Splitting Matchings and the Ryser-Brualdi-Stein Conjecture for Multisets

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Abstract

We study multigraphs whose edge-sets are the union of three perfect matchings, M_1, M_2, and M_3. Given such a graph G and any $a_1, a_2, a_3 \in \mathbb{N}$ with $a_1 + a_2 + a_3 \leqslant n - 2$, we show there exists a matching M of G with $|M \cap M_i| = a_i$ for each $i \in \{1, 2, 3\}$. The bound $n - 2$ in the theorem is best possible in general. We conjecture however that if G is bipartite, the same result holds with $n - 2$ replaced by $n - 1$. We give a construction that shows such a result would be tight. We also make a conjecture generalising the Ryser-Brualdi-Stein conjecture with colour multiplicities.

Mathematics Subject Classifications: 05C35, 05B15

1 Introduction

Let G be a graph on $2n$ vertices whose edge-set is the union of k edge-disjoint perfect matchings. Alternatively, one can also imagine a properly k-edge-coloured k-regular graph, where the matchings are the colour classes. For which sequences $a_1, \dots, a_k$ with $\sum_{i \in [k]} a_i \leqslant n$ does there exist a "colourful" matching M of G with the property that $|M \cap M_i| \geqslant a_i$ for each $i \in [k]$? This question was introduced by Arman, Rödl, and Sales [3, Question 1.1]. In their main result they obtained a couple of sufficient conditions for a relaxed version of the problem, where the base graph is ℓ-regular and ℓ-edge-coloured with a slightly larger $\ell \sim (1 + \varepsilon)k$.

In our paper we are mostly concerned with the original problem for three colours. Arguably, the first natural question is whether there exists a "fairly split" perfect matching M, i.e. one with $|M \cap M_i| = n/3$ for every $i = 1, 2, 3$. Of course n has to be divisible by 3 for this to have a chance of happening. It turns out that even if 3 divides n, a fairly split perfect matching is only guaranteed to exist if $n = 3$. Even more generally, for any $k \leqslant n - 1$ or $k = n$ even, the only colour-multiplicity tuples $(a_1, \dots, a_k)$ with

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$n = a_1 + \cdots + a_k$ which can be realised by a colorful perfect matching in any properly k-edge-coloured k-regular graph on $2n$ vertices are the trivial ones, namely those having a coordinate n.

Proposition 1. *Let $a_1, \ldots a_k \in \{0, 1, \ldots, n-1\}$ and $n = a_1 + \cdots + a_k$. For every $n > k$ or $n = k$ even, there exists a bipartite graph $G = (V, E)$ with n vertices in each side whose edge set is the disjoint union of k perfect matchings $M_1, \ldots, M_k$, and there is no perfect matching M of G with $|M \cap M_i| = a_i$ for each $i \in [k]$.*

The existence of a fairly split perfect matching for odd $k = n$ in bipartite graphs is known as Ryser's Conjecture, a famous and tantalising open problem.

As to the question of Arman, Rödl, and Sales for three colours, we show that a colourful matching of size as large as $n - 2$ can always be found for any colour-multiplicity vector (a_1, a_2, a_3). In fact, this can be guaranteed even when the matchings we start with are not necessarily disjoint.

Theorem 2. *Let G be a (multi-)graph on $2n$ vertices whose edge set is the disjoint union of three perfect matchings M_1, M_2, M_3. Then for any $a_1, a_2, a_3 \in \mathbb{N}$ with $a_1 + a_2 + a_3 \leqslant n - 2$ there exists a matching M in G such that $|M \cap M_1| = a_1$, $|M \cap M_2| = a_2$, and $|M \cap M_3| = a_3$.*

The proofs of the above theorem and Proposition 1 are given in Section 2.

Remark 1. In light of Proposition 1, it is natural to ask how close to a fairly split perfect matching we can get for $k \geqslant 3$. Arman et al. [3] note that their results imply that one can always choose a matching M with $|M \cap M_i| \geqslant n/k - \varepsilon n$ for every $i \in \{1, \ldots, k\}$. In their concluding remarks they also mention that their proof could be modified to establish the existence of a (smallest) constant C_k, depending only on k, such that a matching M with $|M \cap M_i| \geqslant n/k - C_k$ for each $i \in \{1, \ldots, k\}$ can always be found. Proposition 1 shows that $C_k \geqslant 1$ for every k and Theorem 2 shows that $C_3 = 1$. Using Alon's Necklace Theorem, as in [3], in combination with some extra combinatorial ideas, one can obtain a linear bound $C_k \leqslant 4k - 6$ for all k. Since we believe that $C_k = 1$ (cf Conjecture 4), we chose not to include the proof of that bound.

Remark 2. We note that the bound $n-2$ in Theorem 2 cannot be improved for general graphs without extra assumptions. To see this, for any even $n > 2$ one can consider the (unique) decomposition of $n/2$ disjoint copies of K_4 into three perfect matchings M_1, M_2, M_3. Then the intersection of any matching M of G with any K_4 is a subset of some M_i, consequently the size of M is at most n minus the number of indices $i \in \{1, 2, 3\}$ for which $|M \cap M_i|$ is odd. Hence a matching M of size $n - 1 = a_1 + a_2 + a_3$ with colour-multiplicity triple (a_1, a_2, a_3) does not exist if a_1, a_2, a_3 are all odd.

We conjecture that the construction from the previous remark is the only exception, i.e., a split with $a_1 + a_2 + a_3 = n - 1$ should always possible if at least one component of G is not a K_4.

Conjecture 3. Let G be a graph on $2n$ vertices whose edge set is decomposed into perfect matchings M_1, M_2 and M_3 and let a_1, a_2, a_3 be non-negative integers such that

$a_1 + a_2 + a_3 = n - 1$. If G has a component that is not isomorphic to a K_4, then there exists a matching M in G such that $|M \cap M_i| = a_i$ for each $i \in \{1, 2, 3\}$.

A positive answer to this conjecture would in particular complete the resolution of the question of Arman et al. for three colours, as it implies that for a colour-multiplicity triple (a_1, a_2, a_3) with $a_1 + a_2 + a_3 = n - 1$ a colourful matching is guaranteed to exist if and only if at least one of the a_i is even. This would also imply that such a matching always exists if n is odd.

The construction in Proposition 1 is bipartite. We conjecture that the $n - 2$ in Theorem 2 can be replaced with $n - 1$ if G is assumed to be bipartite. (This is actually a special case of Conjecture 3.) Even more generally, we suspect that for bipartite graphs the condition of Proposition 1 on the colour-multiplicities is best possible. More precisely, we conjecture that the following multiplicity version of the Ryser-Brualdi-Stein conjecture is true[1].

Conjecture 4. Let G be a complete bipartite graph on $2n$ vertices whose edge set is decomposed into perfect matchings M_i, $i = 1, \ldots, n$. Let a_i, $i \in \{1, \ldots, n\}$ be a sequence of non-negative integers such that $\sum_i a_i = n - 1$. Then, there exists a matching M in G such that $|M \cap M_i| = a_i$ for each $i \in \{1, \ldots, n\}$.

Note that by König's Theorem any collection of k pairwise disjoint perfect matchings of $K_{n,n}$ can be extended to a collection of n pairwise disjoint perfect matchings. Therefore, if G is bipartite the question of Arman et al for the colour multiplicity-tuple $(a_1, \ldots, a_k)$ is equivalent to the same question for the n-tuple $(a_1, \ldots, a_k, 0, \ldots, 0)$. Conjecture 4 is easy to show when there are at most two non-zero colour-multiplicities. The case of three non-zero colour-multiplicities, that is the strengthening of Theorem 2 for bipartite graphs, is already open. As in Theorem 2, Conjecture 4 could also be true for multigraphs, but for simplicity we restrict ourselves to simple graphs.

Conjecture 4 is quite optimistic, as it implies the Ryser-Brualdi-Stein conjecture (see [6] and the citations therein) by setting $a_i = 1$ for all $i \in \{1, \ldots, n-1\}$ and $a_n = 0$. In fact, Conjecture 4 is also related to the stronger Aharoni-Berger conjecture (see [7]). Several other related generalisations of the Ryser-Brualdi-Stein conjecture have been previously proposed. See for example Conjecture 1.9 in [1], see also [4].

Remark 3. An old result of Hall [5] which was independently discovered by Salzborn and Szekeres [8] (see also [9] for a modern exposition) shows that there can be no counterexample to Conjecture 4 coming from addition tables of abelian groups (as in the proof of Proposition 1). It seems to be a problem of independent interest to generalise such results to non-abelian groups, which would give further evidence for Conjecture 4.

2 Proofs

Proof of Proposition 1. First we show that if $k < n$ or $k = n$ is even then there exist pairwise distinct $x_1, x_2, \ldots$ or $x_k \in \mathbb{Z}_n$ such that $a_1x_1 + \cdots + a_kx_k \not\equiv 0 \pmod{n}$. If

[1]Noga Alon independently also asked this as a question [2].

$\sum_{i=1}^{k} ia_{\pi(i)} \not\equiv 0 \pmod n$ for some $\pi \in S_k$, then the choice $x_{\pi(i)} = i$ for every $i \in [k]$ works. This is certainly the case unless $a_1 = \cdots = a_k = n/k$. In that case, if $n = k$ is even, then $\sum_{i=1}^{n} i \cdot 1 \equiv n/2 \not\equiv 0 \pmod n$. If $n > k$ then, since none of the colour-multiplicities is n, we can assume without loss of generality that $a_k \not\equiv 0 \pmod n$. Then the choice $x_k = k+1$ and $x_i = i$ for every $i < k$ works, as then $\sum_{i=1}^{k} x_i a_i \equiv 0 + a_k \not\equiv 0 \pmod n$. Here note that since k divides n and $k < n$ we have $k \leqslant n/2$, so $k+1 < n$.

Let G be a bipartite graph between two copies of the cyclic group $\mathbb{Z}_n$ consisting of the edges whose endpoints sum to $x_1, x_2, \ldots,$ or x_k. The edges whose endpoints sum to x_i form a perfect matching M_i, and these matchings are pairwise disjoint. Suppose there exists a perfect matching M of G with $|M \cap M_i| = a_i$ for each $i \in [k]$. Summing up the endpoints of M in two different ways, we obtain

$$a_1 \cdot x_1 + a_2 \cdot x_2 + \cdots + a_k \cdot x_k = \sum_{i \in \mathbb{Z}_n} i + \sum_{i \in \mathbb{Z}_n} i.$$

Observe that the right hand side of the above equality is 0 (for example, by pairing up inverses), which contradicts the choice of $x_1, x_2, \ldots, x_k$. □

Proof of Theorem 2. We say that a matching $M \subset E(G)$ is *distributed as* (a_1, a_2, a_3) if it satisfies $|M \cap M_1| = a_1$, $|M \cap M_2| = a_2$, and $|M \cap M_3| = a_3$. It suffices to prove the claim for triples (a_1, a_2, a_3) with $a_1 = \max\{a_1, a_2, a_3\}$ as the roles of the matchings are interchangeable. We will show that given an M that is distributed as (a_1, a_2, a_3) with $a_1 + a_2 + a_3 = n - 2$ we can find a matching M' that is distributed as $(a_1 - 1, a_2 + 1, a_3)$. This also implies the existence of matching distributed as $(a_1 - 1, a_2, a_3 + 1)$. Starting from M_1 minus two arbitrary edges we can then find a matching distributed as (a_1, a_2, a_3) for any such triple satisfying $a_1 + a_2 + a_3 = n - 2$.

For any matching $M \subset E(G)$ of size $n-2$ and any vertex x that is unmatched by M, let $P_{23}(M, x)$ be the maximum $(M_2 \setminus M)$-$(M_3 \cap M)$-alternating path starting at x, and let $\ell_{23}(M, x)$ be its length. Let

$$\ell_{23}(M) := \min_{x \text{ unmatched by } M} \ell_{23}(M, x).$$

For a matching M of G and $v \in V(G)$ denote by $M(v)$ the vertex u that is matched by M to v i.e. $M(v) = u$ if and only if $\{v, u\} \in M$. Choose M such that $\ell_{23}(M)$ is minimised over all matchings that are distributed as (a_1, a_2, a_3). Pick an unmatched vertex x with $\ell_{23}(M, x) = \ell_{23}(M)$ and an unmatched vertex z that is distinct from the endpoints of $P_{23}(M, x)$ and from $M_3(x)$. We can choose such vertices because there are four unmatched vertices in total. If $M_2(x)$ is incident to an edge of $M \cap M_1$ or unmatched we are done since in the former case the matching

$$M \setminus \{M_2(x) M_1(M_2(x))\} \cup \{x M_2(x)\}$$

is distributed as $(a_1 - 1, a_2 + 1, a_3)$ while in the latter we can pick

$$M \setminus \{e\} \cup \{x M_2(x)\}$$

for any $e \in M \cap M_1$. Hence we assume that $M_2(x)$ is incident to an edge of $M \cap M_3$. Now $M_3(z)$ cannot be incident to an edge of $M \cap M_2$ because

$$M' := M \setminus \{M_2(x)M_3(M_2(x)), M_3(z)M_2(M_3(z))\} \cup \{xM_2(x), zM_3(z)\}$$

would be a matching that is distributed as (a_1, a_2, a_3) and in which $P_{23}(M', M_3(M_2(x)))$ would be a path of length $\ell_{23}(M, x) - 2$, which contradicts our choice of M. Here it was important that z is different from the endpoints of $P_{23}(M, x)$ so $P_{23}(M', M_3(M_2(x)))$ is a subpath of $P_{23}(M, x)$ not containing x and therefore $P_{23}(M', M_3(M_2(x)))$ has smaller length than $P_{23}(M, x)$. Therefore $M_3(z)$ is unmatched or incident to an edge of $M \cap M_1$. If $M_3(z)$ is incident to $M \cap M_1$ then

$$M'' := M \setminus \{M_2(x)M_3(M_2(x)), M_3(z)M_1(M_3(z))\} \cup \{xM_2(x), zM_3(z)\}$$

is the desired matching. Should $M_3(z)$ be unmatched then for any $e \in M \cap M_1$,

$$M''' := M \setminus \{M_2(x)M_3(M_2(x)), e\} \cup \{xM_2(x), zM_3(z)\}$$

is distributed as $(a_1 - 1, a_2 + 1, a_3)$. Here we used that $M_3(x) \neq z$, or equivalently that $M_3(z) \neq x$. So under the previous that assumption $M_2(x)$ is incident to an edge in $M \cap M_3$, we have that the edges $xM_2(x), zM_3(z)$ are disjoint. Hence M'' and M''' are indeed matchings of G. □

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