

Graphs of Linear Growth have Bounded Treewidth

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Submitted: Nov 5, 2022; Accepted: May 14, 2023; Published: Jul 14, 2023

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Abstract

A graph class \mathcal{G} has linear growth if, for each graph $G \in \mathcal{G}$ and every positive integer r , every subgraph of G with radius at most r contains $O(r)$ vertices. In this paper, we show that every graph class with linear growth has bounded treewidth.

Mathematics Subject Classifications: 05C83, 05C62

1 Introduction

The *growth* of a (possibly infinite) graph¹ G is the function $f_G: \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ where $f_G(r)$ is the supremum of $|V(H)|$ taken over all subgraphs H of G with radius at most r . Growth in graphs is an important topic in group theory [17, 18, 20, 21, 31, 36, 40], where growth of a finitely generated group is defined through the growth of the corresponding Cayley graphs. Growth of graphs also appears in metric geometry [28], algebraic graph theory [15, 16, 24–26, 39], and in models of random infinite planar graphs [1, 12]. A graph class \mathcal{G} has *linear/quadratic/polynomial/exponential growth* if $\sup\{f_G(r): G \in \mathcal{G}\}$ is bounded from above and below by a linear/quadratic/polynomial/exponential function of r .

This paper focuses on graph classes with linear growth. Linear growth has previously been studied in the context of infinite vertex-transitive graphs [15, 16, 24–26, 39]. Notably, Imrich and Seifert [25] characterised when an infinite vertex-transitive graph has linear

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¹We consider undirected graphs G with vertex-set $V(G)$ and edge-set $E(G)$. For integers $m, n \in \mathbb{Z}$, let $[m, n] := \{z \in \mathbb{Z}: m \leq z \leq n\}$ and $[n] := [1, n]$. Let \mathbb{N} be the set of positive integers. Let \log be the natural logarithm \log_e .

growth in terms of its automorphism group. We take a more structural and less algebraic approach, and prove that graph classes with linear growth have a tree-like structure.

To formalise this result, we need the following definition. A *tree-decomposition* of a graph G is a collection $(B_x \subseteq V(G) : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the nodes of a tree T , such that:

- for every edge $uv \in E(G)$, some bag B_x contains both u and v , and
- for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty subtree of T .

The *width* of a tree-decomposition is the size of the largest bag, minus 1. The *treewidth* of a graph G , denoted by $\text{tw}(G)$, is the smallest integer w for which there is a tree-decomposition of G of width w , or ∞ if no such w exists. Treewidth can be thought of as measuring how structurally similar a graph is to a tree. Indeed, a connected graph has treewidth 1 if and only if it is a tree. Treewidth is of fundamental importance in structural and algorithmic graph theory; see [4, 22, 32] for surveys.

Our main result shows that graphs with linear growth have bounded treewidth.

Theorem 1. *For any $c \geq 1$, every graph G with growth $f_G(r) \leq cr$ has treewidth at most $49c^2 + 30c$.*

It suffices to prove Theorem 1 for finite graphs, since for $k \in \mathbb{N}$, an infinite graph has treewidth at most k if and only if every finite subgraph has treewidth at most k (see [37, 38]).

Theorem 1 is proved in section 2, where we also prove an $\Omega(c \log c)$ lower bound on the treewidth in Theorem 1. section 3 considers graphs with linear growth in proper minor-closed graph classes. In this case, we improve the upper bound on the treewidth in Theorem 1 to $O(c)$. section 4 explores the product structure of graphs with linear growth. Combining Theorem 1 with results from the literature, we show that graphs with linear growth are subgraphs of bounded ‘blow-ups’ of trees with bounded maximum degree, which is a qualitative strengthening of Theorem 1. This section also presents two conjectures about the product structure of graphs with linear and polynomial growth. Finally, section 5 studies the growth of subdivisions of graphs. We show that a finite graph with bounded treewidth and bounded maximum degree has a subdivision with linear growth. We also show that for any superlinear function f with $f(r) \geq 1 + \Delta r$, every finite graph with maximum degree Δ (regardless of its treewidth) has a subdivision with growth bounded above by f . These results show that, for instance, in Theorem 1, ‘treewidth’ cannot be replaced by ‘pathwidth’, while ‘ cr ’ cannot be replaced by ‘ $cr^{1+\varepsilon}$ ’.

Graphs with bounded treewidth have many attractive properties, and Theorem 1 implies that all such properties hold for graphs of linear growth. To conclude this introduction, we give one such example. A *k -stack layout* of a graph G is a pair (\leq, φ) where \leq is a linear ordering on $V(G)$ and $\varphi: E(G) \rightarrow [k]$ is a function such that $\varphi(ux) \neq \varphi(vy)$ for any two edges $ux, vy \in E(G)$ with $u < v < x < y$. The *stack-number* of a (possibly infinite) graph G is the minimum integer $k \geq 0$ such that there exists a k -stack layout of G , or ∞ if no such k exists. This topic is widely studied; see [2, 3, 9, 10, 42, 43] for example.

Eppstein, Hickingbotham, Merker, Norin, Seweryn, and Wood [13] recently showed that $P_n \boxtimes P_n \boxtimes P_n$, which has growth $(2r + 1)^3$, has unbounded stack-number (as $n \rightarrow \infty$). Motivated by this discovery, they asked whether graphs of quadratic or of linear growth have bounded stack-number. Ganley and Heath [14] showed that every finite graph with treewidth k has stack-number at most $k + 1$. Theorem 1 thus implies a positive answer to the second part of this question.

Theorem 2. *For any $c \geq 1$, every graph G with growth $f_G(r) \leq cr$ has stack-number at most $49c^2 + 30c + 1$.*

As before, it suffices to prove Theorem 2 for finite graphs, since a standard compactness argument shows that for $k \in \mathbb{N}$, an infinite graph has stack-number at most k if and only if every finite subgraph has stack-number at most k (see appendix A).

For the remainder of the paper, we assume that every graph is finite.

2 Growth and Treewidth

This section proves our main result, Theorem 1, as well as a lower bound for the growth of the class of graphs of tree-width at most c , Theorem 8.

The key tool we use is that of balanced separations. A *separation* of a graph G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$ and no edge of G has one end in $A \setminus B$ and the other in $B \setminus A$. The *order* of the separation (A, B) is $|A \cap B|$. For $\alpha \in [\frac{2}{3}, 1)$, a separation (A, B) of a graph on n vertices is *α -balanced* if $|A| \leq \alpha n$ and $|B| \leq \alpha n$. The *α -separation number* $\text{sep}_\alpha(G)$ of a graph G is the smallest integer s such that every subgraph of G has an α -balanced separation of order at most s .

Robertson and Seymour [33] showed that $\text{sep}_{2/3}(G) \leq \text{tw}(G) + 1$. Dvořák and Norin [11] established the following converse.

Theorem 3 ([11]). *For every graph G , $\text{tw}(G) \leq 15 \text{sep}_{2/3}(G)$.*

The next two lemmas are folklore. The first one enables us to work in the more general setting of α -balanced separation.

Lemma 4. *For every $\alpha \in [\frac{2}{3}, 1)$ and every graph G , $\text{sep}_{2/3}(G) \leq \lceil \log_\alpha(\frac{2}{3}) \rceil \text{sep}_\alpha(G)$.*

Proof. Let H be a subgraph of G and let $n := |V(H)|$. Let (A_1, B_1) be an α -balanced separation of H with order at most $\text{sep}_\alpha(G)$. For $i \in \mathbb{N}$, we iteratively construct some $\max\{\frac{2}{3}, \alpha^i\}$ -balanced separation (A_i, B_i) of H with order at most $i \text{sep}_\alpha(G)$. If $\max\{|A_i \setminus B_i|, |B_i \setminus A_i|\} \leq \frac{2}{3}n$, then (A_i, B_i) is a $\frac{2}{3}$ -balanced separation of H and we set $(A_{i+1}, B_{i+1}) := (A_i, B_i)$. Otherwise, we may assume that $|B_i \setminus A_i| > \frac{2}{3}n$. Let (C_i, D_i) be an α -balanced separation of $H[B_i \setminus A_i]$ with order at most $\text{sep}_\alpha(G)$. Without loss of generality, assume that $|D_i| \geq |C_i|$ and hence $|D_i| > \frac{n}{3}$. Set $A_{i+1} := A_i \cup C_i$ and $B_{i+1} := D_i \cup (A_i \cap B_i)$. Thus $|B_{i+1} \setminus A_{i+1}| \leq \alpha |B_i \setminus A_i| \leq \alpha^{i+1}n$ and $|A_{i+1} \cap B_{i+1}| \leq |A_i \cap B_i| + \text{sep}_\alpha(G) \leq (i + 1) \text{sep}_\alpha(G)$ and $|A_{i+1} \setminus B_{i+1}| \leq n - |D_i| < \frac{2}{3}n$, so (A_{i+1}, B_{i+1}) is as desired.

Now (A_i, B_i) with $i = \lceil \log_\alpha(\frac{2}{3}) \rceil$ is a $\frac{2}{3}$ -balanced separation of H of order at most $i \text{sep}_\alpha(G)$, as required. \square

Lemma 5. For every $\alpha \in [\frac{2}{3}, 1)$ and every graph G , if every connected subgraph of G has an α -balanced separation of order less than c , then $\text{sep}_\alpha(G) < c$.

Proof. Consider a subgraph H of G . Let $n := |V(H)|$. We prove that H has an α -balanced separation of order less than c by induction on the number of components of H . If H is connected, then the claim holds by assumption. So assume that H has at least two components and let J be the smallest component of H . If $|V(H) \setminus V(J)| \leq \frac{2}{3}n$, then $(V(H) \setminus V(J), V(J))$ is an α -balanced separation of H of order 0. So assume that $|V(H) \setminus V(J)| \geq \frac{2}{3}n$. By induction, $H - V(J)$ has an α -balanced separation (A, B) of order less than c such that $|A| \geq \frac{n}{3}$. Therefore, since $|(B \cup V(J)) \setminus A| \leq \frac{2n}{3} \leq \alpha n$ and $A \cap V(J) = \emptyset$, it follows that $(A, B \cup V(J))$ is an α -balanced separation of H of order less than c , as required. \square

The next lemma is the heart of this paper.

Lemma 6. For $c \geq 1$, every graph G with growth $f_G(r) \leq cr$ satisfies

$$\text{sep}_{(1-\frac{1}{4c})}(G) < 2c.$$

Proof. Consider a connected subgraph H of G and note that $f_H(r) \leq f_G(r) \leq cr$. Let $n := |V(H)|$. Let $v \in V(H)$, let $p := \max\{\text{dist}_H(v, w) : w \in V(H)\}$, and let $V_i := \{w \in V(H) : \text{dist}_H(v, w) = i\}$ for $i \in [0, p]$. Let $R := \{i \in [p] : |V_i| \geq 2c\}$ and $S := \{i \in [p] : |V_i| < 2c\}$. Since H has radius at most p ,

$$2c|R| \leq \sum_{i \in R} |V_i| \leq n \leq cp,$$

Therefore $|R| \leq \frac{p}{2}$ and $|S| = p - |R| \geq \frac{p}{2}$. Let j be the minimum element of S such that $|S \cap [0, j]| \geq \frac{|S|}{2}$. Let $A := \bigcup_{i \in [0, j]} V_i$ and $B := \bigcup_{i \in [j, p]} V_i$. Then $|A \cap B| = |V_j| < 2c$ and

$$|A| \geq \frac{|S|}{2} \geq \frac{p}{4} \geq \frac{n}{4c} \quad \text{and} \quad |B| \geq \frac{|S|}{2} \geq \frac{p}{4} \geq \frac{n}{4c}.$$

Since V_j separates $A \setminus B$ and $B \setminus A$, there is no edge of H with one end in $A \setminus B$ and the other in $B \setminus A$. Moreover, since $|A| \geq \frac{n}{4c}$ and $|B| \geq \frac{n}{4c}$, it follows that $|A \setminus B| \leq (1 - \frac{1}{4c})n$ and $|B \setminus A| \leq (1 - \frac{1}{4c})n$. Thus (A, B) is a $(1 - \frac{1}{4c})$ -balanced separation of H of order less than $2c$. Since $1 - \frac{1}{4c} \geq \frac{2}{3}$, the result follows by Lemma 5. \square

We are now ready to prove our main theorem which we restate for convenience.

Theorem 1. For any $c \geq 1$, every graph G with growth $f_G(r) \leq cr$ has treewidth at most $49c^2 + 30c$.

Proof. Let G be a graph with growth $f_G(r) \leq cr$. By Lemmas 4 and 6,

$$\text{sep}_{2/3}(G) \leq \left\lceil \log_{(1-\frac{1}{4c})} \left(\frac{2}{3} \right) \right\rceil \text{sep}_{(1-\frac{1}{4c})}(G) \leq \left\lceil \log_{(1-\frac{1}{4c})} \left(\frac{2}{3} \right) \right\rceil 2c.$$

Note that $\log_{(1-\frac{1}{4c})}(\frac{2}{3}) = \frac{\log(\frac{2}{3})}{\log(4c) - \log(4c-1)}$. Additionally, by the mean value theorem there is some $x \in (4c-1, 4c)$ such that $x^{-1} = \log(4c) - \log(4c-1)$. Combining these observations with Theorem 3 yields

$$\text{tw}(G) \leq 15 \text{sep}_{2/3}(G) \leq 30 \lceil \log(\frac{3}{2}) x \rceil c \leq 30 (\log(\frac{3}{2}) 4c + 1) c \leq 49c^2 + 30c. \quad \square$$

We conclude this section by showing that the function $49c^2 + 30c$ in Theorem 1 cannot be replaced by any function in $o(c \log c)$. For a vertex v in a graph G , the r -ball at v is the set $B_r(v) := \{w \in V(G) : \text{dist}_G(v, w) \leq r\}$.

Lemma 7. *There is an absolute constant $\beta > 0$ such that, for every integer $k \geq 2$, there is a cubic graph G with treewidth at least k and growth $f_G(r) \leq \frac{\beta kr}{\log k}$.*

Proof. Grohe and Marx [19] proved there is an absolute constant $\alpha \in (0, 1)$ such that for every even integer $n \geq 4$ there is an n -vertex cubic graph with treewidth at least αn . Apply this result with $n := \max\{2\lceil \frac{k}{2\alpha} \rceil, 4\}$ to obtain a cubic graph G with treewidth at least k . Let $v \in V(G)$ and $r \in \mathbb{N}$, and consider the ball $B_r(v)$. Since G is cubic, $\frac{|B_r(v)|}{r} \leq \min\{\frac{n}{r}, 3 \cdot \frac{2^r}{r}\}$, which is maximised when $n = 3 \cdot 2^r$. Thus $\frac{|B_r(v)|}{r} \leq \frac{n}{\log_2(n/3)} \leq \frac{\beta k}{\log k}$, for some absolute constant β , as required. \square

Theorem 8. *If g is any function such that for any $c \geq 1$, every graph G of growth $f_G(r) \leq cr$ has treewidth at most $g(c)$, then $g(c) \in \Omega(c \log c)$.*

Proof. By Lemma 7, there is an absolute constant $\beta > 0$ such that for every $k \in \mathbb{N}$ there is a cubic graph G with treewidth at least k and growth $f_G(r) \leq \frac{\beta kr}{\log k}$. Let k be sufficiently large so that $\log k \geq \beta$. Let $c := \frac{\beta k}{\log k}$. It follows that $k\beta \geq c \log c$ and $f_G(r) \leq cr$. Hence $\frac{c \log c}{\beta} \leq k \leq \text{tw}(G) \leq g(c)$, and $g(c) \in \Omega(c \log c)$, as desired. \square

3 Growth and Minors

This section studies growth in proper minor-closed graph classes. A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A graph class \mathcal{G} is *minor-closed* if for every $G \in \mathcal{G}$ every minor of G is also in \mathcal{G} . A minor-closed class \mathcal{G} is *proper* if some graph is not in \mathcal{G} . A graph parameter λ is *minor-monotone* if $\lambda(H) \leq \lambda(G)$ whenever H is a minor of G .

Grid graphs are the key examples here. For $n \in \mathbb{N}$, the $n \times n$ grid is the graph with vertex set $\{(v_1, v_2) : v_1, v_2 \in [n]\}$ where (v_1, v_2) and (u_1, u_2) are adjacent if $v_1 = u_1$ and $|v_2 - u_2| = 1$, or $v_2 = u_2$ and $|v_1 - u_1| = 1$. This graph has treewidth n (see [22]), and is a canonical example of a graph with large treewidth in the sense that every graph G with sufficiently large treewidth contains the $n \times n$ grid as a minor [34]. Since treewidth is minor-monotone, Theorem 1 implies that any graph G with growth $f_G(r) \leq cr$ cannot contain the $n \times n$ grid as a minor, where $n = \lceil 49c^2 + 30c \rceil$. We prove this directly with $n = \lceil 2c \rceil$.

Theorem 9. *For any $c \geq 1$, every graph G with growth $f_G(r) \leq cr$ does not contain the $[2c] \times [2c]$ grid as a minor.*

Proof. It is sufficient to consider the case when $2c \in \mathbb{N}$. Suppose for contradiction that G is a graph with growth $f_G(r) \leq cr$ that contains a $2c \times 2c$ grid as a minor. Thus, there is a collection $\mathcal{H} := \{H_{i,j} : (i, j) \in [2c]^2\}$ of pairwise vertex-disjoint connected subgraphs of G such that for every $i \in [2c]$ and $j \in [2c - 1]$ there is an edge between $H_{i,j}$ and $H_{i,j+1}$ and an edge between $H_{j,i}$ and $H_{j+1,i}$. For each $i \in [2c]$, let $R_i := \bigcup_{j \in [2c]} V(H_{i,j})$ and $C_i := \bigcup_{j \in [2c]} V(H_{j,i})$. Without loss of generality, there exists $x \in [2c]$ such that $s := |R_x| \leq |C_i|$ for all $i \in [2c]$. Let v be a vertex in R_x .

We claim that $|B_{2s-1}(v) \cap C_i| \geq s$ for each $i \in [2c]$. Since $G[R_x]$ is connected, $R_x \subseteq B_{s-1}(v)$, so $B_{s-1}(v)$ contains a vertex of C_i . If $C_i \subseteq B_{2s-1}(v)$ then $|B_{2s-1}(v) \cap C_i| = |C_i| \geq |R_x| = s$, as claimed. Otherwise, since C_i is connected, C_i intersects $B_j(v) \setminus B_{j-1}(v)$ for each $j \in [s - 1, 2s]$, implying $|B_{2s-1}(v) \cap C_i| \geq s$, which proves the claim. Since C_i is disjoint from $C_{i'}$ for all distinct $i, i' \in [2c]$, we find that $2cs \leq |B_{2s-1}(v)| \leq c(2s - 1)$, which is the desired contradiction. \square

Demaine and Hajiaghayi [5] showed that for any fixed graph H , every H -minor-free graph G with treewidth k contains an $\Omega(k) \times \Omega(k)$ grid as a minor (see [27] for explicit bounds). In this case, Theorem 9 implies the following improvement on Theorem 1.

Corollary 10. *For any $c \geq 1$ and any fixed graph H , every H -minor-free graph G with growth $f_G(r) \leq cr$ has treewidth at most $O(c)$.* \square

In the case of planar graphs, Robertson, Seymour, and Thomas [35] showed that every planar graph containing no $n \times n$ -grid minor has treewidth at most $6n - 5$. Theorem 9 thus implies:

Corollary 11. *For any $c \in \mathbb{N}$, every planar graph G with growth $f_G(r) \leq cr$ has treewidth at most $12c + 1$.* \square

Recall that Theorem 8 provides an $\Omega(c \log c)$ lower bound on the treewidth of graphs G with growth $f_G(r) \leq cr$. Thus to conclude the $O(c)$ upper bounds in Corollaries 10 and 11, it is essential to make some assumption such as excluding a fixed minor.

4 Product Structure

Much of the research on the growth of finite graphs has centred around polynomial growth. In this setting, Krauthgamer and Lee [28] showed that every graph G of growth $f_G(r) \leq r^d$ (for $r \geq 2$) is isomorphic to a subgraph of the strong product of $O(d \log d)$ sufficiently long paths. Here the *strong product* $G \boxtimes H$ of graphs G and H is the graph with vertex-set $V(G) \times V(H)$ with an edge between two vertices (v, w) and (v', w') if $vv' \in E(G)$ and $w = w'$ or $ww' \in E(H)$, or $v = v'$ and $ww' \in E(H)$. Note that $G \boxtimes K_t$ is simply the graph obtained from G by replacing each vertex of G by a copy of K_t and replacing each edge of G by $K_{t,t}$ between the corresponding copies of K_t , sometimes called a *blow-up* of

G . The following result, due to a referee of [7] and refined in [8, 41], allows us to describe graphs of linear growth as subgraphs of blow-ups of trees.

Lemma 12 ([7, 8, 41]). *For $k, \Delta \in \mathbb{N}$, any graph with treewidth less than k and maximum degree Δ is isomorphic to a subgraph of $T \boxtimes K_{18k\Delta}$ for some tree T .*

A graph G with growth $f_G(r) \leq cr$ has maximum degree at most $c - 1$. Thus the following result² is a consequence of Theorem 1 and Lemma 12.

Theorem 13. *For any $c \geq 1$, every graph G with growth $f_G(r) \leq cr$ is isomorphic to a subgraph of $T \boxtimes K_{\lfloor 882c^3 \rfloor}$ for some tree T .*

The graph $T \boxtimes K_{\lfloor 882c^3 \rfloor}$ preserves the boundedness of the treewidth of G . However, the growth of $T \boxtimes K_{\lfloor 882c^3 \rfloor}$ is at least the growth of T which can be exponential, for example if T is a complete binary tree. This leads us to conjecture the following rough characterisation of graphs of linear growth.

Conjecture 14. There exist functions $g: \mathbb{R} \rightarrow \mathbb{N}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ such that for any $c \geq 1$, every graph G with growth $f_G(r) \leq cr$ is isomorphic to a subgraph of $T \boxtimes K_{g(c)}$ for some tree T with growth $f_T(r) \leq h(c)r$.

This conjecture (if true) would approximately characterise graphs of linear growth in the sense that every subgraph H of $T \boxtimes K_{g(c)}$ has growth $f_H(r) \leq g(c)h(c)r \in O(r)$.

More generally, for graphs of polynomial growth, we conjecture the following rough characterisation.

Conjecture 15. There exist functions $g: \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{N}$ and $h: \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ such that for any $c \geq 1$ and $d \in \mathbb{N}$, every graph G with growth $f_G(r) \leq cr^d$ is isomorphic to a subgraph of $T_1 \boxtimes \cdots \boxtimes T_d \boxtimes K_{g(c,d)}$, where each T_i is a tree of growth $f_{T_i}(r) \leq h(c,d)r$.

Again, this conjecture (if true) would approximately characterise graphs of degree- d polynomial growth in the sense that if H is a subgraph of $T_1 \boxtimes \cdots \boxtimes T_d \boxtimes K_{g(c,d)}$, then $f_H(r) \leq g(c,d) (h(c,d)r)^d \in O(r^d)$.

5 Growth and Subdivisions

This section considers the growth of subdivisions of graphs. A graph \tilde{G} is a *subdivision* of a graph G if \tilde{G} can be obtained from G by replacing each edge vw by a path P_{vw} with endpoints v and w (internally disjoint from the rest of \tilde{G}). If each of these paths has the same length, then \tilde{G} is said to be *uniform*.

Let \mathcal{G} be a graph class with bounded degree and bounded treewidth. Theorem 17 below shows that there is a graph class $\tilde{\mathcal{G}}$ with linear growth where for every graph $G \in \mathcal{G}$, there is a subdivision of G contained in $\tilde{\mathcal{G}}$. By Lemma 12, we can obtain Theorem 17 from the following result.

²It follows from a result of Kuske and Lohrey [29] (or Huynh, Mohar, Šámal, Thomassen, and Wood [23] in the countable case) that Theorem 13 also holds for infinite graphs.

Lemma 16. *For any $k, \Delta \in \mathbb{N}$, any $\varepsilon > 0$, any tree T , and any subgraph G of $T \boxtimes K_k$ with maximum degree at most Δ , there is a subdivision \tilde{G} of G with growth $f_{\tilde{G}}(r) \leq (k\Delta + \varepsilon)r + 1$.*

Proof. Let $n := |V(T)|$, let $V(T) = \{v_i : i \in [0, n - 1]\}$, and let $V(K_k) = \{w_i : i \in [k]\}$. For each edge $e = (v_a, w_b)(v_c, w_d) \in E(G)$, let $\gamma(e) := \min\{\text{dist}_T(v_0, v_a), \text{dist}_T(v_0, v_c)\}$. For every $i \in [0, n - 1]$, let $\ell(i)$ be the number of edges e of G with $\gamma(e) > i$. Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $g(n) = 1$ and $\varepsilon g(r) \geq 2g(r + 1)\ell(r) + |V(G)|$ for all $r \in [0, n - 1]$. Let \tilde{G} be the subdivision of G obtained by replacing each edge $e \in E(G)$ by a path of length $2g(\gamma(e))$.

For a vertex $v \in V(\tilde{G})$ and a positive integer r , consider the ball $B_r(v)$ in \tilde{G} . If there is no edge $xy \in E(G)$ such that $x, y \in B_r(v) \setminus \{v\}$, then $\tilde{G}[B_r(v)]$ is a subdivision of a star and $|B_r(v)| \leq 1 + \Delta r$, as required.

Otherwise, let $h := \min\{\gamma(xy) : xy \in E(G), x, y \in B_r(v) \setminus \{v\}\}$. Note that $r \geq g(h)$. Let S_1 be the set of subdivision vertices of edges $e \in E(G)$ with $\gamma(e) \leq h$, and let S_2 be the set of subdivision vertices of edges $e \in E(G)$ with $\gamma(e) > h$. By the definition of g , we have that $|S_2| + |V(G)| \leq \varepsilon r$. Since $\tilde{G}[B_r(v)]$ is connected and by the definition of h , there is no vertex $(v_i, w_j) \in V(G) \cap B_r(v)$ such that v_i is at distance less than h from v_0 in T . Hence $B_r(v)$ contains subdivision vertices of at most $k\Delta$ edges e of G with $\gamma(e) \leq h$.

Suppose that $r < 2g(h)$, and note that v is a subdivision vertex of some edge $x_0y_0 \in E(G)$ with $\gamma(x_0y_0) \leq h$. Consider an edge $xy \in E(G) \setminus \{x_0y_0\}$ such that $\gamma(e) \leq h$. Note that the total number of subdivision vertices of xy or x_0y_0 contained in $B_r(v)$ is at most $2r$, and that $B_r(v)$ contains at least as many subdivision vertices of x_0y_0 as of xy . It follows that $|B_r(v) \cap S_1| \leq k\Delta r$.

Now suppose that $r \geq 2g(h)$. In this case, every edge $e \in E(G)$ with $\gamma(e) = h$ has fewer than r subdivision vertices and $B_r(v)$ contains at most r subdivision vertices of each edge $e \in E(G)$ with $\gamma(e) < h$, and so again $|B_r(v) \cap S_1| \leq k\Delta r$. Hence in both cases $|B_r(v)| \leq (k\Delta + \varepsilon)r$, as required. \square

The following theorem is a direct consequence of Lemmas 12 and 16.

Theorem 17. *For any $k, \Delta \in \mathbb{N}$ and $\varepsilon > 0$, every graph G with maximum degree Δ and treewidth less than k has a subdivision \tilde{G} with growth $f_{\tilde{G}}(r) \leq (18k\Delta^2 + \varepsilon)r + 1$. \square*

Note that the growth of any subdivision \tilde{G} of a graph G depends on the maximum degree Δ of G (since $f_{\tilde{G}}(1) \geq \Delta + 1$). We now show that every graph G with bounded treewidth is a minor of a graph G' with linear growth where the growth of G' does not depend on the maximum degree of G . Markov and Shi [30] proved that every graph G is a minor of some graph G' with maximum degree 3 and treewidth at most $\text{tw}(G) + 1$. Theorem 17 applied to G' gives the following result.

Corollary 18. *For every $k \in \mathbb{N}$ and $\varepsilon > 0$, every graph G with treewidth less than k is a minor of some graph \tilde{G} with growth $f_{\tilde{G}}(r) \leq (162(k + 1) + \varepsilon)r + 1$. \square*

Theorem 17 implies that Theorem 1 is best possible in the sense that treewidth cannot be replaced by any parameter that is unbounded for graphs of bounded treewidth

and bounded maximum degree, and does not decrease when taking subdivisions. As an example, pathwidth³ is a graph parameter that is unbounded on trees with maximum degree 3 and does not decrease when taking subdivisions. In particular, there is a class of trees that has linear growth and unbounded pathwidth. Thus, “treewidth” cannot be replaced by “pathwidth” in Theorem 1.

Finally, consider subdividing graphs with bounded maximum degree without the assumption of bounded treewidth. While Theorem 1 implies that we cannot obtain linear growth in this more general setting, we can get arbitrarily close in the following sense. A function $f: \mathbb{N} \rightarrow \mathbb{R}$ is *superlinear* if $\frac{f(x)}{x} \rightarrow \infty$ as $x \rightarrow \infty$. We now show that for every superlinear function f with $f(r) \geq 1 + \Delta r$, every graph with maximum degree at most Δ admits a subdivision of growth at most f .

Theorem 19. *For any $\Delta \in \mathbb{N}$ and any superlinear function $f: \mathbb{N} \rightarrow \mathbb{R}$ with $f(r) \geq \Delta r + 1$, every graph G with maximum degree Δ has a uniform subdivision \tilde{G} with growth $f_{\tilde{G}}(r) \leq f(r)$.*

Proof. Let $m := |E(G)|$, $n := |V(G)|$, and let $\ell \in \mathbb{N}$ be such that $f(r) \geq 2rm + n$ for all $r \geq \ell$ (which exists since f is superlinear). Let \tilde{G} be obtained from G by subdividing every edge 2ℓ times. Now consider $r \in \mathbb{N}$ and a vertex $v \in V(\tilde{G})$. If $r \geq \ell$, then $|B_r(v)| \leq |V(\tilde{G})| \leq 2\ell m + n \leq f(r)$. If $r \leq \ell$, then $\tilde{G}[B_r(v)]$ is isomorphic to a subdivision of a star, so $|B_r(v)| \leq 1 + \Delta r \leq f(r)$, as required. \square

As mentioned above, every graph is a minor of some graph of maximum degree 3. Hence we have the following immediate corollary of Theorem 19.

Corollary 20. *For any superlinear function $f: \mathbb{N} \rightarrow \mathbb{R}$ with $f(r) \geq 1 + 3r$, every graph G is a minor of a graph \tilde{G} with growth $f_{\tilde{G}}(r) \leq f(r)$.*

Corollary 20 shows that, for any superlinear function f with $f(r) \geq 1 + 3r$, any minor-monotone graph parameter that is unbounded on the class of all graphs is also unbounded on the class of graphs G with $f_G(r) \leq f(r)$. For example, “ cr ” cannot be replaced by “ $cr^{1+\varepsilon}$ ” in Theorem 1.

Acknowledgements

Research of Rutger Campbell and Kevin Hendrey is supported by the Institute for Basic Science (IBS-R029-C1). Research of J. Pascal Gollin is supported by the Institute for Basic Science (IBS-R029-Y3). Bojan Mohar is supported in part by the NSERC grant R611450. Daniel J. Harvey is partially supported by JSPS KAKENHI grant 22H03550. Research of Marc Distel and Robert Hickingbotham is supported by Australian Government Research Training Program scholarships. Research of David Wood is supported by the Australian Research Council.

This research was initiated at the [Structural Graph Theory Downunder II](#) program of the Mathematical Research Institute MATRIX (March 2022).

³A *path-decomposition* is a tree-decomposition that is indexed by the nodes of a path. The *pathwidth* of a graph G is the minimum width of a path-decomposition of G .

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A Stack-number of infinite graphs

The following result is proved via a standard compactness argument.

Proposition 21. *For $k \in \mathbb{N}$, a graph G has stack-number at most k if and only if every finite subgraph of G has stack-number at most k .*

First, we introduce a version of the compactness principle in combinatorics; see [6, Appendix A]. A partially ordered set (\mathcal{P}, \leq) is *directed* if any two elements have a common upper bound; that is, for any $p, q \in \mathcal{P}$ there exists $r \in \mathcal{P}$ with $p \leq r$ and $q \leq r$. A *directed inverse system* consists of a directed poset \mathcal{P} , a family of sets $(S_p : p \in \mathcal{P})$, and for all $p, q \in \mathcal{P}$ with $p < q$ a map $g_{q,p} : S_q \rightarrow S_p$ such that the maps are *compatible*; that is, $g_{q,p} \circ g_{r,q} = g_{r,p}$ for all $p, q, r \in \mathcal{P}$ with $p < q < r$. The *inverse limit* of such a directed inverse system is the set

$$\lim_{\leftarrow} (S_p : p \in \mathcal{P}) = \left\{ (s_p : p \in \mathcal{P}) \in \prod_{p \in \mathcal{P}} S_p : g_{q,p}(s_q) = s_p \text{ for all } p, q \in \mathcal{P} \text{ with } p < q \right\}.$$

Lemma 22 (Generalised Infinity Lemma [6, Appendix A]). *The inverse limit of any directed inverse system of non-empty finite sets is non-empty.*

Proof of Proposition 21. For a linear order \leq on a set X and a subset $Y \subseteq X$, let $\leq \upharpoonright Y$ denote the restriction of \leq to Y . Similarly for a function φ with domain X and a subset $Y \subseteq X$, let $\varphi \upharpoonright Y$ denote the restriction of φ to Y .

(\implies) Clearly $(\leq \upharpoonright V(H), \varphi \upharpoonright E(H))$ is a k -stack layout for any k -stack layout (\leq, φ) of G and any subgraph H of G .

(\impliedby) Let \mathcal{P} be the set of finite subsets of $V(G)$ and consider the directed poset (\mathcal{P}, \subseteq) . For every finite set $X \subseteq V(G)$, let S_X be the set of all k -stack layouts of $G[X]$. For $Y \subseteq X \in \mathcal{P}$ and $(\leq, \varphi) \in S_X$, let $g_{X,Y}(\leq, \varphi) := (\leq \upharpoonright Y, \varphi \upharpoonright E(G[Y]))$, and note that $g_{X,Y}(\leq, \varphi) \in S_Y$. Moreover, for $Z \subseteq Y \subseteq X \in \mathcal{P}$ and $(\leq, \varphi) \in S_X$,

$$\begin{aligned} g_{Y,Z}(g_{X,Y}(\leq, \varphi)) &= ((\leq \upharpoonright Y) \upharpoonright Z, (\varphi \upharpoonright E(G[Y])) \upharpoonright E(G[Z])) = (\leq \upharpoonright Z, \varphi \upharpoonright E(G[Z])) \\ &= g_{X,Z}(\leq, \varphi). \end{aligned}$$

Hence, we have a directed inverse system of non-empty finite sets. By the Generalised Infinity Lemma, there is an element $((\leq_X, \varphi_X) \in S_X : X \in \mathcal{P})$ in the inverse limit. Define a relation \leq on $V(G)$ by setting $v \leq w$ if $v \leq_{\{v,w\}} w$ for $v, w \in V(G)$, and define a function φ on $E(G)$ by setting $\varphi(vw) := \varphi_{\{v,w\}}(vw)$ for $vw \in E(G)$. By the compatibility of the maps $g_{X,Y}$, we have that \leq is a linear order on $V(G)$ and $(\leq \upharpoonright X, \varphi \upharpoonright E(G[X])) \in S_X$ for all $X \in \mathcal{P}$. Now any two edges ux and vy with $u < v < x < y$ are assigned distinct colours since $(\leq \upharpoonright X, \varphi \upharpoonright E(G[X])) \in S_X$ for $X = \{u, v, x, y\}$. Hence (\leq, φ) is a k -stack layout of G . \square