The h-Polynomial and the Rook Polynomial of some Polyominoes

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Abstract

Let X be a convex polyomino such that its vertex set is a sublattice of $\mathbb{N}^2$. Let $\Bbbk[X]$ be the toric ring (over a field $\Bbbk$) associated to X in the sense of Qureshi, *J. Algebra*, 2012. Write the Hilbert series of $\Bbbk[X]$ as $(1+h_1t+h_2t^2+\cdots)/(1-t)^{\dim(\Bbbk[X])}$. For $k \in \mathbb{N}$, let r_k be the number of configurations in X with k pairwise non-attacking rooks. We show that $h_2 < r_2$ if X is not a thin polyomino. This partially confirms a conjectured characterization of thin polyominoes by Rinaldo and Romeo, *J. Algebraic Combin.*, 2021.

Mathematics Subject Classifications: 13F65, 05E40, 13D40

1 Introduction

A polyomino is a finite union of unit squares with vertices at lattice points in the plane that is connected and has not finite cut-set [9, 4.7.18]. (Definitions are given in Section 2.) A. A. Qureshi [6] associated a finitely generated graded algebra $\Bbbk[X]$ (over a field $\Bbbk$) to polyomino X. For $k \in \mathbb{N}$, a k-rook configuration in X is an arrangement of k rooks in pairwise non-attacking positions. The rook polynomial $r(t)$ of X is $\sum_{k\in\mathbb{N}} r_k t^k$ where r_k is the number of k-rook configurations in X. The h-polynomial of $\Bbbk[X]$ is the (unique) polynomial $h(t) \in \mathbb{Z}[t]$ such that the Hilbert series of $\Bbbk[X]$ is $h(t)/(1-t)^d$ where $d = \dim \Bbbk[X]$. A polyomino is thin if it does not contain a 2×2 square of four unit squares (such as the one shown in Figure 2).

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G. Rinaldo and F. Romeo [7, Theorem 1.1] showed that if X is a simple thin polyomino, then $h(t) = r(t)$ and conjectured [7, Conjecture 4.5] that this property characterises thin polyominoes. In this paper, we prove this conjecture in the following case:

Theorem 1. *Let X be a convex polyomino such that its vertex set $V(X)$ is a sublattice of $\mathbb{N}^2$. Let $h(t) = 1+h_1t+h_2t^2+\cdots$ be the h-polynomial of $\Bbbk[X]$ and $r(t) = 1+r_1t+r_2t^2+\cdots$ be the rook polynomial of X. If X is not thin, then $h_2 < r_2$. In particular $h(t) \neq r(t)$.*

Its proof proceeds as follows: we first observe that $\Bbbk[X]$ is the Hibi ring of the distributive lattice $V(X)$ and that the Hilbert series of the Hibi ring of a distributive lattice and that of the Stanley-Reisner ring of its order complex are the same. We then use the results of [1] relate the h-polynomial to descents in maximal chains of $V(X)$, and find an injective map from the set of maximal chains of $V(X)$ to the rook configurations in X, to conclude that $h_k \leqslant r_k$ in general. We then show that if X is not thin, this map is not surjective to show that $h_2 < r_2$. In Corollary 11 we extend our result to L-convex polyominoes.

Section 2 contains the definitions and preliminaries. Proof of the theorem is given in Section 3.

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2 Preliminaries

Definition 2. A *cell* in $\mathbb{R}^2$ is a set of the form $\{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant a+1, b \leqslant y \leqslant b+1\}$ where $(a, b) \in \mathbb{Z}^2$. We identify the cells of X by their top-right corners: For $v \in \mathbb{Z}^2$, $C(v)$ is the cell whose top-right corner is v. A *polyomino* X is a finite union of cells that is connected and has no finite cut-set (i.e., removing finite sets from X leaves X connected) [9, 4.7.18]. We say that a polyomino X is *horizontally convex* if for every line segment ℓ parallel to the x-axis with end-points in X, $\ell \subseteq X$. Similarly we define *vertically convex* polyominoes. We say that a polyomino X is *convex* if it is horizontally convex and vertically convex. The set of cells of X is denoted by $C(X)$. The *vertex set* $V(X)$ of X is $X \cap \mathbb{Z}^2$. By the left-boundary vertices of X, we mean the elements of $\mathbb{Z}^2 \cap \partial X$ that are top-left vertices of the cells of X; the *bottom-boundary vertices* of X are the elements of $\mathbb{Z}^2 \cap \partial X$ that are bottom-right vertices of the cells of X;

Qureshi [6] associated a toric ring to a polyomino.

Definition 3. Let X be a convex polyomino. Let $R = \Bbbk[\{x_v \mid v \in V(X)\}]$ be a polynomial ring. An *interval* in X is a subset of X of the form $[a, b] := \{c \in V(X) \mid a \leqslant c \leqslant b\}$ where $a \leqslant b \in V(X)$ and $\leqslant$ is the partial order on $\mathbb{R}^2$ given by componentwise comparison: $a = (a_1, a_2) \leqslant b = (b_1, b_2)$ if $a_1 \leqslant b_1$ and $a_2 \leqslant b_2$. Let I_X be the R-ideal generated by the binomials of the form $x_a x_b - x_c x_d$ where $a \leqslant b \in V(X)$ and $c, d \in V(X)$ are the other two corners of the interval $[a, b]$. Let $\Bbbk[X] = R/I_X$.

Setup 2.1. Let X be a convex polyomino such that $V(X)$ is a sublattice of $\mathbb{N}^2$. Let $\mathrm{JI}(X)$ be the poset of join-irreducible elements of $V(X)$. After a suitable translation, if necessary, we assume that $(0, 0)$ and (m, n) are the elements $\hat{0}$ and $\hat{1}$ of $V(X)$. Hence $|\,\mathrm{JI}(X)| = m + n$.

Definition 4. Let L be a finite distributive lattice. Let $R = \Bbbk[\{x_a \mid a \in L\}]$. The *Hibi ideal* [4] I_L of L is the R-ideal generated by the binomials of the form $x_a x_b - x_c x_d$ where $a, b \in L$ and c and d are the join and the meet of a and b. The *Hibi ring* of L is $\Bbbk[L] := R/I_L$.

Definition 5. Let R be a standard graded $\Bbbk$-algebra. The *h-polynomial* of R is the polynomial $h(t)$ such that the Hilbert series of R is $h(t)/(1-t)^d$ where $d = \dim R$.

Remark 6. When X is as in Setup 2.1, the polyomino ring $\Bbbk[X]$ is the Hibi ring $\Bbbk[V(X)]$. Hence we are interested in the h-polynomial of the Hibi ring of a distributive lattice. Let L be a distributive lattice. The order complex $\Delta(L)$ is the simplicial complex whose faces are the chains of L. The *Stanley-Reisner ring* $\Bbbk[\Delta(L)]$ of $\Delta(L)$ is the quotient of $\Bbbk[\{x_a \mid a \in L\}]$ by the ideal generated by $\{x_a x_b \mid a, b \text{ incomparable}\}$. There is a flat deformation from $\Bbbk[L]$ to $\Bbbk[\Delta(L)]$; see, e.g., [2, Section 7.1], after noting that Hibi rings are ASLs. Hence the h-polynomials of $\Bbbk[X]$ and of $\Bbbk[\Delta(V(X))]$ are the same. We use the results of [1] to relate the h-polynomial of $\Delta(L)$ to the descents in the maximal chains of L.

Discussion 2.2. We follow the discussion of [1, Section 1]. Let $\omega : \mathrm{JI}(X) \longrightarrow \{1, \ldots, m+n\}$ be a (fixed) order-preserving map. Let $\mathcal{M}(X)$ be the set of maximal chains of $V(X)$. Let $\mu \in \mathcal{M}(X)$. We first write μ as a chain of order ideals of $\mathrm{JI}(X)$: $\hat{0} = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{m+n} = \hat{1}$. Then $|I_i \smallsetminus I_{i-1}| = \{p_i\}$ for some $p_i \in \mathrm{JI}(X)$. Define $\omega(\mu) = (\omega(p_1), \ldots, \omega(p_{m+n}))$. For $1 \leqslant i \leqslant m + n - 1$, we say that i is a *descent* of μ if $\omega(p_i) > \omega(p_{i+1})$. The *descent set* $\mathrm{Des}(\mu)$ of μ is $\{i \mid 1 \leqslant i \leqslant m+n-1,\ i \text{ is a descent of } \mu\}$. For $k \in \mathbb{N}$, define $\mathcal{M}_k(X) = \{\mu \in \mathcal{M}(X) : |\,\mathrm{Des}(\mu)| = k\}$.

We now think of μ as a lattice path from $(0, 0)$ to (m, n) consisting of horizontal and vertical edges. Label the vertices of μ as $(0, 0) = \mu_0, \mu_1, \ldots, \mu_{m+n} = (m, n)$, with $\mu_i - \mu_{i-1}$ a unit vector (when we think of these as elements of $\mathbb{R}^2$) pointing to the right or upwards. Then, if $i \in \mathrm{Des}(\mu)$, then the direction of μ changes at μ_i, i.e, the vectors $\mu_i - \mu_{i-1}$ and $\mu_{i+1} - \mu_i$ are perpendicular to each other. Hence μ_{i-1} and μ_{i+1} are the bottom-left and top-right vertices of a cell (the cell $C(\mu_{i+1})$ in our notation) of X. Thus we get a function

$$\psi : \mathcal{M}(X) \longrightarrow \mathrm{Pow}(C(X)), \qquad \mu \mapsto \{C(\mu_{i+1}) \in C(X) \mid i \in \mathrm{Des}(\mu)\}. \tag{2.3}$$

Proposition 7. *When X is as in Setup 2.1. Write $h(t) = 1 + h_1 t + h_2 t^2 + \cdots$ for the h-polynomial of $\Bbbk[X]$. Then $h_i = |\,\mathcal{M}_i(X)|$.*

Proof. Use [1, Theorems 4.1 and 1.1] with standard grading (i.e. setting $t_i = t$ for all i) to see that the h-polynomial of the Stanley Reisner ring of $\Delta(V(X)))$ is

$$\sum_{i \in \mathbb{N}} |\mathcal{M}_i(X)| t^i.$$

The proposition now follows from Remark 6. □

Discussion 2.4. Let X be as in Setup 2.1. Left-boundary vertices and bottom-boundary vertices are join-irreducible. Let $p \in V(X)$; assume that p is not a left-boundary vertex or a bottom-boundary vertex. If $p \notin \partial X$ then it is the top-right vertex of a cell in X, and hence is not join-irreducible. If $p \in \partial X$ then p is the bottom-left vertex of the unique cell containing it (i.e., the bottom element $\hat{0}$ of $V(X)$) or the top-right vertex of the unique cell containing it (i.e., the top element $\hat{1}$ of $V(X)$); hence $p \notin \mathrm{JI}(X)$. Thus we have established that $\mathrm{JI}(X)$ is the union of the set of the left-boundary vertices and of the set of the bottom-boundary vertices. The sets of the left-boundary vertices and of the bottom-boundary vertices are totally ordered in $V(X)$. Therefore if (p, p') is a pair of incomparable elements of $\mathrm{JI}(X)$, then one of them is a left-boundary vertex and the other is a bottom-boundary vertex.

3 Proof of the theorem

Proposition 8. *Let $\mu \in \mathcal{M}(X)$ and $i \in \mathrm{Des}(\mu)$. Write μ as a chain of order ideals $\hat{0} = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{m+n} = \hat{1}$ and $|I_i \smallsetminus I_{i-1}| = \{p_i\}$ with $p_i \in \mathrm{JI}(X)$. Then*

a. p_i and p_{i+1} are incomparable;

b. $i + 1 \notin \mathrm{Des}(\mu)$.

Proof. (a): Assume, by way of contradiction, that they are comparable. Then $p_i < p_{i+1}$. Hence $\omega(p_i) < \omega(p_{i+1})$, contradicting the hypothesis that $i \in \mathrm{Des}(\mu)$.

(b): By way of contradiction, assume that $i + 1 \in \mathrm{Des}(\mu)$. Then, by (a), p_{i+1} and p_{i+2} are incomparable. We see from Discussion 2.4 and the definition of the p_i that $p_i < p_{i+2}$. Therefore $\omega(p_i) < \omega(p_{i+2})$ contradicting the hypothesis that $\omega(p_i) > \omega(p_{i+1}) > \omega(p_{i+2})$. □

Proposition 9. *The function ψ of (2.3) is injective.*

Proof. Let $\mu, \nu \in \mathcal{M}(X)$ be such that $\psi(\mu) = \psi(\nu)$. As earlier, write μ and ν as chains of order ideals of $\mathrm{JI}(X)$:

$$\mu : \hat{0} = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{m+n} = \hat{1};$$
$$\nu : \hat{0} = I'_0 \subsetneq I'_1 \subsetneq \cdots \subsetneq I'_{m+n} = \hat{1}.$$

For $1 \leqslant i \leqslant m + n$, write $I_i \smallsetminus I_{i-1} = \{p_i\}$ and $I'_i \smallsetminus I'_{i-1} = \{p'_i\}$ with $p_i, p'_i \in \mathrm{JI}(X)$. We will prove by induction on i that $I_i = I'_i$ for all $0 \leqslant i \leqslant m + n$. Since $I_0 = I'_0$, we may assume that $i > 0$ and that $I_j = I'_j$ for all $j < i$.

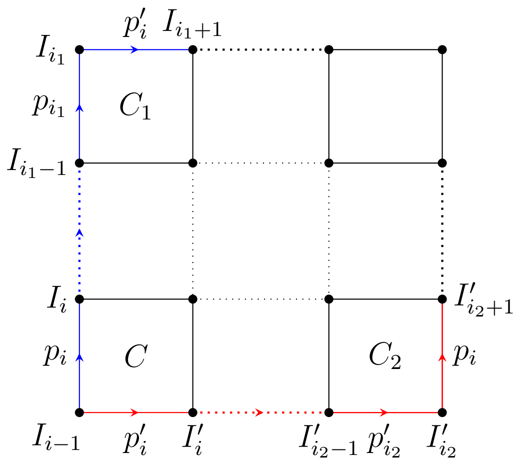

Figure 1: C, C_1, C_2, μ (blue) and ν (red) from the proof of Proposition 9.

Assume, by way of contradiction, that $I_i \neq I'_i$. Then I_{i-1} (which equals I'_{i-1}) is the bottom-left vertex of a cell C. Without loss of generality, we may assume that I_i is the top-left vertex of C and that I'_i is the bottom-right vertex of C. (In other words, μ goes up and ν goes to the right from I_{i-1}, or equivalently, p_i is a left-boundary vertex and p'_i is a bottom-boundary vertex.)

Let

$$i_1 = \min\{j > i : p'_i \in I_j\} - 1;$$
$$i_2 = \min\{j > i : p_i \in I'_j\} - 1.$$

Then the edge (I_{i_1-1}, I_{i_1}) is vertical while (I_{i_1}, I_{i_1+1}) is horizontal; this is the first time μ turns horizontal after I_{i-1}. Let C_1 be the cell with I_{i_1-1}, I_{i_1} and I_{i_1+1} as the bottom-left, the top-left and the top-right vertices respectively. Similarly the edge (I'_{i_2-1}, I'_{i_2}) is vertical while (I'_{i_2}, I'_{i_2+1}) is horizontal; this is the first time ν turns vertical after I'_{i-1}. Let C_2 be the cell with I'_{i_2-1}, I'_{i_2} and I'_{i_2+1} as the bottom-left, the bottom-right and the top-right vertices respectively. (The possibility that $C_1 = C$ or $C_2 = C$ has not been ruled out.) See Figure 1 for a schematic showing the cells C, C_1 and C_2 and the chains μ and ν.

We now prove a sequence of statements from which the proposition follows.

a. If $C_1 \notin \psi(\mu)$, then $C_2 \in \psi(\nu)$. Proof: Note that $p_{i_1+1} = p'_i$ and $p'_{i_2+1} = p_i$. Since $C_1 \notin \psi(\mu)$, we see that

$$\omega(p'_i) = \omega(p_{i_1+1}) > \omega(p_{i_1}) \geqslant \omega(p_i),$$

where the last inequality follows from noting that $p_i < \cdots < p_{i_1}$ since they are left-boundary vertices. Therefore, in the chain ν, we have

$$\omega(p'_{i_2}) \geqslant \omega(p'_i) > \omega(p_i) = \omega(p'_{i_2+1}),$$

i.e., $i_2 \in \mathrm{Des}(\nu)$. Hence $C_2 \in \psi(\nu)$.

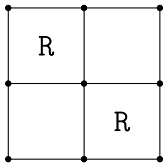

Figure 2: 2-rook (denoted by R) configuration in a non-thin polyomino.

b. If $C_2 \notin \psi(\nu)$, then $C_1 \in \psi(\mu)$. Immediate from (a).

c. If $C_1 \neq C$ then $C \notin \psi(\mu)$ and $C_1 \notin \psi(\nu)$. Proof: Note that μ does not pass through the top-right vertex of C and that ν does not pass through the bottom-left vertex of C_1.

d. If $C_2 \neq C$ then $C \notin \psi(\nu)$ and $C_2 \notin \psi(\mu)$. Proof: Note that ν does not pass through the top-right vertex of C and that μ does not pass through the bottom-left vertex of C_1.

e. If $C_1 \neq C$, then $\psi(\mu) \neq \psi(\nu)$. Proof: If $C_1 \in \psi(\mu)$, use (c) to see that

$$C_1 \in \psi(\mu) \setminus \psi(\nu).$$

Now assume that $C_1 \notin \psi(\mu)$. Then $C_2 \in \psi(\nu)$ by (a). If $C_2 = C$, then $C_2 \notin \psi(\mu)$ by (c); otherwise, $C_2 \notin \psi(\mu)$ by (d).

f. If $C_2 \neq C$, then $\psi(\mu) \neq \psi(\nu)$. Proof: If $C_2 \in \psi(\nu)$, use (d) to see that

$$C_2 \in \psi(\nu) \setminus \psi(\mu).$$

Now assume that $C_2 \notin \psi(\nu)$. Then $C_1 \in \psi(\mu)$ by (b). If $C_1 = C$, then $C_1 \notin \psi(\nu)$ by (d); otherwise, $C_1 \notin \psi(\nu)$ by (c).

g. C belongs to at most one of $\psi(\mu)$ and $\psi(\nu)$. Proof: Suppose $C \in \psi(\mu)$. Then $i_1 = i + 1$, $p_{i_1} = p'_i$ and $\omega(p_i) > \omega(p'_i)$. For C to belong to $\psi(\nu)$, we need that $I'_{i+1} = I_{i+1}$ (i.e., μ and ν are the same up to $i + 1$, except at i); for this to hold, it is necessary that $p'_{i+1} = p_i$, but then $i \notin \mathrm{Des}(\nu)$. The other case is proved similarly.

h. If $C_1 = C_2 = C$ then $\psi(\mu) \neq \psi(\nu)$. Proof: By (g), it suffices to show that $C \in \psi(\mu)$ or $C \in \psi(\nu)$. This follows from (a) and (b).

The proposition is proved by (e), (f), and (h). □

Proposition 10. *Let $k \in \mathbb{N}$ and $\mu \in \mathcal{M}_k(X)$. Then $\psi(\mu)$ is a k-rook configuration in X.*

Proof. Since $|\psi(\mu)| = k$, it suffices to note that the cells of $\psi(\mu)$ are in distinct rows and columns. This follows from Proposition 8(b). □

Proof of Theorem 1. For each $i \in \mathbb{N}$, $h_i = |\mathcal{M}_i(X)|$ by Proposition 7. By Propositions 9 and 10 we see that $h_i \leqslant r_i$ for all i. Since X is not thin, X contains a 2-rook configuration as in Figure 2. Such a rook configuration cannot be in the image of ψ. Hence $h_2 < r_2$. □

Using results of [3], we can extend our result to L-convex polyominoes as follows. Let X be an L-convex polyomino. Then there exists a polyomino X^* (the Ferrer diagram projected by X, in the sense of [3]) such that

a. X^* is a convex polyomino such that $V(X^*)$ is a sublattice of $\mathbb{N}^2$ (since X^* is a Ferrer diagram);

b. If X is not thin, then X^* is not thin;

c. X and X^* have the same rook polynomial [3, Lemma 2.4];

d. $\Bbbk[X]$ and $\Bbbk[X^*]$ are isomorphic to each other [3, Theorem 3.1], so they have the same h-polynomial.

Thus we get:

Corollary 11. *Let X be an L-convex polyomino that is not thin. Let $h(t) = 1 + h_1 t + h_2 t^2 + \cdots$ be the h-polynomial of $\Bbbk[X]$ and $r(t) = 1 + r_1 t + r_2 t^2 + \cdots$ be the rook polynomial of X. Then $h_2 < r_2$.*

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