Ramsey Numbers of Large Even Cycles and Fans

Chunlin You[a] Qizhong Lin[b]

Submitted: Feb 22, 2022; Accepted: Jun 8, 2023; Published: Jul 28, 2023

Abstract

For graphs F and H, the Ramsey number $R(F, H)$ is the smallest positive integer N such that any red/blue edge coloring of K_N contains either a red F or a blue H. Let C_n be a cycle of length n and F_n be a fan consisting of n triangles all sharing a common vertex. In this paper, we prove that for all sufficiently large n,

$$R(C_{2\lfloor an\rfloor}, F_n) = \begin{cases} (2+2a+o(1))n & \text{if } 1/2 \leqslant a < 1, \\ (4a+o(1))n & \text{if } a \geqslant 1. \end{cases}$$

Mathematics Subject Classifications: 05D10, 05C55

1 Introduction

For graphs H_1 and H_2, the Ramsey number $R(H_1, H_2)$ is defined as the smallest integer N such that for any red/blue edge coloring of K_N, there exists either a red H_1 or a blue H_2. The existence of Ramsey number $R(H_1, H_2)$ follows from Ramsey [23].

Let C_n and K_n be a cycle and a complete graph on n vertices, respectively. A fan F_n is a graph on $2n + 1$ vertices with a vertex v, called the *center* of the fan, and $2n$ other vertices $v_1, \ldots, v_{2n}$ such that for $i = 1, \ldots, n$, $vv_{2i-1}v_{2i}$ is a triangle. Each of the n edges $v_{2i-1}v_{2i}$ is called a *blade* of the fan.

For the Ramsey number $R(C_m, C_n)$, it has been studied and completely determined in Bondy and Erdős [7], Faudree and Schelp [14], and Rosta [24]. The Ramsey numbers of fans $R(F_m, F_n)$ have been studied, both in the diagonal case (when $m = n$) and the off-diagonal case. For results in the off-diagonal case, see [18, 19, 20, 29]. In particular, Lin, Li and Dong [20] showed that $R(F_m, F_n) = 4n + 1$ for each fixed $m \geqslant 1$ and large n. Recently, Chen, Yu and Zhao [10] improve the bounds for $R(F_n, F_n)$ significantly and obtain that

$$\frac{9n}{2} - 5 \leqslant R(F_n, F_n) \leqslant \frac{11n}{2} + 6,$$

[a]School of Mathematics and Statistics, Yancheng Teachers University, Yancheng, PR China (chunlin_you@163.com).
[b]Corresponding author. College of Mathematics and Computer Science, Fuzhou University, Fuzhou, PR China (linqizhong@fzu.edu.cn).

and Dvořák and Metrebian [12] make a further improvement on the upper bound by decreasing the coefficient of the main term from 5.5 to about 5.167.

The Ramsey numbers of cycles versus fans also attracted much of attention. For instance, Li and Rousseau [18] obtained that $R(C_3, F_n) = 4n + 1$ for all $n \geqslant 2$, one can also see Bollobas [5, Theorem 13 in Ch. 6]. Generally, for fixed m and large n, Liu and Li [21] showed that $R(C_{2m+1}, F_n) = 4n + 1$. Shi [26] considered the case when the order of cycle is much larger than that of fan, in particular, the author showed that $R(C_n, F_m) = 2n - 1$ holds for all $n > 3m$. A wheel W_n is a graph on $n + 1$ vertices $v_0, v_1, \ldots, v_n$ so that the induced subgraph on v_i with $i > 0$ is the cycle C_n, and also v_0 is adjacent to v_i for each $1 \leqslant i \leqslant n$. From [3, 31, 32], we know that for large n,

$$R(C_{2m+1}, W_{2n}) = \begin{cases} 4n+1 & \text{if } 1 \leqslant m \leqslant 2n/3, \\ 4n+1 & \text{if } 2n/3 < m < n - 251, \\ 4m+1 & \text{if } m \geqslant n + 251. \end{cases}$$

Since $F_n \subset W_{2n}$, we obtain that

$$R(C_{2m+1}, F_n) = \begin{cases} 4n+1 & \text{if } 1 \leqslant m < n - 251, \\ 4m+1 & \text{if } m \geqslant n + 251. \end{cases}$$

For more Ramsey numbers involving fans, we refer the reader to [9, 22, 25, 30], etc.

In this paper, we are concerned with the asymptotic behavior of the Ramsey number $R(C_{2\lfloor an \rfloor}, F_n)$ when n is large and $a \geqslant 1/2$ is fixed.

Theorem 1. *For all sufficiently large n,*

$$R(C_{2\lfloor an \rfloor}, F_n) = \begin{cases} (2 + 2a + o(1))n & \text{if } 1/2 \leqslant a < 1, \\ (4a + o(1))n & \text{if } a \geqslant 1. \end{cases}$$

The following corollary is immediate.

Corollary 2. *We have $R(C_{2n}, F_n) = (4 + o(1))n$, and $R(C_n, F_n) = (3 + o(1))n$ for all sufficiently large even integer n.*

2 Preliminaries

Throughout this paper, all graphs are finite and simple. Let $G = G(V, E)$ be such a graph. For a vertex $v \in V$, let $N_G(v)$ denote the neighborhood of v in G, and $\deg_G(v) = |N_G(v)|$ is the degree of a vertex $v \in V$. We denote by $\delta(G)$ and $\triangle(G)$ the minimum and maximum degrees of the vertices of G. For a vertex $v \in V$ and $U \subset V$, we write $N_G(v, U)$ for the neighbors of v in U in graph G and denote $\deg(v, U) = |N_G(v, U)|$. For a vertex set $X \subset V$ and $U \subset V \setminus X$, we write $N_G(X, U)$ for all neighbors of X in U in graph G. In particular, we write $N_G(X)$ for all neighbors of X in $V \setminus X$.

For disjoint vertex sets $A, B \subseteq V$, let $e_G(A, B)$ denote the number of edges of G with one endpoint in A and the other in B, and the density between A and B is

$$d_G(A, B) = \frac{e_G(A, B)}{|A||B|}.$$

We always delete the subscripts when there is no confusion.

Definition 3 (**ϵ**-regular)**.** For $\epsilon > 0$ and $d \leqslant 1$, a pair (A, B) is ϵ-regular if for all $X \subseteq A$ and $Y \subseteq B$ with $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have $|d(X, Y) - d(A, B)| < \epsilon$.

Definition 4 ((**ϵ, d**)-regular)**.** A pair (A, B) is said to be (ϵ, d)-regular if it is ϵ-regular and $d(A, B) \geqslant d$.

The following property is well-known, see e.g. [17].

Fact 5. *Let (A, B) be an ϵ-regular pair with density d. Then for any $Y \subset B$, $|Y| > \epsilon|B|$ we have*

$$\#\{x \in A : \deg(x, Y) \leqslant (d - \epsilon)|Y|\} \leqslant \epsilon|A|.$$

In this paper, we will use the following regularity lemma.

Lemma 6 (Szemerédi [27])**.** *For every $\epsilon > 0$ and integer $t_0 \geqslant 1$, there exists $T_0 = T_0(\epsilon, t_0)$ such that, for every graph G of large order n, there exists a partition $V(G) = \cup_{i=0}^{t} V_i$ satisfying $t_0 \leqslant t \leqslant T_0$ and*

(i) $|V_0| < \epsilon n$, $|V_1| = |V_2| = \ldots = |V_t|$;

(ii) all but at most ϵt^2 pairs (V_i, V_j), $1 \leqslant i \neq j \leqslant t$, are ϵ-regular.

For a graph G, denote by $\nu(G)$ the size of the largest matching of G. Let us recall the following classical result in graph theory due to Hall, see, e.g., [8, 28].

Lemma 7 (Hall [8])**.** *Let G be a bipartite graph on parts X and Y. For any non-negative integer d, $\nu(G) \geqslant |X| - d$ if and only if $|N(S)| \geqslant |S| - d$ for every $S \subseteq X$.*

For a matching $M \subseteq E$, we call all vertices which are not incident to any edge in M the unmatched vertices in M. Furthermore, we denote by $q(G \setminus S)$ the number of odd components in $G \setminus S$. We use a generalization of Tutte's Theorem in our proof.

Lemma 8 (Berge [4])**.** *Let $G = (V, E)$ be a graph. For any set $S \subseteq V$ and any matching M, the number of unmatched vertices in M is at least $q(G \setminus S) - |S|$. Moreover, there exists a set $S \subseteq V$ such that every maximum matching of G misses exactly $q(G \setminus S) - |S|$ vertices.*

What minimum degree condition guarantees a path of a preassigned length. This question was answered by Erdős and Gallai [13] and again Andrasfai [1].

Lemma 9 (Erdős and Gallai [13])**.** *Let G be a connected graph with minimum degree δ and at least $2\delta + 1$ vertices. Then G contains a path of at least $2\delta + 1$ vertices.*

The complete bipartite graphs $K_{\delta, n-\delta}$ with $n \geqslant 2\delta + 1$ show that the lemma is best possible in the sense that there exist graphs of minimum degree δ with no longer paths. All such extremal graphs are illustrated by Ali and Staton [2].

A graph is called *weakly pancyclic* if it contains cycles of every length between its girth and its circumference. A graph is *pancyclic* if it is weakly pancyclic with girth 3 and circumference $n = |V(G)|$. In particular, if $\delta = \delta(G) \geqslant n/2$, then $c(G) = n$. This is a well-known result for a graph being hamiltonian. For the special case of $\delta \geqslant n/2$, the following result tells us more about the structure of a graph.

Lemma 10 (Bondy [6]). *If a graph G with n vertices satisfies $\delta(G) \geqslant n/2$, then G is pancyclic unless $n = 2r$ and $G = K_{r,r}$.*

Let nK_2 denote a matching of size n, i.e., n pairwise disjoint edges, and let S_t be a star with t edges.

Lemma 11 (Gyárfás and Sárközy [16]). *Suppose that $n_1 \geqslant n_2 \geqslant 1$ and $k \geqslant 1$. Then*

$$R(S_k, n_1K_2, n_2K_2) = \begin{cases} 2n_1 + n_2 - 1 & \text{if } k \leqslant n_1, \\ n_1 + n_2 - 1 + k & \text{if } k \geqslant n_1. \end{cases}$$

We will apply the following result to get a large monochromatic component for every 2-coloring of the edges of graph G with large minimum degree.

Lemma 12 (Gyárfás and Sárközy [16]). *For any 2-color of edges of a graph G with minimum degree $\delta(G) \geqslant \frac{3|V(G)|}{4}$, there is a monochromatic component of order larger than $\delta(G)$. This estimate is sharp.*

We also need the following result, which states that a bipartite graph with high density always contains a large matchings.

Lemma 13 (Figaj and Łuczak [15]). *Let $G = (V, E)$ be a bipartite graph with bipartition $\{V_1, V_2\}$, $|V_1| \geqslant |V_2|$, and at least $(1-\epsilon)|V_1||V_2|$ edges, for some $0 < \epsilon < 0.01$. Then, there is a component in G of at least $(1-3\epsilon)(|V_1| + |V_2|)$ vertices which contains a matching of cardinality at least $(1-3\epsilon)|V_2|$.*

3 Proof of Theorem 1

For a graph $G = G(V, E)$, if $S \subseteq V$, then $G - S$ denotes the subgraph of G induced by $V(G) \setminus S$. For any subset $A \subseteq V$, we use $G[A]$ to denote the subgraph induced by the vertex set A in G. For two subsets $A \subseteq V$ and $B \subseteq V$, we use $G[A, B]$ to denote the subgraph induced by all edges between A and B in G.

For a 2-edge colored graph G, we use G^r (or G^b) to denote the subgraph of G formed by all red (or blue) edges of G. If $A \subseteq V$, then A^r is defined to be $G^r[A]$ and A^b is defined to be $G^b[A]$. For any subset $A \subseteq V$, we also use $\delta^r(A)$ (or $\delta^b(A)$) to denote $\delta(A^r)$ (or $\delta(A^b)$) for convenience.

In the following, we always omit the floors and ceilings when there is no affection on our argument.

Part (I) $1/2 \leqslant a < 1$

Let $\overline{G}$ denote the complement graph of G. Note that the graph $K_{2\lfloor an\rfloor-1} \cup K_{n-2} \cup K_{n-2}$ contains no cycle $C_{2\lfloor an\rfloor}$ and its complement contains no F_n, so we have $R(C_{2\lfloor an\rfloor}, F_n) \geqslant 2\lfloor an\rfloor + 2n - 4$ for $1/2 \leqslant a < 1$.

It remains to show the upper bound. Let $N = (2a + 2 + \gamma)n$, where $1/2 \leqslant a < 1$ and $0 < \gamma < 1/10$ is sufficiently small, we will show $R(C_{2an}, F_n) \leqslant N$ for all large n, i.e., any red/blue edge coloring of K_N yields either a red C_{2an} or a blue F_n. Suppose to the contrary that for large n, there exists a coloring that contains neither a red C_{2an} nor a blue F_n. We aim to find a contradiction.

Consider a 2-edge coloring of $G = K_N$ defined on V. Set

$$\beta = \min\left\{\frac{\gamma}{100}, \frac{1-a}{30(a+1)}\right\}, \text{ and } \epsilon = \frac{\beta^2}{10^4}. \tag{1}$$

We apply the regularity lemma (Lemma 6) with ϵ and sufficiently large t_0 to obtain

$$T_0 = T_0(\epsilon, t_0) = \min\left\{2t_0, \frac{5}{4\epsilon}\right\} \tag{2}$$

such that there exists a partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ satisfying $t_0 \leqslant t \leqslant T_0$ and (i) $|V_0| < \epsilon N$, $|V_1| = |V_2| = \ldots = |V_t|$; (ii) all but at most ϵt^2 pairs (V_i, V_j), $1 \leqslant i \neq j \leqslant t$, are ϵ-regular for G^r and G^b. We construct the reduced graph H with vertex set $\{v_1, v_2, \ldots, v_t\}$ and the edge set formed by pairs $\{v_i, v_j\}$ for which (V_i, V_j) is ϵ-regular with respect to G^r and G^b. Thus we obtain a bijection $f : v_i \to V_i$ between the vertices of H and the clusters of the partition.

Color an edge v_iv_j red if the density of the red edges between V_i and V_j is at least β, and blue otherwise. Let H^r and H^b be the subgraphs induced by all red edges and blue edges of H, respectively. Since there are at most ϵt^2 edges that are uncolored in H, by deleting at most $\sqrt{\epsilon}t$ vertices, we may assume that each vertex has at most $\sqrt{\epsilon}t$ non-neighbors. In what follows, when referring to the reduced graph H, we will assume that these vertices have been removed.

A connected matching in a graph G is a matching M such that all edges of M are in the same connected component of G.

Claim 14. *H^r contains no connected matching of size more than $(\frac{a}{2a+2} - 0.15\beta)t$.*

Proof. On the contrary, suppose that H^r contains a connected matching M on at least $2k = (a/(a+1) - 0.3\beta)t$ vertices. Let F be a minimal connected red subgraph containing M. We may assume that $M = \{v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}\}$ and $f(v_{2i-1}) = V_{2i-1}$ for $1 \leqslant i \leqslant k$. Clearly, F is a tree. Consider a closed walk $W = V_1V_2 \cdots V_{2k-1}V_{2k} \cdots V_1$ that contains all edges of M. By applying a similar argument as in Figaj and Łuczak [15] we can obtain a red cycle of length $2an$, contradicting our assumption that G^r contains no such cycle. □

Claim 15. *H^b contains no fan with at least $k = (\frac{1}{2a+2} - 0.05\beta)t$ blades.*

Proof. If not, H^b contains a fan on $2k+1$ vertices. Suppose that v_a is the center of such fan with k blades, say $v_1v_2, \ldots, v_{2k-1}v_{2k}$. By relabelling the vertices if needed, we may assume that $f(v_a) = V_a$ and $f(v_i) = V_i$ for $i \in [2k] = \{1, 2, \ldots, 2k\}$.

Note that an edge v_iv_j in H is blue if and only if the density $d_{G^b}(V_i, V_j) \geqslant 1 - \beta$, by Fact 5, all but at most $2k\epsilon$ vertices of V_a has degree at least $(1 - \beta - \epsilon)|V_i|$ in each V_i for $1 \leqslant i \leqslant 2k$. Since $2k\epsilon < 1$ from (2), we can choose a vertex $u \in V_a$ such that u has at least $(1-\beta-\epsilon)|V_i|$ neighbors in each V_i for $1 \leqslant i \leqslant 2k$. Let $V_i' = N_{G^b}(u) \cap V_i$ for $1 \leqslant i \leqslant 2k$. Therefore,

$$|V_i'| = |N_{G^b}(u) \cap V_i| \geqslant (1-\beta-\epsilon)|V_i| > \epsilon|V_i| \tag{3}$$

for every $1 \leqslant i \leqslant 2k$. Moreover, we have $e_{G^b}(V_{2i-1}', V_{2i}') \geqslant (1-\beta-\epsilon)|V_{2i-1}'||V_{2i}'|$ for $1 \leqslant i \leqslant k$ since (V_{2i-1}, V_{2i}) are $(\epsilon, 1-\beta)$-regular in G^b. Hence, by Lemma 13, the graph $G^b[V_{2i-1}, V_{2i}]$ contains a matching of cardinality at least $(1-3(\beta+\epsilon))\,|V_{2i-1}'|$. Let $S = \cup_{i=1}^{2k} V_i'$. Therefore, $G^b[S]$ contains a matching of cardinality at least

$$\begin{aligned} k(1-3\,(\beta+\epsilon))\,|V_i'| &\overset{(3)}{\geqslant} k(1-3\,(\beta+\epsilon))(1-(\beta+\epsilon))\,|V_i| \\ &\geqslant k(1-4\,(\beta+\epsilon))\frac{(1-\epsilon)(2a+2+\gamma)n}{t} \\ &\overset{(1)}{>} \left(\frac{1}{2a+2} - 0.05\beta\right)(1-5\beta)\,(2a+2+\gamma)\,n \\ &> \left(\frac{1}{2a+2} - 2\beta\right)(2a+2+\gamma)\,n \overset{(1)}{>} (1+2\beta)\,n, \end{aligned}$$

yielding a blue F_n with center u in G^b, a contradiction. □

Claim 16. $\deg_{H^r}(v) \geqslant \frac{a}{2a+2}t$ *for every* $v \in V(H)$.

Proof. On the contrary, without loss of generality, suppose that H^r contains a vertex v_c such that $\deg_{H^r}(v_c) \leqslant \frac{a}{2a+2}t - 1$. Since $\deg_H(v) \geqslant (1-2\sqrt{\epsilon})t - 1$ for each $v \in V(H)$, we have that $\deg_{H^b}(v_c) \geqslant (\frac{a+2}{2a+2} - 2\sqrt{\epsilon})t$. Denote $H_1 = H[N_{H^b}(v_c)]$, i.e., the subgraph of H induced by the neighborhood of v_c in H^b. Without loss of generality, we may assume that

$$|V(H_1)| = \left(\frac{a+2}{2a+2} - 2\sqrt{\epsilon}\right)t. \tag{4}$$

Note that every vertex in H has at most $\sqrt{\epsilon}t$ non-neighbors. Let C_1 be the vertex set of a largest monochromatic component in H_1. Here and in what follows, we also use C_1 (C_i) to denote its vertex set of the component C_1 (C_i). From Lemma 12,

$$|C_1| > \delta(H_1) \geqslant |V(H_1)| - 1 - \sqrt{\epsilon}t \overset{(4)}{=} \left(\frac{a+2}{2a+2} - 3\sqrt{\epsilon}\right)t - 1. \tag{5}$$

Suppose first that C_1 is a red component. We apply Lemma 11 to the subgraph C_1 with $k = \sqrt{\epsilon}t + 1$, $n_1 = (\frac{1}{2a+2} - 0.05\beta)t$, and $n_2 = (\frac{a}{2a+2} - 0.15\beta)t$ to obtain that

$$R(S_k, n_1K_2, n_2K_2) = 2n_1 + n_2 - 1 = \left(\frac{a+2}{2a+2} - 0.25\beta\right)t - 1 \overset{(5),(1)}{<} |C_1|.$$

Thus, by noting that every vertex in H has at most $\sqrt{\epsilon}t$ non-neighbors, we can get either a red matching of size $(\frac{a}{2a+2} - 0.15\beta)t$ or a blue matching of size $(\frac{1}{2a+2} - 0.05\beta)t$. The first case contradicts Claim 14 since the red matching in C_1 is clearly connected. For the second, we obtain a blue fan with center v_c in H_1^b with at least $(\frac{1}{2a+2} - 0.05\beta)t$ blades, which contradicts Claim 15.

In the following, we assume that C_1 is a largest blue component. Denote $U = V(H_1) \setminus V(C_1)$. Recall that every vertex in H_1 has at most $\sqrt{\epsilon}t$ non-neighbors. If $U \neq \emptyset$, then U is completely covered by a red component C_r due to the minimum degree condition of H_1. Note that there are no edges of H_1 in the bipartite graph $H_1[C_1 \setminus C_r, C_r \setminus C_1]$, we have that $|C_1 \setminus C_r| \leqslant \sqrt{\epsilon}t$ since $C_r \setminus C_1 = U \neq \emptyset$. Thus we have

$$|C_1 \cap C_r| = |C_1| - |C_1 \setminus C_r| \overset{(5)}{>} \left(\frac{a+2}{2a+2} - 4\sqrt{\epsilon}\right)t - 1 \overset{(1)}{>} \left(\frac{a+2}{2a+2} - 0.35\beta\right)t - 1.$$

We apply Lemma 11 to the subgraph induced by $C_1 \cap C_r$ in H_1 to conclude that $H_1[C_1 \cap C_r]$ contains a red connected matching of size at least $(\frac{a}{2a+2} - 0.15\beta)t$ or a blue matching of size at least $(\frac{1}{2a+2} - 0.1\beta)t$ which together with v_c yield a blue fan with more than $(\frac{1}{2a+2} - 0.1\beta)t$ blades, contradicting Claim 14 or Claim 15.

Now we assume $U = \emptyset$. So we have $C_1 = V(H_1)$ and hence

$$|C_1| = |V(H_1)| \overset{(4)}{\geqslant} \left(\frac{a+2}{2a+2} - 2\sqrt{\epsilon}\right)t. \tag{6}$$

Without loss of generality, we define C_2 as a largest red component in H_1. From Claim 15, we know that the largest blue matching has size $m < (\frac{1}{2a+2} - 0.05\beta)t$ in H_1. Applying Lemma 8 to the subgraph induced by all blue edges in H_1, we can find a subset $S \subset V(H_1)$ such that the number of odd components

$$\begin{aligned} q(V(H_1) \setminus S) = |S| + |V(H_1)| - 2m &> \left(\frac{a+2}{2a+2} - 2\sqrt{\epsilon}\right)t - 2\left(\frac{1}{2a+2} - 0.05\beta\right)t \\ &\overset{(1)}{>} \left(\frac{a}{2a+2} + 6\sqrt{\epsilon}\right)t. \end{aligned} \tag{7}$$

Clearly, $q(V(H_1) \setminus S) + |S| \leqslant |V(H_1)|$, which implies that $|S| \leqslant m \leqslant (\frac{1}{2a+2} - 0.05\beta)t - 1$.

Let R be the red subgraph of H_1 whose vertex set is $V(H_1) \setminus S$ and edge set consists of all red edges between blue components of $V(H_1) \setminus S$ in H_1. It is clear that

$$|V(R)| \geqslant |V(H_1)| - |S| \geqslant \left(\frac{a+2}{2a+2} - 2\sqrt{\epsilon}\right)t - \left(\frac{1}{2a+2} - 0.05\beta\right)t + 1 \overset{(1)}{>} \left(\frac{1}{2} + 2\sqrt{\epsilon}\right)t. \tag{8}$$

We will show that R is connected. Otherwise, $V(R)$ can be partitioned into two non-empty sets A and B such that there are no red edges between A and B. Without loss of generality, suppose $|A| \geqslant |B|$. Then $|A| > (1/4 + \sqrt{\epsilon})t$. If A intersects each of these $q(V(H_1) \setminus S)$ blue odd components in $V(H_1) \setminus S$, then any vertex $v \in B$ is non-adjacent to all vertices in the intersecting set of A and those blue components not containing v. Thus, any vertex $v \in B$ is non-adjacent to at least

$$q(V(H_1) \setminus S) - 1 \overset{(7)}{>} \left(\frac{a}{2a+2} + 6\sqrt{\epsilon}\right)t - 1$$

vertices in H_1. On the other hand, if A does not intersect some blue odd component, then any vertex v in this component is non-adjacent to any vertex of A. Therefore, in both cases we can find a vertex v that is non-adjacent to at least $(\frac{a}{2a+2}+6\sqrt{\epsilon})t-1$ vertices, which clearly contradicts the fact that $\delta(H_1) \geqslant |V(H_1)|-1-\sqrt{\epsilon}t$. Thus, R is connected as desired. Since C_2 is the largest red component, it follows from (8) that

$$|C_2| \geqslant |V(R)| > (1/2+2\sqrt{\epsilon})t. \tag{9}$$

Let $p=|C_1 \setminus C_2|$. Then $p \geqslant 23\sqrt{\epsilon}t$. Otherwise, $|C_2| > (\frac{a+2}{2a+2}-0.25\beta)t-1$ from (6), by a similar argument as above by applying Lemma 11, we can get either a red connected matching of size at least $(\frac{a}{2a+2}-0.15\beta)t$, or a blue matching of size at least $(\frac{1}{2a+2}-0.05\beta)t$ which together with v_c yield a blue fan with more than $(\frac{1}{2a+2}-0.05\beta)t$ blades. This again leads to a contradiction from Claim 14 or Claim 15.

We first suppose that $23\sqrt{\epsilon}t \leqslant p < (\frac{1-a}{a+1}-0.05\beta)t$. Thus

$$|C_2| = |C_1| - p \overset{(6)}{=} \left(\frac{a+2}{2a+2} - 2\sqrt{\epsilon}\right)t - p. \tag{10}$$

We apply Lemma 11 to the subgraph induced by vertex set C_2 in H_1 with $k=\sqrt{\epsilon}t+1$, $n_1 = (\frac{1}{2a+2}-0.05\beta)t-p/2$ and $n_2=(\frac{a}{2a+2}-0.15\beta)t$. Note that $n_1 > n_2$ and

$$R(S_k, n_1K_2, n_2K_2) = \left(\frac{a+2}{2a+2} - 0.25\beta\right)t - p - 1 \overset{(10)}{<} |C_2|,$$

so there exists a blue matching M_1 of size $n_1 = (\frac{1}{2a+2}-0.05\beta)t-p/2$ since otherwise a red connected matching of size at least n_2 will lead to a contradiction from Claim 14.

Note that $\delta(H_1)=\delta(H_1[C_1]) \geqslant |C_1|-1-\sqrt{\epsilon}t$ and all (but at most ϵt^2) edges between $C_1 \setminus C_2$ and $C_2 \setminus V(M_1)$ are blue. For any subset $S \subseteq C_2 \setminus V(M_1)$, if

$$|C_2 \setminus V(M_1)| < |C_1 \setminus C_2| = p \leqslant \left(\frac{1-a}{a+1} - 0.05\beta\right)t,$$

then the total number of blue neighbors of S in $C_1 \setminus C_2$ satisfies that

$$|N_{H_1^b}(S, C_1 \setminus C_2)| \geqslant |C_1 \setminus C_2| - \sqrt{\epsilon}t > |C_2 \setminus V(M_1)| - \sqrt{\epsilon}t.$$

Recall that $|C_2| = |C_1| - p$ and $n_1 = (\frac{1}{2a+2}-0.05\beta)t-p/2$. Hence, by Lemma 7, the bipartite graph $H_1^b[C_1 \setminus C_2, C_2 \setminus V(M_1)]$ contains a blue matching of size at least

$$\begin{aligned}|C_2 \setminus V(M_1)| - \sqrt{\epsilon}t &= |C_2| - 2n_1 - \sqrt{\epsilon}t = |C_1| - \left(\frac{1}{a+1} - 0.1\beta\right)t - \sqrt{\epsilon}t \\ &\overset{(6)}{=} \left(\frac{a}{2a+2} + 0.1\beta - 3\sqrt{\epsilon}\right)t \\ &\overset{(1)}{>} \left(\frac{a}{2a+2} + 0.06\beta\right)t.\end{aligned}$$

This matching together with M_1 yield a blue matching of size at least

$$n_1 + \left(\frac{a}{2a+2} + 0.06\beta\right)t = (1/2 + 0.01\beta)t - p/2 > \left(\frac{1}{2a+2} - 0.05\beta\right)t$$

since $p < (\frac{1-a}{a+1} - 0.05\beta)t$. If $|C_2 \setminus V(M_1)| \geqslant p = |C_2|$, then Lemma 7 again implies the bipartite graph $H_1^b[C_1 \setminus C_2, C_2 \setminus V(M_1)]$ contains a blue matching of size at least

$$|C_1 \setminus C_2| - \sqrt{\epsilon}t = p - \sqrt{\epsilon}t,$$

which together with M_1 yield a blue matching of size at least $n_1 + p - \sqrt{\epsilon}t > (\frac{1}{2a+2} - 0.05\beta)t$ in H_1^b (also in H^b). Therefore, for either case, we can get a blue fan with center v_c and at least $(\frac{1}{2a+2} - 0.05\beta)t$ blades, which contradicts Claim 15.

Now we assume that $p \geqslant (\frac{1-a}{a+1} - 0.05\beta)t$. Recall that $C_1 = V(H_1)$ and $|C_2| > (1/2 + 2\sqrt{\epsilon})t$ from (9), so we can upper bound p as that

$$p = |C_1 \setminus C_2| = |C_1| - |C_2| \overset{(6)}{<} \left(\frac{1}{2a+2} - 4\sqrt{\epsilon}\right)t \overset{(1)}{=} \left(\frac{1}{2a+2} - 0.04\beta\right)t.$$

Thus we have $(\frac{1-a}{a+1} - 0.05\beta)t \leqslant p < (\frac{1}{2a+2} - 0.04\beta)t$. We apply Lemma 11 to the subgraph induced by C_2 in H_1 with $k = \sqrt{\epsilon}t + 1$, $n_1 = (\frac{a}{2a+2} + 0.01\beta)t$ and $n_2 = (\frac{1}{2a+2} - 0.04\beta)t - p$. Note that $n_1 \geqslant n_2$ and $1/2 \leqslant a < 1$, hence we have that

$$R(S_k, n_1K_2, n_2K_2) = \left(\frac{2a+1}{2a+2} - 0.02\beta\right)t - p - 1 \overset{(1),(6)}{<} |C_1| - p = |C_2|.$$

Thus there is a blue matching M_2 of size $n_2 = (\frac{1}{2a+2} - 0.04\beta)t - p$ since otherwise a red connected matching of size at least $n_1 = (\frac{a}{2a+2} + 0.01\beta)t$ will again lead to a contradiction from Claim 14.

It is clear that $|C_2 \setminus V(M_2)| > p + \sqrt{\epsilon}t$. Recall that $\delta(H_1) = \delta(H_1[C_1]) \geqslant |C_1| - 1 - \sqrt{\epsilon}t$ and all edges between $C_1 \setminus C_2$ and $C_2 \setminus V(M_2)$ are blue. Thus, by Lemma 7, the bipartite graph $H_1^b[C_1 \setminus C_2, C_2 \setminus V(M_2)]$ contains a blue matching of size p, which together with M_2 yield a blue matching of size $n_2 + p = (\frac{1}{2a+2} - 0.04\beta)t$ in H_1^b. Therefore, we can get a blue fan with center v_c and at least $(\frac{1}{2a+2} - 0.05\beta)t$ blades, which contradicts Claim 15.

This completes the proof of Claim 16. □

Claim 17. *H^r is connected.*

Proof. Suppose H^r is not connected. Let $C_1, C_2, \ldots, C_\tau$ be the vertex sets of red components of H^r, such that $|C_1| \geqslant |C_2| \ldots \geqslant |C_\tau|$. Note that $\delta^r(H) \geqslant \frac{a}{2a+2}t$ by Claim 16, thus for $1 \leqslant i \leqslant \tau$,

$$|C_i| \geqslant \frac{a}{2a+2}t + 1, \text{ and } \delta^r(C_i) = \delta(H^r) \geqslant \frac{a}{2a+2}t \tag{11}$$

for each $i \in [\tau]$. Therefore, there are at most five components in H^r, i.e., $\tau \leqslant 5$, by noting $1/2 \leqslant a < 1$. The proof is divided into four cases according to the number of red components.

Case A: $\tau = 5$

Since each vertex $v \in V(H)$ has at most $\sqrt{\epsilon}t$ non-neighbors, it follows from Lemma 7 that the bipartite graph $H^b[C_1, C_2]$ has a blue matching of size at least $|C_2| - \sqrt{\epsilon}t > (\frac{a}{2a+2} - \sqrt{\epsilon})t$. Similarly, $H^b[C_3, C_4]$ has a blue matching of size at least $|C_4| - \sqrt{\epsilon}t > (\frac{a}{2a+2} - \sqrt{\epsilon})t$. In total, we have that $(\cup_{i=1}^4 C_i)^b$ contains a blue matching M_1 with at least $(\frac{a}{a+1} - 2\sqrt{\epsilon})t$ edges.

Note that all edges joining C_5 and C_i are blue for $1 \leqslant i \leqslant 4$, and each vertex in H has at most $\sqrt{\epsilon}t$ non-neighbors, so there is a blue fan with center $u \in C_5$ and blades in M_1 of size at least

$$|E(M_1)| - \sqrt{\epsilon}t \geqslant \left(\frac{a}{a+1} - 3\sqrt{\epsilon}\right) t \overset{(1)}{>} \left(\frac{1}{2a+2} - 0.05\beta\right) t$$

by noting $1/2 \leqslant a < 1$. This contradicts Claim 15.

Case B: $\tau = 4$

Suppose that there exists a vertex $v_d \in C_1$ such that $\deg_{C_1^b}(v_d) \geqslant (\frac{a}{2a+2} - 0.03\beta)t$, i.e., v_d has at least $(\frac{a}{2a+2} - 0.03\beta)t$ blue neighbors in C_1. Let C_1' be the blue neighbors of v_d in C_1. We may assume that $|C_1'| = (\frac{a}{2a+2} - 0.03\beta)t$. Recall that $|C_2| \geqslant \frac{a}{2a+2}t + 1$ and all edges between C_1' and C_2 are blue. Thus for every $S \subseteq C_1'$,

$$|N_{(H-K)^b}(S, C_2)| \geqslant |C_2| - \sqrt{\epsilon}t > \left(\frac{a}{2a+2} - \sqrt{\epsilon}\right) t > |S|.$$

We apply Lemma 7 to the subgraph induced by $C_1' \cup C_2$ in H^b with $X = C_1'$ and $Y = C_2$ to obtain a blue matching of size $|C_1'| = (\frac{a}{2a+2} - 0.03\beta)t$. Since all (but at most ϵt^2) edges between C_3 and C_4 are blue, by a similar argument, we can find a blue matching of size $(\frac{a}{2a+2} - \sqrt{\epsilon})t$ in $H^b[C_3, C_4]$. Recall that every vertex in H has at most $\sqrt{\epsilon}t$ non-neighbors and $1/2 \leqslant a < 1$, so we can find a matching of size at least

$$\left(\frac{a}{2a+2} - 0.03\beta\right) t + \left(\frac{a}{2a+2} - \sqrt{\epsilon}\right) t - \sqrt{\epsilon}t \overset{(1)}{\geqslant} \left(\frac{a}{a+1} - 0.05\beta\right) t \geqslant \left(\frac{1}{2a+2} - 0.05\beta\right) t$$

in $[N(v_d)]^b$, which together with v_d forms a blue fan with blades more than $(\frac{1}{2a+2} - 0.05\beta)t$. This contradicts Claim 15.

In the following, we may assume that $\deg_{C_1^b}(v) < (\frac{a}{2a+2} - 0.03\beta)t$ for every vertex $v \in C_1$. We claim that $|C_1| \leqslant (\frac{a}{a+1} - 0.04\beta)t$. Otherwise, for every vertex $v \in C_1$,

$$\deg_{C_1^r}(v) \geqslant |C_1| - 1 - \sqrt{\epsilon}t - \deg_{C_1^b}(v) \geqslant |C_1| - \left(\frac{a}{2a+2} - 0.02\beta\right) t > \frac{|C_1|}{2}.$$

According to Lemma 10, we obtain that C_1^r hence H^r contains a red cycle with more than $(\frac{a}{a+1} - 0.3\beta)t$ vertices. This contradicts Claim 14. Note that C_1 is the largest red component, so we have

$$\frac{(1-\sqrt{\epsilon})t - 1}{4} \leqslant |C_1| \leqslant \left(\frac{a}{a+1} - 0.04\beta\right) t \tag{12}$$

and $|C_i| \geqslant \frac{a}{2a+2}t+1$ for every $2 \leqslant i \leqslant 4$. Note that $|C_4| \leqslant (|C_1|+|C_2|+|C_3|+|C_4|)/4 \leqslant t/4$, so we have

$$|C_1| + |C_4| \leqslant \left(\frac{a}{a+1} - 0.04\beta\right)t + t/4 = \left(\frac{5a+1}{4a+4} - 0.04\beta\right)t.$$

Thus, $|C_2| + |C_3| \geqslant |V(H)| - |K| - (|C_1| + |C_4|) \geqslant (\frac{3-a}{4a+4} + 0.03\beta)t - 1$. It follows that

$$|C_2| > \left(\frac{3-a}{8a+8} + 0.01\beta\right)t \tag{13}$$

as $|C_2| \geqslant |C_3|$. Since $|C_4| \leqslant t/4$, we have $|C_1| + |C_2| + |C_3| \geqslant 3t/4 - \sqrt{\epsilon}t - 1$. Therefore, we can take two disjoint subsets of C_3, say C_3' and C_3'', such that

$$|C_1| + |C_3'| = |C_2| + |C_3''| = \left(\frac{1}{2a+2} - 0.03\beta\right)t.$$

By (12) and (13), we get that

$$|C_3'| < \left(\frac{1-a}{4a+4} - 0.03\beta + \frac{\sqrt{\epsilon}}{4}\right)t \overset{(1)}{<} \left(\frac{1-a}{4a+4} - 0.02\beta\right)t$$

and

$$|C_3''| < \left(\frac{1}{2a+2} - 0.03\beta\right)t - \left(\frac{3-a}{8a+8} + 0.01\beta\right)t \overset{(1)}{<} \left(\frac{1}{8} - 2\sqrt{\epsilon}\right)t.$$

Therefore, $|C_3'| + \sqrt{\epsilon}t < |C_2|$ and $|C_3''| + \sqrt{\epsilon}t < |C_1|$.

Denote $A = C_1 \cup C_3'$ and $B = C_2 \cup C_3''$. Then the bipartite graphs $H^b[A, C_2]$ and $H^b[B, C_1]$ are almost blue complete bipartite graphs.

We claim that the bipartite graph $H^b[A, B]$ contains a blue matching of cardinality at least $(\frac{1}{2a+2} - 0.04\beta)t$. Indeed, if $S \subseteq C_3' \subseteq A$, then we have

$$|N_{(H-K)^b}(S, B)| \geqslant |C_2| - \sqrt{\epsilon}t > |C_3'| \geqslant |S|$$

by noting that every vertex in H has at most $\sqrt{\epsilon}t$ non-neighbors, and if $S \subseteq A$ and $S \cap C_1 \neq \emptyset$, then

$$|N_{(H-K)^b}(S, B)| \geqslant |B| - \sqrt{\epsilon}t \geqslant |S| - \sqrt{\epsilon}t.$$

Therefore, by Lemma 7, the bipartite graph $H^b[A, B]$ and hence H^b contains a blue matching M_2 of cardinality at least $|A| - \sqrt{\epsilon}t = (\frac{1}{2a+2} - 0.04\beta)t$ by (1). The claim follows.

Note that all (but at most ϵt^2) edges between C_4 and $A \cup B$ are blue, and every vertex in H has at most $\sqrt{\epsilon}t$ non-neighbors. Then there exists a vertex $w \in C_4$ whose blue neighborhood contains a blue matching of cardinality at least $|E(M_2)| - \sqrt{\epsilon}t = (\frac{1}{2a+2} - 0.05\beta)t$ in M_2. Thus we get a blue fan with center w and at least $(\frac{1}{2a+2} - 0.05\beta)t$ blades. This leads to a contradiction by Claim 15.

Case C: $\tau = 3$

Note that all (but at most ϵt^2) edges joining C_3 and C_i are blue for $1 \leqslant i \leqslant 2$. Since each vertex in H has at most $\sqrt{\epsilon}t$ non-neighbors and $|C_1| \geqslant |C_2| \geqslant |C_3| \geqslant \frac{a}{2a+2}t + 1$, we may assume that $|C_2| < (\frac{1}{2a+2} - 0.02\beta)t$. Otherwise, by Lemma 7, the bipartite graph $H^b[C_1, C_2]$ contains a blue matching M_3 of size at least $|C_2| - \sqrt{\epsilon}t \geqslant (\frac{1}{2a+2} - 0.03\beta)t$ by noting (1). Thus we can find a blue fan with center $v \in C_3$ and blades in M_3 of size at least $(\frac{1}{2a+2} - 0.03\beta)t - \sqrt{\epsilon}t \geqslant (\frac{1}{2a+2} - 0.04\beta)t$, which contradicts Claim 15. So $|C_2| < (\frac{1}{2a+2} - 0.02\beta)t$ follows, and we have

$$\left(\frac{a}{a+1} + 0.01\beta\right)t < t - \sqrt{\epsilon}t - 1 - 2|C_2| \leqslant |C_1| \leqslant \frac{t}{a+1} - 2. \tag{14}$$

Since $\delta^r(C_1) \overset{(11)}{\geqslant} \frac{a}{2a+2}t$, which implies that $(C_1)^r$ contains a path of length $2\delta^r(C_1)$ due to Lemma 9. This implies that C_1 contains a red path with more than $(\frac{a}{a+1} - 0.3\beta)t$ vertices in H^r, which leads to a contradiction by Claim 14.

Case D: $\tau = 2$

For this case, $|C_1| \geqslant |C_2| \geqslant \frac{a}{2a+2}t + 1$, so we have

$$\frac{(1-\sqrt{\epsilon})t - 1}{2} \leqslant |C_1| \leqslant \frac{a+2}{2a+2}t - 1.$$

Moreover, it is clear that $\delta^r(C_1) \geqslant \frac{a}{2a+2}t$ by (11), thus $(C_1)^r$ contains a path on at least

$$2\delta^r(C_1) + 1 > \frac{a}{a+1}t$$

vertices by Lemma 9. This agian contradicts Claim 14.

This completes the proof of Claim 17. □

Now, by Lemma 9 and Claim 17, we conclude that H^r contains a path on at least

$$2\delta(H^r) + 1 > \frac{a}{a+1}t$$

vertices, where the last inequality follows from Claim 16. Thus we obtain a red path with more than $\frac{a}{a+1}t$ vertices, which contradicts Claim 14.

The proof of Part (I) is complete. □

Part (II) $a \geqslant 1$

The lower bound $R(C_{2\lfloor an \rfloor}, F_n) \geqslant 4\lfloor an \rfloor - 1$ is clear for every fixed $a \geqslant 1$. Let $N = (4a + \gamma)n$, where $\gamma > 0$ is a sufficiently small real number. Therefore, it suffices to show $R(C_{2\lfloor an \rfloor}, F_n) \leqslant N$. Thus we shall show that any red-blue edge coloring of K_N on vertex set V yields either a red $C_{2\lfloor an \rfloor}$ or a blue F_n. Suppose to the contrary that for fixed $a \geqslant 1$ and large n, there exists a coloring that contains neither a red $C_{2\lfloor an \rfloor}$ nor a blue F_n. We aim to find a contradiction.

Similar as above, we apply the regularity lemma to obtain a partition of V with the corresponding properties, and H, H^r and H^b are defined similarly. By a similar argument as Claim 14 and Claim 15, we get the following claims.

Claim 18. *H^r contains no connected matching of size more than $\left(\frac{1}{4} - 0.15\beta\right) t$.*

Claim 19. *H^b contains no fan with at least $(\frac{1}{4a} - 0.05\beta)t$ blades.*

We will also have the following claims.

Claim 20. *For each vertex $v \in V(H)$, $\deg_{H^r}(v) \geqslant \frac{2a-1}{4a}t$.*

Proof. On the contrary, we assume that H^r contains a vertex u such that $\deg_{H^r}(u) \leqslant \frac{2a-1}{4a}t - 1$. Since $\delta(H) \geqslant (1 - 2\sqrt{\epsilon})t - 1$, we have

$$\deg_{H^b}(u) \geqslant \left(1 - 2\sqrt{\epsilon}\right) t - \frac{2a-1}{4a}t = \left(\frac{2a+1}{4a} - 2\sqrt{\epsilon}\right) t.$$

Denote $H_1 = H[N_{H^b}(u)]$. Note that every vertex in H_1 has at most $\sqrt{\epsilon}t$ non-neighbors. Let C_1 and C_2 be the vertex sets of the largest blue and red components in H_1 respectively. Set $p = |C_1 \setminus C_2|$. By the same argument as Claim 16 step by step, we must have that $C_1 = V(H_1)$ and

$$|C_1| = |V(H_1)| \geqslant \left(\frac{2a+1}{4a} - 2\sqrt{\epsilon}\right) t, \;\; |C_2| > (1/2 + 2\sqrt{\epsilon})t, \;\text{ and }\; p > 20\sqrt{\epsilon}t.$$

Then we have

$$|C_2| = |C_1| - |C_1 \setminus C_2| \geqslant \left(\frac{2a+1}{4a} - 2\sqrt{\epsilon}\right) t - p. \tag{15}$$

We first assume that $p = |C_1 \setminus C_2| \geqslant (\frac{1}{4a} - 0.05\beta)t$. Note that $|C_2| > (1/2 + 2\sqrt{\epsilon})t$ and all (but at most ϵt^2) edges between $C_1 \setminus C_2$ and C_2 are blue. Since each vertex in H has at most $\sqrt{\epsilon}t$ non-neighbors, we conclude that the bipartite graph $H_1^b[C_1 \setminus C_2, C_2]$ contains a blue matching of size at least $(\frac{1}{4a} - 0.05\beta)t$ by Lemma 7. Thus we can get a blue fan with center u and at least $(\frac{1}{4a} - 0.05\beta)t$ blades, which contradicts Claim 19.

Thus we may assume $20\sqrt{\epsilon}t < p < (\frac{1}{4a} - 0.05\beta)t$. We apply Lemma 11 to the subgraph spanned by C_2 in H_1 with parameters $k = \sqrt{\epsilon}t + 1$, $n_1 = (\frac{1}{4} - 0.15\beta)t$, and $n_2 = (\frac{1}{4a} - 0.05\beta)t - p$ to obtain that

$$R(S_k, n_1K_2, n_2K_2) = 2n_1 + n_2 - 1 = \left(\frac{2a+1}{4a} - 0.35\beta\right) t - p - 1 \overset{(15),(1)}{<} |C_2|.$$

Since every vertex in H_1 has at most $\sqrt{\epsilon}t$ non-neighbors, we can get a blue matching M of size at least $n_2 = (\frac{1}{4a} - 0.05\beta)t - p$ otherwise a red connected matching of size at least $(\frac{1}{4} - 0.15\beta)t$ will lead to a contradiction from Claim 18. Note that

$$|C_2 \setminus V(M)| = |C_1| - p - 2n_2 \geqslant \left(\frac{2a-1}{4a} - 2\sqrt{\epsilon} + 0.1\beta\right) t + p \overset{(1)}{>} \left(\frac{2a-1}{4a} + 6\sqrt{\epsilon}\right) t + p,$$

so we have $|C_2 \setminus V(M)| > p + \sqrt{\epsilon}t = |C_1 \setminus C_2| + \sqrt{\epsilon}t$. Since all (but at most ϵt^2) edges between $C_1 \setminus C_2$ and $C_2 \setminus V(M)$ are blue, according to the minimum degree of H_1, we obtain that the bipartite graph $H_1^b[C_1 \setminus C_2, C_2 \setminus V(M)]$ contains a blue matching of size p by Lemma 7, which together with M yield a blue matching of size at least $n_2 + p = (\frac{1}{4a} - 0.05\beta)t$ in H_1^b. Again, we can get a blue fan with center u and at least $(\frac{1}{4a} - 0.05\beta)t$ blades in H^b, which contradicts Claim 19. □

Claim 21. H^r *is connected.*

Proof. On the contrary, we assume that H^r is disconnected. By Claim 20, we have

$$\delta^r(H) \geqslant \frac{2a-1}{4a}t,$$

and all red components of size at least $\frac{2a-1}{4a}t + 1$. Thus there are at most three red components in H^r.

If H^r has three components C_1, C_2 and C_3 with $|C_1| \geqslant |C_2| \geqslant |C_3|$, then we have

$$|C_1| \geqslant |C_2| \geqslant |C_3| \geqslant \frac{2a-1}{4a}t + 1, \text{ and } \delta^r(C_i) \geqslant \frac{2a-1}{4a}t.$$

By a similar argument as Case A of Part (I), H^b contains a blue fan with more than $(\frac{1}{4a} - 0.05\beta)t$ blades for $a \geqslant 1$. This is a contradiction by Claim 19.

Therefore, we may assume that H^r has two components C_1 and C_2 with $|C_1| \geqslant |C_2|$. It is clear that

$$|C_2| \geqslant \frac{2a-1}{4a}t + 1, \text{ and } \frac{1-\sqrt{\epsilon}}{2}t - \frac{1}{2} \leqslant |C_1| \leqslant \frac{2a+1}{4a}t - 1.$$

Suppose that there exists a vertex $u \in C_1$ such that

$$\deg_{C_1^b}(u) \geqslant d := \frac{|C_1|}{2} - \sqrt{\epsilon}t - 1 \geqslant \left(\frac{1}{4} - 2\sqrt{\epsilon}\right)t.$$

Note that all (but at most ϵt^2) edges between C_1 and C_2 are blue and each vertex in H has at most $\sqrt{\epsilon}t$ non-neighbors. Then u together with $(\frac{1}{4} - 2\sqrt{\epsilon})t$ blue neighbors in C_1 and $(\frac{2a-1}{4a} - \sqrt{\epsilon})t$ blue neighbors in C_2 form a blue fan with at least $(\frac{1}{4a} - 0.05\beta)t$ blades by Lemma 7. This leads to a contradiction from Claim 19. Thus we have $\deg_{C_1^b}(v) \leqslant d - 1$ for every vertex $v \in C_1$. It follows that

$$\delta^r(C_1) \geqslant |C_1| - 1 - \sqrt{\epsilon}t - d + 1 > \frac{|C_1|}{2}.$$

By Lemma 10, $(C_1)^r$ is pancyclic, which implies that $(C_1)^r$ contains a red cycle of length

$$|C_1| \geqslant \frac{1-\sqrt{\epsilon}}{2}t - \frac{1}{2} \overset{(1)}{>} \left(\frac{1}{2} - 0.3\beta\right)t,$$

which contradicts Claim 18. □

Now note that $(1 - \sqrt{\epsilon})t \leqslant v(H) \leqslant t$ and $\delta(H^r) \geqslant \frac{2a-1}{4a}t$ from Claim 20, it follows from Claim 21 and Lemma 9 that H^r contains a path of length

$$2\delta(H^r) \geqslant \frac{2a-1}{2a}t > \left(\frac{1}{2} - 0.3\beta\right)t,$$

where the last inequality holds since $a \geqslant 1$. This leads to a contradiction from Claim 18.

The proof of Part (II) is complete. □

4 Concluding remarks

In this paper, we are concerned with the asymptotic behavior of the Ramsey number $R(C_{2\lfloor an\rfloor}, F_n)$ when n is large and $a \geqslant 1/2$ is fixed. For fixed $0 < a < 1/2$ and large n, we also expect to give a uniform asymptotic behavior of $R(C_{2\lfloor an\rfloor}, F_n)$, but we encounter more obstacles for fixed $0 < a < 1/2$. The graph $G = 3K_{2\lfloor an\rfloor-1}$ implies that $R(C_{2\lfloor an\rfloor}, F_n) \geqslant 6\lfloor an\rfloor - 2$ for $2/5 \leqslant a < 1/2$. For $0 < a < 2/5$, the graph $G = K_{\lfloor an\rfloor-1} + \overline{K_{2n}}$ shows that $R(C_{2\lfloor an\rfloor}, F_n) \geqslant \lfloor an\rfloor + 2n$. When we consider the upper bound, our method will encounter more obstacles for $0 < a < 1/2$. For example, the minimum degree of H^r maybe small and so we cannot find sufficiently large connected matches in H^r for $0 < a < 1/2$. Therefore, it would be interesting to determine the values of $R(C_{2\lfloor an\rfloor}, F_n)$ when $0 < a < 1/2$.

Acknowledgements

The authors would like to thank the anonymous referees for their invaluable comments which greatly improve the presentation of this paper. Qizhong Lin is supported by NSFC grant 11671088 and 12226401, and NSFFP(2022J02018).

In particular, the authors would like to thank Dr. Suyun Jiang for pointing out that almost all values of $R(C_{2m+1}, F_n)$ are already known from $R(C_{2m+1}, W_{2n})$; moreover, the authors would like to thank Dr. Suyun Jiang for pointing out that "H^r is 2-connected" in the proof of Theorem 1 can be substituted by "Lemma 9", which makes the proofs of Case D more simpler.

References

[1] B. Andrasfai. Paths, Circuits, and loops of graphs. *Mat. Lapok*, 13:95–107, 1962.

[2] A. A. Ali, and W. Staton. On extremal graphs with no long paths. *Electron. J. Combin.* 3(1):# R20, 1996.

[3] R. Alweiss. Ramsey numbers of odd cycles versus larger even wheels. *Discrete Math.*, 341(4):981–989, 2018.

[4] C. Berge. Sur le couplage maximum d'un graphe. *C. R. Acad. Sci. Paris*, 247: 258–259, 1958.

[5] B. Bollobás. Modern graph theory. Springer, New York, 1998.

[6] J. A. Bondy. Pancyclic graphs I. *J. Combin. Theory Ser. B*, 11:80–84, 1971.

[7] J. A. Bondy, and P. Erdős. Ramsey numbers for cycles in graphs. *J. Combin. Theory Ser. B*, 14:46–54, 1973.

[8] J. A. Bondy. and U.S.R. Murty, Graph theory, Springer, 2008.

[9] M. Brennan. Ramsey numbers of trees and unicyclic graphs versus fans. *Discrete Math.*, 340(5):969–983, 2017.

[10] G. Chen, X. Yu, and Y. Zhao. Improved bounds on the Ramsey number of fans. *European J. Combin.*, 96:103347, 2021.

[11] G. A. Dirac. Some theorems on abstract graphs. *Proc. London Math. Soc.*, 2(3):69–81, 1952.

[12] V. Dvořák, and H. Metrebian. A new upper bound for the Ramsey number of fans. *European J. Combin.*, 110:103680, 2023.

[13] P. Erdős, and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Hungar.*, 10:337–356, 1959.

[14] R. J. Faudree, and R. H. Schelp. All Ramsey numbers for cycles in graphs. *Discrete Math.*, 8:313–329, 1974.

[15] A. Figaj, and T. Łuczak. The Ramsey number for a triple of long even cycles. *J. Combin. Theory Ser. B*, 97(4):584–596, 2007.

[16] A. Gyárfás, and G. Sárközy. Star versus two stripes Ramsey numbers and a conjecture of Schelp. *Combin. Probab. Comput.*, 21:179–186, 2012.

[17] J. Komlós, and M. Simonovits. Szemerédi's regularity lemma and its applications in graph theory, in: Combinatorics, Paul Erdős is Eighty, Vol.2 (Keszthely,1993), Bolyai Soc.Math.Stud.,2, János Bolyai Math. Soc., Budapest, 295–352, 1996.

[18] Y. Li, and C. C. Rousseau. Fan-complete graph Ramsey numbers. *J. Graph Theory*, 23(4):413–420, 1996.

[19] Q. Lin, and Y. Li. On Ramsey numbers of fans. *Discrete Appl. Math.*, 157(1):191–194, 2009.

[20] Q. Lin, Y. Li, and L. Dong. Ramsey goodness and generalized stars. *European J. Combin.*, 31(5):1228–1234, 2010.

[21] M. Liu, and Y. Li. Ramsey numbers of a fixed odd-cycle and generalized books and fans. *Discrete Math.*, 339(10):2481–2489, 2016.

[22] S. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, #DS1 16, 2021.

[23] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 30(2):264–286, 1929.

[24] V. Rosta. On a Ramsey-type problem of J. A. Bondy and P. Erdős. I, II. *J. Combin. Theory Ser. B*, 15:105–120, 1973.

[25] A. Salman, and H. Broerma. Path-fan Ramsey numbers. *Discrete Appl. Math.*, 154(9):1429–1436, 2006.

[26] L. Shi. Ramsey numbers of long cycles versus books or wheels. *European J. Combin.*, 31(3):828–838, 2010.

[27] E. Szemerédi. Regular partitions of graphs, Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, 1976), 399–401, Colloq. Internat. CNRS, 260, CNRS, Paris, 1978.

[28] D. B. West. Introduction to Graph Theory. 2nd Edition, Prentice-Hall, Englewood Cliffs, NJ, 2001.

[29] Y. Zhang, H. Broersma, and Y. Chen. A note on Ramsey numbers for fans. *Bull. Aust. Math. Soc.*, 92(1):19–23, 2015.

[30] Y. Zhang, H. Broersma, and Y. Chen. Ramsey numbers of trees versus fans. *Discrete Math.*, 338(6):994–999, 2015.

[31] Y. Zhang, H. Broersma, and Y. Chen. Three results on cycle-wheel Ramsey numbers. *Graphs Combin.*, 31(6):2467–2479, 2015.

[32] Y. Zhang, Y. Zhang, and Y. Chen. The Ramsey numbers of wheels versus odd cycles. *Discrete Math.*, 323:76–80, 2014.