A New Feasibility Condition for the AT4 Family

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Submitted: Jun 16, 2022; Accepted: Mar 21, 2023; Published: Apr 7, 2023

Abstract

Let Γ be an antipodal distance-regular graph with diameter 4 and eigenvalues $\theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$. Then Γ is tight in the sense of Jurišić, Koolen, and Terwilliger (J. Algebraic Combin, 2000) whenever Γ is locally strongly regular with nontrivial eigenvalues $p := \theta_2$ and $-q := \theta_3$. Assume that Γ is tight. Then the intersection numbers of Γ are expressed in terms of p, q, and r, where r is the size of the antipodal classes of Γ. We denote Γ by $\mathrm{AT4}(p, q, r)$ and call this an antipodal tight graph of diameter 4 with parameters p, q, r. In this paper, we give a new feasibility condition for the $\mathrm{AT4}(p, q, r)$ family. We determine a necessary and sufficient condition for the second subconstituent of $\mathrm{AT4}(p, q, 2)$ to be an antipodal tight graph. Using this condition, we prove that there does not exist $\mathrm{AT4}(q^3 -$

*Supported by Anhui Natural Science Foundation (2108085MA01), NSFC (No.11601002), and Key Projects in Natural Science Research of Anhui Provincial Department of Education (Nos. KJ2020A0015 and KJ2018A0438)

†Partially supported by the National Natural Science Foundation of China (No. 12071454), Anhui Initiative in Quantum Information Technologies (No. AHY150000) and the National Key R and D Program of China (No. 2020YFA0713100)

$2q, q, 2)$ for $q \equiv 3 \pmod 4$. We discuss the $\mathrm{AT4}(p, q, r)$ graphs with $r = (p+q^3)(p+q)^{-1}$.

Mathematics Subject Classifications: 05E30

1 Introduction

Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. Let $k = \theta_0 > \theta_1 > \cdots > \theta_d$ denote the eigenvalues of Γ. Jurišić et al. [12, 7] showed that the intersection numbers a_1, b_1 of Γ satisfy the following inequality

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) \geqslant -\frac{ka_1b_1}{(a_1+1)^2}, \qquad (1)$$

and defined Γ to be *tight* whenever Γ is not bipartite, and equality holds in (1). The tight distance-regular graphs have been studied in many papers; see [5, 6, 7, 8, 9, 10, 12, 15] and also see [3, Section 6.1]. A number of characterizations of the tightness property resulted; for instance, Γ is tight if and only if $a_1 \neq 0$, $a_d = 0$, and Γ is 1-homogeneous in the sense of Nomura [14]. In addition, Γ is tight if and only if each local graph of Γ is connected strongly regular, with nontrivial eigenvalues $b^+ = -1 - b_1/(1+\theta_d)$, and $b^- = -1 - b_1/(1+\theta_1)$; cf. [12]. Jurišić and Koolen [7] proved that tight distance-regular graphs with diameter three are precisely Taylor graphs, which are distance-regular graphs with intersection array $\{k, c, 1; 1, c, k\}$; cf [1, Section 1.5]. Moreover, by the results of [12, Section 7], the Terwilliger algebra of a Taylor graph does not give new feasibility conditions. For further information on Taylor graphs and their tightness, see [1, Section 7.6.C], [7, Section 3], [12], and [16].

We assume that Γ is tight with diameter four. We further assume that Γ is an antipodal r-cover. Let p and $-q$ denote the nontrivial eigenvalues of a local graph of Γ, where we assume $p > -q$. Then all intersection numbers and eigenvalues of Γ are expressed in terms of p, q, and r; cf. [8]. We denote the graph Γ by $\mathrm{AT4}(p, q, r)$ and call it an *antipodal tight graph* of diameter 4. Jurišić et al. [8, 7, 12, 10, 9] have investigated the $\mathrm{AT4}(p, q, r)$ graphs and showed various feasibility conditions for p, q, and r. Note that the family of antipodal tight graphs $\mathrm{AT4}(sq, q, q)$ are classified; cf. [10]. Additionally, Koolen et al. [12] showed that $\mathrm{AT4}(p, q, 2)$ is pseudo-vertex-transitive by using its Terwilliger algebra.

In the present paper, we study the $\mathrm{AT4}(p, q, r)$ graphs and give a new feasibility condition for the $\mathrm{AT4}(p, q, r)$ family. In Section 2, we review some preliminaries concerning the $\mathrm{AT4}(p, q, r)$ graphs. In Section 3 we show a new feasibility condition for the $\mathrm{AT4}(p, q, r)$ graphs; see Theorem 8. The μ-graph will play an important role in this section. Using the feasibility condition, we show that for a graph $\mathrm{AT4}(qs, q, q)$ we have $s \leqslant q$. In Section 4 we discuss $\mathrm{AT4}(p, q, 2)$ and its second subconstituent Δ_2. We give a necessary and sufficient condition for the graph Δ_2 to be an antipodal tight graph; see Theorem 17. From this result, we show the nonexistence of $\mathrm{AT4}(q^3 - 2q, q, 2)$ when $q \equiv 3 \pmod 4$. In particular, we show that the $\mathrm{AT4}(21, 3, 2)$ graph does not exist. The paper ends in Section 5 with some comments on $\mathrm{AT4}(p, q, r)$ with $r = (p+q^3)(p+q)^{-1}$ and an open problem for $\mathrm{AT4}(p, q, 3)$.

2 Preliminaries

In this section, we recall some definitions and results concerning the AT4(p,q,r) graphs that we need later in the paper. For more background information we refer the reader to [1, 3]. Throughout this section, let Γ denote a simple connected graph with vertex set $V(\Gamma)$ and diameter d. For $0 \leqslant i \leqslant d$ and for $x \in V(\Gamma)$ we set $\Gamma_i(x) = \{y \in V(\Gamma) : \partial(x,y) = i\}$, where $\partial = \partial_\Gamma$ denotes the shortest path-length distance function. For notational convenience, we define $\Gamma_{-1}(x) = \emptyset$ and $\Gamma_{d+1}(x) = \emptyset$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. The *i-th subconstituent* $\Delta_i(x)$ of Γ with respect to $x \in V(\Gamma)$ is the subgraph of Γ induced by $\Gamma_i(x)$. We abbreviate $\Delta(x) := \Delta_1(x)$ and call this the *local graph* of Γ at x. We say that Γ is *locally* Δ whenever all local graphs of Γ are isomorphic to Δ. We say that Γ is *regular with valency* k (or k-regular) whenever $|\Gamma(x)| = k$ for all $x \in V(\Gamma)$. We say that Γ is *distance-regular* whenever for all integers $0 \leqslant i \leqslant d$ and for all vertices $x, y \in V(\Gamma)$ with $\partial(x,y) = i$, the numbers

$$a_i = |\Gamma_i(x) \cap \Gamma(y)|, \qquad b_i = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \qquad c_i = |\Gamma_{i-1}(x) \cap \Gamma(y)|$$

are independent of x, y, where we define $b_d := 0$ and $c_0 := 0$. Observe that Γ is regular with valency $k = b_0$ and $a_i + b_i + c_i = k$ for $0 \leqslant i \leqslant d$. The array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is called the *intersection array of* Γ.

Suppose that Γ is k-regular with n vertices. We say that Γ is *strongly regular* with parameters (n,k,a,c) whenever each pair of adjacent vertices has the same number a of common neighbors, and each pair of distinct non-adjacent vertices has the same number c of common neighbors. Note that a connected strongly regular graph is distance-regular with diameter two and parameters (n, b_0, a_1, c_2). For $x, y \in V(\Gamma)$ with $\partial(x,y) = 2$, the subgraph of Γ induced by $\Gamma(x) \cap \Gamma(y)$ is called the *$\mu(x,y)$-graph* of Γ. If the $\mu(x,y)$-graph of Γ does not depend on the choice of x and y, then we simply call it the *μ-graph* of Γ.

Lemma 1 (cf. [7, Theorem 3.1])**.** *Let Γ be a distance-regular graph. Suppose that all local graphs of Γ are strongly regular with parameters (n', k', a', c'). Then the following* (i), (ii) *hold:*

(i) *μ-graphs of Γ are c'-regular.*

(ii) *$c_2 c'$ is even.*

The graph Γ is said to be *antipodal* whenever for any vertices x, y, z such that $\partial(x,y) = \partial(x,z) = d$, it follows that $\partial(y,z) = d$ or $y = z$. The property of being at distance d or zero induces an equivalence relation on $V(\Gamma)$, and the equivalence classes are called *antipodal classes.* We say that Γ is an *antipodal r-cover* if the equivalence classes have size r.

Lemma 2 (cf. [8, Section 4])**.** *Let Γ be an antipodal distance-regular graph with diameter four, n vertices, valency k, and antipodal class size r. Then the intersection array of Γ is determined by parameters (k, a_1, c_2, r), and has the following form:*

$$\{b_0, b_1, b_2, b_3; c_1, c_2, c_3, c_4\} = \{k, k - a_1 - 1, (r-1)c_2, 1; 1, c_2, k - a_1 - 1, k\}. \qquad (2)$$

Let $k = \theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$ denote the eigenvalues of Γ. Then the parameters a_1, c_2 are expressed in terms of the eigenvalues and r:

$$a_1 = \theta_1 + \theta_3, \qquad c_2 = \frac{\theta_0 + \theta_2\theta_4}{r}. \tag{3}$$

Let Ω denote the set of triples of vertices (x, y, z) of Γ such that $\partial(x, y) = 1$ and $\partial(x, z) = \partial(y, z) = 2$. For $(x, y, z) \in \Omega$, we define the number $\alpha(x, y, z) := |\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)|$, called the (*triple*) *intersection number* of Γ. We say that the intersection number α of Γ *exists* whenever $\alpha = \alpha(x, y, z)$ is independent of all $(x, y, z) \in \Omega$. If Γ is a 1-homogeneous graph with diameter $d \geqslant 2$ and $a_2 \neq 0$, then the intersection number α of Γ exists. This is because, according to the definition of 1-homogeneity, for any two adjacent vertices x and y and for any vertex $z \in \Gamma_2(x) \cap \Gamma_2(y)$, the scalar $\alpha = |\Gamma(z) \cap \Gamma(x) \cap \Gamma(y)|$ is constant; see [12, Lemma 11.5]. A strongly regular graph with $a_2 \neq 0$, that is locally strongly regular is 1-homogeneous if and only if α exists; cf. [12].

We now recall an antipodal tight graph AT4(p, q, r). In the following three lemmas, we review some properties concerning AT4(p, q, r) from [8], which will be used later.

Lemma 3 (cf. [8])**.** *Let Γ denote an antipodal tight graph* AT4(p, q, r)*. Then the following* (i)–(iv) *hold.*

(i) *The graph Γ has nontrivial eigenvalues $\theta_1 > \theta_2 > \theta_3 > \theta_4$, where*

$$\theta_1 = pq + p + q, \quad \theta_2 = p, \quad \theta_3 = -q, \quad \theta_4 = -q^2, \tag{4}$$

and its intersection array is

$$\left\{ q(pq + p + q), (q^2 - 1)(p + 1), \frac{(r - 1)q(p + q)}{r}, 1; \right. \\ \left. 1, \frac{q(p + q)}{r}, (q^2 - 1)(p + 1), q(pq + p + q) \right\}.$$

(ii) *The local graph of Γ at each vertex is strongly regular with parameters $(n', k', a', c') = (q(pq + p + q), p(q + 1), 2p - q, p)$, and its spectrum is given by*

$$\begin{pmatrix} p(q+1) & p & -q \\ 1 & \ell_1 & \ell_2 \end{pmatrix}, \tag{5}$$

where

$$\ell_1 := \frac{(q^2 - 1)(pq + p + q)}{p + q} \quad \text{and} \quad \ell_2 := \frac{pq(p + 1)(q + 1)}{p + q}. \tag{6}$$

(iii) *The graph Γ is 1-homogeneous. In particular, $\alpha = (p + q)/r$.*

(iv) *The parameters p, q, r are integers such that $p \geqslant 1, q \geqslant 2, r \geqslant 2$ and*

(1) *$pq(p + q)/r$ is even, $r(p + 1) \leqslant q(p + q)$, and $r | p + q$,*

(2) $p \geqslant q-2$, *with equality if and only if the Krein parameter* $q^4_{4,4}=0$,

(3) $(p+q)|q^2(q^2-1)$ *and* $(p+q^2)|q^2(q^2-1)(q^2+q-1)(q+2)$.

We remark that by Lemma 3(i) one readily finds the intersection numbers $\{a_i\}_{i=0}^4$ of Γ:

$$a_0=a_4=0, \qquad a_1=a_3=p(q+1), \qquad a_2=pq^2. \tag{7}$$

Lemma 4 (cf. [9, Theorem 4.3])**.** *Let* Γ *be an antipodal tight graph* $\mathrm{AT4}(p,q,r)$ *with* $p>1$. *Then its* μ*-graphs are complete multipartite if and only if there exists an integer* s *such that* $(p,q,r)=(qs,q,q)$.

Lemma 5 (cf. [9, Corollary 4.5])**.** *Let* Γ *denote an antipodal tight graph* $\mathrm{AT4}(p,q,r)$. *Then exactly one of the following statements holds.*

(i) Γ *is the unique* $\mathrm{AT4}(1,2,3)$ *graph (and* $\alpha=1$*), i.e., the Conway-Smith graph.*

(ii) Γ *is an* $\mathrm{AT4}(q-2,q,q-1)$ *graph (and* $\alpha=1$*).*

(iii) Γ *is an* $\mathrm{AT4}(qs,q,q)$ *graph, where* s *is an integer (and* $\alpha=s+1$*).*

(iv) $(p+q)(2q+1)\geqslant 3r(p+2)$ *and* $\alpha\geqslant 3$, *in particular,* $r\leqslant q-1$.

Lastly, we recall the spectral excess theorem [2]. Recall the graph Γ with vertex set $V(\Gamma)$ and diameter d. Denote the spectrum of Γ by $\mathrm{Spec}(\Gamma)=\{{\lambda_0}^{m_0},{\lambda_1}^{m_1},\ldots,{\lambda_d}^{m_d}\}$. Let $\mathcal{P}$ denote the vector space of polynomials of degree at most d. With reference to $\mathrm{Spec}(\Gamma)$ define an inner product on $\mathcal{P}$ by

$$\langle p,q\rangle=\frac{1}{n}\sum_{i=0}^{d} m_i p(\lambda_i)q(\lambda_i). \tag{8}$$

With respect to (8), there exists a unique system of orthogonal polynomials $\{p_i\}_{i=0}^d$ such that p_i has degree i and $\langle p_i,p_i\rangle=p_i(\lambda_0)$ for $0\leqslant i\leqslant d$.

Lemma 6 (cf. [2, Theorem 1])**.** *Let* Γ *be a connected* k*-regular graph on* n *vertices with diameter* d. *Let* $\{p_i\}_{i=0}^d$ *be the orthogonal polynomials corresponding to* Γ. *If* $k_d(x)$ *is the number of vertices at distance* d *from a vertex* x *in* Γ, *then*

$$n-p_d(k)\leqslant n\left[\sum_{x\in V(\Gamma)}\frac{1}{n-k_d(x)}\right]^{-1}, \tag{9}$$

with equality if and only if Γ *is distance-regular.*

3 A new feasibility condition

In this section, we introduce a new feasibility condition for the AT4(p,q,r) family. We use the following notation. Let Γ denote an antipodal tight graph AT4(p,q,r). Fix a vertex x in Γ. Choose a vertex y in Γ with $\partial(x,y)=2$. Consider the antipodal class containing y, denoted by $\{y=y_1,y_2,\ldots,y_r\}$. We define the subgraph H of Γ as the union of the μ-graphs $\Gamma(x)\cap\Gamma(y_i)$ of Γ for all $1\leqslant i\leqslant r$:

$$H:=\bigcup_{i=1}^{r}\Gamma(x)\cap\Gamma(y_i). \tag{10}$$

Observe that H is p-regular and $|V(H)|=q(p+q)$.

Lemma 7. *Let H be the graph as in* (10)*. Then the following* (i)–(iii) *hold.*

(i) *H has p as an eigenvalue of multiplicity r.*

(ii) *H has $-q$ as an eigenvalue of multiplicity at least $pq(1+p+q-q^2)/(p+q)$.*

(iii) *H has at least three distinct eigenvalues.*

Proof. (i) Since H has r connected components and each component is p-regular, the result follows.
(ii) Consider the local graph $\Delta=\Delta(x)$ of Γ. By Lemma 3(ii), Δ is strongly regular with parameters (n',k',a',c') and the spectrum (5). Denote the eigenvalues of Δ by $\delta_1\geqslant\delta_2\geqslant\cdots\geqslant\delta_{n'}$, where $n'=q(pq+p+q)$. By (5), we find that $\delta_i=-q$ for all $2+\ell_1\leqslant i\leqslant n'$, where ℓ_1 is from (6). Denote the eigenvalues of H by $\varepsilon_1\geqslant\varepsilon_2\geqslant\cdots\geqslant\varepsilon_m$, where $m=|V(H)|=q(p+q)$. Since H is a subgraph of Δ, by interlacing we have $\delta_i\geqslant\varepsilon_i\geqslant\delta_{n'-m+i}$ for $1\leqslant i\leqslant m$. Evaluate these inequalities at $i=2+\ell_1$ and $i=m$, respectively, and combine the two results to get

$$-q=\delta_{2+\ell_1}\geqslant\varepsilon_{2+\ell_1}\geqslant\varepsilon_m\geqslant\delta_{n'}=-q.$$

From this, it follows that $\varepsilon_j=-q$ for all $2+\ell_1\leqslant j\leqslant m$. Thus, the multiplicity of $-q$ is at least $q(p+q)-1-\ell_1$. Simplify this quantity to get the desired result.
(iii) Since the μ-graph is a subgraph of H, it suffices to show that the μ-graph has diameter at least 2. If the μ-graph has diameter 1, then it must be the complete graph K_{p+1}. Since $|V(H)|=q(p+q)=r(p+1)$ and by Lemma 4, we have $q=1$, a contradiction. Therefore, the μ-graph has diameter at least 2, as desired. □

Theorem 8. *Let Γ be an antipodal tight graph* AT4(p,q,r)*. Then*

$$r\leqslant\frac{p+q^3}{p+q}. \tag{11}$$

If the equality holds, then the μ-graph of Γ is strongly regular with parameters

$$\left(\frac{q(p+q)}{r},p,(q-1)(q-2)+\frac{2(p-1)}{q+1},\frac{p+q^3}{q+1}\right). \tag{12}$$

Proof. Let H be the subgraph of Γ as in (10). By Lemma 7(i), (ii), H has eigenvalues p with multiplicity r and $-q$ with multiplicity at least $pq(1+p+q-q^2)/(p+q)$, denoted by σ. By Lemma 7(iii), H has (possibly repeated) eigenvalues distinct from p and $-q$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_\tau$ for some τ. Since $|V(H)| = q(p+q)$, we have

$$\tau = q(p+q) - r - \sigma. \tag{13}$$

Let B denote the adjacency matrix of H. Then one readily checks that $\operatorname{tr}(B) = 0$ and $\operatorname{tr}(B^2) = pq(p+q)$. Using these equations and linear algebra, we have

$$\sum_{i=1}^{\tau} \lambda_i = \sigma q - rp, \qquad \sum_{i=1}^{\tau} \lambda_i^2 = pq(p+q) - rp^2 - \sigma q^2. \tag{14}$$

By the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{\tau} \frac{\lambda_i}{\tau}\right)^2 \leqslant \sum_{i=1}^{\tau} \frac{\lambda_i^2}{\tau}. \tag{15}$$

Evaluate (15) using (13) and (14) and simplify the result to get

$$\frac{q^2-1}{p+q} + \frac{q(p+q) - r(p+1)}{pq - r(p+q) + q^3} \geqslant 1. \tag{16}$$

Verify $pq - r(p+q) + q^3 > 0$ by considering each case of Lemma 5. Using this inequality, solve (16) for r and simplify the result to obtain (11).

For the second assertion, the equality in (11) holds if and only if the equality in (15) holds if and only if there exists $\nu \in \mathbb{R}$ such that $\lambda_i = \nu\tau^{-1}$ for all i $(1 \leqslant i \leqslant \tau)$. Thus, if the equality holds in (11), then we find that the μ-graph has three distinct eigenvalues: p, $-q$, and $\nu\tau^{-1}$. Therefore, the μ-graph is strongly regular. The parameters (12) follow routinely. □

Example 9. Consider the AT4$(351, 9, 3)$ graph. One checks that $r = (p+q^3)(p+q)^{-1}$ with $p = 351, q = 9, r = 3$. By Theorem 8, the μ-graph of AT4$(351, 9, 3)$ is strongly regular with parameters $(1080, 351, 126, 108)$; see Table 1.

Remark 10. The converse of the second statement in Theorem 8 does not hold in general. For example, the μ-graph of AT4$(2, 2, 2)$ is $K_{2,2}$, which is a strongly regular graph. However, equality in (11) does not hold since $r = 2 \neq 5/2 = (p+q^3)(p+q)^{-1}$.

Remark 11. In the proof of Theorem 8, we saw that the μ-graph has the eigenvalue $\lambda := \nu\tau^{-1}$ when equality holds in (11). By the equation on the left in (14) it follows that $\lambda = (\sigma q - rp)\tau^{-1}$. Evaluate this equation using (13) to get $\lambda = (p-q^2)(1+q)^{-1}$. Observe that $p > \lambda > -q$. The spectrum of the μ-graph is

$$\begin{pmatrix} p & \frac{p-q^2}{1+q} & -q \\ 1 & \frac{p(q-1)(q+1)^2}{p+q^3} & \frac{pq(1+p+q-q^2)}{p+q^3} \end{pmatrix}.$$

Corollary 12. *For an antipodal tight graph* AT4(qs, q, q), *we have* $s \leqslant q$.

Proof. The result follows from (11). $\square$

We have a comment on a bound for p. By the first expression in Lemma 3(iv)(3), we find an upper bound for p, namely, $p \leqslant q^4 - q^2 - q$. In the following, by using Theorem 8 we obtain a better bound for p.

Corollary 13. *For an antipodal tight graph* AT4(p, q, r), *we have* $p \leqslant q^3 - 2q$.

Proof. By Theorem 8 and since $r \geqslant 2$, we have $2 \leqslant (p+q^3)(p+q)^{-1}$. The result follows. $\square$

4 The graph AT4$(q^3 - 2q, q, 2)$

In this section we discuss the antipodal tight graphs AT4$(p, q, 2)$ and their second subconstituent graphs. We find the spectrum of this second subconstituent of AT4$(p, q, 2)$. We then give a necessary and sufficient condition for this second subconstituent to be an antipodal tight graph with diameter four. Let Γ denote an antipodal tight graph AT4$(p, q, 2)$. For $0 \leqslant i \leqslant 4$, consider the i-th subconstituent $\Delta_i = \Delta_i(x)$ of Γ with respect to a vertex x in Γ. Note that Δ_1 is isomorphic to Δ_3. For $0 \leqslant i \leqslant 4$ let k_i denote the cardinality of the vertex set of Δ_i. One readily finds that

$$k_0 = k_4 = 1, \qquad k_1 = k_3 = q(pq + p + q), \tag{17}$$

$$k_2 = \frac{2(pq + p + q)(q^2 - 1)(p + 1)}{p + q}. \tag{18}$$

By Lemma 3(ii), we see that Δ_i $(i = 1, 3)$ is strongly regular and its spectrum is given by (5). We now discuss the spectrum of Δ_2 in detail. To this end, we begin with the following lemma that will be used shortly.

Lemma 14. *For a given vertex x of Γ, let B denote the adjacency matrix of the second subconstituent $\Delta_2(x)$ of Γ. Then*

(i) $\operatorname{tr}(B) = 0$,

(ii) $\operatorname{tr}(B^2) = \dfrac{2pq^2(pq + p + q)(q^2 - 1)(p + 1)}{p + q}$,

(iii) $\operatorname{tr}(B^3) = \dfrac{2pq^3(pq + p + q)(q^2 - 1)(p^2 - 1)}{p + q}$.

Proof. (i) Clear.
(ii) Observe that $\operatorname{tr}(B^2)$ is the total number of closed 2-walks in Δ_2. This number is equal to a_2k_2. Evaluate this using (7) and (18).
(iii) Observe that $\operatorname{tr}(B^3)$ is the total number of directed 3-cycles in Δ_2. This number is equal to a_2k_2h, where h is the number of triangles containing one given edge in Δ_2. By construction, we find $h = a_1 - \alpha r$, where $r = 2$ and $\alpha = (p + q)/2$ by Lemma 3(iii). Evaluate $a_2k_2(a_1 - 2\alpha)$ using (7) and (18). $\square$

Lemma 15. *Let Γ denote an antipodal tight graph* $\mathrm{AT4}(p,q,2)$. *For each vertex $x \in V(\Gamma)$, the spectrum of the second subconstituent $\Delta_2(x)$ is*

$$\begin{pmatrix} pq^2 & pq & p+q-q^2 & p & -q & -q^2 \\ 1 & \dfrac{(q^2-1)(pq+p+q)}{p+q} & \dfrac{pq(p+1)(q+1)}{p+q} & m_1 & m_2 & m_3 \end{pmatrix}, \qquad (19)$$

where

$$m_1 = \frac{-q(p+1)(p-q^3+2q)(pq+p+q)}{(p+q)(p+q^2)}, \qquad (20)$$

$$m_2 = \frac{p(q^2-1)(pq+p+q)}{p+q}, \qquad (21)$$

$$m_3 = \frac{p(p-q+2)(q^2-1)(pq+p+q)}{(p+q)(p+q^2)}. \qquad (22)$$

Proof. By the proof of [13, Lemma 8.5], $\Delta_2 = \Delta_2(x)$ has at most seven distinct eigenvalues, denoted by

$$pq^2, \quad pq, \quad p+q-q^2, \quad pq+p+q, \quad p, \quad -q, \quad -q^2.$$

Also, by the proof of [13, Lemma 8.5] the multiplicity of pq (resp. $p+q-q^2$) is equal to the multiplicity of the eigenvalue p (resp. $-q$) of $\Delta(x)$. By these comments and Lemma 3(ii), we may denote the spectrum of Δ_2 by

$$\begin{pmatrix} pq^2 & pq & p+q-q^2 & \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ 1 & \ell_1 & \ell_2 & m_0 & m_1 & m_2 & m_3 \end{pmatrix},$$

where $\{\theta_i\}_{i=1}^4$ are from (4) and ℓ_1, ℓ_2 are from (6).

We find the multiplicities m_i $(0 \leqslant i \leqslant 3)$. Let B denote the adjacency matrix of Δ_2. By linear algebra, we have

$$\mathrm{tr}(B^j) = (pq^2)^j + (pq)^j \ell_1 + (p+q-q^2)^j \ell_2 + \sum_{i=0}^{3} m_i \theta_{i+1}^j, \qquad (23)$$

for a nonnegative integer j. For each $j = 0,1,2,3$, evaluate (23) using (18) and Lemma 14 to get a system of four linear equations in four variables m_0, m_1, m_2, m_3. Solve this system of equations to get

$$m_0 = 0,$$

$$m_1 = \frac{-q(p+1)(p-q^3+2q)(pq+p+q)}{(p+q)(p+q^2)},$$

$$m_2 = \frac{p(q^2-1)(pq+p+q)}{p+q},$$

$$m_3 = \frac{p(p-q+2)(q^2-1)(pq+p+q)}{(p+q)(p+q^2)}.$$

The result follows. ☐

Remark 16. We verify that the expressions of $\{m_i\}_{i=1}^3$ in (20)–(22) are integers. Recall the integers ℓ_1, ℓ_2 from (6). Observe that the expression on the right in (21) is equal to $p\ell_1$, which is a positive integer. Note that the expression on the right in (22) can be expressed as

$$(q^2-1)(pq+p-q^3-3q^2+q+2) - \frac{2q^2(q^2-1)}{p+q} + \frac{q^2(q^2-1)(q^2+q-1)(q+2)}{p+q^2}. \quad (24)$$

By Lemma 3(iv)(3), the expression (24) is an integer. By these comments, it follows that the expression on the right in (20) becomes an integer.

We now give a necessary and sufficient condition for the second subconstituent of Γ to be antipodal tight.

Theorem 17. *Let Γ denote an antipodal tight graph* AT4$(p, q, 2)$*. For each vertex $x \in V(\Gamma)$, the second subconstituent $\Delta_2(x)$ of Γ is an antipodal distance-regular graph with diameter four if and only if $p = q^3 - 2q$. Moreover, $\Delta_2(x)$ is an* AT4$(q^3 - q^2 - q, q, 2)$ *graph when $p = q^3 - 2q$.*

Proof. Abbreviate $\Delta_2 = \Delta_2(x)$. Suppose that Δ_2 is an antipodal distance-regular graph with diameter four. Then Δ_2 has precisely five distinct eigenvalues, which implies that one of m_i $(i = 1, 2, 3)$ in (19) must be zero. We claim $m_1 = 0$. As we saw in Remark 16, $m_2 > 0$. If $m_3 = 0$, by (22) and since $p \geqslant 1$ and $q \geqslant 2$ we have $p = q - 2$; this is a contradiction to [4, Theorem 2]. Therefore, we need to have $m_1 = 0$, as claimed. Then, by (20) we have $p = q^3 - 2q$.

Conversely, suppose that $p = q^3 - 2q$. Then, from (20)–(22) we find that $m_1 = 0$ and each m_i $(i = 2, 3)$ is nonzero. By this and Lemma 15, Δ_2 has precisely five distinct eigenvalues. It follows that Δ_2 has diameter at most 4. Next, recall the vertex set $\Gamma_2 = \Gamma_2(x)$ of Δ_2 and pick a vertex $v \in \Gamma_2$. Let $k_4(v)$ denote the number of vertices in Γ_2 at distance 4 from v. We claim that $k_4(v) \geqslant 1$. Consider the antipodal vertex $u \in \Gamma_2$ of v. Then $\partial(u, v) = 4$ in Γ, which implies that the distance between u and v in Δ_2 is at least 4. However, since the diameter of Δ_2 is at most 4, the distance between u and v in Δ_2 must be 4. From this, we find that Δ_2 has diameter 4 and $k_4(v) \geqslant 1$, as claimed.

We apply Lemma 6 to Δ_2, and then apply the above claim to the right-hand side of (9) to get

$$n - p_4(\kappa) \leqslant n \left[\sum_{v \in \Gamma_2} \frac{1}{n - k_4(v)} \right]^{-1} \leqslant n - 1, \quad (25)$$

where $n = |\Gamma_2|$ and κ is the valency of Δ_2. Calculate $p_4(\kappa)$ using [2, Section 6] with (19). Then we find $p_4(\kappa) = 1$. By this, the equality in (25) holds. Therefore, by Lemma 6 we have that Δ_2 is distance-regular. Moreover, $k_4(v) = 1$ for all $v \in \Gamma_2$, and hence Δ_2 is an antipodal 2-cover with diameter 4.

Next, we show that Δ_2 is an antipodal tight graph AT4$(q^3 - q^2 - q, q, 2)$ when $p = q^3 - 2q$. Recall the spectrum of Δ_2 from (19). Since $m_1 = 0$, for notational convenience we denote the eigenvalues of Δ_2 by

$$\theta_0 := pq^2 \;>\; \theta_1 := pq \;>\; \theta_2 := p + q - q^2 \;>\; \theta_3 := -q \;>\; \theta_4 := -q^2. \quad (26)$$

Since Δ_2 is antipodal distance-regular with diameter 4, express a_1 and c_2 in (3) in terms of q using (26) and $p = q^3 - 2q$. Using (2) together with parameters $k = \theta_0$, a_1, c_2, $r = 2$, we find the intersection array of Δ_2:

$$\left\{ q^3(q^2-2), (q-1)^3(q+1)^2, \frac{q^3(q-1)}{2}, 1; \right. \\ \left. 1, \frac{q^3(q-1)}{2}, (q-1)^3(q+1)^2, q^3(q^2-2) \right\}. \tag{27}$$

Compare (27) with the intersection array in [8, Theorem 5.4(ii)]. The result follows. □

Corollary 18. *If* AT4$(q^3 - q^2 - q, q, 2)$ *does not exist, neither does* AT4$(q^3 - 2q, q, 2)$.

Proof. It directly follows from Theorem 17. □

Corollary 19. *Neither* AT4$(q^3 - q^2 - q, q, 2)$ *nor* AT4$(q^3 - 2q, q, 2)$ *exists when* $q \equiv 3 \pmod 4$.

Proof. By Corollary 18, it suffices to show that AT4$(q^3 - q^2 - q, q, 2)$ does not exist for $q \equiv 3 \pmod 4$. Suppose that there exists $\Gamma =$ AT4$(q^3 - q^2 - q, q, 2)$, where $q \equiv 3 \pmod 4$. By Lemma 5(ii), a local graph of Γ is strongly regular with parameters (n', k', a', c'), where we notice that $c' = q^3 - q^2 - q$. By (27), the intersection number c_2 of Γ is $q^3(q-1)/2$. Thus, $c_2 c' = q^3(q-1)(q^3 - q^2 - q)/2$. This quantity is odd since $q \equiv 3 \pmod 4$, which contradicts Lemma 1(ii). Therefore, such a graph Γ does not exist. □

We have some comments. Jurišić presented a table that lists some known examples and open cases of the AT4 family; see [6, Table 2]. We improve this table by using the results of the present paper and the result in [4] as follows. By [4, Theorem 2] B2, B6, and B11 in [6, Table 2] are ruled out. Since A10 and B8 in [6, Table 2] satisfy the equality in Theorem 8, their μ-graphs are strongly regular; however, B8 should be ruled out by Corollary 19. Note that the smallest eigenvalue of the μ-graph of AT4(p, q, r) is $-q$ by Lemma 7. By this note, we find that the μ-graph of B4 in [6, Table 2] cannot be $K_{9,9}$, and the μ-graph of B5 in [6, Table 2] cannot be $2 \cdot K_{8,8}$. Based on these comments above, [6, Table 2] has been updated; see the new version, Table 1.

We finish this section with a comment.

Theorem 20. *Let* Γ *denote an antipodal tight graph* AT4$(p, 3, r)$. *Then* Γ *must be one of the following* (i) – (iii)*:*

(i) AT4$(3, 3, 3)$, *that is* Γ *is the* $3.O_6^-(3)$ *graph;* cf. [1, Section 13.2C].

(ii) AT4$(9, 3, 3)$, *that is* Γ *is the* $3.O_7(3)$ *graph;* cf. [1, Section 13.2D].

(iii) AT4$(9, 3, 2)$.

Proof. By Lemma 5, we have $r = 2$ or $r = 3$. If $r = 3$, Γ is either AT4$(3, 3, 3)$ or AT4$(9, 3, 3)$ by [10, Theorem 5.1]. The uniqueness of $3.O_7(3)$ refers to [11]. If $r = 2$, then $p \leqslant 21$ by Corollary 13. All possible parameters are listed in Table 1, in which AT4$(9, 3, 2)$ is the unique open case. □

Table 1: The AT4 family, $\alpha = (p+q)/r$, $c_2 = q\alpha$.

(a) Known examples, where "!" indicates the uniqueness of the corresponding graph.

#	graph	k	p	q	r	α	c_2	μ-graph
A1	! Conway-Smith	10	1	2	3	1	2	K_2
A2	! J(8,4)	16	2	2	2	2	4	$K_{2,2}$
A3	! halved 8-cube	28	4	2	2	3	6	$K_{3\times 2}$
A4	! $3.O_6^-(3)$	45	3	3	3	2	6	$K_{3,3}$
A5	! Soicher1	56	2	4	3	2	8	$2 \cdot K_{2,2}$
A6	! $3.O_7(3)$	117	9	3	3	4	12	$K_{4\times 3}$
A7	Meixner1	176	8	4	2	6	24	$2 \cdot K_{3\times 4}$
A8	! Meixner2	176	8	4	4	3	12	$K_{3\times 4}$
A9	Soicher2	416	20	4	3	8	32	$\overline{K_2}$-ext. of $\frac{1}{2}Q_5$
A10	$3.Fi_{24}^-$	31671	351	9	3	120	1080	SRG$(1080, 351, 126, 108)$

(b) Remaining open cases of small members of the AT4 family on at most 4096 vertices (with valency $k \leqslant 416$) and some ideas for their μ-graphs (whose valency is p).

#	graph	k	p	q	r	α	c_2	μ-graph
B1		96	4	4	2	4	16	$2 \cdot K_{4,4}$
B2	does not exist	115	3	5	2	4	20	$2 \cdot$ Petersen
B3		115	3	5	4	2	10	Petersen
B4		117	9	3	2	6	18	unknown; not $K_{9,9}$
B5		176	8	4	3	4	16	unknown; not $2 \cdot K_{8,8}$
B6	does not exist	204	4	6	2	5	30	$5 \cdot K_{3\times 2}$
B7		204	4	6	5	2	12	$2 \cdot K_{3\times 2}$
B8	does not exist	261	21	3	2	12	36	SRG$(36, 21, 12, 12)$
B9		288	6	6	2	6	36	$3 \cdot K_{6,6}$
B10		288	6	6	3	4	24	$2 \cdot K_{6,6}$
B11	does not exist	329	5	7	2	6	42	$7 \cdot K_6$
B12		336	16	4	2	10	40	$2 \cdot K_{5\times 4}$
B13		416	20	4	2	12	48	$2 \cdot K_{6\times 4}$

5 Case $r = (p+q^3)(p+q)^{-1}$

In Section 3, we gave a new feasibility condition (11) for the AT4(p,q,r) family. In this section, we give some comments on the graphs AT4(p,q,r) when equality holds in (11), that is, $r = (p+q^3)(p+q)^{-1}$. Let Γ denote an antipodal tight graph AT4(p,q,r) with $r = (p+q^3)(p+q)^{-1}$. Recall $r \geqslant 2$. If $r = 2$, then we have $p = q^3 - 2q$. It turns out that Γ is AT4$(q^3-2q, q, 2)$, which has been treated in Section 4. Assume that $r > 2$. Then we have the following feasibility condition for Γ.

Lemma 21. *With the above notation, we have* $(q+r)|r(r-2)(r-1)^2(r^2-r-1)$.

Proof. Solve the equation $r=(p+q^3)(p+q)^{-1}$ for p to get $p=(q^3-rq)(r-1)^{-1}$. By Lemma 3(iv)(3), we know that

$$(p+q^2)|q^2(q^2-1)(q^2+q-1)(q+2). \tag{28}$$

Substitute $p=(q^3-rq)(r-1)^{-1}$ in (28) and simplify the result to get

$$(q+r)|(r-1)q(q+1)(q+2)(q^2+q-1).$$

Set $h(q):=(r-1)q(q+1)(q+2)(q^2+q-1)$. Then there exist polynomials $f(q)$ and $g(r)$ such that $h(q)=f(q)(q+r)+g(r)$. By this comment, we find that $(q+r)|h(q)$ if and only if $(q+r)|g(r)$. Put $q=-r$ in $h(q)=f(q)(q+r)+g(r)$ to get $g(r)$. The result follows. □

Corollary 22. *For* $r>2$*, the set of feasible parameters* $\{(p,q):r=(p+q^3)(p+q)^{-1}\}$ *is finite.*

Proof. By Lemma 21, $q+r$ divides $r(r-2)(r-1)^2(r^2-r-1)$. Since the set of such q is finite, the result follows. □

Example 23. Consider an antipodal tight graph AT4$(p,q,3)$. If $(p+q^3)(p+q)^{-1}=3$, by Lemma 21 we have $(q+3)|60$. The possible values for q are

$$2,3,7,9,12,17,27,57.$$

Table 1 shows that the AT4$(p,q,3)$ exists for $q=2$, 3, and 9, which are AT4$(1,2,3)$, AT4$(9,3,3)$, and AT4$(351,9,3)$, respectively. Note that the existence for other values of q is unknown.

We finish the paper with the open problem.

Problem 24. Classify all AT4$(p,q,3)$ graphs with $(p+q^3)(p+q)^{-1}=3$.

Acknowledgements

The authors express their gratitude to the referees for carefully reading and providing valuable comments on the paper.

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