KD}(\mathbf{b} + \mathbf{e}_k)$. Proposition 9 implies $\mathrm{key}_{\mathbf{b}} \in \mathrm{KD}(\mathbf{a})$, and since Kohnert moves preserve column weights, we have $\mathbf{cwt}(\mathbf{b}) = \mathbf{cwt}(\mathbf{a})$. Therefore the column index $b_k + 1$ satisfies the conditions of the proposition for the diagram $\mathrm{key}_{\mathbf{b}+\mathbf{e}_k}$. Since $U \in \mathrm{KD}(\mathbf{b}+\mathbf{e}_k)$, we have $\boldsymbol{\theta}(U) \preceq \mathbf{b}+\mathbf{e}_k$ by Lemma 8, and hence $\mathrm{key}_{\boldsymbol{\theta}(U)} \in \mathrm{KD}(\mathbf{b}+\mathbf{e}_k)$ by Proposition 9. Again, since Kohnert moves preserve column weights, it follows that $b_k + 1$ satisfies the given conditions.

Notice $\mathrm{key}_{\mathbf{b}+\mathbf{e}_k} \in \mathcal{D}(\mathbf{a}, n)$, and since $\mathbf{b} \preceq \mathbf{a}$, we have $\mathbf{cwt}(\mathbf{b}) = \mathbf{cwt}(\mathbf{a})$. It follows that $b_k + 1$ is the unique column index satisfying the conditions above for $\mathrm{key}_{\mathbf{b}+\mathbf{e}_k}$. Now, since $U \in \mathrm{KD}(\mathbf{b} + \mathbf{e}_k)$ and since Kohnert moves preserve column weights, we have $\mathbf{cwt}(U) = \mathbf{cwt}(\mathrm{key}_{\mathbf{b}+\mathbf{e}_k}) = \mathbf{cwt}(\mathbf{b} + \mathbf{e}_k)$. Therefore, $c = b_k + 1$. □

Definition 34. For $\mathbf{a}$ a weak composition and $U \in \mathcal{D}(\mathbf{a}, n)$, the *added column* (with respect to $\mathbf{a}$) of U is the unique column index satisfying Lemma 33.

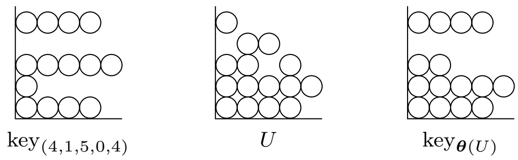

Figure 6: A Kohnert diagram U in $\mathcal{D}((4, 1, 5, 0, 4), n)$ for $n \geqslant 3$ with added column 2.

Example 35. Let $\mathbf{a} = (4, 1, 5, 0, 4)$, and consider the generic Kohnert diagram U in Fig. 6. We have $\mathbf{cwt}(U) = (4, 4, 3, 3, 1) = \mathbf{cwt}(\mathbf{a}) + \mathbf{e}_2$, and so U has added column $c = 2$. Furthermore, $\boldsymbol{\theta}(U) = (4, 5, 2, 0, 4) = \mathbf{b} + \mathbf{e}_3$ where $\mathbf{b} = (4, 5, 1, 0, 4) \prec \mathbf{a}$. Thus we have $U \in \mathrm{KD}(\mathbf{b} + \mathbf{e}_3)$ and indeed $c = b_3 + 1$.

Theorem 36. *For each weak composition* $\mathbf{a}$*, the map* Δ_1 *induces a weight-preserving bijection*

$$\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_1) \xrightarrow{\sim} \mathcal{D}(\mathbf{a}, 1). \tag{15}$$

In particular, Theorem 10 is proved for $k = 1$*.*

Proof. Let $T \in \mathrm{KD}(\mathbf{a})$, and consider the matching sequence $M = \mathcal{M}_{\boldsymbol{\theta}} T$ on T. Since $(c, 1)$ is the left-most empty position of T in row 1, the thread decomposition algorithm implies that the path

$$P = (1, 1) \leftarrow (2, 1) \leftarrow \cdots \leftarrow (c - 1, 1)$$

is a (weakly) connected component of M. We can extend P by appending the matching $(c - 1, 1) \leftarrow (c, 1)$ to it. It follows that the directed graph

$$M' = M \cup ((c - 1, 1) \leftarrow (c, 1))$$

is a matching sequence with anchor weight

$$\mathbf{wt}(M') = \boldsymbol{\theta}(T) + \mathbf{e}_1.$$

Thus, $\Delta_1(T) = T \sqcup \{(c, 1)\} \in \mathrm{KD}(\boldsymbol{\theta}(T) + \mathbf{e}_1)$ (by Theorem 31), and since $\boldsymbol{\theta}(T) \preceq \mathbf{a}$ (by Lemma 8), we have $\Delta_1(T) \in \mathcal{D}(\mathbf{a}, 1)$.

Since $T \in \mathrm{KD}(\mathbf{a})$ and Kohnert moves preserve column weights, $\mathbf{cwt}(T) = \mathbf{cwt}(\mathbf{a})$. So by Lemma 33, c is the added column of $\Delta_1(T)$, and we may recover T from $\Delta_1(T)$.

On the other hand, for every diagram $U \in \mathcal{D}(\mathbf{a}, 1)$, we have $U \in \mathrm{KD}(\mathbf{b} + \mathbf{e}_1)$ for some weak composition $\mathbf{b} \in \mathrm{lswap}(\mathbf{a})$. Now consider the thread decomposition $M = \mathcal{M}_{\boldsymbol{\theta}}(U)$

on U. By Theorem 31, $\boldsymbol{\theta}(M) \preceq \mathbf{b} + \mathbf{e}_1$. In particular, U must occupy every position in row 1 at every column 1 to $b_1 + 1$.

Let $x \in U$ be the cell at position $(b_1 + 1, 1)$. Then x must be in the same thread at the cell in position $(1, 1)$, and it must be the rightmost cell of that thread. In particular, removing x from both U and M gives a matching sequence on $U \setminus \{x\}$ with anchor weight $\mathbf{b}$. By Theorem 31, we have $U \setminus \{x\} \in \mathrm{KD}(\mathbf{b}) \subset \mathrm{KD}(\mathbf{a})$. Since $\mathrm{row}(x) = 1$ and the diagram $U \setminus \{x\}$ occupies every position in row 1 left of x, we have $\Delta_1(U \setminus \{x\}) = U$. It also follows from Lemma 33 that $\mathrm{col}(x) = b_1 + 1$ is the added column of U.

Hence, the map $\Delta_1 : \mathrm{KD}(\mathbf{a}) \longrightarrow \mathcal{D}(\mathbf{a}, 1)$ has an inverse that is well-defined on the target space. The desired bijection follows. □

4.2 Rectification

Not every diagram is a generic Kohnert diagram. The criterion of Proposition 5 extends to a measurable way of identifying where and to what extent a diagram fails to be a generic Kohnert diagram.

Definition 37 ([3])**.** Let T be an arbitrary diagram. For each position (c, r) with $c > 1$, define

$$\mathfrak{m}_T(c, r) = \#\{(c-1, s) \in T \mid s \geqslant r\} - \#\{(c, s) \in T \mid s \geqslant r\}. \tag{16}$$

Proposition 5 states T is a generic Kohnert diagram if and only if $\mathfrak{m}_T(c, r) \geqslant 0$ for all positions (c, r) with $c > 1$. When this fails, say in some column $c > 1$, we may take r to be the highest row index such that

$$\mathfrak{m}_T(c, r) = \min_{r'}\{\mathfrak{m}_T(c, r')\} < 0.$$

Then T has a cell at position (c, r) and no cell at position $(c-1, r)$. Thus, following Assaf [3], we may correct, or *rectify*, T by moving certain cells left.

Definition 38. For T an arbitrary diagram, define the diagram $\varrho(T)$ by

- if $\mathfrak{m}_T(c, r) \geqslant 0$ at every position (c, r) with $c > 1$, then $\varrho(T) = T$;
- otherwise, let $c > 1$ be the leftmost column index satisfying

  $$\mathfrak{m}_T(c, r') < 0$$

  for some row index r', let r be the maximum such index that achieves the minimum value of $\mathfrak{m}_T$ in column c, and set $\varrho(T)$ to be the diagram obtained by pushing the cell in position (c, r) left to the empty position $(c-1, r)$.

By our earlier observations ϱ is well-defined over all diagrams, and so it can be composed with itself repeatedly until the result is a generic Kohnert diagram.

Proposition 39. *Given an arbitrary diagram T, there exists some integer $m \geqslant 0$ such that for all $N > m$, $\varrho^N(T) = \varrho^m(T)$.*

Proof. Cells move only left under ϱ, so since a diagram has finitely many cells and lies in the first quadrant, the procedure must ultimately terminate when the characterization of Kohnert diagrams in Proposition 5 is satisfied. Weight preservation is immediate since cells never change rows. $\square$

Definition 40. The *rectification* of an arbitrary diagram T is the result of iteratively applying ϱ as needed until we have a generic Kohnert diagram. That is,

$$\mathrm{rectify}(T) = \varrho^m(T), \tag{17}$$

for m sufficiently large such that $\varrho^{m+1}(T) = \varrho^m(T)$.

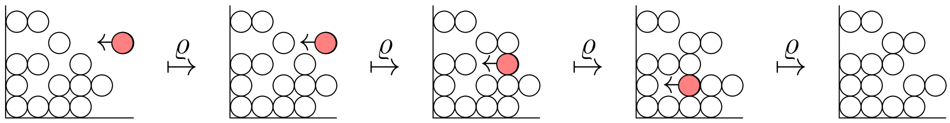

Figure 7: Rectification, where ϱ acts by moving the colored cell left.

Example 41. In Fig. 7 is a Kohnert diagram for the weak composition $(4, 1, 5, 0, 4)$, with an additional cell in row 4 beyond the last occupied column. Here $\mathfrak{m}(c, r) \geqslant 0$ for $c < 6$ or $r > 4$, and $\mathfrak{m}(6, r) = -1$ for $r \leqslant 4$. Therefore ϱ will act on this additional cell in position $(6, 4)$, moving it one column to the left within its row to position $(5, 4)$. Iterating, ϱ acts by moving the colored cell (●) left four times until arriving at the Kohnert diagram for $(4, 2, 5, 0, 4)$ on the right side of Fig. 7.

We use rectification in a limited sense in this paper, though it is worth noting the sense in which we use it gives a generalization of RSK insertion; see Fig. 8

Theorem 42. *Let $\mathbb{D} : \mathrm{SSYT}_n(\lambda) \to \mathrm{KD}(\mathrm{rev}(\lambda))$ denote the bijection from semistandard Young tableaux to generic Kohnert diagrams obtained by moving the cells in column c with entry r to position $(c, n+1-r)$. Then for $T \in \mathrm{SSYT}_n(\lambda)$ and $1 \leqslant j \leqslant n$, we have*

$$\mathbb{D}(\mathrm{RSK}(T, j)) = \mathrm{rectify}(\mathbb{D}(T) \sqcup \{(\lambda_1 + 1, n + 1 - j)\}), \tag{18}$$

where $\mathrm{RSK}(T, j)$ denotes the result of inserting j into T via RSK.

Proof. We assume familiarity with RSK insertion; see [14] for details. Recall that when an entry k bumps an entry l in some column c, we have $k < l$ and there is no entry k in column c. Suppose the RSK algorithm generates a sequence $(T, j) = (T_1, j_1), \ldots, (T_p, j_p), (T_{p+1}, \varnothing)$ where $T_i \in \mathrm{SSYT}(\lambda)$ is obtained by inserting j_{i-1} into row $i-1$ for $1 \leqslant i \leqslant p$, j_i is the bumped entry for $i \leqslant p$, and $T_{p+1} = \mathrm{RSK}(T, j)$. Then we may consider the diagrams $U_i = \mathbb{D}(T_i) \sqcup \{(c_i, n + 1 - j_i)\}$ where c_i is the column in which j_i bumps j_{i+1} in T_i for $1 \leqslant i \leqslant p$ and $c_{p+1} = \lambda_{p+1} + 1$. Also set $U_{p+1} = \mathbb{D}(T_{p+1})$, which is a generic Kohnert

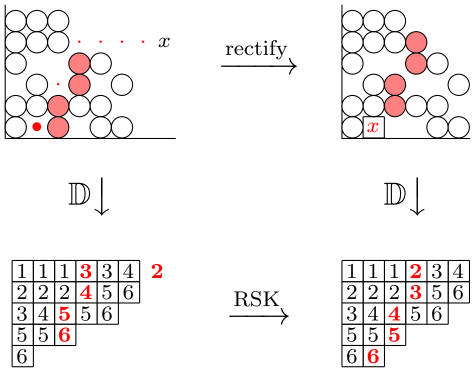

Figure 8: An example of RSK as rectification, where the red highlighted cells/numbers denote the rectification/bumping path.

diagram. It suffices to show for each $i > 1$ there exists some integer $s_i \geqslant 0$ such that $U_i = \varrho^{c_i - c_{i+1}}(U_{i-1})$.

The difference between U_k and U_{k+1} is that the cell in row $n+1-j_k$ has moved left from column c_k to column c_{k+1}. If $c_{k+1} = c_k$, then $U_{k+1} = U_k$, and we are done. If $c_{k+1} < c_k$, then since moving entries in column c_k in rows $r \geqslant j_k$ will not result in a semistandard Young tableaux, U_k is not a generic Kohnert diagram. Moreover, this properties persist even as the cell y in row $n+1-j_k$ column c_k of U_k moves left provided it stays strictly right of column c_{k+1}. Since any matching on $\mathbb{D}(T_k)$ gives a matching between all columns, the unmatched cell y of U_k must be the cell that moves under ϱ, and again, this persists as it moves left until reaching column c_{k+1}. Thus $U_i = \varrho^{c_i - c_{i+1}}(U_{i-1})$ as desired. $\square$

Rectification is the heart of our bijection for Theorem 10 for sufficiently large values of k and plays a role in the general case as well.

Definition 43. Let T be a generic Kohnert diagram, let

$$c = \max_{x \in T}\{\mathrm{col}(x)\}$$

be the rightmost occupied column of T, and let $j \leqslant n$ be a positive integer. Then the *top insertion map* Δ_∞ sends the tuple (T, j) to the diagram

$$\Delta_\infty(T, j) = \mathrm{rectify}(T \sqcup \{(c+1, j)\}). \tag{19}$$

Remark 44. Note the maps Δ_1 and $\Delta_\infty(-, 1)$ differ in general. For instance, we have $\Delta_1(\mathrm{key}_{(0,2,1)}) = \mathrm{key}_{(1,2,1)}$ whereas $\Delta_\infty(\mathrm{key}_{(0,2,1)}, 1)$ is not a key diagram.

The top insertion map Δ_∞ induces the bijection of Theorem 10 for sufficiently large values of k. Note, however, that Δ_∞ is itself independent of k.

Theorem 45. *Let* $\mathbf{a}$ *be a weak composition, and set* $\ell = \max_i\{a_i > 0\}$*. For every positive integer* $\ell \leqslant k \leqslant n$*, the map* Δ_∞ *induces a weight preserving bijection*

$$\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_k) \xrightarrow{\sim} \mathcal{D}(\mathbf{a}, k). \tag{20}$$

In particular, Theorem 10 is proved for $k \geqslant \ell$*.*

Proof. Assaf and Searles [1, Definition 4.5] give a weight-preserving, injective map from Kohnert diagrams for $\mathbf{a}$ to semistandard Young tableaux of shape $\mathrm{sort}(\mathbf{a})$ [1, Theorem 4.6]. In the proof of [5][Theorem 6.1], Haglund, Luoto, Mason and van Willigenburg prove RSK insertion establishes the desired bijection on this subset of tableaux arising as the image of the Assaf–Searles bijection. Thus the result now follows from Theorem 42. □

5 Stratification

We now complete the proof of the existence of a weight-preserving bijection as asserted in Theorem 10. Given a weak composition $\mathbf{a}$, the target spaces $\mathcal{D}(\mathbf{a}, k)$ for each positive integer k form a nested sequence of spaces of Kohnert diagrams,

$$\cdots \supset \mathcal{D}(\mathbf{a}, k+1) \supset \mathcal{D}(\mathbf{a}, k) \supset \mathcal{D}(\mathbf{a}, k-1) \supset \cdots \supset \mathcal{D}(\mathbf{a}, 1).$$

We use this to *stratify* the target space of the desired bijection by

$$\overline{\mathcal{D}}(\mathbf{a}, k) = \mathcal{D}(\mathbf{a}, k) \setminus \mathcal{D}(\mathbf{a}, k-1). \tag{21}$$

In Section 5.1, for each integer $k > 1$, we define an injective map

$$\partial_{\mathbf{a},k} \ : \ \overline{\mathcal{D}}(\mathbf{a}, k) \longrightarrow \mathrm{KD}(\mathbf{a}) \tag{22}$$

satisfying $\mathbf{wt}(U) = \mathbf{wt}(\partial_{\mathbf{a},k}(U)) + \mathbf{e}_k$, and we show how these maps together with bijections for top and bottom insertion prove Theorem 10. In Section 5.2, we prove the image of $\partial_{\mathbf{a},k}$ is a Kohnert diagram for $\mathbf{a}$, and in Section 5.3 we prove $\partial_{\mathbf{a},k}$ is injective.

5.1 Stratum maps

Up to this point, given a Kohnert diagram T, we have been primarily interested in the matching $\mathcal{M}_{\boldsymbol{\theta}}(T)$ corresponding to the thread decomposition $\boldsymbol{\theta}(T)$. To study the stratum maps, we consider also matchings corresponding to the *Kohnert labeling of* T *with respect to* $\mathbf{a}$ [1, Definition 2.5].

Definition 46 ([1])**.** Let $\mathbf{a}$ be a weak composition and $T \in \mathrm{KD}(\mathbf{a})$. The *Kohnert labeling of* T *with respect to* $\mathbf{a}$, denoted by $\mathcal{L}_{\mathbf{a}}(T)$, is defined as follows. Assuming all columns right of column j have been labeled, bijectively assign labels $\{i \mid a_i \geqslant j\}$ to cells of column j from bottom to top by choosing at each cell the smallest unused label i such that the i in column $j+1$, if it exists, is weakly lower.

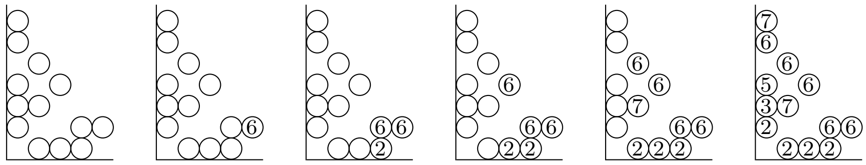

Figure 9: The Kohnert labeling with respect to $\mathbf{a} = (0, 4, 1, 0, 1, 5, 2)$.

The *Kohnert matching of T with respect to* $\mathbf{a}$, denoted by $\mathcal{M}_{\mathbf{a}}(T)$, is the matching sequence on T defined by x matching to y for cells $x \in T$ in column $i+1$ and $y \in T$ in column i if and only if $\mathcal{L}_{\mathbf{a}}(x) = \mathcal{L}_{\mathbf{a}}(y)$. For example, see Fig. 9.

Assaf and Searles [1, Theorem 2.8] prove this is well-defined and use it to define and establish basic properties of *Kohnert tableaux* [1, Definition 2.3].

Proposition 47. *For $T \in \mathrm{KD}(\mathbf{a})$, we have $\mathbf{wt}(\mathcal{M}_{\mathbf{a}}) \preceq \mathbf{a}$.*

Proof. We may regard the Kohnert matching sequence $\mathcal{M}_{\mathbf{a}}$ as a labeling L where $L(x) = k$ whenever the cell on the component of x in the first column is in row k. If this labeling agrees with the Kohnert labeling $\mathcal{L}_{\mathbf{a}}$, then $\mathbf{wt}(\mathcal{M}_{\mathbf{a}}) = \mathbf{a}$. Otherwise, consider the key diagram $\mathrm{key}_{\mathbf{wt}(\mathcal{M}_{\mathbf{a}})}$ with the labeling L' where $L'(x)$ is the label of the cell in the first column of T in the same row as x. By [1, Theorem 2.8], this is the Kohnert labeling of $\mathrm{key}_{\mathbf{wt}(\mathcal{M}_{\mathbf{a}})}$ with respect to $\mathbf{a}$, and so $\mathrm{key}_{\mathbf{wt}(\mathcal{M}_{\mathbf{a}})} \in \mathrm{KD}(\mathbf{a})$. Thus by Proposition 9, $\mathbf{wt}(\mathcal{M}_{\mathbf{a}}) \preceq \mathbf{a}$. ☐

The matching sequence on T corresponding to the thread decomposition of T is a special case of a Kohnert matching on T; for example, see Fig. 10.

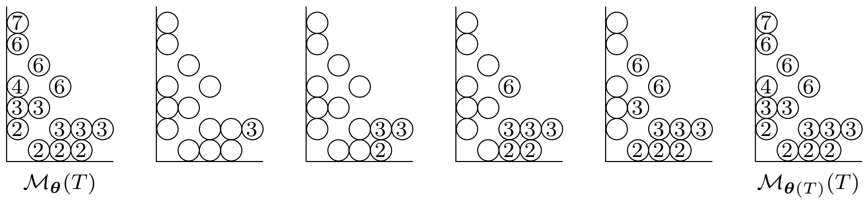

Figure 10: An example illustrating Proposition 48, where $\boldsymbol{\theta}(T) = (0, 4, 5, 1, 0, 3, 1)$.

Proposition 48. *For T a generic Kohnert diagram, $\mathcal{M}_{\boldsymbol{\theta}}(T) = \mathcal{M}_{\boldsymbol{\theta}(T)}(T)$.*

Proof. By Lemma 8, we have $T \in \mathrm{KD}(\boldsymbol{\theta}(T))$, so we may consider the Kohnert matching of T with respect to $\boldsymbol{\theta}(T)$. Note $\mathcal{M}_{\boldsymbol{\theta}}(T)$ and $\mathcal{M}_{\boldsymbol{\theta}(T)}(T)$ have the same weight $\boldsymbol{\theta}(T)$, and so the same number of threads. Let $i \geqslant 1$ be the smallest index for which $\boldsymbol{\theta}(T)_i > 0$. We will show that the components for $\mathcal{M}_{\boldsymbol{\theta}}(T)$ and $\mathcal{M}_{\boldsymbol{\theta}(T)}(T)$ anchored in row i coincide, from which the result follows by induction on the number of threads since the base case of one component is trivial.

Consider the cell in column $\boldsymbol{\theta}(T)_i$ anchored in row i. Since i is minimal, this is the lowest cell of T in column $\boldsymbol{\theta}(T)_i$ for $\mathcal{M}_{\boldsymbol{\theta}(T)}(T)$. If for $\mathcal{M}_{\boldsymbol{\theta}}(T)$ there is a cell below this, then that cell must belong to an earlier thread, and so it will be threaded before the cells anchored at row i. However, since it is lower in column $\boldsymbol{\theta}(T)_i$, it will always take weakly lower cells, contradicting the minimality of i. Thus both threads begin with the lowest cell of T in column $\boldsymbol{\theta}(T)_i$.

We may assume the components anchored in row i for $\mathcal{M}_{\boldsymbol{\theta}}(T)$ and $\mathcal{M}_{\boldsymbol{\theta}(T)}(T)$ agree weakly right of column $c+1 \leqslant \boldsymbol{\theta}(T)_i$. Then for $\mathcal{M}_{\boldsymbol{\theta}(T)}(T)$, the chosen cell in column c will be the lowest cell of T that sits weakly above the cell labeled i in column $c+1$. In order for $\mathcal{M}_{\boldsymbol{\theta}}(T)$ not to choose the same cell, there must be an earlier thread that takes the cell chosen by $\mathcal{M}_{\boldsymbol{\theta}(T)}(T)$, but then again this thread will have priority of the one anchored in row i and will continue to select lower cells, once again contradicting the minimality of i. Therefore the components anchored in row i for $\mathcal{M}_{\boldsymbol{\theta}}(T)$ and $\mathcal{M}_{\boldsymbol{\theta}(T)}(T)$ coincide, and the result follows. $\square$

Recall Lemma 33 associates to each $U \in \mathcal{D}(\mathbf{a}, n)$ a unique added column with respect to $\mathbf{a}$, called the added column of U (Definition 34).

Lemma 49. *For a weak composition* $\mathbf{a}$ *and a positive integer* $k > 1$, *let* $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$, *and let* c *be the added column of* U. *Then*

1. *the key diagram* $\mathrm{key}_{\boldsymbol{\theta}(U)}$ *has a cell* y *in position* (c, k), *and*
2. *the diagram* $\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\}$ *is a Kohnert diagram for* $\mathbf{a}$.

Proof. By Lemma 8, since $U \in \mathcal{D}(\mathbf{a}, k)$, we have $\boldsymbol{\theta}(U) \preceq \mathbf{b}+\mathbf{e}_k$ for some weak composition $\mathbf{b} \in \mathrm{lswap}(\mathbf{a})$, and, by Lemma 33, $c = b_k + 1$. Thus $\mathrm{key}_{\boldsymbol{\theta}(U)} \in \mathrm{KD}(\mathbf{b} + \mathbf{e}_k)$, and we may consider the Kohnert labeling $L = \mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(\mathrm{key}_{\boldsymbol{\theta}(U)})$. Let y be the cell in column c of $\mathrm{key}_{\boldsymbol{\theta}(U)}$ with $L(y) = k$. Note y is the rightmost cell with label k. Thus we may define a matching M on $\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\}$ by matching cells in adjacent columns if and only if they have the same label under L, and $\mathbf{wt}(M) = \mathbf{b}$. Thus, by Theorem 31 and Proposition 9, we have $\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\} \in \mathrm{KD}(\mathbf{b}) \subset \mathrm{KD}(\mathbf{a})$. To prove the lemma, we have only to show y is in row k.

Let $\mathbf{b}' = \boldsymbol{\theta}(\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\})$. Since $\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\} \in \mathrm{KD}(\mathbf{a})$, Lemma 8 ensures $\mathbf{b}' \preceq \mathbf{a}$. Let k' be the row index of y. Since $L(y) = k$, we know by [1, Theorem 2.8] that $k' \leqslant k$. Since $\mathrm{key}_{\boldsymbol{\theta}(U)}$ is a key diagram, for every position (i, j) occupied by $\mathrm{key}_{\boldsymbol{\theta}(U)}$, the position $(i-1, j)$ is occupied by $\mathrm{key}_{\boldsymbol{\theta}(U)}$ as well. Therefore, considering the thread decomposition of $\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\}$, we have $b'_{k'} = c - 1$, where the path in the thread decomposition that is anchored at row k', if it exists, only occupies row k'. Hence, either by appending y to the end of the path anchored at row k' (if $c > 1$) or having y form its own path (if $c = 1$), we obtain from $\mathcal{M}_{\boldsymbol{\theta}}(\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\})$ a matching sequence on $\mathrm{key}_{\boldsymbol{\theta}(U)}$ with weight $\mathbf{b}' + \mathbf{e}_k$. So by Theorem 31, we have $U \in \mathrm{KD}(\mathbf{b}' + \mathbf{e}_{k'})$. If $k' < k$, then $U \in \mathcal{D}(\mathbf{a}, k-1)$, which directly contradicts how U is picked from $\overline{\mathcal{D}}(\mathbf{a}, k)$. Thus, $k' = k$, as desired. $\square$

Lemma 49 motivates the following notation; see Fig. 11.

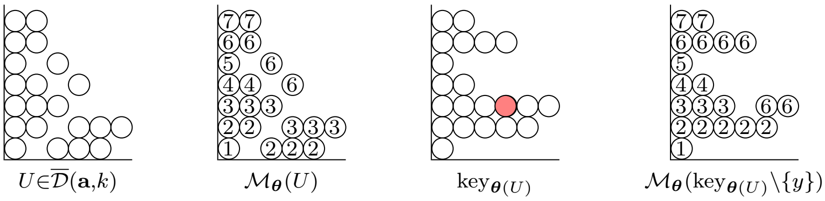

Figure 11: A generic Kohnert diagram U (left) in $\overline{\mathcal{D}}(\mathbf{a}, k)$ for $\mathbf{a} = (1,5,2,1,2,6,3)$ and $k = 3$ with added column $c = 4$, its thread decomposition (middle), and the key diagram $\mathrm{key}_{\boldsymbol{\theta}(U)}$ (right) with the cell y in position (c, k) highlighted in $\mathrm{key}_{\boldsymbol{\theta}(U)}$.

Definition 50. For $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$, the *excised weight of* U (with respect to $\mathbf{a}$ and k) is the weak composition $\boldsymbol{\theta}(\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\})$, where y is the cell of $\mathrm{key}_{\boldsymbol{\theta}(U)}$ in position (c, k) for c the added column of U.

Example 51. Consider the weak composition $\mathbf{a} = (1,5,2,1,2,6,3)$, and the generic Kohnert diagram U on the left side of Fig. 11. The thread weight of U is $\boldsymbol{\theta}(U) = (1,5,6,2,1,4,2)$. Note that $\boldsymbol{\theta}(U) \preceq \mathbf{b} + \mathbf{e}_k$, where $\mathbf{b} = (1,5,3,2,1,6,2) \preceq \mathbf{a}$ and $k = 3$, and so $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$. Comparing column weights, the added column of U with respect to $\mathbf{a}$ is $c = 4$. The key diagram $\mathrm{key}_{\boldsymbol{\theta}(U)}$ shown on the right side of Fig. 11 has a cell (●) in position $(4, 3)$, and removing it gives a diagram whose thread weight is $\mathbf{b}$, which makes $\mathbf{b}$ the excised weight of U.

For $\mathbf{b}$ the excised weight of a generic Kohnert diagram $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$ with respect to a weak composition $\mathbf{a}$ and row index k, we infer from Lemmas 49(2) and 8 that $\mathrm{key}_{\boldsymbol{\theta}(U)} \in \mathrm{KD}(\mathbf{b} + \mathbf{e}_k)$ and $\mathbf{b} \preceq \mathbf{a}$. Thus, $U \in \mathrm{KD}(\mathbf{b} + \mathbf{e}_k)$, and we may consider the Kohnert labeling $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(U)$. Through this labeling, we partition U into sub-diagrams, which we transform and glue back together to obtain the image of U under the map $\partial_{\mathbf{a},k}$.

The simplicity of the statement of the following lemma belies its utility.

Lemma 52. *For* $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$ *and* $\mathbf{b}$ *the excised weight of* U*, each cell* x *in the first column of* U *has* $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(x)$ *equal to the row index of* x*. In particular,* $\mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}(U)) = \mathbf{b} + \mathbf{e}_k$.

Proof. Recall $\mathbf{b} = \boldsymbol{\theta}(\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\})$ for y the cell in position (c, k) of U, where c is the added column of U. Thus, for all $j \neq k$, we have $b_j > 0$ if and only if $\boldsymbol{\theta}(U)_j > 0$. On the other hand, since y is in row k, the key diagram $\mathrm{key}_{\boldsymbol{\theta}(U)}$ occupies positions in row k. Therefore, $\boldsymbol{\theta}(U)_k > 1$, and in particular, U has a cell at position $(1, k)$. By construction, the set of labels used in $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}$ coincides with the row indices of the cells in column 1 of U. Since each cell x has $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(U)$ at least equal to the row index of x, each cell in U in column 1 must have its row as its label. □

In particular, by Lemma 52, U has a cell in column 1 of row k, and this cell belongs to a component of $\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}(U)$ of length c.

Definition 53. Let M be a matching sequence on a generic Kohnert diagram T. For each cell $x \in T$, the *matching path length for x in M*, denoted by $\mu_M(x)$, is the number of cells in the connected component of M containing x.

Definition 54. For $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$, let c denote the added column of U and $\mathbf{b}$ the excised weight of U. Define sets

$$U^+ \;=\; \{x \in U \mid \mu_{\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}}(x) \geqslant c\}, \tag{23}$$

$$U^- \;=\; \{x \in U \mid \mu_{\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}}(x) < c\}, \tag{24}$$

and let $U^+_* = U^+ \setminus \{w\}$, where w is the cell in column 1 of row k.

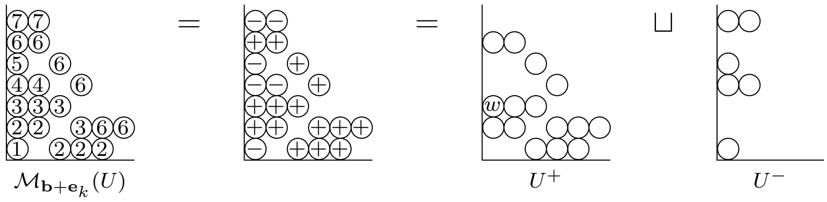

Figure 12: The partitioning of the generic Kohnert diagram U (left) in $\overline{\mathcal{D}}(\mathbf{a}, k)$ for $\mathbf{a} = (1, 5, 2, 1, 2, 6, 3)$ and $k = 3$ using the Kohnert labeling to obtain U^+ and U^-; here $c = 4$.

Example 55. Continuing with Ex. 51, the Kohnert labeling of the diagram U with respect to $\mathbf{b} + \mathbf{e}_k = (1, 5, 4, 2, 1, 6, 2)$, where $\mathbf{b}$ is the excised weight of U and $k = 3$, as shown on the left side of Fig. 12. The added column $c = 4$ dictates which labels are included for each half of the partitioning giving the decomposition.

Notice $U = U^+ \sqcup U^-$. We can now define the map $\partial_{\mathbf{a},k}$.

Definition 56. Given a weak composition $\mathbf{a}$ and an integer $k > 1$, the *kth stratum map of* $\mathbf{a}$, denoted by $\partial_{\mathbf{a},k}$, acts on $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$ by

$$\partial_{\mathbf{a},k}(U) = \mathrm{rectify}(U^+_*) \cup U^-. \tag{25}$$

Note the union in (25) remains *disjoint*. This is shown in Theorem 65 below.

Example 57. Continuing with Ex. 55, we remove the cell in position $(1, k)$ of U^+ and rectify to obtain the diagram on the right side of Fig. 13. Notice this is disjoint from U^-: we prove later in Section 5.2 that this will always be the case. Their (disjoint) union is a Kohnert diagram for $\mathbf{a} = (1, 5, 2, 1, 2, 6, 3)$, and is the result of applying the stratum map $\partial_{\mathbf{a},k}$ to the diagram U.

The map $\partial_{\mathbf{a},k}$ is well-defined with $\mathbf{wt}(U) = \mathbf{wt}(\partial_{\mathbf{a},k}(U)) + \mathbf{e}_k$ by Lemma 52. In Section 5.2, we study properties of rectification to prove the following.

Theorem 58. *The diagram $\partial_{\mathbf{a},k}(U)$ is a Kohnert diagram for* $\mathbf{a}$.

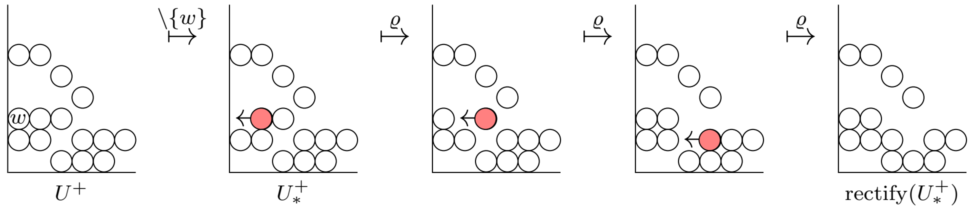

Figure 13: The rectification of the diagram U^+_* obtained from U^+ by removing the cell in position $(1,3)$.

Therefore $\partial_{\mathbf{a},k}$ is a map from the stratum $\overline{\mathcal{D}}(\mathbf{a},k)$ into the Kohnert space $\mathrm{KD}(\mathbf{a})$. In Section 5.3, we prove this map is reversible and thus injective.

Theorem 59. *For $U \in \overline{\mathcal{D}}(\mathbf{a},k)$, the diagram U is the unique pre-image of the diagram $\partial_{\mathbf{a},k}(U)$ under the kth stratum map $\partial_{\mathbf{a},k}$ of $\mathbf{a}$. That is, $\partial_{\mathbf{a},k}$ is injective.*

Using the injectivity of the stratum maps together with the bijectivity of the top and bottom insertion maps, we now prove there exists a weight-preserving bijection

$$\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_k) \xrightarrow{\sim} \bigcup_{\substack{\mathbf{b} \preceq \mathbf{a} \\ 1 \leqslant j \leqslant k}} \mathrm{KD}(\mathbf{b}+\mathbf{e}_j).$$

In the proof, we extend our notion of row-weights on sets of Kohnert diagrams to Cartesian products of such sets, e.g. if we have $(S,T) \in \mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{b})$ for some weak compositions $\mathbf{a}$ and $\mathbf{b}$, then we define $\mathbf{wt}(S,T) = \mathbf{wt}(S) + \mathbf{wt}(T)$. In this way, we can discuss the row-weight spaces of said sets, e.g. given another weak composition $\mathbf{d}$, the set $[\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{b})]_{\mathbf{d}}$ is the set of all tuples (S,T) with $S \in \mathrm{KD}(\mathbf{a})$ and $T \in \mathrm{KD}(\mathbf{b})$ such that $\mathbf{wt}(S) + \mathbf{wt}(T) = \mathbf{d}$.

Proof of Theorem 10. Let $\mathbf{a}$ be a weak composition. It suffices to show

$$\#[\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_k)]_{\mathbf{b}} = \#[\mathcal{D}(\mathbf{a},k)]_{\mathbf{b}}, \tag{26}$$

for every positive integer $k \leqslant n$ and every weak composition $\mathbf{b}$. To start, we fix a weak composition $\mathbf{b}$. We immediately get

$$\#[\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_n)]_{\mathbf{b}} = \#[\mathcal{D}(\mathbf{a},n)]_{\mathbf{b}}$$

via the weight-preserving bijection in Eq. (20) of Theorem 45. We will now show

$$\#[\mathcal{D}(\mathbf{a},k)]_{\mathbf{b}} = \#[\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_k)]_{\mathbf{b}}$$

for every positive integer $k \leqslant n$. By stratification, we have

$$\mathcal{D}(\mathbf{a},n) = \mathcal{D}(\mathbf{a},1) \sqcup \bigsqcup_{1<k\leqslant n} \overline{\mathcal{D}}(\mathbf{a},k).$$

The weight-preserving bijection in Eq. (14) of Theorem 36 gives us

$$\#[\mathcal{D}(\mathbf{a},1)]_{\mathbf{b}} = \#[\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_1)]_{\mathbf{b}} = \#[\mathrm{KD}(\mathbf{a}) \times \{\mathrm{key}_{\mathbf{e}_1}\}]_{\mathbf{b}}. \tag{27}$$

On the other hand, for every $1 < k \leqslant n$, the stratum map $\partial_{\mathbf{a},k}$ which is injective by Theorem 59 and excises a cell in row k, gives us

$$\#[\overline{\mathcal{D}}(\mathbf{a},k)]_{\mathbf{b}} \leqslant \#[\mathrm{KD}(\mathbf{a})]_{\mathbf{b}-\mathbf{e}_k} = \#[\mathrm{KD}(\mathbf{a}) \times \{\mathrm{key}_{\mathbf{e}_k}\}]_{\mathbf{b}} \tag{28}$$

Eq. (27) and (28) together imply

$$\begin{aligned}
\#[\mathcal{D}(\mathbf{a},n)]_{\mathbf{b}} &= \#[\mathcal{D}(\mathbf{a},1)]_{\mathbf{b}} + \textstyle\sum_{1<k\leqslant n} \#[\overline{\mathcal{D}}(\mathbf{a},k)]_{\mathbf{b}} \\
&\leqslant \#[\mathrm{KD}(\mathbf{a}) \times \{\mathrm{key}_{\mathbf{e}_1}\}]_{\mathbf{b}} + \textstyle\sum_{1<k\leqslant n} \#[\mathrm{KD}(\mathbf{a}) \times \{\mathrm{key}_{\mathbf{e}_k}\}]_{\mathbf{b}} \\
&\leqslant \#[\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_n)]_{\mathbf{b}}.
\end{aligned}$$

Thus, since $\#[\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_n)]_{\mathbf{b}} = \#[\mathcal{D}(\mathbf{a},n)]_{\mathbf{b}}$, we have for each $1 < k \leqslant n$, the inequality in Eq. (28) is in fact an equality. So for each $1 < k \leqslant n$, we have

$$\begin{aligned}
\#[\mathcal{D}(\mathbf{a},k)]_{\mathbf{b}} &= \#[\mathcal{D}(\mathbf{a},1)]_{\mathbf{b}} + \textstyle\sum_{1<j\leqslant k} \#[\overline{\mathcal{D}}(\mathbf{a},k)]_{\mathbf{b}} \\
&= \#[\mathrm{KD}(\mathbf{a}) \times \{\mathrm{key}_{\mathbf{e}_1}\}]_{\mathbf{b}} + \textstyle\sum_{1<j\leqslant k} \#[\mathrm{KD}(\mathbf{a}) \times \{\mathrm{key}_{\mathbf{e}_j}\}]_{\mathbf{b}} \\
&= \#[\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{e}_k)]_{\mathbf{b}}.
\end{aligned}$$

In conclusion, Eq. (26) holds for every positive integer $k \leqslant n$ and every weak composition $\mathbf{b}$, as desired. □

5.2 Image of the stratum maps

To prove the image of U under $\partial_{\mathbf{a},k}$ is a Kohnert diagram in $\mathrm{KD}(\mathbf{a})$, we first show $\mathrm{rectify}(U_*^+)$ and U^- are disjoint, from which it follows that $\partial_{\mathbf{a},k}(U)$ is a generic Kohnert diagram. To this end, we study the rectification process in detail.

Definition 60. For $U \in \overline{\mathcal{D}}(\mathbf{a},k)$, let $y_i \in U_*^+$ be the cell that moves from column $i+1$ of $\varrho^{i-1}(U_*^+)$ to column i of $\varrho^i(U_*^+)$, and set $U^0 = \{y_1, y_2, \cdots, y_{c-1}\} \subset U_*^+$.

For example, these are the highlighted cells in Fig 13.

To describe the general movement of cells of U_*^+ under rectification, it is helpful to understand their labels under $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(U)$. The cells with label k under $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(U)$ are of particular importance, so we introduce the following notation. Recall $b_k + 1 = c$, and so there are c of these cells forming a component of the matching induced by $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}$.

Definition 61. For $U \in \overline{\mathcal{D}}(\mathbf{a},k)$ and $\mathbf{b}$ the excised weight of U, define sets

$$U^{=k} = \{x \in U \mid \mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(x) = k\}, \tag{29}$$
$$U^{<k} = \{x \in U \mid \mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(x) < k\} \tag{30}$$
$$U^{\leqslant k} = \{x \in U \mid \mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(x) \leqslant k\}. \tag{31}$$

Set $U_*^{=k} = U^{=k} \setminus \{w\}$ for w the cell in column 1, row k.

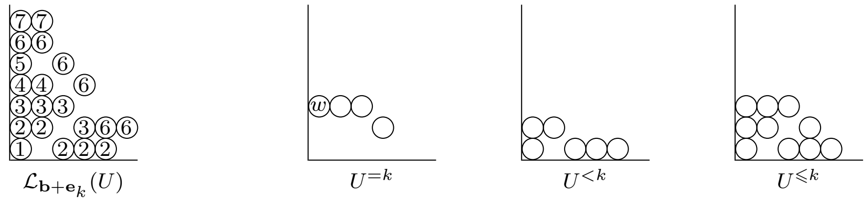

Figure 14: The partitioning of $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$ for $k = 3$ and $\mathbf{a} = (1, 5, 2, 1, 2, 6, 3)$ based on its Kohnert labeling for $\mathbf{b} + \mathbf{e}_k = (1, 5, 4, 2, 1, 6, 2)$ for $\mathbf{b}$ is the excised weight of U.

For example, see Fig. 14. Notice we have $U^{=k} \subset U^+$ and, of course, $U^{=k}_* \subset U^+_*$.

The running example illustrates the following description of rectification.

Lemma 62. *Let $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$ with added column c. The diagram $\varrho^{i-1}(U^+_*)$ is not a generic Kohnert diagram if and only if $i < c$. Moreover, for r_i the row of the cell that moves from column $i+1$ of $\varrho^{i-1}(U^+_*)$ to column i of $\varrho^i(U^+_*)$, we have*

1. *r_i is weakly less than the row of the cell of $U^{=k}$ in column $i+1$, and*
2. *$k \geqslant r_1 \geqslant r_2 \geqslant \cdots \geqslant r_{c-1}$.*

Proof. If $c = 1$, the result is trivial, so we assume $c > 1$. Since U^+ consists of all components of $\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}(U)$ of length $\geqslant c$, U^+ has the same number of cells for each column weakly left of column c, and so for each column $1 < c' \leqslant c$, we have

$$\mathfrak{m}_{U^+}(c', 1) = 0. \tag{32}$$

Let $U^{=k} = \{w_1, \ldots, w_c\}$ with w_j the cell in column j.

We proceed by induction on i, with base case of $i = 1$ in which we consider U^+_* itself. By Eq. (32), since U^+_* has one fewer cell in the first column than U^+, we must have $\mathfrak{m}_{U^+_*}(2, 1) = -1$. In particular, $\varrho^0(U^+_*) = U^+_*$ is not a generic Kohnert diagram. Letting y_1 denote the cells that moves under ϱ, it must do so from column 2 of U^+_* to column 1 of $\varrho(U^+_*)$. Among the cells of U^+_* above w_2, $\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}$ gives a matching from the cells in column 2 to the cells in column 1, and so $\mathfrak{m}_{U^+_*}(2, j) \geqslant 0$ for all rows j above the row of w_2. By definition of ϱ, the cell y_1 must lie weakly below w_2 and so weakly below row k as well. This proves all statements for $i = 1$.

For $1 \leqslant i < c$, we now assume the result for all $h \leqslant i$, so that $\varrho^{h-1}(U^+_*)$ is not a generic Kohnert diagram, giving a sequence of cells $y_1, \ldots, y_i$ where y_h moves from column $h+1$ of $\varrho^{h-1}(U^+_*)$ to column h of $\varrho(\varrho^h(U^+_*))$ and satisfies (1) y_h lies weakly below w_{h+1} for $h \leqslant i$ and (2) $k \geqslant \mathrm{row}(y_1) \geqslant \cdots \geqslant \mathrm{row}(y_i)$.

Since $\varrho^{i-1}(U^+_*)$ and U^+_* coincide for all columns strictly right of column i, and differ in column i only in the presence of y_{i-1} in U^+_*, it follows from Eq. (32) that $\mathfrak{m}_{\varrho^{i-1}(U^+_*)}(i+1, j) \geqslant -1$ for all rows j and $\mathfrak{m}_{\varrho^{i-1}(U^+_*)}(i+1, j) \geqslant 0$ for all $j > \mathrm{row}(y_{i-1})$. Thus when y_i moves from column $i+1$ to column i, we have $\mathfrak{m}_{\varrho^i(U^+_*)}(i+1, j) \geqslant 0$ for every row j.

Therefore $\varrho^i(U_*^+)$ is a generic Kohnert diagram if and only if $\mathfrak{m}_{\varrho^i(U_*^+)}(i+2,j) \geqslant 0$ for every row j. We consider two cases.

First suppose $i = c-1$, so that $i+2 = c+1$. Since $c > 1$, the Kohnert matching sequence $\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}(U)$ matches each cell in column $c+1$ of U_*^+ with a cell in column c of U_*^+. Using this, we have a matching from each cell in column $c+1$ of $\varrho^i(U_*^+)$ with a cell in column c of U_*^+ *except* for the cell, say x, that was matched to y_{c-1} in U_*^+. Since y_{c-1} is weakly below w_c, we may match x with w_c to complete the matching, thereby showing $\mathfrak{m}_{\varrho^i(U_*^+)}(i+2,j) \geqslant 0$ for every row j., and so $\varrho^i(U_*^+)$ is a generic Kohnert diagram.

Second suppose $i < c-1$, so that $i+2 \leqslant c$. By Eq. (32), we have $\mathfrak{m}_{\varrho^i(U_*^+)}(i+2,1) = -1$, and so $\varrho^i(U_*^+)$ is not a generic Kohnert diagram. Thus let y_{i+1} be the cell of $\varrho^i(U_*^+)$ that moves in passing to $\varrho(\varrho^i(U_*^+))$. Now let x be the cell in column $i+1$ of U_*^+ such that $\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}(x) = y_i$. Since $\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}$ gives a matching from column $i+2$ to column $i+1$ in U_*^+, it gives a matching for cells weakly above the row of x from column $i+2$ to column $i+1$. In particular, $\mathfrak{m}_{\varrho^i(U_*^+)}(i+2,j) \geqslant 0$ for all rows $j > \mathrm{row}(x)$. In particular, y_{i+1} is weakly below x, and so too weakly below y_i, proving (1). If w_{i+2} is weakly above x, then y_{i+1} is weakly below w_{i+2} as well, proving (2). Otherwise, if w_{i+2} is strictly below x, then, similar to the case $i = c-1$, we construct a matching for cells above $\mathrm{row}(w_{i+1})$ from column $i+2$ to column $i+1$ using $\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}(U)$ for every cell except x and then matching x to w_{i+1}, since x is weakly below y_i which is below w_{i+1}. Thus $\mathfrak{m}_{U_i}(i+2,j) \geqslant 0$ for all rows $j > \mathrm{row}(w_{i+2})$. So y_{i+1} is again weakly below w_{i+2} proving (2) for this case as well. Therefore all statements hold for $i+1$, and so the result follows by induction. □

Notice the labels of the cells of the rectification path remain weakly smaller than k.

Lemma 63. *For $U \in \overline{\mathcal{D}}(\mathbf{a},k)$, let c be the added column and $\mathbf{b}$ the excised weight of U. Then for $i < c$, we have $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(y_i) \leqslant k$, where $U^0 = \{y_1, \ldots, y_{c-1}\}$.*

Proof. Let $U^{=k} = \{w_1, \ldots, w_c\}$ with w_i the cell in column i. We claim each cell $x \in U^+ \setminus U^{\leqslant k}$ in column $i \leqslant c$ lies strictly above the cell w_i. Indeed, in column c, if some such x lies below w_c, then since $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(x) > k = \mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(w_i)$, the labeling algorithm will assign k to x instead of w_c, since x must also have been available and there is no cell labeled k to the right. Continuing left, consider the largest column index $i < c$ such that some cell $x \in U^+ \setminus U^{\leqslant k}$ in column i lies below w_i, then in order for x not to have been selected, we must have w_{i+1} in a row strictly above that of x. However, since $\mu_{\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}}(x) \geqslant c = \mu_{\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}}(w_i)$, there is also a cell x' in column $i+1$ in a row weakly below that of x, and hence below w_{i+1}, a contradiction to the choice of i. Thus each x in column $i \leqslant c$ must indeed lie above w_i.

By Lemma 62(1), in U each cell y_i for $i < c$ is weakly below the cell w_{i+1} in its column. Thus, from the claim, y_i cannot be in $U^+ \setminus U^{\leqslant k}$. The result follows. □

One final lemma necessary to prove $\mathrm{rectify}(U_*^+)$ and U^- are disjoint relates the thread decomposition of $U^{\leqslant k}$ with the Kohnert labeling used to define it. For an illustration continuing the running example from Fig. 14, see Fig. 15.

Lemma 64. *For $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$ and $\mathbf{b}$ the excised weight of U, we have*

$$\begin{aligned} \mathcal{M}_{\boldsymbol{\theta}}(U^{\leqslant k}) &= \mathcal{M}_{\mathbf{b}+\mathbf{e}_k}(U)\big|_{U^{\leqslant k}} = \mathcal{M}_{(b_1,\ldots,b_k)+\mathbf{e}_k}(U^{\leqslant k}), & (33) \\ \mathcal{M}_{\boldsymbol{\theta}}(U^{+} \cap U^{\leqslant k}) &= \mathcal{M}_{\mathbf{b}+\mathbf{e}_k}(U)\big|_{U^{+}\cap U^{\leqslant k}}. & (34) \end{aligned}$$

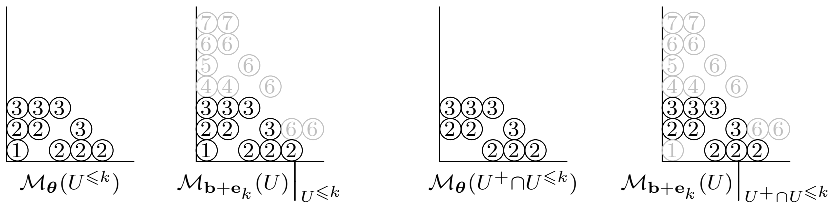

Figure 15:

Proof. Since Definition 46 prioritizes the smallest labels first, it follows that $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(U)$ restricted to labels $\leqslant k$ is precisely $\mathcal{L}_{(b_1,\ldots,b_k)+\mathbf{e}_k}(U^{\leqslant k})$. By Lemma 52, this means

$$\mathbf{wt}(\mathcal{M}_{(b_1,\ldots,b_k)+\mathbf{e}_k}(U^{\leqslant k})) = (b_1, \ldots, b_k) + \mathbf{e}_k$$

By Proposition 48, to prove the lemma, it suffices to show $\boldsymbol{\theta}(U^{\leqslant k}) = (b_1, \ldots, b_k) + \mathbf{e}_k$.

Since $\mathcal{M}_{(b_1,\ldots,b_k)+\mathbf{e}_k}(U^{\leqslant k})$ is a matching sequence on $U^{\leqslant k}$ of weight $(b_1, \ldots, b_k) + \mathbf{e}_k$, we have $U^{\leqslant k} \in \mathrm{KD}((b_1, \ldots, b_k) + \mathbf{e}_k)$ by Theorem 31. Hence by Lemma 8, we have $\boldsymbol{\theta}(U^{\leqslant k}) \preceq (b_1, \ldots, b_k) + \mathbf{e}_k$.

For the reverse inequality, we first show $\boldsymbol{\theta}(U)_k = c$. Since $\boldsymbol{\theta}(U^{\leqslant k}) \preceq (b_1, b_2, \cdots, b_k)+\mathbf{e}_k$, we have $\boldsymbol{\theta}(U^{\leqslant k})_k \leqslant b_k + 1 = c$. On the other hand,

$$\boldsymbol{\theta}(U^{\leqslant k}) + (0^k, b_{k+1}, b_{k+2}, \cdots) \preceq \mathbf{b} + \mathbf{e}_k,$$

so by Proposition 9, we have $\mathrm{key}_{\boldsymbol{\theta}(U^{\leqslant k})+(0^k,b_{k+1},b_{k+1},\cdots)} \in \mathrm{KD}(\mathbf{b} + \mathbf{e}_k) \subset \mathcal{D}(\mathbf{a}, k)$. Combining the labeling $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(x)$ on cells of $U \setminus U^{\leqslant k}$ with the thread matching on $U^{\leqslant k}$ gives a matching sequence on U of weight $\boldsymbol{\theta}(U^{\leqslant k}) + (0^k, b_{k+1}, b_{k+2}, \cdots)$. Thus $U \in \mathrm{KD}(\boldsymbol{\theta}(U^{\leqslant k}) + (0^k, b_{k+1}, b_{k+2}, \cdots))$ by Theorem 31. Since $U \notin \mathcal{D}(\mathbf{a}, k-1)$, neither is $\mathrm{key}_{\boldsymbol{\theta}(U^{\leqslant k})+(0^k,b_{k+1},b_{k+1},\cdots)}$. Moreover, $\mathrm{key}_{\boldsymbol{\theta}(U^{\leqslant k})+(0^k,b_{k+1},b_{k+1},\cdots)}$ has the same added column c as U, since they share the same column weight. So by Lemma 49, the diagram $\mathrm{key}_{\boldsymbol{\theta}(U^{\leqslant k})+(0^k,b_{k+1},b_{k+1},\cdots)}$ has a cell y at position (c, k), and consequently, $\boldsymbol{\theta}(U^{\leqslant k})_k \geqslant c$. Therefore $(\boldsymbol{\theta}(U^{\leqslant k}))_k = c$ as claimed.

Since $\mathbf{b} = \boldsymbol{\theta}(\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\})$, where y is in row k, we necessarily have $b_j = \boldsymbol{\theta}(U)_j$ for all $j < k$. Moreover, since $\boldsymbol{\theta}(U) \preceq \mathbf{b} + \mathbf{e}_k$, we have

$$(0^{k-1}, \boldsymbol{\theta}(U)_k, \boldsymbol{\theta}(U)_{k+1}, \cdots) \preceq (0^{k-1}, b_k + 1, b_{k+1}, b_{k+2}, \cdots).$$

Meanwhile, since $U \in \mathrm{KD}(\boldsymbol{\theta}(U^{\leqslant k}) + (0^k, b_{k+1}, b_{k+2}, \cdots))$, by Lemma 8 we have $\boldsymbol{\theta}(U) \preceq \boldsymbol{\theta}(U^{\leqslant k}) + (0^k, b_{k+1}, b_{k+2}, \cdots)$. Combining these, we have

$$\begin{aligned}(b_1, \cdots, b_{k-1}, \boldsymbol{\theta}(U)_k, \boldsymbol{\theta}(U)_{k+1}, \cdots) &= \boldsymbol{\theta}(U) \\ &\preceq (\boldsymbol{\theta}(U^{\leqslant k})_1, \cdots, \boldsymbol{\theta}(U^{\leqslant k})_{k-1}, b_k + 1, b_{k+1}, b_{k+2}, \cdots).\end{aligned}$$

So since $(0^{k-1}, (\boldsymbol{\theta}(U))_k, (\boldsymbol{\theta}(U))_{k+1}, \cdots) \preceq (0^{k-1}, b_k + 1, b_{k+1}, b_{k+2}, \cdots)$, it follows that $(b_1, b_2, \cdots, b_{k-1}) \preceq (\boldsymbol{\theta}(U^{\leqslant k})_1, \boldsymbol{\theta}(U^{\leqslant k})_2, \cdots, \boldsymbol{\theta}(U^{\leqslant k})_{k-1})$. This together with $(\boldsymbol{\theta}(U^{\leqslant k}))_k = c$ proves $(b_1, \ldots, b_k) + \mathbf{e}_k \preceq \boldsymbol{\theta}(U^{\leqslant k})$ as desired. Eq. (34) follows from Eq. (33) since the threading algorithm begins with the longest threads. $\square$

Theorem 65. *For $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$, the diagrams* $\mathrm{rectify}(U^+_*)$ *and U^- are disjoint. Consequently, the diagram $\partial_{\mathbf{a},k}(U)$ is a generic Kohnert diagram.*

Proof. Continue notation from Lemma 62, with c the added column of U. Since U^+_* and U^- are disjoint, it suffices for the theorem to show no cell in U^- lies immediately left of any cell y_i in $U^0 \subset U^+_*$.

Suppose, for contradiction, $x \in U^-$ lies immediately left of some y_i. By Lemma 63, $y_i \in U^{\leqslant k}$, and so by Lemma 64, the cell to which y_i threads in the thread decomposition of $U^{\leqslant k}$ must be labeled the same as y_i under $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(U)$, and the thread length of y must be at least c since $y \in U^+$.

If $x \in U^{\leqslant k}$, then Lemma 64 also applies to x, so its thread has length less than c since $x \in U^-$. In particular, y does not thread into x. However, since x is immediately left of y_i, the threading algorithm would thread y_i into x, since x has a shorter thread and so is still available when y_i is threaded. This contradiction means $x \notin U^{\leqslant k}$, and also y_i does not thread into x. Since y_i is immediately right of x, it threads into a higher cell. Thus since x has a longer thread than y_i, it has priority in the threading algorithm, and so must terminate in a lower cell in column 1. However, by Lemmas 63 and 52, the thread of y_i terminates in row k forcing the thread of x to terminate in a lower row, contradicting $x \notin U^{\leqslant k}$. Thus there cannot be a cell immediately left of any y_i. $\square$

To prove the diagram $\partial_{\mathbf{a},k}(U)$ is, in particular, a Kohnert diagram for $\mathbf{a}$, we observe that $\partial_{\mathbf{a},k}(U)$ can be described in a more refined way; see Fig. 16.

Lemma 66. *For $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$, we have*

$$\mathrm{rectify}(U^+_*) = (U^+ \setminus U^{\leqslant k}) \sqcup \mathrm{rectify}(U^+_* \cap U^{\leqslant k}).$$

In particular, for each i, the cell that moves from $\varrho^{i-1}(U^+_*)$ to $\varrho^i(U^+_*)$ is also the cell that moves from $\varrho^{i-1}(U^+_* \cap U^{\leqslant k})$ to $\varrho^i(U^+_* \cap U^{\leqslant k})$.

Proof. If $c = 1$, then $U^{=k}_* = \varnothing$, and so U^+ is a generic Kohnert diagram leaving nothing to prove. Thus we may assume $c > 1$. We show by induction on $i < c$ that

$$\varrho^i(U^+_*) = (U^+ \setminus U^{\leqslant k}) \sqcup \varrho^i(U^+_* \cap U^{\leqslant k})$$

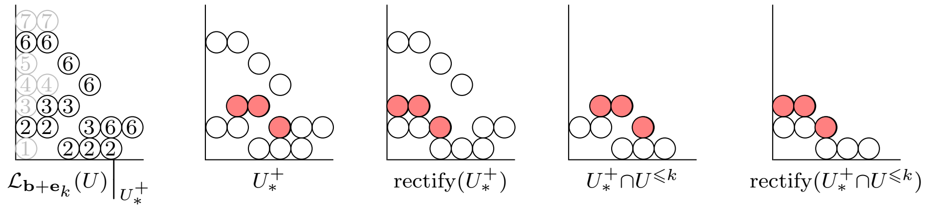

Figure 16: The refined partitioning of U_*^+, wherein U^0 is highlighted.

with the same cell moving in both cases. The base case $i = 0$ follows from the decomposition $U_*^+ = (U^+ \setminus U^{\leqslant k}) \sqcup (U_*^+ \cap U^{\leqslant k})$.

Assume the result for $h < i$ for some $1 \leqslant i < c$. Since $\varrho^i(U_*^+)$ is not a generic Kohnert diagram by Lemma 62, and since $U^+ \setminus U^{\leqslant k}$ is, it follows that $\varrho^i(U_*^+ \cap U^{\leqslant k})$ is not a generic Kohnert diagram. Moreover, the cell y that moves from $\varrho^i(U_*^+ \cap U^{\leqslant k})$ to $\varrho^{i+1}(U_*^+ \cap U^{\leqslant k})$ must do so from column $i+2$ to column $i+1$, and so must be the same cell as that which moves from $\varrho^i(U_*^+)$ to $\varrho^{i+1}(U_*^+)$ by Lemma 63. Finally, in the case $i = c-1$, we have

$$\mathrm{rectify}(U_*^+) = U_{c-1} = (U^+ \setminus U^{\leqslant k}) \sqcup \varrho^{c-1}(U_*^+ \cap U^{\leqslant k}),$$

with all the moving cells coinciding on each side. Since the last cell to move lies weakly below w_c by Lemma 62, we may construct a matching from the cells of $\varrho^{c-1}(U_*^+ \cap U^{\leqslant k})$ exactly the same as in the proof of Lemma 62, showing this is also a generic Kohnert diagram and so equals $\mathrm{rectify}(U_*^+ \cap U^{\leqslant k})$. □

Our final lemma before Theorem 58 equates the thread decompositions of $U^+ \cap U^{\leqslant k}$ when we remove either $U_*^{=k}$ or the cells that move during rectification.

Lemma 67. *For $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$, the diagram $\mathrm{rectify}(U_*^+ \cap U^{\leqslant k}) \setminus \mathrm{rectify}(U^0)$ is a generic Kohnert diagram with thread weight*

$$\boldsymbol{\theta}(\mathrm{rectify}(U_*^+ \cap U^{\leqslant k}) \setminus \mathrm{rectify}(U^0)) = \boldsymbol{\theta}(U^+ \cap U^{<k}).$$

Proof. As usual, if the added column c of U satisfies $c = 1$, then the result is trivial, so we assume $c > 1$. Letting $U^{=k} = \{w_1, \ldots, w_c\}$ with w_i in column i, we consider successive steps in the rectification of $U_*^+ \cap U^{\leqslant k}$. Define sets V_i by

$$V_i = \varrho(V_{i-1} \cup \{w_{i+1}\} \setminus \{y_i\}$$

for $1 \leqslant i < c$, where $V_0 = U^+ \cap U^{\leqslant k}$. Then we have

$$\varrho^i(U_*^+ \cap U^{\leqslant k}) = V_i \sqcup \{y_s, w_t \mid 1 \leqslant s < i+1 < t \leqslant c\}.$$

In particular, $\varrho^{c-1}(U_*^+ \cap U^{\leqslant k}) = \mathrm{rectify}(U_*^+ \cap U^{\leqslant k}) = V_{c-1} \sqcup \mathrm{rectify}(U^0)$. By construction, for each i, the cells in columns i and $i+1$ of diagrams V_i and $\varrho^i(U_*^+ \cap U^{\leqslant k})$ coincide. We prove by induction on i that V_i is a generic Kohnert diagram with thread weight

$$\boldsymbol{\theta}(V_i) = \boldsymbol{\theta}(U^+ \cap U^{<k}).$$

The base case is immediate since $V_0 = U^+ \cap U^{<k}$, so assume the result for some $0 \leqslant i < c-1$. Since cells of $V_i \sqcup \{w_{i+2}\}$ and $\varrho^i(U^+_* \cap U^{\leqslant k})$ coincide in columns $i+1$ and $i+2$, since $\varrho^i(U^+_* \cap U^{\leqslant k})$ is not a generic Kohnert diagram, neither is $V_i \sqcup \{w_{i+2}\}$. Therefore $V_i \sqcup \{w_{i+2}\}$ is a weak but not generic Kohnert diagram.

We claim w_{i+2} is the *highest* cell of $V_i \sqcup \{w_{i+2}\}$ whose removal results in a Kohnert diagram. The diagrams V_i and $U^+ \cap U^{<k}$ consist of the same cells weakly to the right of column $i+2$. By Lemma 64, in the thread decomposition of $(U^+ \cap U^{<k}) \sqcup U^{=k}$, the cells of $U^{=k}$ are threaded last. Thus w_{i+2} is the last cell of its column to be threaded in $(U^+ \cap U^{<k}) \sqcup U^{=k}$, and so w_{i+2} is the last cell of its column to be threaded in $V_i \sqcup \{w_{i+2}\}$ as well. Therefore $\mathcal{M}_{\boldsymbol{\theta}}(V_i) \subset \mathcal{M}_{\boldsymbol{\theta}}(V_i \sqcup \{w_{i+2}\})$, and so w_{i+2} is the highest removable cell of $V_i \sqcup \{w_{i+2}\}$. Since y_{i+1} moves from $V_i \sqcup \{w_{i+2}\} \overset{\varrho}{\mapsto} V_{i+1}$, it is the *lowest* removable cell of $V_i \sqcup \{w_{i+2}\}$. Therefore, V_{i+1} is a generic Kohnert diagram, with thread weight $\boldsymbol{\theta}(V_{i+1}) = \boldsymbol{\theta}(V_i) = \boldsymbol{\theta}(U^+ \cap U^{<k})$. □

Finally, we prove Theorem 58, showing $\partial_{\mathbf{a},k}(U) \in \mathrm{KD}(\mathbf{a})$ for $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$.

Proof of Theorem 58. By Lemma 66, $\mathrm{rectify}(U^+_*) = (U^+ \setminus U^{\leqslant k}) \sqcup \mathrm{rectify}(U^+_* \cap U^{\leqslant k})$, so there exists a matching sequence M on $\mathrm{rectify}(U^+_*)$ satisfying $\mathbf{wt}(M) = \boldsymbol{\theta}(U^+ \setminus U^{\leqslant k}) + \boldsymbol{\theta}(U^+ \cap U^{\leqslant k})$. By Lemma 62, the cells of $\mathrm{rectify}(U^0)$ are weakly below row k, and so by Lemma 67, we have

$$\mathbf{wt}(M) \preceq \boldsymbol{\theta}(U^+ \setminus U^{\leqslant k}) + \boldsymbol{\theta}(U^+ \cap U^{<k}) + (c-1)\mathbf{e}_k.$$

Both U^+ and $U^{<k}$ are constructed by taking subsets of path components of $\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}$, and so restricting gives a matching sequence on $U \setminus U^{<k}$ and on $U \cap U^{<k}$. By Theorem 31, $U^+ \setminus U^{\leqslant k}$ and $U^+ \cap U^{<k}$ are Kohnert diagrams of $\mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+ \setminus U^{\leqslant k}})$ and $\mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+ \cap U^{<k}}))$, respectively. Thus by Lemma 8, we have

$$\begin{aligned}
\boldsymbol{\theta}(U^+ \setminus U^{\leqslant k}) &\preceq \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+ \setminus U^{\leqslant k}}) \\
\boldsymbol{\theta}(U^+ \cap U^{\leqslant k}) &\preceq \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+ \cap U^{<k}}).
\end{aligned}$$

By definition, all labels in $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+ \setminus U^{\leqslant k}}$ (respectively, all labels in $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+ \cap U^{<k}}$) are strictly greater than (respectively, strictly less than) k. Thus there exists a matching on the key diagram $\mathrm{key}_{\boldsymbol{\theta}(U^+ \setminus U^{\leqslant k}) + \boldsymbol{\theta}(U^+ \cap U^{<k}) + (c-1)\mathbf{e}_k}$ with weight

$$\begin{aligned}
&\mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+ \setminus U^{\leqslant k}}) + \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+ \cap U^{<k}}) + (c-1)\mathbf{e}_k \\
&\qquad = \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+}) + \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^{=k}}) - \mathbf{e}_k = \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+}) - \mathbf{e}_k.
\end{aligned}$$

So then Theorem 31 implies

$$\mathbf{wt}(M) \preceq \boldsymbol{\theta}(U^+ \setminus U^{\leqslant k}) + \boldsymbol{\theta}(U^+ \cap U^{<k}) + (c-1)\mathbf{e}_k \preceq \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+}) - \mathbf{e}_k.$$

Thus $\partial_{\mathbf{a},k}(U) = U^- \sqcup \mathrm{rectify}(U^+_*)$ has a matching sequence $M' = (\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^-}) \sqcup M$. By Lemma 52, we have $\mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^-}) = \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^-})$, and so

$$\mathbf{wt}(M') = \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^-}) + \mathbf{wt}(M) \preceq \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^-}) + \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}\big|_{U^+}) - \mathbf{e}_k.$$

Since $\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}$ is a matching on $U = U^- \sqcup U^+$ with $\mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}) = \mathbf{b} + \mathbf{e}_k$, we have

$$\mathbf{wt}(M') \preceq \mathbf{wt}(\mathcal{M}_{\mathbf{b}+\mathbf{e}_k}) - \mathbf{e}_k = \mathbf{b} + \mathbf{e}_k - \mathbf{e}_k = \mathbf{b} \preceq \mathbf{a}.$$

Since M' is a matching sequence on $\partial_{\mathbf{a},k}(U)$, by Theorem 31 we conclude $\partial_{\mathbf{a},k}(U)$ is a Kohnert diagram for $\mathbf{a}$. $\square$

5.3 Stratum maps are injective

We have shown $\partial_{\mathbf{a},k}$ is well-defined from $\overline{\mathcal{D}}(\mathbf{a},k)$ to $\mathrm{KD}(\mathbf{a})$. We now show injectivity.

Definition 68. For $U \in \overline{\mathcal{D}}(\mathbf{a},k)$, we partition the stratum map by

$$\begin{aligned}\partial^+_{\mathbf{a},k}(U) &= U^+_* \setminus U^0 = \mathrm{rectify}(U^+_*) \setminus \mathrm{rectify}(U^0)\\ \partial^-_{\mathbf{a},k}(U) &= U^- \sqcup \mathrm{rectify}(U^0).\end{aligned}$$

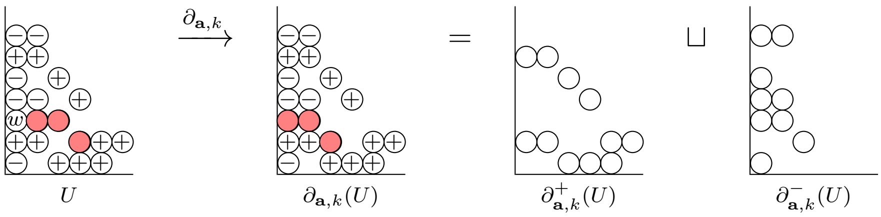

Figure 17: The partitioning of the generic Kohnert diagram U (left) in $\overline{\mathcal{D}}(\mathbf{a},k)$ for $\mathbf{a} = (1,5,2,1,2,6,3)$ and $k = 3$ using the Kohnert labeling to obtain U^+ and U^-; here $c = 4$.

Example 69. Recalling Ex. 55 developed in Figs. 12 and 13, the partitioning of the diagram U into $U^+_* - U^0$ (indicated by $\oplus$), the cell in column 1, row k (indicated by ⓦ), U^0 (indicated by a red filled circle), and U^- (indicated by $\ominus$) is shown in Fig. 17. Using this, we apply the stratum map and partition the image as shown. Anticipating Proposition 70, notice $\partial^-_{\mathbf{a},k}(U)$ has no cell right of column $c = 4$.

By construction we have $\partial_{\mathbf{a},k}(U) = \partial^+_{\mathbf{a},k}(U) \sqcup \partial^-_{\mathbf{a},k}(U)$. We will show this partitioning is natural with respect to the thread decomposition.

Proposition 70. *For $U \in \overline{\mathcal{D}}(\mathbf{a},k)$, the subdiagrams $\partial^+_{\mathbf{a},k}(U)$ and $\partial^-_{\mathbf{a},k}(U)$ are generic Kohnert diagrams. For M^+ and M^- matching sequences on $\partial^+_{\mathbf{a},k}(U)$ and $\partial^-_{\mathbf{a},k}(U)$, respectively, for every cell $x \in \partial_{\mathbf{a},k}(U)$, we have*

$$\begin{aligned}\mu_{M^+}(x) &\geqslant c \quad \text{if } x \in \partial^+_{\mathbf{a},k}(U)\\ \mu_{M^-}(x) &< c \quad \text{if } x \in \partial^-_{\mathbf{a},k}(U).\end{aligned}$$

In particular, we have

$$\mathcal{M}_{\boldsymbol{\theta}}(\partial_{\mathbf{a},k}(U)) = \mathcal{M}_{\boldsymbol{\theta}}(\partial^+_{\mathbf{a},k}(U)) \sqcup \mathcal{M}_{\boldsymbol{\theta}}(\partial^-_{\mathbf{a},k}(U)).$$

Proof. We first consider the diagram $\partial^{+}_{\mathbf{a},k}(U)$. By Lemma 66,

$$\partial^{+}_{\mathbf{a},k}(U) = (U^{+} \setminus U^{\leqslant k}) \sqcup \mathrm{rectify}(U^{+}_{*} \cap U^{\leqslant k}) \setminus \mathrm{rectify}(U^{0}).$$

By definition, $U^{+} \setminus U^{\leqslant k}$ is a generic Kohnert diagram with the same number of cells at each column weakly left of column c. On the other hand, $(\mathrm{rectify}(U^{+}_{*} \cap U^{\leqslant k}) \setminus \mathrm{rectify}(U^{0}))$ is a generic Kohnert diagram by Lemma 67, with thread weight $\boldsymbol{\theta}(U^{+} \cap U^{<k})$, and hence must also have the same number of cells at each column weakly left of column c. Thus, $\partial^{+}_{\mathbf{a},k}(U)$ is a generic Kohnert diagram with the same number of cells at each column weakly left of column c. It follows that $\mu_{M^{+}}(x) \geqslant c$ for every cell $x \in \partial^{+}_{\mathbf{a},k}(U)$ and every matching sequence on $\partial^{+}_{\mathbf{a},k}(U)$.

Now consider $\partial^{-}_{\mathbf{a},k}(U)$. By definition, U^{-} is a generic Kohnert diagram occupying only positions strictly left of column c. By Lemma 62, $\mathrm{rectify}(U^{0})$ is a generic Kohnert diagram whose cells all lie strictly left of column c. Thus, $\partial^{-}_{\mathbf{a},k}(U) = U^{-} \sqcup \mathrm{rectify}(U^{0})$ is a generic Kohnert diagram occupying only positions strictly left of column c. Thus $\mu_{M^{-}}(x) < c$ for every cell $x \in \partial^{-}_{\mathbf{a},k}(U)$ and every matching sequence on $\partial^{-}_{\mathbf{a},k}(U)$. ☐

The following is helpful in studying thread decompositions for U and for $\partial_{\mathbf{a},k}(U)$.

Lemma 71. *For $U \in \overline{\mathcal{D}}(\mathbf{a},k)$, c the added column of U, and $\mathbf{b}$ the excised weight of U, for any weak composition $\mathbf{d}$ such that $\boldsymbol{\theta}(U) \preceq \mathbf{d} \prec \mathbf{b}+\mathbf{e}_k$, we have $d_k > c$.*

Proof. Since $\mathbf{b} = \boldsymbol{\theta}(\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\})$, where by Lemma 49 the cell y is in position (c,k), we have $(\boldsymbol{\theta}(U))_j = b_j$ for all indices $j < k$. Hence, $d_j = b_j$ for all $j < k$ as well. So since $\mathbf{d} \preceq \mathbf{b}+\mathbf{e}_k$, we must have $d_k \geqslant b_k + 1 = c$.

Suppose, for contradiction, $d_k = c$. Since $\boldsymbol{\theta}(U) \preceq \mathbf{d}$, we may consider the Kohnert labeling $L = \mathcal{L}_{\mathbf{d}}(\mathrm{key}_{\boldsymbol{\theta}(U)})$. Since $d_k = b_j = (\boldsymbol{\theta}(U))_j$ for all indices $j < k$, we must have $L(x) = \mathrm{row}(x)$ for every cell x below row k. Therefore for every cell in position (i,k) for $1 \leqslant i \leqslant d_k$, must have label $L(x) = k$. In particular, $L(y) = k$ since y is in position (c,k). Since $d_k = b_k + 1 = c$, y is the rightmost cell in $\mathrm{key}_{\boldsymbol{\theta}(U)}$ with label k under L. Therefore, restricting L to the diagram $\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\}$ gives us a labeling with content weight $\mathbf{d}-\mathbf{e}_k$. By Lemma 8, $\mathbf{b} = \boldsymbol{\theta}(\mathrm{key}_{\boldsymbol{\theta}(U)} \setminus \{y\}) \preceq \mathbf{d}-\mathbf{e}_k$. So since $b_j = d_j$ for all indices $j < k$, and since $b_k = d_k - 1$, we necessarily have $(0^k, b_{k+1}, b_{k+2}, \cdots) \preceq (0^k, d_{k+1}, d_{k+2}, \cdots)$. Hence, since $b_j = d_j$ for all indices $j < k$, and since $b_k + 1 = d_k$, we ultimately get $\mathbf{b}+\mathbf{e}_k \preceq \mathbf{d}$. Since $\mathbf{d} \preceq \mathbf{b}+\mathbf{e}_k$ we must have $\mathbf{d} = \mathbf{b}+\mathbf{e}_k$, which contradicts our assumption that $\mathbf{d} \neq \mathbf{b}+\mathbf{e}_k$. ☐

Fig. 18 illustrates the next result on the running example.

Lemma 72. *For $U \in \overline{\mathcal{D}}(\mathbf{a},k)$ and $\mathbf{b}$ the excised weight of U, we have*

$$\mathcal{M}_{\boldsymbol{\theta}}(U \setminus U^{=k}) = \mathcal{M}_{\mathbf{b}+\mathbf{e}_k}(U)\big|_{U \setminus U^{=k}}. \tag{35}$$

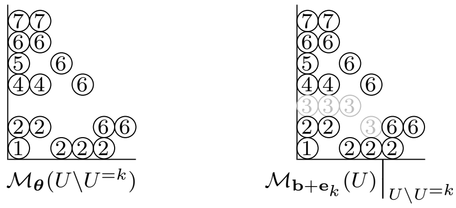

Figure 18: An illustration of Lemma 72.

Proof. For brevity, we let $M = \mathcal{M}_{\mathbf{b}+\mathbf{e}_k}(U)\big|_{U\setminus U^{=k}}$. Since the cells of $U^{=k}$ form the path labeled k in $\mathcal{L}_{\mathbf{b}+\mathbf{e}_k}(U)$, restricting to cells of $U \setminus U^{=k}$ is equivalent to $\mathcal{L}_{\mathbf{b}-b_k\mathbf{e}_k}(U \setminus U^{=k})$. By Proposition 48, in order to prove $M = \mathcal{M}_{\boldsymbol{\theta}}(U \setminus U^{=k})$, it suffices to show that $\boldsymbol{\theta}(U \setminus U^{=k}) = \mathbf{wt}(M) = \mathbf{b} - b_k\mathbf{e}_k$, by Lemma 52.

By Theorem 31, the fact that $(U \setminus U^{=k})$ is the underlying diagram for the matching sequence M implies that $(U \setminus U^{=k})$ is a Kohnert diagram for $\mathbf{wt}(M)$. Thus $\boldsymbol{\theta}(U \setminus U^{=k}) \preceq \mathbf{wt}(M)$.

To prove $\mathbf{wt}(M) \preceq \boldsymbol{\theta}(U \setminus U^{=k})$, note that $M \sqcup (w_1 \leftarrow w_2 \leftarrow \cdots w_c)$ is a matching sequence on U, so that by Theorem 31, we have $\boldsymbol{\theta}(U) \preceq \mathbf{wt}(M) + c\mathbf{e}_k$. Since $\boldsymbol{\theta}(U \setminus U^{=k}) \preceq \mathbf{wt}(M)$ with $(\boldsymbol{\theta}(U \setminus U^{=k}))_k = 0 = (\mathbf{wt}(M))_k$, we have

$$\boldsymbol{\theta}(U) \preceq \boldsymbol{\theta}(U \setminus U^{=k}) + c\mathbf{e}_k \preceq \mathbf{wt}(M) + c\mathbf{e}_k = \mathbf{b} - b_k\mathbf{e}_k + c\mathbf{e}_k = \mathbf{b} + \mathbf{e}_k.$$

If $\boldsymbol{\theta}(U \setminus U^{=k}) \neq \mathbf{wt}(M)$, then $\boldsymbol{\theta}(U \setminus U^{=k}) + c\mathbf{e}_k \neq \mathbf{b} + \mathbf{e}_k$, and it follows from Lemma 71 that $c = (\boldsymbol{\theta}(U \setminus U^{=k}) + c\mathbf{e}_k)_k > c$, a contradiction. $\square$

Corollary 73. *For $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$, c the added column and $\mathbf{b}$ the excised weight of U, the weight of the thread decomposition of $\partial^+_{\mathbf{a},k}(U)$ is*

$$\boldsymbol{\theta}(\partial^+_{\mathbf{a},k}(U))_j = \begin{cases} b_j & \text{if } b_j \geqslant c \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have $\#\{j > k \mid b_j \geqslant c\} = \#\{j > k \mid (\boldsymbol{\theta}(\partial_{\mathbf{a},k}(U))_j \geqslant c\}$.

We now have our first major step toward reversing the stratum maps.

Theorem 74. *For $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$ with added column c, let y_0 be the position $(1, k)$ and let $U^0 = \{y_1, y_2, \cdots, y_{c-1}\} \subset U^+_*$ be the rectification path of U. Then for each $1 < i \leqslant c-1$, y_i is the highest cell in column i weakly below y_{i-1} with $\mu_{\mathcal{M}_{\boldsymbol{\theta}(\partial_{\mathbf{a},k}(U))}}(y_i) < c$.*

Proof. By Proposition 70, we have $\partial^-_{\mathbf{a},k}(U) = \{x \in \partial_{\mathbf{a},k}(U) \mid \mu_M(x) < c\}$. Thus it suffices to show at every column $1 \leqslant i < c$, the cells of $U^- = \partial^-_{\mathbf{a},k}(U) \setminus \mathrm{rectify}(U^0)$ in column i that are below y_{i-1} are also below y_i.

By definition of ϱ, we have $\mathfrak{m}_T(i, \mathrm{row}(y_i)) = 0$, and by Lemma 62, we have $\mathfrak{m}_T(i, j) > 0$ for all rows j such that $\mathrm{row}(y_i) < j \leqslant \mathrm{row}(y_{i-1})$. By the greedy choice of the threading algorithm, this means the cells weakly above $\mathrm{row}(y_i)$ and strictly below y_{i-1} in columns $i-1, i$ are matched with lengths at least c. Thus by Proposition 70, they cannot lie in U^-, and the result follows. $\square$

By Theorem 74, if we know the added column c of U, then we can recover rectify(U^0) to obtain $\partial_{\mathbf{a},k}(U)$. Our next and final task for proving the injectivity of $\partial_{\mathbf{a},k}(U)$ is to show that the added column c is unique.

The following elementary lemma allows us to adjust a matching sequence on a key diagram to pass through a certain cell at the end of its row.

Lemma 75. *Let T be a generic Kohnert diagram, and let c and r be positive integers such that T occupies every position weakly left of column c in row r. Let z be the cell in position (c, r), and let w be a cell in row r weakly to the left of z.*

If there exists a matching sequence M on T with $\mu_M(w) \geqslant c$, then there exists a matching sequence M' on T with $\mathbf{wt}(M') \preceq \mathbf{wt}(M)$ with $\mu_{M'}(z) \geqslant c$.

Proof. Let M be a matching sequence on T and suppose w is the rightmost cell in row r weakly to the left of z such that $\mu_M(w) \geqslant c$. If $w = z$, then there is nothing to show, so assume $w \neq z$ in which case we may let x be the cell of T to the immediate right of w in row k. By induction, it suffices to show that there exists a matching sequence M' on T such that $\mu_{M'}(x) \geqslant c$.

If $M(x) = w$, then M is such a matching, so we may assume this is not the case. Therefore $M(x)$ lies strictly above row k and the cell y for which $M(y) = w$ lies strictly below row k. Consider the underlying diagram S of the path components of w and x in M, and let $\mathcal{M}_{\boldsymbol{\theta}}(S) = P \sqcup Q$, where P has length $\mu_M(w) \geqslant c$ and Q has length $\mu_M(x) < c$. Since $M(x)$ is above w, the latter is threaded first, and so belongs to P. If x also belongs to P, then we set N to be the thread matching on S. Otherwise, x belongs to Q, which is strictly shorter than P, and so there must exist some pair of cells u and v in some column i weakly to the right of x, with the cell u of Q above the cell v of P, such that in column $(i + 1)$, either the cell of Q is below the cell of P or there is no cell of Q at all. In either case, we may construct a matching sequence N on S from $\mathcal{M}_{\boldsymbol{\theta}}(S)$ by swapping the cells into which u and v thread, setting $N(x) = w$, and, for y the cell that threads into w, setting $N(y)$ to be the cell into which x threads. Then $\mathbf{wt}(N) = \boldsymbol{\theta}(S)$, and $\mu_N(x) = \mu_N(w) \geqslant c$. Use this to define a matching sequence $M' = (M\,|_{T\setminus S}) \sqcup N$ on T, so that $\mu_{M'}(x) \geqslant c$ and via Lemma 8 $\mathbf{wt}(M') = \mathbf{wt}(M\,|_{T\setminus S}) + \boldsymbol{\theta}(S) \preceq \mathbf{wt}(M)$. □

The next simple lemma is the foundation for uniqueness of the added column. To see the result on the running example, recall $\mathbf{a} = (1, 5, 2, 1, 2, 6, 3)$, $k = 3$, the added column for U is $c = 4$, and the excised weight of U is $\mathbf{b} = (1, 5, 3, 2, 1, 6, 2)$.

Lemma 76. *For any $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$, the added column c of U satisfies $a_k < c$. Moreover, for $\mathbf{b}$ the excised weight of U, we have*

$$\#\{j > k \mid b_j \geqslant c\} = \#\{j > k \mid a_j \geqslant c\} \tag{36}$$

Proof. From the relation $\mathbf{b} \preceq \mathbf{a}$, for any value c we have inequalities

$$\begin{aligned} \#\{j > k \mid b_j \geqslant c\} &\leqslant \#\{j > k \mid a_j \geqslant c\}, \\ \#\{j \geqslant k \mid b_j \geqslant c\} &\leqslant \#\{j \geqslant k \mid a_j \geqslant c\}. \end{aligned}$$

Further, since $b_k = c-1$ for c the added column of U, the left hand sides above must be equal. Therefore if equality holds for the lower expression, then $a_k < c$, and so Eq. (36) follows. Thus it suffices to show

$$\#\{j > k \mid b_j \geqslant c\} = \#\{j \geqslant k \mid a_j \geqslant c\}. \tag{37}$$

Suppose Eq. (37) is false. We will show this implies $U \in \mathcal{D}(\mathbf{a}, k-1)$, contradicting $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$, thereby proving Eq. (37) and so, too, Eq. (36).

Let y be the cell in position (b_k, k) of the key diagram $\mathrm{key}_{\mathbf{b}}$. We claim there exists a matching sequence M on $\mathrm{key}_{\mathbf{b}}$ with $\mathbf{wt}(M) \preceq \mathbf{a}$ such that $\mu_M(y) \geqslant c$. The key diagram $\mathrm{key}_{\mathbf{b}}$ is a Kohnert diagram for $\mathbf{a}$ by Proposition 9, so by Definition 46, we may consider the labeling $\mathcal{L}_{\mathbf{a}}$ on $\mathrm{key}_{\mathbf{b}}$, with corresponding matching sequence $\mathbf{wt}(\mathcal{M}_{\mathbf{a}}) \preceq \mathbf{a}$. If $\mu_{\mathcal{M}_{\mathbf{a}}}(y) \geqslant c$, then we may take $M = \mathcal{M}_{\mathbf{a}}$. Otherwise, by Lemma 75, we may assume every cell in row k of $\mathrm{key}_{\mathbf{b}}$ belongs to a path component in $\mathcal{M}_{\mathbf{a}}$ strictly shorter than c. If Eq. (37) is false, then there exists a cell z in column c of $\mathrm{key}_{\mathbf{b}}$ such that z is below row k but $L(z) \geqslant k$. Let i be the leftmost column such that the cell on the path component of $\mathcal{M}_{\mathbf{a}}$ containing z lies strictly below row k. If $i = c$, then we may take $M = \mathcal{M}_{\mathbf{a}}$ for all cells other than z and set $M(z) = y$. Then $\mathbf{wt}(M) \prec \mathbf{wt}(\mathcal{M}_{\mathbf{a}}) \preceq \mathbf{a}$ and $\mu_M(y) = \mu_{M_0}(z) \geqslant c$. Else if $i < c$, then let w denote the cell in position (i, k). By our assumption about the lengths of paths in $\mathcal{M}_{\mathbf{a}}$ for cells in row k, we have $\mu_{\mathcal{M}_{\mathbf{a}}}(w) < c \leqslant \mu_{\mathcal{M}_{\mathbf{a}}}(z)$. Thus there exists some column maximal column $t \geqslant i$ such that the cells on the path component of $\mathcal{M}_{\mathbf{a}}$ containing w lie above the cells on the path component of $\mathcal{M}_{\mathbf{a}}$ containing z in every column s between i and t. Define L' to be the labeling derived from L by swapping the labels of the cells in the path components of $\mathcal{M}_{\mathbf{a}}$ containing w and z for every column s between i and t. Then $\mathbf{wt}(L') = \mathbf{wt}(L) = \mathbf{a}$. The same argument used in the proof of Proposition 47 applies to the underlying matching sequence of L', and so we obtain a matching sequence $\mathcal{M}_{L'}$ on $\mathrm{key}_{\mathbf{b}}$ with $\mathbf{wt}(\mathcal{M}_{L'}) \preceq \mathbf{a}$ such that $\mu_M(w) = \mu_{\mathcal{M}_{\mathbf{a}}}(z) \geqslant c$, proving the claim.

Let M be any matching sequence on $\mathrm{key}_{\mathbf{b}}$ with $\mathbf{wt}(M) \preceq \mathbf{a}$ and $\mu_M(y) \geqslant c$, and let z be the cell in column c for which $M(z) = y$. Let z' denote the cell in position (c, k) of the key diagram $\mathrm{key}_{\mathbf{b}+\mathbf{e}_k}$. Define a matching sequence M' on $\mathrm{key}_{\mathbf{b}+\mathbf{e}_k} \setminus \{z\}$ from M by setting $M'(z') = y$, $M'(x) = z'$ if $M(x) = z$, and $M'(x) = M(x)$ otherwise. Then $\mathbf{wt}(M') = \mathbf{wt}(M) \preceq \mathbf{a}$. Thus, by Theorem 31, the diagram $\mathrm{key}_{\mathbf{b}+\mathbf{e}_k} \setminus \{z\}$ is a Kohnert diagram for $\mathbf{a}$, so that by Lemma 8, we have $\boldsymbol{\theta}(\mathrm{key}_{\mathbf{b}+\mathbf{e}_k} \setminus \{z\}) \preceq \mathbf{a}$. Define another matching sequence N on $\mathrm{key}_{\mathbf{b}+\mathbf{e}_k}$ with anchor weight $\boldsymbol{\theta}(\mathrm{key}_{\mathbf{b}+\mathbf{e}_k} \setminus \{z\}) + \mathbf{e}_{\mathrm{row}(z)}$ by taking $\mathcal{M}_{\boldsymbol{\theta}}(\mathrm{key}_{\mathbf{b}+\mathbf{e}_k} \setminus \{z\})$ and then matching z into the cell to its immediate left in $\mathrm{key}_{\mathbf{b}+\mathbf{e}_k}$. By Theorem 31, $\mathrm{key}_{\mathbf{b}+\mathbf{e}_k}$ is a Kohnert diagram for $\boldsymbol{\theta}(\mathrm{key}_{\mathbf{b}+\mathbf{e}_k} \setminus \{z\}) + \mathbf{e}_{\mathrm{row}(z)}$, and so by Lemma 8,

$$\boldsymbol{\theta}(U) \preceq \mathbf{b} + \mathbf{e}_k \preceq \boldsymbol{\theta}(\mathrm{key}_{\mathbf{b}+\mathbf{e}_k} \setminus \{z\}) + \mathbf{e}_{\mathrm{row}(z)}.$$

Since $\boldsymbol{\theta}(\mathrm{key}_{\mathbf{b}+\mathbf{e}_k} \setminus \{z\}) \preceq \mathbf{a}$ and $\mathrm{row}(z) < k$, we conclude that $U \in \mathcal{D}(\mathbf{a}, k-1)$. $\square$

Finally, we combine these results to prove for $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$, the diagram U is the unique pre-image of the diagram $\partial_{\mathbf{a},k}(U)$ under the kth stratum map $\partial_{\mathbf{a},k}$ of $\mathbf{a}$.

Proof of Theorem 59. Let $U \in \overline{\mathcal{D}}(\mathbf{a}, k)$. By Lemma 62, we know exactly how each $y_i \in \mathrm{rectify}(U^0)$ moved in constructing $\partial_{\mathbf{a},k}(U)$. Moreover, by Lemma 52, the excised cell w

of U is in position $(1,k)$. Thus, by Theorem 74, we may completely recover the Kohnert diagram $U \in \overline{\mathcal{D}}(\mathbf{a},k)$ from its image under $\partial_{\mathbf{a},k}$ provided we know the added column c of U. It follows that if we have a different diagram $U' \in \overline{\mathcal{D}}(\mathbf{a},k)$ such that $\partial_{\mathbf{a},k}(U') = \partial_{\mathbf{a},k}(U)$, then U' must come equipped with a distinct added column $c' \neq c$ and consequently have an excised weight $\mathbf{b}' \neq \mathbf{b}$, since by construction we would get $b'_k = c' - 1 \neq c - 1 = b_k$.

Suppose, for contradiction, $U' \in \overline{\mathcal{D}}(\mathbf{a},k)$ with added column $c' > c$ and excised weight $\mathbf{b}' \neq \mathbf{b}$ satisfies $\partial_{\mathbf{a},k}(U') = \partial_{\mathbf{a},k}(U)$. By Lemma 76 and Corollary 73, we have

$$\#\{j > k \mid (\boldsymbol{\theta}(\partial_{\mathbf{a},k}(U)))_j \geqslant c\} = \#\{j > k \mid b_j \geqslant c\} = \#\{j > k \mid a_j \geqslant c\}.$$

By Lemma 49 (2), $\mathbf{b}' \preceq \mathbf{a}$, and by Lemma 76, we have $c > a_k$, so that by our assumption, we have $b' = c' - 1 \geqslant c > a_k$, and so

$$\#\{j > k \mid b'_j \geqslant c\} < \#\{j > k \mid a_j \geqslant c\}.$$

However, since $\boldsymbol{\theta}(\partial_{\mathbf{a},k}(U)) \preceq \mathbf{b}'$, by Corollary 73, we have

$$\#\{j > k \mid (\boldsymbol{\theta}(\partial_{\mathbf{a},k}(U)))_j \geqslant c\} \geqslant \#\{j > k \mid a_j \geqslant c\}.$$

These two inequalities directly contradict the earlier one, and so c is unique. □

References

[1] Sami Assaf and Dominic Searles. Kohnert tableaux and a lifting of quasi-Schur functions. *J. Combin. Theory Ser. A*, 156:85–118, 2018.

[2] Sami H. Assaf. An insertion algorithm for multiplying Demazure characters by Schur polynomials. arXiv:2109.05651.

[3] Sami H. Assaf. Demazure crystals for Kohnert polynomials. *Trans. Amer. Math. Soc.*, 375(3):2147–2186, 2022.

[4] Michel Demazure. Une nouvelle formule des caractères. *Bull. Sci. Math. (2)*, 98(3):163–172, 1974.

[5] J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg. Refinements of the Littlewood-Richardson rule. *Trans. Amer. Math. Soc.*, 363(3):1665–1686, 2011.

[6] Donald E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.*, 34:709–727, 1970.

[7] Axel Kohnert. Weintrauben, Polynome, Tableaux. *Bayreuth. Math. Schr.*, (38):1–97, 1991. Dissertation, Universität Bayreuth, Bayreuth, 1990.

[8] Alain Lascoux and Marcel-Paul Schützenberger. Keys & standard bases. In *Invariant theory and tableaux (Minneapolis, MN, 1988)*, volume 19 of *IMA Vol. Math. Appl.*, pages 125–144. Springer, New York, 1990.

[9] Sarah Mason. An explicit construction of type A Demazure atoms. *J. Algebraic Combin.*, 29(3):295–313, 2009.

[10] Danjoseph Keeny B. Quijada. *A Pieri rule for key polynomials*. ProQuest LLC, Ann Arbor, MI, 2021. Thesis (Ph.D.)–University of Southern California.

[11] Victor Reiner and Mark Shimozono. Key polynomials and a flagged Littlewood-Richardson rule. *J. Combin. Theory Ser. A*, 70(1):107–143, 1995.

[12] G. de B. Robinson. On the Representations of the Symmetric Group. *Amer. J. Math.*, 60(3):745–760, 1938.

[13] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.

[14] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.