Combinatorics of Centers of 0-Hecke Algebras in Type A

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Submitted: Mar 18, 2022; Accepted: Jul 18, 2023; Published: Aug 11, 2023

Abstract

A basis of the center of the 0-Hecke algebra of an arbitrary finite Coxeter group was described by He in 2015. This basis corresponds to certain equivalence classes of the Coxeter group. We consider the case of the symmetric group $\mathfrak{S}_n$. Building on work of Geck, Kim and Pfeiffer, we obtain a complete set of representatives of the equivalence classes. This set is naturally parametrized by certain compositions of n called *maximal*. We develop an explicit combinatorial description for the equivalence classes that are parametrized by the maximal compositions whose odd parts form a hook.

Mathematics Subject Classifications: 05E16, 20B30, 20C08

1 Introduction

Let W be a finite Coxeter group. The Iwahori-Hecke algebra $H_W(q)$ of W is a deformation of the group algebra of W with nonzero parameter q. Iwahori-Hecke algebras arise in the representation theory of finite groups of Lie type and Knot theory [9]. Setting $q = 0$ results in the 0-Hecke algebra $H_W(0)$. A first (and thorough) study of $H_W(0)$ was carried out by Norton [23]. Its structure diverges considerably from the generic $q \neq 0$ case [3]. The 0-Hecke algebras appear in the modular representation theory of finite groups of Lie type [4, 23]. The Grothendieck ring of the finitely generated modules of the 0-Hecke algebras of the symmetric groups is isomorphic to the Hopf algebra of quasisymmetric functions [19]. This article is related to the center $Z(H_n(0))$ of the 0-Hecke algebra $H_n(0)$ of the symmetric group $\mathfrak{S}_n$.

Fayers mentions the description of $Z(H_W(0))$ as an open problem in [5]. Brichard gives a formula for the dimension of the center in type A [2]. Yang and Li obtain a lower bound for the dimension of $Z(H_W(0))$ for irreducible W in several types other than A [24]. Moreover, they specify the dimension in type $I_2(n)$ for $n \geqslant 5$. In [13] He describes a basis of $Z(H_W(0))$ in arbitrary type indexed by a set of equivalence classes $W_{\max}/\approx$ of W.

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 https://doi.org/10.37236/11126

Motivated by the connection to the center of $H_n(0)$, we are interested in the quotient set $(\mathfrak{S}_n)_{\max}/{\approx}$. We want to develop a combinatorial description for certain elements of $(\mathfrak{S}_n)_{\max}/{\approx}$. To this end, we introduce a complete set of representatives of $(\mathfrak{S}_n)_{\max}/{\approx}$. Other sets of representatives can be deduced from [12] or [2].

Let S be the set of Coxeter generators of W and ℓ be the length function of W. Define $W_{\min}$ and $W_{\max}$ to be the set of elements of W whose length is minimal and maximal in their conjugacy class, respectively. Geck and Pfeiffer introduce in [8] a relation $\to$ on W. It is the reflexive and transitive closure of the relations $\xrightarrow{s}$ for $s \in S$ where we have $w \xrightarrow{s} w'$ if $w' = sws$ and $\ell(w') \leqslant \ell(w)$. By setting $w \approx w'$ if and only if $w \to w'$ and $w' \to w$ one obtains an equivalence relation $\approx$ on W. The $\approx$-equivalence classes of W are known as *cyclic shift classes*.

In the case where W is a Weyl group, Geck and Pfeiffer show that $W_{\min}$ in conjunction with the relation $\to$ has remarkable properties and how these properties can be used in order to define a character table of $H_W(q)$ with $q \neq 0$ [8]. Since then their results have been generalized to finite [6], affine [15] and finally to all Coxeter groups [21]. The relation $\to$ can also be used to describe the conjugacy classes of Coxeter groups [9, 12, 20] in particular for computational purposes [6, 9]. Geck, Kim and Pfeiffer introduce a twisted version $\to_\delta$ of the relation belonging to twisted conjugacy classes of W in [7]. Building on the results of [8], Geck and Rouquier define a basis of $Z(H_W(q))$ for $q \neq 0$ and W a finite Weyl group, which is naturally indexed by the conjugacy classes of W [10]. A generalization of cyclic shift classes related to parabolic character sheaves was given by He [14]. On $W/{\approx}$ the relation $\to$ gives rise to a partial order. Gill considers the corresponding subposets $\mathcal{O}/{\approx}$ where $\mathcal{O}$ is a conjugacy class of W [11].

For an element Σ of the quotient set $W_{\max}/{\approx}$, He defines the element $T_{\leqslant\Sigma} := \sum_x T_x$ where x runs over the order ideal in Bruhat order of W generated by Σ [13]. Then he shows that the elements $T_{\leqslant\Sigma}$ for $\Sigma \in W_{\max}/{\approx}$ form a basis of $Z(H_W(0))$. We consider He's approach in Section 2.

For each composition $\alpha \vDash n$, Kim defines the *element in stair form* $\sigma_\alpha \in \mathfrak{S}_n$ [17]. Moreover, she calls $\alpha \vDash n$ *maximal* if there is a $k \geqslant 0$ such that the first k parts of α are even and the remaining parts are odd and weakly decreasing. In this case we write $\alpha \vDash_e n$. We show in Theorem 18 that the elements in stair form σ_α for $\alpha \vDash_e n$ form a system of representatives of $(\mathfrak{S}_n)_{\max}/{\approx}$. For $\alpha \vDash_e n$ let $\Sigma_\alpha \in (\mathfrak{S}_n)_{\max}/{\approx}$ be the equivalence class of the element in stair form σ_α. It follows that the elements $T_{\leqslant\Sigma_\alpha}$ for $\alpha \vDash_e n$ form a basis of $Z(H_n(0))$. This leads to an alternative proof of Brichard's dimension formula from [2], which she obtained by considering braid diagrams on the Möbius strip. The system of representatives of the elements in stair form is the topic of Section 3.

Since $T_{\leqslant\Sigma_\alpha}$ depends on the order ideal generated by Σ_α, a description of the elements of Σ_α is desirable. This is the subject of Section 4. We obtain combinatorial characterizations of the equivalence classes $\Sigma_{(n)}$ (Theorem 49) and $\Sigma_{(k,1^{n-k})}$ with k odd (Theorem 69) and a decomposition rule $\Sigma_{(\alpha_1,\ldots,\alpha_l)} = \Sigma_{(\alpha_1)} \odot \Sigma_{(\alpha_2,\ldots,\alpha_l)}$ if α_1 is even, given by an injective operator $\odot$ which we call the *inductive product* (Theorem 84). From the combination of Theorem 49 and Theorem 84 it follows that the only unknown part in the description of Σ_α is $\Sigma_{\alpha'}$ where α' is the maximal composition consisting of the odd parts of α. As we know $\Sigma_{\alpha'}$ in case where α' is an odd hook from Theorem 69, we can describe Σ_α for all $\alpha \vDash_e n$ whose odd parts form a hook (Remark 87). In particular, this includes a characterization of $\Sigma_{(k,1^{n-k})}$ for even k as well (Theorem 93). The case of describing Σ_α in the case where α has only odd parts and is not a hook remains as an open problem (see Remark 94).

Let $n \geqslant 3$. Norton showed that $H_n(0)$ has exactly three blocks: one nontrivial block B and two blocks of dimension one [23]. The author used the results of this paper in order to show that for each maximal composition $\alpha \neq (1^n)$ whose odd parts form a hook, the basis element $T_{\leqslant \Sigma_\alpha}$ annihilates all the simple modules belonging to block B [18].

The structure is as follows. In Section 2 we present the background material and review He's basis of the center of $H_W(0)$ and the connection to the quotient set $W_{\max}/\approx$. In Section 3 we obtain the system of representatives of $(\mathfrak{S}_n)_{\max}/\approx$ given by the elements in stair form. There we encounter several intermediate results which are also applied in Section 4, where we consider the equivalence classes $\Sigma_\alpha \in (\mathfrak{S}_n)_{\max}/\approx$.

2 Preliminaries

Let $\mathbb{K}$ be an arbitrary field. We set $\mathbb{N} := \{1, 2, \ldots\}$ and always assume that $n \in \mathbb{N}$. For $a, b \in \mathbb{Z}$ we define the *discrete interval* $[a, b] := \{c \in \mathbb{Z} \mid a \leqslant c \leqslant b\}$ and use the shorthand $[a] := [1, a]$.

2.1 Coxeter groups

We consider basic concepts from the theory of finite Coxeter groups. Our motivation is the application to the symmetric groups. Refer to [1, 16] for details.

Let S be a set. A *Coxeter matrix* is a map $m \colon S \times S \to \mathbb{N} \cup \{\infty\}$ such that (1) $m(s, s') = 1$ if and only if $s' = s$ and (2) $m(s, s') = m(s', s)$ for all $s, s' \in S$. The corresponding *Coxeter graph* is the undirected graph with vertex set S containing the edge $\{s, s'\}$ if and only if $m(s, s') \geqslant 3$. If $m(s, s') \geqslant 4$ then the edge $\{s, s'\}$ is labeled with $m(s, s')$. A group W is called a *Coxeter group* with *Coxeter generators* S if W is generated by S subject to the relations

(1) $s^2 = 1$ for all $s \in S$,

(2) $(ss's\cdots)_{m(s,s')} = (s'ss'\cdots)_{m(s,s')}$ for all $s, s' \in S$ with $s \neq s'$ and $m(s, s') < \infty$

where $(ss's\cdots)_p$ denotes the alternating product of s and s' with p factors.

Let W be a Coxeter group with Coxeter generators S. We always assume that W is finite. For $I \subseteq S$ the *parabolic subgroup* W_I is the subgroup of W generated by I. It is a Coxeter group with Coxeter generators I.

Each $w \in W$ can be written as a product $w = s_1 \cdots s_k$ with $s_i \in S$. Then $s_1 \cdots s_k$ is called a *word* for w. If k is minimal among all words for w, $s_1 \cdots s_k$ is a *reduced word* for w and $\ell(w) := k$ is the *length* of w. The *left* and the *right descent set* of $w \in W$ are given by

$$\begin{aligned} D_L(w) &:= \{s \in S \mid \ell(sw) < \ell(w)\}\,, \\ D_R(w) &:= \{s \in S \mid \ell(ws) < \ell(w)\}\,. \end{aligned} \tag{2.1}$$

The *Bruhat order* $\leqslant$ is the partial order on W given by $u \leqslant w$ if and only if there exists a reduced word for w which contains a reduced word of u as a subsequence. The Bruhat poset is graded by the length function ℓ. Since W is finite, there exists a greatest element $w_0 \in W$ in Bruhat order. This element is called the *longest element* of W. It has the following useful properties.

Lemma 1 ([1, Proposition 2.3.2 and Corollary 2.3.3]). *Let w_0 be the longest element of W. Then we have*

(1) $w_0^2 = 1$,

(2) $\ell(ww_0) = \ell(w_0w) = \ell(w_0) - \ell(w)$ *for all* $w \in W$,

(3) $\ell(w_0ww_0) = \ell(w)$ *for all* $w \in W$.

Lemma 2 ([1, Propositions 2.3.4 and 3.1.5]). *For the Bruhat order on W, we have that*

(1) $w \mapsto ww_0$ *and* $w \mapsto w_0w$ *are antiautomorphisms,*

(2) $w \mapsto w_0ww_0$ *is an automorphism.*

We now define the 0-Hecke algebra of W. Refer to Chapter 1 of [22] for background information on $H_W(0)$.

Definition 3. The *0-Hecke algebra* $H_W(0)$ of W is the unital associative $\mathbb{K}$-algebra generated by the elements T_s for $s \in S$ subject to the relations

(1) $T_s^2 = -T_s$,

(2) $(T_sT_{s'}T_s\cdots)_{m(s,s')} = (T_{s'}T_sT_{s'}\cdots)_{m(s,s')}$ for all $s, s' \in S$ with $s \neq s'$.

For $w \in W$ define $T_w := T_{s_1}\cdots T_{s_k}$ where $s_1\cdots s_k$ is a reduced word for w. The word property ensures that this is well defined [1, Theorem 3.3.1]. We have that $\{T_w \mid w \in W\}$ is a $\mathbb{K}$-basis of $H_W(0)$ with multiplication given by

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ -T_w & \text{if } \ell(sw) < \ell(w) \end{cases}$$

for $w \in W$ and $s \in S$ [22, Theorem 1.13].

2.2 The symmetric group

For a finite set X we define $\mathfrak{S}(X)$ to be the group formed by all bijections from X to itself. The *symmetric group* $\mathfrak{S}_n$ is the group $\mathfrak{S}([n])$. Its elements are called *permutations*.

Let S be the set of adjacent transpositions $s_i := (i, i+1) \in \mathfrak{S}_n$ for $i = 1, \dots, n-1$. The elements of S generate $\mathfrak{S}_n$ as a Coxeter group subject to the relations

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i s_j &= s_j s_i \text{ if } |i-j| \geqslant 2 \end{aligned}$$

[1, Proposition 1.5.4]. For $n \geqslant 2$, $\mathfrak{S}_n$ is an irreducible Coxeter group of type A_{n-1}. In the case of the symmetric group $\mathfrak{S}_n$, we always assume that S is the set of adjacent transpositions. For $\sigma \in \mathfrak{S}_n$ we have

$$\begin{aligned} D_L(\sigma) &= \left\{ s_i \in S \mid \sigma^{-1}(i) > \sigma^{-1}(i+1) \right\}, \\ D_R(\sigma) &= \left\{ s_i \in S \mid \sigma(i) > \sigma(i+1) \right\} \end{aligned} \tag{2.2}$$

[1, Proposition 1.5.3]. The longest element w_0 of $\mathfrak{S}_n$ is given by $w_0(i) = n - i + 1$ for $i \in [n]$. We denote the 0-*Hecke algebra of the symmetric group* $\mathfrak{S}_n$ with $H_n(0) := H_{\mathfrak{S}_n}(0)$ and use the shorthand $T_i := T_{s_i}$ for $i \in [n-1]$.

2.3 Combinatorics

A *composition* $\alpha = (\alpha_1, \dots, \alpha_l)$ is a finite sequence of positive integers. The *length* and the *size* of α are given by $\ell(\alpha) := l$ and $|\alpha| := \sum_{i=1}^{l} \alpha_i$, respectively. The α_i are called *parts* of α. If α has size n, α is called *composition of* n and we write $\alpha \vDash n$. A *weak* composition of n is a finite sequence of nonnegative integers that sum up to n. We write $\alpha \vDash_0 n$ if α is a weak composition of n. The *empty composition* $\emptyset$ is the unique composition of length and size 0. A *partition* is a composition whose parts are weakly decreasing. We write $\lambda \vdash n$ if λ is a partition of size n. For example, $(1, 4, 3) \vDash 8$ and $(4, 3, 1) \vdash 8$. Partitions of n of the form $(k, 1^{n-k})$ with $k \in [n]$ are called *hooks*.

A permutation $\sigma \in \mathfrak{S}_n$ can be represented in cycle notation where cycles of length one may be omitted. The *cycle type* (or simply *type*) of a permutation $\sigma \in \mathfrak{S}_n$ is the partition of n whose parts are the sizes of all the cycles of σ. If σ has cycle type $(k, 1^{n-k})$ for a $k \in [n]$ we also call it a k-cycle. A k-cycle is *trivial* if $k = 1$. Writing σ in cycle notation is the same as expanding σ into a product $\sigma_1 \cdots \sigma_r$ of disjoint cycles where the trivial cycles may be omitted in the expansion. On the other hand, in order to describe the cycle notation of a permutation combinatorially, it can be useful to include them. In Section 4 we will characterize the elements of certain equivalence classes of $\mathfrak{S}_n$ by considering them in cycle notation.

2.4 Centers of 0-Hecke algebras

In this section we introduce He's basis of the center of $H_n(0)$. Following his approach in [13], we take a more general point of view and consider the center of $H_W(0)$ for a finite Coxeter group W twisted by an automorphism δ. This enables us to prove a useful invariance property in Corollary 13. By setting $W = \mathfrak{S}_n$ and $\delta = \mathrm{id}$, we recover the desired results on the center of $H_n(0)$.

Let W be a finite Coxeter group with Coxeter generators S and δ be a automorphism of W with $\delta(S) = S$. For instance, we can choose $\delta = \mathrm{id}$. Another example is given by the conjugation with w_0. For $u, w \in W$ we use the shorthand $w^u = uwu^{-1}$. Define $\nu\colon W \to W$, $w \mapsto w^{w_0}$. Then ν is a group automorphism and from Lemma 1 it follows that $\ell(\nu(w)) = \ell(w)$ for all $w \in W$ so that $\nu(S) = S$. In general, each graph automorphism of the Coxeter graph of W gives rise to a W-automorphism that fixes S. By the next lemma, the converse direction is also true. The result is not new. For instance, it was already used implicitly in [7, Section 2.10].

Lemma 4. *Let δ be a group automorphism of W with $\delta(S) = S$. Then*

(*1*) *δ is an automorphism of the Coxeter graph of W,*

(*2*) *δ is an automorphism of the Bruhat order of W.*

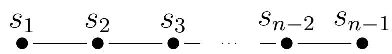
Proof. For $w \in W$ denote the order of w with $\mathrm{ord}(w)$. Let m be the Coxeter matrix and Γ be the Coxeter graph of W. Then $m(s, s') = \mathrm{ord}(ss')$ for all $s, s' \in S$. Since δ is a group automorphism, we have $\mathrm{ord}(\delta(w)) = \mathrm{ord}(w)$ for all $w \in W$. Hence for all $s, s' \in S$

$$m(\delta(s), \delta(s')) = \mathrm{ord}(\delta(s)\delta(s')) = \mathrm{ord}(ss') = m(s, s').$$

Thus, δ is an automorphism of Γ.

By a comment following [1, Proposition 2.3.4], we have that multiplicatively extending a graph automorphism of Γ yields a Bruhat order automorphism of W. Hence, δ is such an automorphism. $\square$

Example 5. The Coxeter graph of $\mathfrak{S}_n$ is shown below.



This graph has at most two automorphisms: the identity and the mapping given by $s_i \mapsto s_{n-i}$. For $n \geqslant 3$ these maps are distinct. Let w_0 be the longest element of $\mathfrak{S}_n$. Then $w_0(j) = n - j + 1$ for all $j \in [n]$ and therefore $s_i^{w_0} = (n - i + 1, n - i) = s_{n-i}$. Hence the second map is ν. Thus, id and ν are the only possibilities for δ if $W = \mathfrak{S}_n$.

Two elements $w, w' \in W$ are called *δ-conjugate* if there is an $x \in W$ such that $w' = xw\delta(x)^{-1}$. The set of δ-conjugacy classes of W is denoted by $\mathrm{cl}(W)_\delta$. For $\mathcal{O} \in \mathrm{cl}(W)_\delta$ the set of elements of minimal length in $\mathcal{O}$ and the set of elements of maximal length in $\mathcal{O}$ is denoted by $\mathcal{O}_{\min}$ and $\mathcal{O}_{\max}$, respectively.

We want to decompose these sets using an equivalence relation. Let $w, w' \in W$. For $s \in S$ we write $w \xrightarrow{s}_\delta w'$ if $w' = sw\delta(s)$ and $\ell(w') \leqslant \ell(w)$. We write $w \to_\delta w'$ if there is a sequence $w = w_1, w_2, \ldots, w_{k+1} = w'$ of elements of W such that for each $i \in [k]$ there exists an $s \in S$ such that $w_i \xrightarrow{s}_\delta w_{i+1}$. If $w \to_\delta w'$ and $w' \to_\delta w$ we write $w \approx_\delta w'$. Then $\approx_\delta$ is an equivalence relation. The equivalence classes of W under $\approx_\delta$ are known as *δ-cyclic shift classes*. If $w \approx_\delta w'$ then $\ell(w) = \ell(w')$. Thus, for all $\mathcal{O} \in \mathrm{cl}(W)_\delta$, $\mathcal{O}_{\min}$ and $\mathcal{O}_{\max}$ decompose into equivalence classes of $\approx_\delta$. Define $W_{\delta,\min} := \bigcup_{\mathcal{O} \in \mathrm{cl}(W)_\delta} \mathcal{O}_{\min}$ and $W_{\delta,\min}/{\approx_\delta}$ to be the quotient set of $W_{\delta,\min}$ by $\approx_\delta$. Analogously, define the sets $W_{\delta,\max} := \bigcup_{\mathcal{O} \in \mathrm{cl}(W)_\delta} \mathcal{O}_{\max}$ and $W_{\delta,\max}/{\approx_\delta}$. In the case $\delta = \mathrm{id}$ we may omit the index δ.

Example 6. We have $(1,2,3) \overset{(1,2)}{\to} (1,3,2) \overset{(1,2)}{\to} (1,2,3)$ so that $(1,2,3) \approx (1,3,2)$. Moreover, $\ell((1,2)) = \ell((2,3)) = 1$ and $\ell((1,3)) = 3$. Hence,

$$\{1\}\,,\ \{(1,2,3),(1,3,2)\} \text{ and } \{(1,3)\}$$

are the elements of $(\mathfrak{S}_3)_{\max}/{\approx}$.

Since δ is a Bruhat order automorphism of W by Lemma 4, we obtain an algebra automorphism of $H_W(0)$ by setting $T_s \mapsto T_{\delta(s)}$ for all $s \in S$ and extending multiplicatively and linearly. This algebra automorphism is also denoted by δ. The *δ-center* of $H_W(0)$ is given by

$$Z(H_W(0))_\delta := \{z \in H_W(0) \mid az = z\delta(a) \text{ for all } a \in H_W(0)\}\,.$$

We now come to He's basis of $Z(H_W(0))_\delta$. For $\Sigma \in W_{\delta,\max}/{\approx_\delta}$ set

$$W_{\leqslant\Sigma} := \{x \in W \mid x \leqslant w \text{ for some } w \in \Sigma\}$$

and

$$T_{\leqslant\Sigma} := \sum_{x \in W_{\leqslant\Sigma}} T_x.$$

Theorem 7 ([13, Theorem 5.4])**.** *The elements $T_{\leqslant\Sigma}$ for $\Sigma \in W_{\delta,\max}/{\approx_\delta}$ form a $\mathbb{K}$-basis of $Z(H_W(0))_\delta$.*

We are concerned with the following special case.

Corollary 8. *The elements $T_{\leqslant\Sigma}$ for $\Sigma \in (\mathfrak{S}_n)_{\max}/{\approx}$ form a basis of $Z(H_n(0))$.*

Example 9. Note that in $\mathfrak{S}_3$

$$(1,2,3) = s_1s_2,\ (1,3,2) = s_2s_1 \text{ and } (1,3) = w_0.$$

Thus, Example 6 and Corollary 8 yield that the elements

$$1,\ 1 + T_1 + T_2 + T_1T_2 + T_2T_1 \text{ and } \sum_{w \in \mathfrak{S}_3} T_w$$

form a basis of $Z(H_3(0))$.

The basis of $Z(H_n(0))$ from Corollary 8 depends on $(\mathfrak{S}_n)_{\max}/{\approx}$. This is the motivation for considering $(\mathfrak{S}_n)_{\max}/{\approx}$ in this paper. The remainder of this section is devoted to show that $\nu(\Sigma) = \Sigma$ for all $\Sigma \in (\mathfrak{S}_n)_{\max}/{\approx}$ in Corollary 13. This result will be useful in Section 4. In order to obtain it, we further study the quotient sets of $W_{\delta,\min}$ and $W_{\delta,\max}$ under $\approx_\delta$.

Define $\delta' := \nu \circ \delta$. Then δ' is a W-automorphism with $\delta'(S) = S$ as well. The Bruhat order antiautomorphism $w \mapsto ww_0$ from Lemma 2 relates $W_{\delta,\min}/{\approx_\delta}$ to $W_{\delta',\max}/{\approx_{\delta'}}$.

Lemma 10 ([13, Section 2.2])**.** *We have a bijection*

$$W_{\delta,\min}/{\approx_\delta} \to W_{\delta',\max}/{\approx_{\delta'}}, \quad \Sigma \mapsto \Sigma w_0.$$

We now come to parametrizations of $W_{\delta,\min}/{\approx_\delta}$ and $W_{\delta,\max}/{\approx_\delta}$ which are due to He. A δ-conjugacy class $\mathcal{O} \in \mathrm{cl}(W)_\delta$ is called *elliptic* (or *cuspidal*) if $\mathcal{O} \cap W_I = \emptyset$ for all $I \subsetneq S$ such that $\delta(I) = I$. Define

$$\Gamma_\delta := \{(I, C) \mid I \subseteq S,\ I = \delta(I) \text{ and } C \in \mathrm{cl}(W_I)_\delta \text{ is elliptic}\}.$$

Proposition 11 ([13, Corollaries 4.2 and 4.3])**.** *The maps*

$$\begin{aligned} \Gamma_\delta &\to W_{\delta,\min}/{\approx_\delta} \\ (I,C) &\mapsto C_{\min} \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \Gamma_{\delta'} &\to W_{\delta,\max}/{\approx_\delta} \\ (I,C) &\mapsto C_{\min} w_0 \end{aligned}$$

are bijections.

A complete set of representatives of the elliptic ν-conjugacy classes of $\mathfrak{S}_n$ is given by [12, Lemma 7.14]. This result can be combined with [12, §7.12] in order to obtain Γ_ν and, by Proposition 11, representatives for $(\mathfrak{S}_n)_{\max}/{\approx}$. Our aim is to characterize the elements of Σ for certain $\Sigma \in (\mathfrak{S}_n)_{\max}/{\approx}$ combinatorially. To this end, we introduce another set of representatives in the next section.

From Proposition 11 we deduce the following invariance properties.

Lemma 12.

(*1*) *We have $\delta(\Sigma) = \Sigma$ for each $\Sigma \in W_{\delta,\min}/{\approx_\delta}$.*

(*2*) *We have $\delta'(\Sigma) = \Sigma$ for each $\Sigma \in W_{\delta,\max}/{\approx_\delta}$.*

Proof. (1) Let $\Sigma \in W_{\delta,\min}/{\approx_\delta}$ and $w \in \Sigma$. By Proposition 11 there exists a tuple $(I, C) \in \Gamma_\delta$ such that $C \in \mathrm{cl}(W_I)_\delta$ and $\Sigma = C_{\min}$. Hence $w \in W_I$ and therefore $w^{-1} \in W_I$. It follows that

$$\delta(w) = w^{-1} w \delta(w^{-1})^{-1} \in C.$$

Moreover, $\ell(\delta(w)) = \ell(w)$ because δ is a Bruhat order automorphism by Lemma 4. Therefore, $\delta(w) \in C_{\min} = \Sigma$. Hence, $\delta(\Sigma) = \Sigma$.

(2) Let $\Sigma \in W_{\delta,\max}/{\approx_\delta}$. From Lemma 10 it follows that $\Sigma w_0 \in W_{\delta',\min}/{\approx_{\delta'}}$. Hence,

$$\delta'(\Sigma) w_0 = \delta'(\Sigma w_0) = \Sigma w_0,$$

where we use that δ' is a group homomorphism with $\delta'(w_0) = w_0$ for the first and Part (1) for the second equality. Now multiply from the right with w_0. $\square$

Setting $W = \mathfrak{S}_n$ and $\delta = \mathrm{id}$ in the second part of Lemma 12 yields the desired result on $(\mathfrak{S}_n)_{\max}/{\approx}$ and ν.

Corollary 13. *We have $\nu(\Sigma) = \Sigma$ for each $\Sigma \in (\mathfrak{S}_n)_{\max}/{\approx}$.*

3 Elements in stair form

The goal of this section is to obtain a new set of representatives of $(\mathfrak{S}_n)_{\max}/{\approx}$ in Theorem 18. As announced in the introduction, the set consists of the *elements in stair form* indexed by the *maximal compositions* of n. This is the foundation of the combinatorial description of the elements of Σ for the equivalence classes $\Sigma \in (\mathfrak{S}_n)_{\max}/{\approx}$ considered in Section 4.

The section is structured as follows. We first define *elements in stair form* and *maximal compositions*. Then Theorem 18 is stated. We proceed with consequences of Theorem 18 before we come its proof. Along the way, we encounter several intermediary results which are also important for Section 4. There Lemmas 23 and 25 are used directly. Moreover, from Lemmas 26 and 27 we infer Proposition 28 at the end of this section, and this result is then applied in Section 4.

We now begin with the definition of the *elements in stair form*.

Definition 14 (Kim, [17])**.** Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash n$. Define the list $(x_1, x_2, \ldots, x_n)$ by setting $x_{2i-1} := i$ and $x_{2i} := n - i + 1$. The *element in stair form* $\sigma_\alpha \in \mathfrak{S}_n$ *corresponding to* α is given by

$$\sigma_\alpha := \sigma_{\alpha_1}\sigma_{\alpha_2}\cdots\sigma_{\alpha_l}$$

where σ_{α_i} is the α_i-cycle

$$\sigma_{\alpha_i} := \left(x_{\alpha_1+\cdots+\alpha_{i-1}+1}, x_{\alpha_1+\cdots+\alpha_{i-1}+2}, \ldots, x_{\alpha_1+\cdots+\alpha_{i-1}+\alpha_i}\right).$$

For instance, $\sigma_{(4,2)} = (1,6,2,5)(3,4)$. We obtain σ_α for $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash n$ as follows. Let $d_i := \sum_{j=1}^{i} \alpha_i$ for $i = 1, \ldots, l$ and consider the list $(x_1, x_2, \ldots, x_n)$ given as above. Then split the list between x_{d_i} and x_{d_i+1} for $i = 1, \ldots, l-1$. The resulting sublists are the cycles of σ_α. In particular, if α and β are compositions with $\sigma_\alpha = \sigma_\beta$ then $\alpha = \beta$. We continue with the maximal compositions.

Definition 15 (Kim, [17])**.** Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash n$. We call α *maximal* and write $\alpha \vDash_e n$ if there exists a k with $0 \leqslant k \leqslant l$ such that α_i is even for $i \leqslant k$, α_i is odd for $i > k$ and $\alpha_{k+1} \geqslant \alpha_{k+2} \geqslant \ldots \geqslant \alpha_l$.

For example, among the two compositions $(4,6,2,3,1,1)$ and $(6,4,3,2,1,1)$ of 17 only the first one is maximal. The term *maximal* is justified by the following result, which goes back to Kim [17]. A proof is given in [7, Theorem 3.3].

Lemma 16. *Let $\alpha \vDash n$. Then $\sigma_\alpha \in (\mathfrak{S}_n)_{\max}$ if and only if α is a maximal composition.*

Thanks to Lemma 16 the following is well defined.

Definition 17. For $\alpha \vDash_e n$ define $\Sigma_\alpha \in (\mathfrak{S}_n)_{\max}/{\approx}$ to be the $\approx$-equivalence class of the element in stair form σ_α.

We now state the main result of the section.

Theorem 18. *A complete system of representatives of* $(\mathfrak{S}_n)_{\max}/\approx$ *is given by*

$$\{\sigma_\alpha \mid \alpha \vDash_e n\}.$$

That is, we have a bijection

$$\{\alpha \vDash_e n\} \to (\mathfrak{S}_n)_{\max}/\approx, \quad \alpha \mapsto \Sigma_\alpha.$$

Before we begin proving Theorem 18, we discuss some immediate consequences. One of them is that $(\mathfrak{S}_n)_{\max}/\approx$ is parametrized by the maximal compositions of n. This can also be shown by using the representatives of $(\mathfrak{S}_n)_{\max}/\approx$ from [12] mentioned after Proposition 11.

Example 19. For $n = 3$ we have

$\alpha \vDash_e 3$	(3)	$(2,1)$	(1^3)
σ_α	$(1,3,2)$	$(1,3)$	1

which by Theorem 18 is a complete set of representatives of $(\mathfrak{S}_3)_{\max}/\approx$.

By combining Corollary 8 and Theorem 18, we obtain the following.

Corollary 20. *The elements* $T_{\leqslant \Sigma_\alpha}$ *for* $\alpha \vDash_e n$ *form a basis of* $Z(H_n(0))$.

This leads to an alternative proof of Brichard's dimension formula.

Corollary 21 ([2, Section 5.1])**.** *The dimension of* $Z(H_n(0))$ *equals*

$$\sum_{\lambda \vdash n} \frac{n_\lambda!}{m_\lambda}$$

where for $\lambda = (1^{k_1}, 2^{k_2}, \dots) \vdash n$, $m_\lambda := \prod_{i \geqslant 1} k_{2i}!$ *and* $n_\lambda := \sum_{i \geqslant 1} k_{2i}$ *is the number of even parts of* λ.

Proof. Each summand is the number of maximal compositions that have the same multiset of parts as $\lambda \vdash n$. Hence, the sum is the number of maximal compositions of n. By Corollary 20 this is the dimension of $Z(H_n(0))$. $\square$

Remark 22. From Theorem 18 and Lemma 10 it follows that the elements $\sigma_\alpha w_0$ for $\alpha \vDash_e n$ form a system of representatives of $(\mathfrak{S}_n)_{\nu,\min}/\approx_\nu$. A basis of the cocenter of $H_n(0)$ twisted by ν is given by such a system [13, Theorem 6.5].

We now come to the proof of Theorem 18. Because of Lemma 16, it remains to show the following.

(a) For each $\Sigma \in (\mathfrak{S}_n)_{\max}/\approx$ there is an $\alpha \vDash_e n$ such that $\sigma_\alpha \in \Sigma$.

(b) If $\alpha, \beta \vDash_e n$ and $\sigma_\alpha \approx \sigma_\beta$ then $\alpha = \beta$.

There are two ways of showing

$$|(\mathfrak{S}_n)_{\max}/{\approx}| = |\{\alpha \vDash_e n\}|$$

which are independent of Theorem 18: (1) Combining Corollary 8 with Brichard's dimension formula (Corollary 21) and (2) using the representatives of [12]. Using one of these, it would suffice to prove (a). However, we choose to include a prove of (b) because it follows a direct combinatorial approach and involves the intermediate results Lemmas 25 to 27 that we need for Section 4.

In order to prove Statement (a) we use the following result. It is also applied in the argumentation leading to Theorem 49.

Lemma 23. *Let W be a finite Coxeter group and $w, w' \in W$ be such that $w \to w'$ and $\ell(w) = \ell(w')$. Then $w \approx w'$.*

Proof. Let S be the set of Coxeter generators of W. It suffices to consider the case where $w \xrightarrow{s} w'$ for some $s \in S$ because by definition $\to$ is the transitive closure of all the relations $\xrightarrow{t}$ with $t \in S$. Then $w' = sws$. Thus, $w = sw's$ and since $\ell(w) = \ell(w')$, we have $w' \xrightarrow{s} w$. Hence $w \approx w'$. $\square$

Proof of Statement (a). Let $\Sigma \in (\mathfrak{S}_n)_{\max}/{\approx}$ and $\sigma \in \Sigma$. In [17, Section 3] it is shown that there is a $\beta \vDash n$ such that $\sigma_\beta \to \sigma$. Moreover, Statement (a$''$) of Section 3.1 in [7] provides the existence of an $\alpha \vDash_e n$ such that $\sigma_\alpha \to \sigma_\beta$. Therefore, $\sigma_\alpha \to \sigma$. Hence, σ_α and σ are conjugate and $\ell(\sigma_\alpha) \geqslant \ell(\sigma)$. But the length of σ is maximal in its conjugacy class. Hence, $\ell(\sigma_\alpha) = \ell(\sigma)$ and Lemma 23 yields $\sigma_\alpha \approx \sigma$. $\square$

We begin working towards Statement (b). It will follow from Lemmas 24 to 27. As before, we will trace the relation $\approx$ back to the elementary steps $\xrightarrow{s_i}$ with $i \in [n-1]$. Consider $\sigma \in \mathfrak{S}_n$ and $\tau = s_i \sigma s_i$. Then we have $\tau \xrightarrow{s_i} \sigma$ or $\sigma \xrightarrow{s_i} \tau$ depending on $\ell(s_i\sigma s_i) - \ell(\sigma)$. Moreover $\sigma \approx \tau$ if and only if the difference vanishes. Thus our first goal is to determine $\ell(s_i\sigma s_i) - \ell(\sigma)$ depending on σ and s_i in Lemma 25.

Lemma 24. *Let $\sigma \in \mathfrak{S}_n$ and $i, j \in [n-1]$. Then $\{\sigma(i), \sigma(i+1)\} \neq \{j, j+1\}$ if and only if $(s_j \in D_L(\sigma) \iff s_j \in D_L(\sigma s_i))$.*

Proof. We consider all permutations in one-line notation. From Equation (2.2) it follows for each $\sigma \in \mathfrak{S}_n$ that $j \in D_L(\sigma)$ if and only if $j+1$ is left of j in σ.

Now fix $\sigma \in \mathfrak{S}_n$. Observe that we obtain σs_i from σ by swapping $\sigma(i)$ and $\sigma(i+1)$. Since these are two consecutive letters in the the one-line notation of σ, the relative positioning of j and $j+1$ is affected by this interchange if and only if $\{\sigma(i), \sigma(i+1)\} = \{j, j+1\}$. Now use the above remark on left descents to deduce the claim. $\square$

We now come to the result on $\ell(s_i\sigma s_i) - \ell(\sigma)$. Determining this difference will also be of interest in Section 4. It comes up when we trace $\approx$ down to elementary steps (as explained above) or when we interchange i and $i+1$ in the cycle notation of σ. Lemma 25 is invoked several times in order to obtain Theorem 49 and Theorem 69.

Lemma 25. *Let $\sigma \in \mathfrak{S}_n$ and $i \in [n-1]$.*

(*1*) *If $\{\sigma(i), \sigma(i+1)\} \neq \{i, i+1\}$ then*

$$\ell(s_i \sigma s_i) = \begin{cases} \ell(\sigma) - 2 & \text{if } \sigma(i) > \sigma(i+1) \text{ and } \sigma^{-1}(i) > \sigma^{-1}(i+1), \\ \ell(\sigma) + 2 & \text{if } \sigma(i) < \sigma(i+1) \text{ and } \sigma^{-1}(i) < \sigma^{-1}(i+1), \\ \ell(\sigma) & \text{else.} \end{cases}$$

(*2*) *If $\{\sigma(i), \sigma(i+1)\} = \{i, i+1\}$ then i and $i+1$ either are fixed points of σ or form a 2-cycle in σ. In particular, $s_i \sigma s_i = \sigma$.*

Proof. Part (2) should be clear. For Part (1) assume that $\{\sigma(i), \sigma(i+1)\} \neq \{i, i+1\}$. We have that

$$\ell(s_i \sigma s_i) - \ell(\sigma) = \ell(s_i \sigma s_i) - \ell(\sigma s_i) + \ell(\sigma s_i) - \ell(\sigma).$$

Equation Equation (2.1) yields that each of the two differences on the right hand side is -1 or 1 depending on the truth value of the statements $s_i \in D_L(\sigma s_i)$ and $s_i \in D_R(\sigma)$, respectively. From Lemma 24 we have that $s_i \in D_L(\sigma s_i)$ if and only if $s_i \in D_L(\sigma)$. That is, the first difference depends on whether $s_i \in D_L(\sigma)$ or not. Thus, Equation (2.2) implies the claim. □

We now show for each $\alpha \vDash_e n$ that all elements of Σ_α have the same orbits of even length on $[n]$.

Lemma 26. *Let $\alpha \vDash_e n$ and $\sigma \in \mathfrak{S}_n$ such that $\sigma_\alpha \approx \sigma$. Then we have the following.*

(*1*) *The orbits of even length of σ and σ_α on $[n]$ coincide.*

(*2*) *Let $\mathcal{O}$ be an σ-orbit on $[n]$ of even length. Then the orbits of σ^2 and σ_α^2 on $\mathcal{O}$ coincide.*

Proof. Since $\sigma_\alpha \approx \sigma$, we have $\sigma_\alpha \to \sigma$ and $\ell(\sigma_\alpha) = \ell(\sigma)$. Using induction on the minimal number of elementary steps $w \xrightarrow{s} w'$ (with some $w, w' \in \mathfrak{S}_n$ and $s \in S$) necessary to relate σ_α to σ, we may assume that there are $\tau \in \mathfrak{S}_n$ and $s_i \in S$ such that $\sigma_\alpha \to \tau \xrightarrow{s_i} \sigma$ and τ satisfies (1) and (2) (σ_α certainly does). Then $\ell(\sigma_\alpha) \geqslant \ell(\tau) \geqslant \ell(\sigma)$ so that in fact $\ell(\sigma_\alpha) = \ell(\tau) = \ell(\sigma)$ and $\sigma_\alpha \approx \tau \approx \sigma$ by Lemma 23.

It remains to show that $\xrightarrow{s_i}$ transfers Properties (1) and (2) from τ to σ. Because $\sigma = s_i \tau s_i$, we obtain σ from τ by interchanging i and $i+1$ in the cycle notation of τ. If i and $i+1$ both appear in orbits of uneven length of τ then (1) and (2) are not affected by this interchange. Thus, we are left with two cases.

Case 1. Assume that i and $i+1$ appear in different orbits of τ, say $\mathcal{O}_1$ and $\mathcal{O}_2$ such that at least one of them, say $\mathcal{O}_1$, has even length. We show that this case does not occur. To do this, let m_1 and m_2 be the minimal elements of $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively. If $\mathcal{O}_2$ also has even length, we assume $m_1 < m_2$.

For $w \in \mathfrak{S}_n$ and $j \in [n]$ let $\langle w \rangle$ denote the subgroup of $\mathfrak{S}_n$ generated by w and $\langle w \rangle j$ be the orbit of j under the natural action of $\langle w \rangle$ on $[n]$. Since τ satisfies Property (2) and

$\mathcal{O}_1$ has even length, there is a $p_1 \geqslant m_1$ such that

$$\begin{aligned} \mathcal{O}_1^< &:= \langle \tau^2 \rangle m_1 = \langle \sigma_\alpha^2 \rangle m_1 = \{m_1, m_1+1, \ldots, p_1\}\,, \\ \mathcal{O}_1^> &:= \langle \tau^2 \rangle \tau(m_1) = \langle \sigma_\alpha^2 \rangle \sigma_\alpha(m_1) = \{n-m_1+1, n-m_1, \ldots, n-p_1+1\}\,. \end{aligned} \tag{3.1}$$

We claim the following:

$$\text{Let } a \in \mathcal{O}_1^<, b \in \mathcal{O}_2 \text{ and } c \in \mathcal{O}_1^>. \text{ Then } a < b < c.$$

To prove the claim, consider the positions of elements of $[n]$ in the cycle notation $\sigma_\alpha = \sigma_{\alpha_1} \cdots \sigma_{\alpha_l}$ given by Definition 14. The elements on odd positions $1, 2, 3, \ldots$ form an strictly increasing sequence. The elements on even positions $n, n-1, \ldots$ form an strictly decreasing sequence but they are always greater than the entries on odd positions.

We want to show that the elements of $\mathcal{O}_2$ all appear right of the cycle consisting of the elements of $\mathcal{O}_1$. If $\mathcal{O}_2$ has even length this is clear. If $\mathcal{O}_2$ has odd length, we can use that by Property (1), the unions of odd orbits of τ and σ_α coincide and that in σ_α the elements of odd orbits are all located right of the elements of the even orbits.

Let $a \in \mathcal{O}_1^<$. Then a is on an odd position and thus it is smaller than any entry right of it. On the other hand, $c \in \mathcal{O}_1^>$ implies that c is on an even position and thus is greater then any entry right of it. Finally, in the last paragraph we have shown that each $b \in \mathcal{O}_2$ is located right of a and c. This establishes the claim.

Now, we have to deal with two cases.

If $i \in \mathcal{O}_1$ and $i+1 \in \mathcal{O}_2$ then the claim implies $i \in \mathcal{O}_1^<$. Then $\tau^{-1}(i), \tau(i) \in \mathcal{O}_1^>$. Since $\tau^{-1}(i+1), \tau(i+1) \in \mathcal{O}_2$, our claim yields $\tau^{-1}(i) > \tau^{-1}(i+1)$ and $\tau(i) > \tau(i+1)$. In addition, since $\mathcal{O}_1$ has even length and $i+1 \notin \mathcal{O}_1$, $\tau(i) \neq i, i+1$. Thus, we obtain from Lemma 25 that $\ell(\sigma) < \ell(\tau)$, a contradiction to $\ell(\tau) = \ell(\sigma)$.

If $i+1 \in \mathcal{O}_1$ and $i \in \mathcal{O}_2$ then the claim implies $i+1 \in \mathcal{O}_1^>$ and similarly as before we obtain $\tau^{-1}(i) > \tau^{-1}(i+1)$ and $\tau(i) > \tau(i+1)$ and thus the same contradiction using Lemma 25. That is, we have shown that i and $i+1$ cannot appear in two different orbits if one of the latter has even length.

Case 2. Assume that i and $i+1$ appear in the same orbit with even length $\mathcal{O}_1$ of τ. Then (1) also holds for σ.

To show (2), assume $i+1 \in \langle \tau^2 \rangle i$ first. Then both elements appear in the same cycle of τ^2. As we obtain σ^2 from τ^2 by swapping i and $i+1$ in cycle notation, (2) also holds for σ.

Lastly, we show that $i+1 \in \langle \tau^2 \rangle i$ is always true. For the sake of contradiction, assume $i+1 \notin \langle \tau^2 \rangle i$.

Suppose in addition that $|\mathcal{O}_1| = 2$. Then $\{\tau(i), \tau(i+1)\} = \{i, i+1\}$ and from Lemma 25 we obtain $\sigma = s_i \tau s_i = \tau$. This contradicts the minimality of the sequence of arrow relations from σ_α to σ.

Now suppose $|\mathcal{O}_1| > 2$. Then $\{\tau(i), \tau(i+1)\} \neq \{i, i+1\}$. Since $i+1 \notin \langle \tau^2 \rangle i$, it follows from Equation (3.1) that $i = \max \mathcal{O}_1^<$ and $i+1 = \min \mathcal{O}_1^>$. Consequently, $\tau^{-1}(i), \tau(i) \in \mathcal{O}_1^>$ and $\tau^{-1}(i+1), \tau(i+1) \in \mathcal{O}_1^<$. But this means that

$$\tau^{-1}(i) > \tau^{-1}(i+1) \text{ and } \tau(i) > \tau(i+1).$$

Because $\{\tau(i), \tau(i+1)\} \neq \{i, i+1\}$, we can now apply Lemma 25 and obtain that $\ell(\sigma) < \ell(\tau)$. Again, we end up with a contradiction. $\square$

Let $\sigma \in \mathfrak{S}_n$. Then the set of orbits of σ on $[n]$ is a set partition of $[n]$. We denote this partition by $P(\sigma)$. The set of even orbits of σ is given by

$$P_e(\sigma) := \{\mathcal{O} \in P(\sigma) \mid |\mathcal{O}| \text{ is even}\}.$$

If $P(\sigma) = P(\sigma')$ for $\sigma, \sigma' \in \mathfrak{S}_n$ then σ and σ' have the same type, i.e. they are conjugate.

Lemma 27. *Let $\alpha, \beta \vDash_e n$ such that σ_α and σ_β are conjugate. If $P_e(\sigma_\alpha) = P_e(\sigma_\beta)$ then $\alpha = \beta$.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_l), \beta = (\beta_1, \ldots, \beta_{l'}) \vDash_e n$ and $(x_1, x_2, \ldots, x_n)$ be the sequence with $x_{2i-1} = i$ and $x_{2i} = n - i + 1$. Since α is maximal, there is a $k \in [0, l]$ such that α_i is even for $i \leqslant k$ and odd for $i > k$. Assume that σ_α and σ_β are conjugate and $P_e(\sigma_\alpha) = P_e(\sigma_\beta)$.

Because σ_α and σ_β are conjugate, α and β have the same multiset of parts. In particular, $l = l'$. Since α and β are maximal, the odd parts of α and β form an weakly decreasing sequence at the end of α and β, respectively. As both compositions have the same length and multiset of parts, it follows that $\alpha_i = \beta_i$ for $i = k+1, \ldots, l$.

We show that $\alpha_i = \beta_i$ for $i = 1, \ldots, k$ with induction. Assume that $i \in [k]$ and $\alpha_j = \beta_j$ for all $1 \leqslant j < i$. Define $d := \sum_{j=1}^{i-1} \alpha_i$. Then by assumption $d = \sum_{j=1}^{i-1} \beta_i$. Moreover, let $\mathcal{O}_{\alpha_i}$ and $\mathcal{O}_{\beta_i}$ be the orbits of x_{d+1} under σ_α and σ_β, respectively. From the definition of elements in stair form it follows that

$$\begin{aligned}
\mathcal{O}_{\alpha_i} &= \{x_{d+1}, x_{d+2}, \ldots, x_{d+\alpha_i}\}, \\
\mathcal{O}_{\beta_i} &= \{x_{d+1}, x_{d+2}, \ldots, x_{d+\beta_i}\}.
\end{aligned}$$

In particular $|\mathcal{O}_{\alpha_i}| = \alpha_i$ and $|\mathcal{O}_{\beta_i}| = \beta_i$. Since $i \leqslant k$, α_i and β_i are even. Consequently, $\mathcal{O}_{\alpha_i}$ and $\mathcal{O}_{\beta_i}$ both have even length. Moreover, they have the element x_{d+1} in common. Hence, $P_e(\sigma_\alpha) = P_e(\sigma_\beta)$ implies $\mathcal{O}_{\alpha_i} = \mathcal{O}_{\beta_i}$. Thus, $\alpha_i = |\mathcal{O}_{\alpha_i}| = |\mathcal{O}_{\beta_i}| = \beta_i$. $\square$

We are now in the position to prove Statement (b), and finish the proof of Theorem 18.

Proof of Statement (b). Let $\alpha, \beta \vDash_e n$ such that $\sigma_\alpha \approx \sigma_\beta$. Then σ_α and σ_β are conjugate. Moreover, Lemma 26 implies $P_e(\sigma_\alpha) = P_e(\sigma_\beta)$. Hence $\alpha = \beta$ by Lemma 27. $\square$

We now use Lemmas 26 and 27 in order to prepare result for the proof of Theorem 84.

Proposition 28. *Let $\alpha \vDash_e n$ and $\sigma \in \mathfrak{S}_n$. Then $\sigma \in \Sigma_\alpha$ if and only if*

(1) σ and σ_α are conjugate in $\mathfrak{S}_n$,

(2) $\ell(\sigma) = \ell(\sigma_\alpha)$,

(3) $P_e(\sigma) = P_e(\sigma_\alpha)$.

Proof. First, assume $\sigma \in \Sigma_\alpha$. Because $\sigma_\alpha \in \Sigma_\alpha$ and $\Sigma_\alpha \in (\mathfrak{S}_n)_{\max}/\!\approx$, σ satisfies (1) and (2). By Lemma 26, (3) holds as well.

Second, assume that σ satisfies (1) – (3). By (1), σ is in the same conjugacy class as σ_α. From (2) it follows, that σ is maximal in its conjugacy class. Then Theorem 18 provides the existence of a $\beta \vDash_e n$ such that $\sigma \in \Sigma_\beta$. Using the already proven implication from left to right, we obtain that σ and σ_β are conjugate and $P_e(\sigma) = P_e(\sigma_\beta)$. But as σ satisfies (1) and (3), it follows that σ_β and σ_α are conjugate and $P_e(\sigma_\beta) = P_e(\sigma_\alpha)$. Thus, Lemma 27 yields $\beta = \alpha$ as desired. □

We end this section with a remark on conjugacy classes.

Remark 29. The conjugacy classes of $\mathfrak{S}_n$ are parametrized by the partitions of n via the cycle type. For a composition α we denote the partition obtained by sorting the parts of α in decreasing order by $\widetilde{\alpha}$. Let $\lambda \vdash n$ and $\mathcal{O}$ be the conjugacy class whose elements have cycle type λ. From Definition 14 it follows that for $\alpha \vDash_e n$ the element in stair form σ_α is contained in $\mathcal{O}$ if and only if $\widetilde{\alpha} = \lambda$. Hence, Theorem 18 implies that $\{\sigma_\alpha \mid \alpha \vDash_e n, \widetilde{\alpha} = \lambda\}$ is a complete set of representatives of $\mathcal{O}_{\max}/\!\approx$. In particular, we have that

$$|\mathcal{O}_{\max}/\!\approx| = 1 \text{ if and only if the even parts of } \lambda \text{ are all equal.}$$

4 Equivalence classes of $(\mathfrak{S}_n)_{\max}$ under $\approx$

Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$. Recall that Σ_α is the $\approx$-equivalence class of the element in stair form σ_α. From Theorem 18 we have that $(\mathfrak{S}_n)_{\max}/\!\approx\; = \{\Sigma_\alpha \mid \alpha \vDash_e n\}$. In Corollary 20 we concluded that the elements $T_{\leqslant \Sigma_\alpha}$ for $\alpha \vDash_e n$ form a basis of $Z(H_n(0))$. We emphasize that $T_{\leqslant \Sigma_\alpha}$ directly depends on Σ_α since $T_{\leqslant \Sigma_\alpha} = \sum_x T_x$ where x runs over the order ideal in Bruhat order generated by Σ_α. Motivated by this connection, the current section is devoted to the description of equivalence classes Σ_α and bijections between them.

Let $(\alpha_1, \ldots, \alpha_j)$ be the even and $\alpha' := (\alpha_{j+1}, \ldots, \alpha_l)$ be the odd parts of α. We will characterize Σ_α combinatorially in the case where α is a hook (Theorem 93). Moreover, will see how an injective operator $\odot$, the *inductive product*, can be used to decompose Σ_α into $\Sigma_{(\alpha_i)}$ for $i = 1, \ldots, j$ and $\Sigma_{\alpha'}$ (Theorem 84). As we know the $\Sigma_{(\alpha_i)}$ from Theorem 93, the only unknown in the description of Σ_α is $\Sigma_{\alpha'}$. If α' is a hook, we can use Theorem 93 again and obtain a description of Σ_α for all α whose odd parts form a hook (Remark 87). As an open problem, the case where α has only odd parts but is not a hook remains (see Remark 94).

The section is structured as follows. In Section 4.1 we consider the case where α has only one part. The first important result is the characterization of the elements of $\Sigma_{(n)}$ by properties of their cycle notation (Theorem 49). From this we obtain bijections relating $\Sigma_{(n-1)}$ with $\Sigma_{(n)}$ for $n \geqslant 4$ (Theorem 50) and a closed formula for the cardinality of $\Sigma_{(n)}$ (Corollary 53).

In Section 4.2 we generalize the characterization of $\Sigma_{(n)}$ to odd hooks, where a hook $\alpha := (k, 1^{n-k})$ is called *odd* if k is odd and *even* otherwise (Theorem 69). Moreover, we define a bijection $\Sigma_{(k)} \times [m+1, n-m] \to \Sigma_{(k,1^{n-k})}$ where k is odd and $m := \frac{k-1}{2}$ (Corollary 70). From this we obtain the cardinality of $\Sigma_{(k,1^{n-k})}$ for odd k (Corollary 71).

Table 1: The elements of $\Sigma_{(n)}$ for small n with the element in stair form $\sigma_{(n)}$ in the top row.

α	(1)	(2)	(3)	(4)	(5)	(6)
Σ_α	(1)	$(1,2)$	$(1,3,2)$	$(1,4,2,3)$	$(1,5,2,4,3)$	$(1,6,2,5,3,4)$
			$(1,2,3)$	$(1,3,2,4)$	$(1,5,2,3,4)$	$(1,6,2,4,3,5)$
					$(1,5,3,2,4)$	$(1,6,3,4,2,5)$
					$(1,4,2,3,5)$	$(1,5,2,4,3,6)$
					$(1,4,3,2,5)$	$(1,5,3,4,2,6)$
					$(1,3,4,2,5)$	$(1,4,3,5,2,6)$

In Section 4.3 we consider the inductive product $\odot$ that allows the decomposition $\Sigma_{(\alpha_1,\ldots,\alpha_l)} = \Sigma_{(\alpha_1)} \odot \Sigma_{(\alpha_2,\ldots,\alpha_l)}$ if α_1 is even (Theorem 84). This yields the reduction mentioned above. Combining it with the results of the other sections, we then infer the description of Σ_α for all $\alpha \vDash_e n$ whose odd parts form a hook (Remark 87). This includes the characterization of Σ_α in the case where α is an even hook (Theorem 93).

4.1 Equivalence classes of n-cycles

In this section we seek a combinatorial description of the elements of $\Sigma_{(n)}$. Examples are given in Table 1. Our main goal is to show in Theorem 49 that the elements of $\Sigma_{(n)}$ are characterized by two properties: *being oscillating* and *having connected intervals*. From this we infer in Theorem 50 a recursive rule for determining $\Sigma_{(n)}$. The intermediate results leading to Theorem 49 can be structured as follows: We first consider the property of being oscillating in Lemmas 33 to 35 and Corollary 36. Then the second property of having connected intervals comes into play. We show in Lemma 40 that the element in stair form $\sigma_{(n)}$ is oscillating and has connected intervals. Proving in Lemma 46 that the relation $\approx$ preserves the two properties is a major step towards Theorem 49. The final ingredient is an algorithm considered in Lemma 47. This algorithm takes an arbitrary n-cycle which is oscillating and has connected intervals as input and computes a sequence of $\approx$-equivalent n-cycles ending up at the element in stair form $\sigma_{(n)}$.

We now begin with the property of being oscillating.

Definition 30. We call the n-cycle $\sigma \in \mathfrak{S}_n$ *oscillating* if there exists a positive integer $m \in \left\{\frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\right\}$ such that $\sigma([m]) = [n-m+1, n]$.

In Corollary 36 we will obtain a more descriptive characterization of oscillating n-cycles. It turns out that the n-cycle σ of $\mathfrak{S}_n$ (represented in cycle notation) is oscillating if n is even and the entries of σ alternate between the sets $[1, \frac{n}{2}]$ and $[\frac{n}{2}+1, n]$ or n is odd and after deleting the entry $\frac{n+1}{2}$ from σ the remaining entries alternate between the sets $[1, \frac{n-1}{2}]$ and $[\frac{n+3}{2}, n]$.

Example 31. (1) Recall that for $n \in \mathbb{N}$ the element in stair form $\sigma_{(n)}$ is an n-cycle of $\mathfrak{S}_n$. For

$$\sigma_{(5)} = (1,5,2,4,3), \quad \sigma_{(5)}^{-1} = (1,3,4,2,5) \quad \text{and} \quad \sigma_{(6)} = (1,6,2,5,3,4)$$

we have

$$\sigma_{(5)}([2]) = [4,5], \quad \sigma_{(5)}^{-1}([3]) = [3,5] \quad \text{and} \quad \sigma_{(6)}([3]) = [4,6].$$

Hence, they are oscillating and the integer m used in Definition 30 is given by

$$m = 2 = \frac{5-1}{2}, \quad m = 3 = \frac{5+1}{2} \quad \text{and} \quad m = 3 = \frac{6}{2},$$

respectively. Note that the entries in the cycles alternate as described after Definition 30.

(2) All the elements shown in Table 1 are oscillating.

We explicitly write down the three cases for m in Definition 30.

Remark 32. Let σ be an oscillating n-cycle $\sigma \in \mathfrak{S}_n$ with parameter m from Definition 30. Then we have

(1) n is even and $\sigma([\frac{n}{2}]) = [\frac{n}{2}+1, n]$ if $m = \frac{n}{2}$,
(2) n is odd and $\sigma([\frac{n-1}{2}]) = [\frac{n+3}{2}, n]$ if $m = \frac{n-1}{2}$,
(3) n is odd and $\sigma([\frac{n+1}{2}]) = [\frac{n+1}{2}, n]$ if $m = \frac{n+1}{2}$.

Our next aim is to give a characterization of the term *oscillating* in Lemma 35. By considering complements in $[n]$ we obtain the following.

Lemma 33. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle and $m \in [n]$. Then $\sigma([m]) = [n-m+1, n]$ if and only if $\sigma([m+1, n]) = [n-m]$.*

Lemma 33 implies that an n-cycle $\sigma \in \mathfrak{S}_n$ is oscillating with parameter m if and only if $\sigma([m+1,n]) = [n-m]$.

Lemma 34. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. Then σ is oscillating if and only if σ^{-1} is oscillating.*

Proof. Let $M := \mathbb{N} \cap \left\{\frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\right\}$. If $n = 1$ then $\sigma = \mathrm{id} = \sigma^{-1}$ (which is oscillating). Thus assume $n \geqslant 2$. It suffices to show the implication from left to right. Suppose that σ is oscillating. Then there is an $m \in M$ such that $\sigma([m]) = [n-m+1, n]$. Consequently, $\sigma([m+1, n]) = [n-m]$ by Lemma 33 and hence

$$\sigma^{-1}([n-m]) = [m+1, n].$$

Moreover, $m+1 = n-(n-m)+1$ and we have $n-m \in M$ since $m \in M$ and $n \geqslant 2$. Therefore, σ^{-1} is oscillating. □

In the following lemma we rephrase Definition 30 from a more local point of view. The result looks rather technical but its main idea is, that an n-cycle is oscillating if and only if its entries in cycle notation alternate between entries smaller than $\frac{n+1}{2}$ and entries greater than $\frac{n+1}{2}$ with an extra rule for the neighbors of $\frac{n+1}{2}$ if n is odd. This result will also be used in the argumentation leading to the characterization of Σ_α for α an odd hook in Section 4.2.

Lemma 35. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. We consider the four implications for all $i \in [n]$*

(i) $i < \frac{n+1}{2} \implies \sigma(i) \geqslant \frac{n+1}{2}$,

(ii) $i < \frac{n+1}{2} \implies \sigma^{-1}(i) \geqslant \frac{n+1}{2}$,

(iii) $i > \frac{n+1}{2} \implies \sigma(i) \leqslant \frac{n+1}{2}$,

(iv) $i > \frac{n+1}{2} \implies \sigma^{-1}(i) \leqslant \frac{n+1}{2}$,

and if n is odd the statement

(A) *either* $\sigma^{-1}(\frac{n+1}{2}) > \frac{n+1}{2}$ *or* $\sigma(\frac{n+1}{2}) > \frac{n+1}{2}$.

Then the following are equivalent.

(1) *σ is oscillating.*

(2) *One of (i) – (iv) is true and if n is odd and $n \geqslant 3$ then also (A) is true.*

(3) *Each one of (i) – (iv) is true and if n is odd and $n \geqslant 3$ then also (A) is true.*

Proof. First suppose that n is odd. If $n = 1$ then $\sigma = \mathrm{id}$ is oscillating and the implications (i) – (iv) are trivially satisfied.

Assume $n \geqslant 3$. We show for each of the implications (x) that (A) and (x) is true if and only if σ is oscillating. As n is odd and $n \geqslant 3$, Statement (A) can be expanded as

$$\begin{array}{cl} \text{either} & \sigma^{-1}(\frac{n+1}{2}) > \frac{n+1}{2} \text{ and } \sigma(\frac{n+1}{2}) < \frac{n+1}{2} \\ \text{or} & \sigma^{-1}(\frac{n+1}{2}) < \frac{n+1}{2} \text{ and } \sigma(\frac{n+1}{2}) > \frac{n+1}{2} \ . \end{array}$$

Moreover, (i) can be rephrased as $\sigma([\frac{n-1}{2}]) \subseteq [\frac{n+1}{2}, n]$. Hence, we have (A) and (i) if and only if

$$\begin{array}{cll} \text{either} & \sigma([\frac{n-1}{2}]) = [\frac{n+3}{2}, n] & (\text{if } \sigma^{-1}(\frac{n+1}{2}) > \frac{n+1}{2} \text{ and } \sigma(\frac{n+1}{2}) < \frac{n+1}{2}) \\ \text{or} & \sigma([\frac{n+1}{2}]) = [\frac{n+1}{2}, n] & (\text{if } \sigma^{-1}(\frac{n+1}{2}) < \frac{n+1}{2} \text{ and } \sigma(\frac{n+1}{2}) > \frac{n+1}{2}). \end{array}$$

In other words, $\sigma([m]) = [n-m+1, n]$ for either $m = \frac{n-1}{2}$ or $m = \frac{n+1}{2}$, i.e. σ is oscillating.

Similarly, we have (A) and (iii) if and only if

$$\text{either} \quad \sigma([\tfrac{n+1}{2}, n]) = [\tfrac{n+1}{2}] \quad \text{or} \quad \sigma([\tfrac{n+3}{2}, n]) = [\tfrac{n-1}{2}].$$

That is, $\sigma([m+1, n]) = [n-m]$ for either $m = \frac{n-1}{2}$ or $m = \frac{n+1}{2}$. This is equivalent to σ being oscillating by Lemma 33.

So far we have shown that

$$\text{(A) and (i)} \iff \sigma \text{ is oscillating} \iff \text{(A) and (iii)}. \tag{4.1}$$

By Lemma 34 we therefore also have

$$\text{(A) and (ii)} \iff \sigma \text{ is oscillating} \iff \text{(A) and (iv)}. \tag{4.2}$$

This finishes the proof for odd n.

Suppose now that n is even. Note that $\frac{n+1}{2} \notin [n]$ as it is not an integer. It is not hard to see that the equivalences from Equation (4.1) and therefore those from Equation (4.2) hold if we drop Statement (A). □

We continue with two consequences of Lemma 35. We first infer the description of oscillating n-cycles mentioned at the beginning of the section.

Corollary 36. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. We consider σ in cycle notation. Then σ is oscillating if and only if one of the following is true.*

(1) *n is even and the entries of σ alternate between the sets $\left[\frac{n}{2}\right]$ and $\left[\frac{n}{2}+1, n\right]$.*

(2) *n is odd and after deleting the entry $\frac{n+1}{2}$ from σ, the remaining entries alternate between the sets $\left[\frac{n-1}{2}\right]$ and $\left[\frac{n+3}{2}, n\right]$.*

Proof. With (A), (i) and (iii) we refer to the statements of Lemma 35.

Suppose that n is even. By Lemma 35, σ is oscillating if and only if the implications (i) and (iii) are satisfied which is the case if and only if the entries of σ alternate between $[\frac{n}{2}]$ and $[\frac{n}{2}+1, n]$.

Suppose that n is odd. If $n \geqslant 3$ then property (A) states that one of the neighbors $\sigma^{-1}(\frac{n+1}{2})$ and $\sigma(\frac{n+1}{2})$ of $\frac{n+1}{2}$ in σ is an element of $[\frac{n-1}{2}]$ and the other one is an element of $[\frac{n+3}{2}, n]$. Therefore, σ satisfies (A), (i) and (iii) if and only if after deleting $\frac{n+1}{2}$ from the cycle notation of σ, the remaining entries alternate between the sets $[\frac{n-1}{2}]$ and $[\frac{n+3}{2}, n]$. Thus, Lemma 35 yields that the latter property is satisfied if and only if σ is oscillating. $\square$

Consider an n-cycle σ in cycle notation such that 1 is the leftmost entry in the cycle. Then we can rephrase Corollary 36 in a more formal way.

Corollary 37. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. If n is odd, let $0 \leqslant l \leqslant n-1$ be such that $\sigma^l(1) = \frac{n+1}{2}$. If n is even, set $l := \infty$. Then σ is oscillating if and only if for all $0 \leqslant k \leqslant n-1$ we have*

$$\sigma^k(1) < \frac{n+1}{2} \quad \text{if } k < l \text{ and } k \text{ is even or } k > l \text{ and } k \text{ is odd,}$$
$$\sigma^k(1) > \frac{n+1}{2} \quad \text{if } k < l \text{ and } k \text{ is odd or } k > l \text{ and } k \text{ is even.}$$

We now come to the second property in the characterization of $\Sigma_{(n)}$: the property of *having connected intervals*. Roughly speaking, an n-cycle of $\mathfrak{S}_n$ has connected intervals if in its cycle notation for each $1 \leqslant k \leqslant \frac{n}{2}$ the elements of the interval $[k, n-k+1]$ are grouped together.

Definition 38. (1) Let $\sigma \in \mathfrak{S}_n$ and $M \subseteq [n]$. We call M *connected* in σ if there is an $m \in M$ such that

$$M = \left\{m, \sigma(m), \sigma^2(m), \ldots, \sigma^{|M|-1}(m)\right\}.$$

(2) Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. We say that σ has *connected intervals* if the interval $[k, n-k+1]$ is connected in σ for all integers k with $1 \leqslant k \leqslant \frac{n}{2}$.

Example 39. All elements shown in Table 1 have connected intervals. In particular, the element in stair form $\sigma_{(6)} = (1, 6, 2, 5, 3, 4)$ has connected intervals. In contrast, in $(1, 5, 2, 6, 3, 4)$ the set $[2, 5]$ is not connected.

The main result of this section is that an n-cycle $\sigma \in \mathfrak{S}_n$ is an element of $\Sigma_{(n)}$ if and only if σ is oscillating and has connected intervals. We now show that the element in stair form has these properties.

Lemma 40. *The element in stair form $\sigma_{(n)} \in \mathfrak{S}_n$ is oscillating and has connected intervals.*

Proof. By Definition 14,

$$\sigma_{(n)} = \begin{cases} (1, n, 2, n-1, \ldots, \frac{n}{2}, n - \frac{n}{2} + 1) & \text{if } n \text{ is even} \\ (1, n, 2, n-1, \ldots, \frac{n-1}{2}, n - \frac{n-1}{2} + 1, \frac{n+1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

Thus, $\sigma_{(n)}([\frac{n}{2}]) = [\frac{n}{2} + 1, n]$ if n is even and $\sigma_{(n)}([\frac{n-1}{2}]) = [\frac{n+3}{2}, n]$ if n is odd. That is, $\sigma_{(n)}$ is oscillating.

For all $k \in \mathbb{N}$ with $1 \leqslant k \leqslant \frac{n}{2}$ the rightmost $|[k, n-k+1]|$ elements in the cycle of $\sigma_{(n)}$ from above form $[k, n-k+1]$. Thus, $\sigma_{(n)}$ has connected intervals. □

Let $\sigma \in \mathfrak{S}_n$. Sometimes it will be convenient to consider σ^{w_0} instead of σ. We will now show that conjugation with the longest element w_0 of $\mathfrak{S}_n$ preserves the properties of being oscillating and having connected intervals.

Lemma 41. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle.*

(*1*) *If σ is oscillating then σ^{w_0} is oscillating.*

(*2*) *If σ has connected intervals then σ^{w_0} has connected intervals.*

Proof. If $n = 1$ the result is trivial. Thus suppose $n \geqslant 2$.

(1) Set $M := \mathbb{N} \cap \left\{\frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\right\}$ and assume that σ is oscillating. Then there is an $m \in M$ such that $\sigma([m]) = [n-m+1, n]$ and from Lemma 33 it follows that $\sigma([m+1, n]) = [n-m]$. Using $w_0(i) = n - i + 1$ for $i \in [n]$, we obtain

$$\begin{aligned} \sigma^{w_0}([n-m]) &= w_0 \sigma w_0([n-m]) \\ &= w_0 \sigma([m+1, n]) \\ &= w_0([n-m]) \\ &= [n - (n-m) + 1, n]. \end{aligned}$$

As $n - m \in M$, it follows that σ^{w_0} is oscillating.

(2) Let $I := [k, n-k+1]$ be given by an integer k with $1 \leqslant k \leqslant \frac{n}{2}$. Then $w_0(I) = I$. Hence, if I is connected in σ then it is also connected in σ^{w_0}. □

In the following result we study the interplay between the conjugation with w_0 and the relation $\approx$. The generalization to all finite Coxeter groups is straight forward.

Lemma 42. *Let $w, w' \in \mathfrak{S}_n$ and ν be the automorphism of $\mathfrak{S}_n$ given by $x \mapsto x^{w_0}$.*

(*1*) *If $w \xrightarrow{s_i} w'$ then $\nu(w) \xrightarrow{s_{n-i}} \nu(w')$.*

(*2*) *If $w \approx w'$ then $\nu(w) \approx \nu(w')$.*

Proof. Assume $w \xrightarrow{s_i} w'$. Then $w' = s_i w s_i$ and $\ell(w') \leqslant \ell(w)$. Since $\nu(s_i) = s_{n-i}$, we have $\nu(w') = s_{n-i}\nu(w)s_{n-i}$. Moreover, $\ell(\nu(w')) \leqslant \ell(\nu(w))$ because $\ell(x) = \ell(\nu(x))$ for all $x \in \mathfrak{S}_n$. Thus, $\nu(w) \xrightarrow{s_{n-i}} \nu(w')$. Now, use the definition of $\approx$ to obtain (2) from (1). $\square$

Consider $n = 5$, the oscillating n-cycle $\sigma = (\mathbf{1}, 4, 2, 3, \mathbf{5})$ and its connected interval $I = \{2, 3, 4\}$. In the cycle notation of σ, this interval is enclosed by the two elements $a = 1$ and $b = 5$. Note that $\frac{n+1}{2} = 3$, $a < 3$ and $b > 3$. This illustrates a property of oscillating n-cycles which is the subject of the next lemma.

Lemma 43. *Assume that $\sigma \in \mathfrak{S}_n$ is an oscillating n-cycle with a connected interval $I := [i, n-i+1]$ such that $i \in \mathbb{N}$ and $2 \leqslant i \leqslant \frac{n+1}{2}$. Let $r := |I|$ and $m \in I$ be such that $I = \left\{\sigma^k(m) \mid k = 0, \ldots, r-1\right\}$. Moreover, set $a := \sigma^{-1}(m)$ and $b := \sigma^r(m)$. Then $a, b \neq \frac{n+1}{2}$ and*

$$a < \frac{n+1}{2} \iff b > \frac{n+1}{2}.$$

Proof. Let $p \in [n-1]$ be such that $\sigma^p(1) = a$. Then $\sigma^{p+r+1}(1) = b$. Since $i > 1$, $1 \notin I$ and thus $p + r + 1 \leqslant n - 1$. We have $r = n - 2i + 2$. Hence, r has the same parity as n.

We want to apply Corollary 37. If n is odd, let $l \in [0, n-1]$ be such that $\sigma^l(1) = \frac{n+1}{2}$. Then $\frac{n+1}{2} \in I$ so that $p < l < p + r + 1$. In particular, $a, b \neq \frac{n+1}{2}$. Clearly, if n is even then $a, b \neq \frac{n+1}{2}$.

Therefore,

$$\begin{aligned} a = \sigma^p(1) < \frac{n+1}{2} &\iff p \text{ is even} \\ &\iff \begin{cases} p + r + 1 \text{ is odd} & \text{if } n \text{ even} \\ p + r + 1 \text{ is even} & \text{if } n \text{ odd} \end{cases} \\ &\iff b = \sigma^{p+r+1}(1) > \frac{n+1}{2}. \end{aligned}$$

where we use Corollary 37 (and $p < l < p + r + 1$ if n is odd) for the first and third equivalence. $\square$

Since the $\to$ relation is the transitive closure of the $\xrightarrow{s_i}$ relations, we are interested in the circumstances under which the conjugation with s_i preserves the property of being oscillating with connected intervals.

Lemma 44. *Let $\sigma \in \mathfrak{S}_n$ be an oscillating n-cycle with connected intervals, $i \in [n-1]$ with $i \leqslant \frac{n+1}{2}$ and $\sigma' := s_i \sigma s_i$. Then σ' is oscillating and has connected intervals if and only if*

(*1*) *if $i = \frac{n}{2}$ then $n = 2$,*

(*2*) *if $i = \frac{n-1}{2}$ or $i = \frac{n+1}{2}$ then $\sigma(i) = i + 1$ or $\sigma^{-1}(i) = i + 1$,*

(3) *if* $i < \frac{n-1}{2}$ *then*

$$\sigma(i) \in I \text{ and } \sigma(i+1) \notin I \text{ or } \sigma^{-1}(i) \in I \text{ and } \sigma^{-1}(i+1) \notin I$$

where $I := [i+1, n-i]$.

Proof. We will use Lemma 35 without further reference. Note that $\sigma' = s_i \sigma s_i$ means that we obtain σ' from σ by interchanging i and $i+1$ in cycle notation. We show the equivalence case by case, depending on i.

Case 1. Suppose $i = \frac{n}{2}$. In this case n is even. If $n = 2$ then $(1,2)$ is the only 2-cycle in $\mathfrak{S}_n$. Thus, $\sigma = \sigma' = (1,2)$. This element is oscillating and has connected intervals.

Assume now that $n > 2$. Since σ is oscillating,

$$\sigma(i) > \frac{n}{2} \text{ and } \sigma^{-1}(i) > \frac{n}{2}.$$

Moreover as $n > 2$, at most one of $\sigma(i)$ and $\sigma^{-1}(i)$ equals $i+1$. Since we obtain σ' from σ by swapping i and $i+1$ in cycle notation we infer

$$\sigma'(i+1) > \frac{n}{2} \text{ or } \sigma'^{-1}(i+1) > \frac{n}{2}.$$

As $i+1 > \frac{n}{2}$, this means that σ' is not oscillating

Case 2. Suppose $i = \frac{n-1}{2}$ or $i = \frac{n+1}{2}$. In this case n is odd and $n \geqslant 3$. Moreover, $i, i+1 \in [k, n-k+1]$ for $k = 1, \dots, \frac{n-1}{2}$. Hence, each of the intervals remains connected if we interchange i and $i+1$. Therefore, σ' has connected intervals. It remains to determine in which cases σ' oscillates. We do this for $i = \frac{n-1}{2}$. The proof for $i = \frac{n+1}{2}$ is similar.

For $i = \frac{n-1}{2}$ we have $i+1 = \frac{n+1}{2}$. Since σ is oscillating,

$$\sigma(i) \geqslant \frac{n+1}{2} \text{ and } \sigma^{-1}(i) \geqslant \frac{n+1}{2}.$$

Because $n \geqslant 3$, there is at most one equality among these two inequalities. Assume that there is no equality at all. Then

$$\sigma'\left(\frac{n+1}{2}\right) > \frac{n+1}{2} \text{ and } \sigma'^{-1}\left(\frac{n+1}{2}\right) > \frac{n+1}{2}$$

since $\sigma' = s_i \sigma s_i$. Hence, σ' is not oscillating.

Conversely, assume that $\sigma(i) = i+1$ or $\sigma^{-1}(i) = i+1$. In other words, there exists an $\varepsilon \in \{-1, 1\}$ such that $\sigma^{\varepsilon}(i) = i+1$. Since $i+1 = \frac{n+1}{2}$ and σ is oscillating, we then have $a := \sigma^{-\varepsilon}(i) > \frac{n+1}{2}$. Moreover, $\sigma^{-\varepsilon}(i+1) = i < \frac{n+1}{2}$. Thus σ being oscillating implies that $b := \sigma^{\varepsilon}(i+1) > \frac{n+1}{2}$. By definition of a and b,

$$\sigma^{\varepsilon} = (a, i, i+1, b, \dots).$$

As a consequence,

$$\sigma'^{\varepsilon} = (a, i+1, i, b, \dots)$$

and σ^{ε} and σ'^{ε} coincide on the part represented by the dots because $\sigma' = s_i \sigma s_i$. From $a > \frac{n+1}{2}$, $i+1 = \frac{n+1}{2}$, $i < \frac{n+1}{2}$ and $b > \frac{n+1}{2}$ it now follows that σ' is oscillating.

Case 3. Suppose $i < \frac{n-1}{2}$. Note that then $n \geqslant 4$. Define $I := [i+1, n-i]$ as in the theorem and set $r := |I|$. Since $i+1 < \frac{n+1}{2}$, we have $r > 1$. We show the implication from left to right first. Assume that σ' is oscillating and has connected intervals. Note that

$$\tau^{\varepsilon}(j) \neq i, i+1 \text{ for all } \tau \in \{\sigma, \sigma'\}, \varepsilon \in \{-1, 1\} \text{ and } j \in \{i, i+1\}$$

since σ and σ' are oscillating and $i, i+1 < \frac{n+1}{2}$. Because I is connected in σ', $i+1 \in I$ and $r > 1$, we have that

$$\exists \varepsilon \in \{-1, 1\} \text{ such that } {\sigma'}^{\varepsilon}(i+1) \in I.$$

Therefore,

$$\exists \varepsilon \in \{-1, 1\} \text{ such that } \sigma^{\varepsilon}(i) \in I$$

as $\sigma' = s_i \sigma s_i$ and ${\sigma'}^{\varepsilon}(i+1) \neq i, i+1$. In fact, the statement

$$\exists \varepsilon \in \{-1, 1\} \text{ such that } \sigma^{\varepsilon}(i) \in I \text{ and } \sigma^{-\varepsilon}(i) \notin I \tag{4.3}$$

is true since otherwise we would have

$$\sigma = (n+i-1, \dots, \sigma^{-1}(i), i, \sigma(i), \dots)$$

with $\sigma^{-1}(i), \sigma(i) \in I$ and $i, n+i-1 \notin I$ in which case I would not be connected in σ.

By interchanging the roles played by σ and σ' in the argumentation leading to Equation (4.3), we get that

$$\exists \varepsilon \in \{-1, 1\} \text{ such that } {\sigma'}^{\varepsilon}(i) \in I \text{ and } {\sigma'}^{-\varepsilon}(i) \notin I.$$

From this we obtain that

$$\exists \varepsilon \in \{-1, 1\} \text{ such that } \sigma^{\varepsilon}(i+1) \in I \text{ and } \sigma^{-\varepsilon}(i+1) \notin I \tag{4.4}$$

by swapping i and $i+1$ in cycle notation and using that $\sigma'(i), {\sigma'}^{-1}(i) \neq i, i+1$.

Now, let $\varepsilon \in \{-1, 1\}$ be such that $\sigma^{\varepsilon}(i) \in I$ and $\sigma^{-\varepsilon}(i) \notin I$. Then

$$I = \left\{\sigma^{\varepsilon k}(i) \mid k = 1, \dots, r\right\} \tag{4.5}$$

since I is connected in σ and $i \notin I$. From Equation (4.4) it follows that $i+1$ appears at the border of I in the cycle notation of σ. Hence, Equation (4.5) implies that

$$\sigma^{\varepsilon}(i) = i+1 \text{ or } \sigma^{\varepsilon r}(i) = i+1.$$

As $\sigma^{\varepsilon}(i) \neq i+1$, it follows that $i+1 = \sigma^{\varepsilon r}(i)$. Thus, Equation (4.5) yields that $\sigma^{-\varepsilon}(i+1) \in I$ and $\sigma^{\varepsilon}(i+1) \notin I$. Therefore, we have $\sigma^{\varepsilon}(i) \in I$ and $\sigma^{\varepsilon}(i+1) \notin I$ for an $\varepsilon \in \{-1, 1\}$ as desired.

Lastly, we prove the direction from right to left of the equivalence. We are still in the case $i < \frac{n-1}{2}$. Thus, assume that there is an $\varepsilon \in \{-1, 1\}$ such that $\sigma^{\varepsilon}(i) \in I$ and $\sigma^{\varepsilon}(i+1) \notin I$. Since σ is oscillating and we interchange two elements $i, i+1 < \frac{n+1}{2}$ in σ in order to obtain σ' from σ, σ' is also oscillating.

It remains to show that σ' has connected intervals. Since $i \notin I$, $\sigma^{\varepsilon}(i) \in I$ and I is connected in σ, we have Equation (4.5). Moreover, from $i+1 \in I$, $\sigma^{\varepsilon}(i+1) \notin I$ and I being connected in σ, it follows that $\sigma^{\varepsilon r}(i) = i+1$. Thus,

$$I = \left\{\sigma'^{\varepsilon k}(i+1) \mid k = 0, \dots, r-1\right\}$$

because $\sigma' = s_i \sigma s_i$. That is, I is connected in σ'. Let $J := [k, n-k+1]$ for $k \in \mathbb{N}$ with $1 \leqslant k \leqslant \frac{n}{2}$ and $k \neq i+1$ be an interval different from I. Then either $i, i+1 \in J$ or $i, i+1 \notin J$. As J is connected in σ and $\sigma' = s_i \sigma s_i$, it follows that J is connected in σ'. Therefore, σ' has connected intervals. □

Example 45. Consider $\sigma = \sigma_{(6)} = (1, 6, 2, 5, 3, 4)$ and $\sigma_i := s_i \sigma s_i$ for $i = 1, 2$. Then σ is oscillating with connected intervals.

Since $\sigma^{-1}(1) \in [2, 5]$ and $\sigma^{-1}(2) \notin [2, 5]$, Lemma 44 yields that σ_1 is oscillating with connected intervals. In contrast, σ_2 is not oscillating with connected intervals because of $\sigma(2), \sigma^{-1}(2) \notin [3, 4]$ and Lemma 44. This can also be checked directly. We have

$$\sigma_1 = (1, 5, 3, 4, 2, 6) \quad \text{and} \quad \sigma_2 = (1, 6, 3, 5, 2, 4).$$

For instance, $[3, 4]$ is not connected in σ_2.

In the next result we show that the relation $\approx$ is compatible with the concept of oscillating n-cycles with connected intervals.

Lemma 46. *Let $\sigma \in \mathfrak{S}_n$ be an oscillating n-cycle with connected intervals, $i \in [n-1]$ and $\sigma' := s_i \sigma s_i$. If $\sigma \approx \sigma'$ then σ' is oscillating and has connected intervals.*

Proof. We do a case analysis depending on i.

Case 1. Suppose $i = \frac{n}{2}$. Then n is even. By Lemma 44, σ' is oscillating with connected intervals if and only if $n = 2$. Thus, we have to show that $\sigma \not\approx \sigma'$ if $n \geqslant 4$. In this case we have $\sigma(i), \sigma^{-1}(i) > \frac{n}{2}$ and $\sigma(i+1), \sigma^{-1}(i+1) \leqslant \frac{n}{2}$ because σ is oscillating. But then Lemma 25 yields $\ell(\sigma') < \ell(\sigma)$ so that $\sigma' \not\approx \sigma$.

Case 2. Suppose $i = \frac{n-1}{2}$ or $i = \frac{n+1}{2}$. We only do the case $i = \frac{n-1}{2}$. The other one is similar. Let $I := [i, n-i+1] = \{i, i+1, i+2\}$. We show the contraposition and assume that σ' is not oscillating or that it does not have connected intervals. Then from Lemma 44 it follows that $\sigma(i) \neq i+1$ and $\sigma^{-1}(i) \neq i+1$. Furthermore, there is an $m \in I$ such that

$$I = \left\{\sigma^{-1}(m), m, \sigma(m)\right\}$$

since I is connected in σ. Thus, $m = i+2$. Assume $\sigma^{-1}(i+2) = i$ and $\sigma(i+2) = i$ (the proof of the other case with $\sigma(i+2) = i$ is analogous). Then $\sigma^{-1}(i) > i+2$ as

σ is oscillating and $\sigma^{-1}(i) \neq i+1, i+2$. Moreover, Lemma 43 applied to I in σ and $\sigma^{-1}(i) > \frac{n+1}{2}$ yields $\sigma(i+1) < \frac{n+1}{2} = i+1$. Therefore,

$$\sigma(i) = i+2 > \sigma(i+1) \quad \text{and} \quad \sigma^{-1}(i) > i+2 = \sigma^{-1}(i+1)$$

so that $\ell(\sigma') < \ell(\sigma)$ by Lemma 25 and hence $\sigma' \not\approx \sigma$.

Case 3. Suppose $i < \frac{n-1}{2}$. Then for all $j \in \{i, i+1\}$ we have $\sigma(j), \sigma^{-1}(j) \geqslant \frac{n+1}{2}$ since $j < \frac{n+1}{2}$ and σ is oscillating. We assume $\sigma \approx \sigma'$ and show that σ' is oscillating and has connected intervals. Define $I_k := [k, n-k+1]$ for all $k \leqslant \frac{n+1}{2}$ and $I := I_{i+1} = [i+1, n-i]$. Thanks to Lemma 44 it suffices to show

$$\sigma(i) \in I \text{ and } \sigma(i+1) \notin I \text{ or } \sigma^{-1}(i) \in I \text{ and } \sigma^{-1}(i+1) \notin I.$$

Since $\sigma \approx \sigma'$, $\ell(\sigma) = \ell(\sigma')$. Hence, Lemma 25 implies that either $\sigma(i) < \sigma(i+1)$ or $\sigma^{-1}(i) < \sigma^{-1}(i+1)$. We assume $\sigma(i) < \sigma(i+1)$ and $\sigma^{-1}(i) > \sigma^{-1}(i+1)$. The other case is similar.

First we show $\sigma(i) \in I$. Assume $\sigma(i) \notin I$ instead. Then $\sigma(i) \geqslant \frac{n+1}{2}$ implies $\sigma(i) > \max I$. Now we use that $\sigma(i) < \sigma(i+1)$ to obtain $\sigma(i+1) \notin I$. From this it follows that

$$I = \left\{\sigma^{-k}(i+1) \mid k = 0, \ldots, r-1\right\}$$

where $r := |I|$ since I is connected in σ and $i+1 \in I$. Now we consider the interval $I_i = [i, n-i+1]$ in σ. Because σ is oscillating, $\sigma(i+1) > \frac{n+1}{2}$. An application of Lemma 43 to I in σ yields $\sigma^{-r}(i+1) < \frac{n+1}{2}$. In particular, $\sigma^{-r}(i+1) \neq n-i+1$. But we also have $i \neq \sigma^{-r}(i+1)$ because $\sigma(i) \notin I$. That is $\sigma^{-r}(i+1) \notin I_i$. As a consequence,

$$I_i = \left\{\sigma^{-k}(i+1) \mid k = 0, \ldots, r-1\right\} \cup \left\{\sigma(i+1), \sigma^2(i+1)\right\}$$

since $I \subseteq I_i$ and I_i is connected in σ. Hence

$$\left\{\sigma(i+1), \sigma^2(i+1)\right\} = \{i, n-i+1\}.$$

As $\sigma(i+1) > \frac{n+1}{2}$, it follows that $\sigma(i+1) = n-i+1$ and $\sigma^2(i+1) = i$. Consequently,

$$\sigma(i) > \max I_i = n-i+1 = \sigma(i+1).$$

This is a contradiction to $\sigma(i) < \sigma(i+1)$ and shows that $\sigma(i) \in I$.

It remains to show that $\sigma(i+1) \notin I$. Because $i \notin I$, $\sigma(i) \in I$ and I is connected,

$$I = \left\{\sigma^k(i) \mid k = 1, \ldots, r\right\}.$$

We can apply Lemma 43 to I in σ and $i < \frac{n+1}{2}$ to obtain $\sigma^{r+1}(i) > \frac{n+1}{2}$. Thus $\sigma^r(i) \leqslant \frac{n+1}{2}$. In particular, $\sigma^r(i) \neq n-i$.

If $i = \frac{n}{2} - 1$ then $I = \{i+1, n-i\}$ and it follows that $\sigma(i) = n-i$ and $\sigma^2(i) = i+1$. That is, $\sigma(i+1) \notin I$ as desired.

Now suppose $i < \frac{n}{2} - 1$. Then $i + 2 \leqslant \frac{n+1}{2}$ and we consider $I_{i+2} = [i+2, n-i-1]$. Assume for the sake of contradiction that $\sigma(i+1) \in I$. This means that $\sigma^r(i) \neq i+1$. In addition, we have already seen that $\sigma^r(i) \neq n-i$. Therefore, $\sigma^r(i) \in I_{i+2}$. Since I_{i+2} is connected in σ and $I_{i+2} \subseteq I$, we have

$$I_{i+2} = \left\{ \sigma^k(i) \mid k = 3, \ldots, r \right\}.$$

and hence $\{\sigma(i), \sigma^2(i)\} = \{i+1, n-i\}$. As $i < \frac{n+1}{2}$, it follows that $\sigma(i) = n-i$ and $\sigma^2(i) = i+1$. But then

$$\sigma(i) = n - i > n - i - 1 = \max I_{i+2} \geqslant \sigma(i+1)$$

which again contradicts the assumption $\sigma(i) < \sigma(i+1)$ and thus shows that $\sigma(i+1) \notin I$.

Case 4. Suppose $i > \frac{n+1}{2}$. Assume $\sigma \approx \sigma'$ and let $\nu\colon \mathfrak{S}_n \to \mathfrak{S}_n, x \mapsto x^{w_0}$, $\tau := \nu(\sigma)$ and $\tau' := \nu(\sigma')$. Since σ is oscillating and has connected intervals, Lemma 41 implies that τ is oscillating and has connected intervals. In addition, from Lemma 42 we have $\tau \approx \tau'$. Because $\tau' = s_{n-i}\tau s_{n-i}$ with $n - i < \frac{n+1}{2}$, we now obtain from the already proven cases that τ' is oscillating and has connected intervals. Hence, $\sigma' = \nu(\tau')$ and Lemma 41 yield that σ' is oscillating with connected intervals. □

From Lemma 40 we know that the element in stair form $\sigma_{(n)}$ is oscillating and has connected intervals. Recall that $\sigma_{(n)} \in \Sigma_{(n)}$. Thus by Lemma 46 the relation $\approx$ propagates these properties to all elements of $\Sigma_{(n)}$. In order to prove Theorem 49, it hence remains to show that each n-cycle which is oscillating and has connected intervals is $\approx$-equivalent to $\sigma_{(n)}$. To this end, we now use an algorithm that takes an oscillating n-cycle $\sigma \in \mathfrak{S}_n$ with connected intervals as input and successively conjugates σ with simple reflections until we obtain $\sigma_{(n)}$. This algorithm has the property that all permutations appearing as interim results are oscillating with connected intervals and $\approx$-equivalent to σ. Eventually, it follows that $\sigma \approx \sigma_{(n)}$.

The mechanism of the algorithm is due to Kim [17]. She used it in order to show that for each $\alpha \vDash_e n$ the element in stair form σ_α has maximal length in its conjugacy class. The next lemma corresponds to one step of the algorithm.

Lemma 47. *Let $\alpha = (n)$ and $\sigma \in \mathfrak{S}_n$ be an oscillating n-cycle with connected intervals which is different from the element in stair form σ_α. Then there exists a minimal integer p such that $1 \leqslant p \leqslant n-1$ and $\sigma^p(1) \neq \sigma_\alpha^p(1)$. Set $a := \sigma^p(1)$, $b := \sigma_\alpha^p(1)$ and*

$$\sigma' := \begin{cases} s_{a-1}\sigma s_{a-1} & \text{if } a > b \\ s_a \sigma s_a & \text{if } a < b. \end{cases}$$

Then $\sigma' \approx \sigma$ and σ' is oscillating and has connected intervals.

Proof. Set $I_k := [k, n-k+1]$ for all $k \in \mathbb{N}$ with $k \leqslant \frac{n+1}{2}$. Because $\sigma \neq \sigma_\alpha$ and both permutations are n-cycles, we have $p \leqslant n-2$. Recall that by Definition 14,

$$\sigma_\alpha = \begin{cases} (1, n, 2, n-1, \ldots, \frac{n}{2}, \frac{n}{2}+1) & \text{if } n \text{ is even} \\ (1, n, 2, n-1, \ldots, \frac{n-1}{2}, \frac{n+3}{2}, \frac{n+1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

If n is odd then $\frac{n+1}{2} = \sigma^{n-1}(1)$ and hence $p \leqslant n-2$ implies $b \neq \frac{n+1}{2}$. If n is even then $b \neq \frac{n+1}{2}$ anyway.

We assume $b < \frac{n+1}{2}$. The proof in the case $b > \frac{n+1}{2}$ is similar and therefore omitted. By the choice of p, we have $b \neq 1$ so that $1 < b < \frac{n+1}{2}$. The definition of σ_α implies

$$\begin{aligned} \left\{\sigma_\alpha^k(1) \mid k = 0, \ldots, p-1\right\} &= [n] \setminus I_b, \\ \left\{\sigma_\alpha^k(1) \mid k = p, \ldots, n-1\right\} &= I_b. \end{aligned} \tag{4.6}$$

Again by the choice of p, the same equalities hold for σ. Hence, $b < a$ as $a \in I_b$ and $b = \min I_b$. Therefore, we consider $\sigma' = s_{a-1}\sigma s_{a-1}$ and show that $\sigma \approx \sigma'$. Then Lemma 46 implies that σ' also is oscillating and has connected intervals.

It follows from the definition of σ_α and $b < \frac{n+1}{2}$ that

$$\sigma^{-1}(a) = \sigma_\alpha^{-1}(b) = n - b + 2 > \frac{n+1}{2}. \tag{4.7}$$

As σ is oscillating, we obtain that $a \leqslant \frac{n+1}{2}$ from Lemma 35. Since Equation (4.6) holds for σ and $p > 0$,

$$\sigma^{-1}(a) \notin I_b \supseteq I_{a-1} \supseteq I_a.$$

Let $r := |I_a|$. Because I_a is connected in σ, $a \in I_a$ and $\sigma^{-1}(a) \notin I_a$, we have

$$\left\{\sigma^k(a) \mid k = 0, \ldots, r-1\right\} = I_a.$$

Now we can use that $I_{a-1} = I_a \cup \{a-1, n-a+2\}$ is connected in σ and that $\sigma^{-1}(a) \notin I_{a-1}$ to obtain

$$\left\{\sigma^k(a) \mid k = 0, \ldots, r+1\right\} = I_{a-1}$$

The descriptions of I_a and I_{a-1} imply that

$$\left\{\sigma^r(a), \sigma^{r+1}(a)\right\} = \{a-1, n-a+2\}.$$

Lemma 43 applied to I_a in σ and $\sigma^{-1}(a) > \frac{n+1}{2}$ now imply that $\sigma^r(a) < \frac{n+1}{2}$. Thus, $\sigma^r(a) = a-1$ and $\sigma^{r+1}(a) = n-a+2$. That is,

$$\sigma(a-1) = n - a + 2 \tag{4.8}$$

Moreover, $\sigma^{-1}(a-1) \in I_a$ implies

$$\sigma^{-1}(a-1) \leqslant n - a + 1. \tag{4.9}$$

We now show

$$\sigma(a) \leqslant n - a + 1. \tag{4.10}$$

and deal with two cases. If $a = \frac{n+1}{2}$ then $n - a + 1 = a$. Furthermore, we then have $r = 1$ and therefore $\sigma(a) = a - 1 < n - a + 1$. If $a < \frac{n+1}{2}$ then $r > 1$ so that $\sigma(a) \in I_a$ and thus $\sigma(a) \leqslant n - a + 1$ as desired.

From eqs. (4.7) and (4.9) it follows that

$$\sigma^{-1}(a-1) \leqslant n - a + 1 < n - b + 2 = \sigma^{-1}(a).$$

Moreover, eqs. (4.8) and (4.10) imply

$$\sigma(a-1) = n - a + 2 > n - a + 1 \geqslant \sigma(a).$$

Since $\sigma' = s_{a-1}\sigma s_{a-1}$, Lemma 25 now yields $\ell(\sigma') = \ell(\sigma)$. Hence, $\sigma' \approx \sigma$ by Lemma 23. $\square$

Example 48. Let $n = 5$ and $\alpha = (n)$. The n-cycle $\sigma = (1, 3, 4, 2, 5) \in \mathfrak{S}_n$ is oscillating and has connected intervals. We can successively use Lemma 47 in order to obtain the sequence

$$\begin{aligned}
\sigma = \sigma^{(0)} &= (1,3,4,2,5),\\
\sigma^{(1)} &= (1,4,3,2,5) = s_3\sigma^{(0)}s_3,\\
\sigma^{(2)} &= (1,5,3,2,4) = s_4\sigma^{(1)}s_4,\\
\sigma^{(3)} &= (1,5,2,3,4) = s_2\sigma^{(2)}s_2,\\
\sigma_\alpha = \sigma^{(4)} &= (1,5,2,4,3) = s_3\sigma^{(3)}s_3.
\end{aligned}$$

Moreover, Lemma 47 ensures that each $\sigma^{(j)}$ is oscillating with connected intervals and all $\sigma^{(j)}$ are $\approx$-equivalent. Therefore, $\sigma \in \Sigma_\alpha$ by Theorem 18.

We now come to the main result of this section, the characterization of $\Sigma_{(n)}$.

Theorem 49. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. Then $\sigma \in \Sigma_{(n)}$ if and only if σ is oscillating and has connected intervals.*

Proof. Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. Recall that $\sigma \in \Sigma_{(n)}$ if and only if $\sigma \approx \sigma_{(n)}$ by Theorem 18. Assume that $\sigma \in \Sigma_{(n)}$. Then $\sigma \approx \sigma_{(n)}$ which by definition of $\approx$ implies that there are sequences $\sigma_\alpha = \sigma^{(0)}, \sigma^{(1)}, \dots, \sigma^{(m)} = \sigma \in \mathfrak{S}_n$ and $i_1, \dots, i_m \in [n-1]$ such that $\sigma^{(j-1)} \approx \sigma^{(j)}$ and $\sigma^{(j)} = s_{i_j}\sigma^{(j-1)}s_{i_j}$ for $j \in [m]$. From Lemma 40 we have that $\sigma_{(n)}$ is oscillating and has connected intervals. Moreover, Lemma 46 yields that $\sigma^{(j)}$ is oscillating with connected intervals if $\sigma^{(j-1)}$ is oscillating with connected intervals. Hence, σ is oscillating and has connected intervals by induction.

Conversely, assume that σ is oscillating and has connected intervals. Then we can use Lemma 47 iteratively to obtain a sequence of $\approx$-equivalent n-cycles starting with σ and eventually ending with σ_α. Thus $\sigma \approx \sigma_\alpha$. $\square$

The goal of the remainder of this section is to find bijections that relate $\Sigma_{(n-1)}$ to $\Sigma_{(n)}$. From this we will obtain a recursive description of $\Sigma_{(n)}$ and a formula for the cardinality of $\Sigma_{(n)}$. To achieve our goal, we define two operators ins and del.

$$\begin{tikzcd}
(1,3,4,2,5) \arrow[dr, shift left, "\mathrm{del}_3"] & (1,4,3,2,5) \arrow[d, shift left, "\mathrm{del}_3"] & (1,4,2,3,5) \arrow[dl, shift left, "\mathrm{del}_3"] \\
& (1,3,2,4) \arrow[ul, shift left, "\mathrm{ins}_{3,1}"] \arrow[u, shift left, "\mathrm{ins}_{3,2}"] \arrow[ur, shift left, "\mathrm{ins}_{3,3}"] \arrow[d, shift left, "\mathrm{del}_3"] & \\
& (1,2,3) \arrow[u, shift left, "\mathrm{ins}_{3,1}"] &
\end{tikzcd}$$

Figure 1: Examples for the operators del_k and $\mathrm{ins}_{k,p}$ appearing in Theorem 50 and its proof. The lower part of the picture serves as an example for the operators used in the case when n is even. The upper part is an example for those used in the case when n is odd. Note that for the integer m from the theorem we have $m = \frac{n}{2} + 1 = 3$ if $n = 4$ and $m = \frac{n+1}{2} = 3$ if $n = 5$.

Assume that the n-cycle $\sigma \in \mathfrak{S}_n$ is given in cycle notation starting with 1. Then for $k \in [2, n+1]$ $\mathrm{ins}_{k,p}(\sigma) \in \mathfrak{S}_{n+1}$ is the $(n+1)$-cycle obtained from σ by adding 1 to each element greater or equal to k in σ and then inserting k behind the pth element in the resulting cycle. Likewise, for $k \in [2, n]$, $\mathrm{del}_k(\sigma) \in \mathfrak{S}_{n-1}$ is the $(n-1)$-cycle obtained by first deleting k from σ and then decreasing each element greater than k by 1. See Figure 1 for examples.

We now define ins and del more formally. Let $\sigma \in \mathfrak{S}_n$ be an n-cycle and $k \in \mathbb{N}$. Set

$$\varepsilon_r := \begin{cases} 0 & \text{if } \sigma^r(1) < k \\ 1 & \text{if } \sigma^r(1) \geqslant k \end{cases}$$

for $r = 0, \ldots, n-1$. In the following we will assume $k > 1$. The operators could also be defined for $k = 1$ but this is not necessary for our purposes and would only make the exposition less transparent.

For $k \in [2, n+1]$ and $p \in [n]$, define $\mathrm{ins}_{k,p}(\sigma)$ to be the $(n+1)$-cycle of $\mathfrak{S}_{n+1}$ given by

$$\mathrm{ins}_{k,p}(\sigma)^r(1) := \begin{cases} \sigma^r(1) + \varepsilon_r & \text{if } r < p \\ k & \text{if } r = p \\ \sigma^{r-1}(1) + \varepsilon_{r-1} & \text{if } r > p \end{cases}$$

for $r = 0, \ldots, n$. For $k \in [2, n]$, define $\mathrm{del}_k(\sigma)$ to be the $(n-1)$-cycle of $\mathfrak{S}_{n-1}$ given by

$$\mathrm{del}_k(\sigma)^r(1) := \begin{cases} \sigma^r(1) - \varepsilon_r & \text{if } r < p \\ \sigma^{r+1}(1) - \varepsilon_{r+1} & \text{if } r \geqslant p \end{cases}$$

for $r = 0, \ldots, n-2$ where p is the element of $[0, n-1]$ with $\sigma^p(1) = k$.

The next results relates $\Sigma_{(n)}$ with $\Sigma_{(n-1)}$ via a bijection for $n \geqslant 4$.

Theorem 50. *Suppose $n \geqslant 4$. If n is even then set $m := \frac{n}{2} + 1$ and*

$$\psi \colon \Sigma_{(n-1)} \to \Sigma_{(n)}, \quad \sigma \mapsto \mathrm{ins}_{m,p}(\sigma)$$

where p is the element of $[n-1]$ with $\sigma^{p-1}(1) = \min\left\{\sigma^{-1}(\frac{n}{2}), \frac{n}{2}\right\}$. If n is odd then set $m := \frac{n+1}{2}$ and

$$\psi \colon \Sigma_{(n-1)} \times \{0,1,2\} \to \Sigma_{(n)}, \quad (\sigma, q) \mapsto \mathrm{ins}_{m,p+q}(\sigma)$$

where p is the element of $[n-3]$ with $\sigma^{p-1}(1) \notin \{m-1, m\}$ and $\sigma^{p}(1) \in \{m-1, m\}$. Then ψ is a bijection.

Corollary 51. *Suppose $n \geqslant 4$. Then*

$$\left|\Sigma_{(n)}\right| = \begin{cases} \left|\Sigma_{(n-1)}\right| & \text{if } n \text{ is even} \\ 3\left|\Sigma_{(n-1)}\right| & \text{if } n \text{ is odd.} \end{cases}$$

Proof of Theorem 50. Theorem 49 states that for all $n \in \mathbb{N}$, $\Sigma_{(n)}$ is the set of oscillating n-cycles of $\mathfrak{S}_n$ with connected intervals. In this proof we repeatedly use this result without further notice.

Let $n \geqslant 4$. We consider all permutations in the cycle notation where 1 is the leftmost entry in its cycle. In particular, deleting an entry from a permutation or inserting an entry into a permutation means that we do this in the chosen cycle notation. We distinguish two cases depending on the parity of n.

Case 1. Assume that n is even. Then $m = \frac{n}{2} + 1$. For $\tau \in \Sigma_{(n-1)}$ let p be given as in the definition of ψ. Then $\min\left\{\tau^{-1}(\frac{n}{2}), \frac{n}{2}\right\}$ is the pth element in the cycle notation of τ. Hence, we obtain $\psi(\tau)$ by increasing each element in τ greater or equal to m by one and then inserting m behind the element at position p.

Set $\varphi \colon \Sigma_{(n)} \to \Sigma_{(n-1)}, \sigma \mapsto \mathrm{del}_m(\sigma)$. That is, for $\sigma \in \Sigma_{(n)}$ we obtain $\varphi(\sigma)$ by first deleting m from σ and then decreasing each entry greater than m by 1.

We show that φ and ψ are well defined and inverse to each other.

(1) We prove that φ is well defined. Let $\sigma \in \Sigma_{(n)}$ and $\tau := \varphi(\sigma)$. We have to show that $\tau \in \Sigma_{(n-1)}$. That is, we have to prove that τ is oscillating and has connected intervals.

To show the latter, let $1 \leqslant i \leqslant \frac{n-1}{2} < \frac{n}{2}$. As $[i, n-i+1]$ is connected in σ there is a $0 \leqslant q \leqslant n-1$ such that

$$\left\{\sigma^{q+1}(1), \ldots, \sigma^{q+r}(1)\right\} = [i, n-i+1]$$

where $r := |[i, n-i+1]|$. Moreover, $m \in [i, n-i+1]$. Thus, $\tau = \mathrm{del}_m(\sigma)$ implies

$$\left\{\tau^{q+1}(1), \ldots, \tau^{q+r-1}(1)\right\} = [i, n-i].$$

Hence, $[i, (n-1)-i+1]$ is connected in τ. It follows that τ has connected intervals.

We now show that τ is oscillating. Note that $n-1$ is odd and $\frac{(n-1)+1}{2} = \frac{n}{2}$. By Lemma 35, it suffices to show that $\tau(i) \geqslant \frac{n}{2}$ for all $i \in [\frac{n}{2} - 1]$ and that either $\tau^{-1}\left(\frac{n}{2}\right) > \frac{n}{2}$ or $\tau\left(\frac{n}{2}\right) > \frac{n}{2}$.

Let $i \in [\frac{n}{2} - 1]$. Since $i < \frac{n}{2}$ and σ is oscillating, we infer $\sigma(i) > \frac{n}{2}$ from Lemma 35. If $\sigma(i) \neq m$ then $\tau(i) = \sigma(i) - 1 \geqslant \frac{n}{2}$. If $\sigma(i) = m$ then $\sigma^2(i) = \frac{n}{2}$ since $m = \frac{n}{2} + 1$, $\left\{\frac{n}{2}, \frac{n}{2} + 1\right\}$ is connected in σ and $i \notin \left\{\frac{n}{2}, \frac{n}{2} + 1\right\}$. Thus, $\tau(i) = \frac{n}{2}$.

We now show that either $\tau^{-1}\left(\frac{n}{2}\right) > \frac{n}{2}$ or $\tau\left(\frac{n}{2}\right) > \frac{n}{2}$. Since $\left\{\frac{n}{2}, \frac{n}{2} + 1\right\}$ is connected in σ there is a $0 \leqslant q \leqslant n - 1$ such that

$$\left\{\sigma^q(1), \sigma^{q+1}(1)\right\} = \left\{\frac{n}{2}, \frac{n}{2} + 1\right\}.$$

Hence, $\tau = \mathrm{del}_{\frac{n}{2}+1}(\sigma)$ implies $\tau^q(1) = \frac{n}{2}$. Because $n \geqslant 4$, we can apply Lemma 43 to $\left\{\frac{n}{2}, \frac{n}{2} + 1\right\}$ in σ and obtain that there are $a < \frac{n}{2}$ and $b > \frac{n}{2} + 1$ such that

$$\left\{\sigma^{q-1}(1), \sigma^q(1), \sigma^{q+1}(1), \sigma^{q+2}(1)\right\} = \left\{a, b, \frac{n}{2}, \frac{n}{2} + 1\right\}.$$

Therefore, $\tau^q(1) = \frac{n}{2}$ and $\tau = \mathrm{del}_{\frac{n}{2}+1}(\sigma)$ yield $\left\{\tau^{-1}\left(\frac{n}{2}\right), \tau\left(\frac{n}{2}\right)\right\} = \{a, b - 1\}$. That is, either $\tau^{-1}\left(\frac{n}{2}\right) > \frac{n}{2}$ or $\tau\left(\frac{n}{2}\right) > \frac{n}{2}$. Thus, τ is oscillating.

(2) We check that ψ is well defined. Let $\tau \in \Sigma_{(n-1)}$ and $\sigma := \psi(\tau)$. We have to show $\sigma \in \Sigma_{(n)}$.

The definition of ψ implies that $\frac{n}{2} + 1$ is a neighbor of $\frac{n}{2}$ in σ. In addition, $[i, n - i]$ is connected in τ for $i \in [\frac{n}{2} - 1]$. Therefore, $[i, n - i + 1]$ is connected in σ for $i \in [\frac{n}{2}]$. That is, σ has connected intervals.

We now show that σ is oscillating. By Lemma 35, it suffices to show that $\sigma(i) > \frac{n}{2}$ for all $i \in [\frac{n}{2}]$. For $i < \frac{n}{2}$ this can be done as before. Thus, we only consider $i = \frac{n}{2}$. As τ is oscillating, Lemma 35 implies that one of the neighbors of $\frac{n}{2}$ is smaller than $\frac{n}{2}$ and the other one is greater than $\frac{n}{2}$. Let a be the smaller and b be the bigger neighbor of $\frac{n}{2}$. In the definition of ψ, p is chosen such that $\frac{n}{2} + 1$ is inserted in τ between a and $\frac{n}{2}$. Thus, $\frac{n}{2}$ has neighbors $\frac{n}{2} + 1$ and $b + 1$ in σ. Consequently, $\sigma\left(\frac{n}{2}\right) > \frac{n}{2}$.

(3) We now show that $\psi \circ \varphi = \mathrm{id}$. Let $\sigma \in \Sigma_{(n)}$. Since $\left\{\frac{n}{2}, \frac{n}{2} + 1\right\}$ is connected in σ, these two elements are neighbors in σ. As σ is oscillating, there is an $a < \frac{n}{2}$ such that $\frac{n}{2} + 1$ has neighbors a and $\frac{n}{2}$. We obtain $\varphi(\sigma)$ from σ by deleting $\frac{n}{2} + 1$ so that a and $\frac{n}{2}$ are neighbors in $\varphi(\sigma)$. On the other hand, we obtain $\psi(\varphi(\sigma))$ from $\varphi(\sigma)$ by inserting $\frac{n}{2} + 1$ between a and $\frac{n}{2}$. Thus $\psi(\varphi(\sigma)) = \sigma$.

(4) Finally, we show that $\varphi \circ \psi = \mathrm{id}$. Let $\tau \in \Sigma_{(n-1)}$. Then we obtain $\psi(\tau)$ from τ by inserting $\frac{n}{2} + 1$ at some position and get $\varphi(\psi(\tau))$ from $\psi(\tau)$ by deleting it again. Hence, $\varphi(\psi(\tau)) = \tau$.

Case 2. Assume that n is odd. Then $m = \frac{n+1}{2}$. For $\tau \in \Sigma_{(n-1)}$ the set $\{m - 1, m\}$ is connected. Thus, there is a unique integer p with $1 \leqslant p \leqslant n - 3$ such that $\tau^{p-1}(1) \notin \{m - 1, m\}$ and $\tau^p(1) \in \{m - 1, m\}$. That is, the integer p from the definition of ψ in the theorem is well defined. Note that p is the position of the left neighbor of the set $\{m - 1, m\}$ in τ.

Conversely, for $\sigma \in \Sigma_{(n)}$, $I := \{m - 1, m, m + 1\}$ is connected in σ. Hence, there is a unique $0 \leqslant p \leqslant n - 1$ such that $I = \left\{\sigma^{p+k}(1) \mid k = 0, 1, 2\right\}$ and a unique $q \in \{0, 1, 2\}$

such that $\sigma^{p+q}(1) = m$. We define the map $\varphi\colon \Sigma_{(n)} \to \Sigma_{(n-1)} \times \{0,1,2\}$ by setting $\varphi(\sigma) := (\mathrm{del}_m(\sigma), q)$. Again, we show that φ and ψ are well defined and inverse to each other.

(1) First we show that the two maps are inverse to each other. Let $\sigma \in \Sigma_{(n)}$ and $\varphi(\sigma) = (\tau, q)$. Then we have

$$q = \begin{cases} 0 & \text{if } m \text{ is the left neighbor of } \{m-1, m+1\} \text{ in } \sigma, \\ 1 & \text{if } m \text{ is located between } m-1 \text{ and } m+1 \text{ in } \sigma, \\ 2 & \text{if } m \text{ is the right neighbor of } \{m-1, m+1\} \text{ in } \sigma. \end{cases}$$

Conversely, let $\tau \in \Sigma_{(n-1)}$, $q \in \{0,1,2\}$ and $\sigma = \psi(\tau, q)$ then

$$m \text{ is } \begin{cases} \text{the left neighbor of } \{m-1, m+1\} \text{ in } \sigma & \text{if } q = 0, \\ \text{located between } m-1 \text{ and } m+1 \text{ in } \sigma & \text{if } q = 1, \\ \text{the right neighbor of } \{m-1, m+1\} \text{ in } \sigma & \text{if } q = 2. \end{cases} \tag{4.11}$$

From this it follows that φ and ψ are inverse to each other.

(2) In order to prove that φ is well defined one has to show that $\mathrm{del}_m(\sigma) \in \Sigma_{(n-1)}$. This can be done similarly as in Case 1.

(3) To see that ψ is well defined, let $\tau \in \Sigma_{(n-1)}$, $q \in \{0,1,2\}$ and $\sigma := \psi(\tau, q)$. We first show that σ has connected intervals. Recall that $m = \frac{n+1}{2}$. Let $i \leqslant \frac{n-1}{2} = m - 1$. Then $[i, n-i]$ is connected in τ since τ has connected intervals. By the definition of ψ, we obtain the entries $[i, n-i+1]$ in σ by adding 1 to each entry $\geqslant m$ of $[i, n-i]$ in τ and then inserting m such that by Equation (4.11) at least one of the neighbors of m is $m-1$ or $m+1$. Since $m-1, m, m+1 \in [i, n-i+1]$ it follows that $[i, n-i+1]$ is connected in σ. Therefore, σ has connected intervals.

In order to show that σ is oscillating, let τ' be the $(n-1)$-cycle of $\mathfrak{S}_n$ obtained by adding 1 to each entry of τ which is greater or equal than m. Since τ is oscillating, the entries in τ' alternate between the sets $[m-1]$ and $[m+1, n]$. Furthermore, we obtain σ from τ' by inserting m somewhere in τ'. Thus, Corollary 36 implies that σ is oscillating. $\square$

From Table 1 we know $\Sigma_{(n)}$ for $n = 1, 2, 3$. That is, Theorem 50 allows us to compute $\Sigma_{(n)}$ recursively for each $n \in \mathbb{N}$. This is illustrated in the following.

Example 52. We want to compute $\Sigma_{(n)}$ for $n = 4, 5$. To do this we use the bijections ψ and the related notation introduced in Theorem 50.

(1) Consider $n = 4$. We have

$$\Sigma_{(4)} = \left\{ \psi(\sigma) \mid \sigma \in \Sigma_{(3)} \right\}$$

by Theorem 50. From Table 1 we obtain $\Sigma_{(3)} = \{(1,3,2), (1,2,3)\}$.

For $\sigma = (1,3,2)$ we have $p = 3$ since

$$\sigma^{3-1}(1) = 2 = \min\{2, 3\} = \min\left\{ \sigma^{-1}\left(\frac{4}{2}\right), \frac{4}{2} \right\}.$$

Thus,

$$\psi(\sigma) = \mathrm{ins}_{3,3}((1,3,2)) = (1,3+1,2,3) = (1,4,2,3).$$

For $\sigma = (1,2,3)$ we have $p = 1$ and

$$\psi(\sigma) = \mathrm{ins}_{3,1}((1,2,3)) = (1,3,2,3+1) = (1,3,2,4).$$

Therefore, $\Sigma_{(4)} = \{(1,4,2,3),(1,3,2,4)\}$.

(2) Consider $n = 5$. Theorem 50 yields

$$\Sigma_{(5)} = \left\{\psi(\sigma,q) \mid \sigma \in \Sigma_{(4)}, q \in \{0,1,2\}\right\}. \tag{4.12}$$

Let $m = \frac{5+1}{2} = 3$ and $I = \{m-1,m\} = \{2,3\}$.

For $\sigma = (1,4,2,3)$ we have $p = 2$ since $\sigma^{2-1}(1) = 4 \notin I$ and $\sigma^2(1) = 2 \in I$. Thus, for instance we have

$$\psi(\sigma,1) = \mathrm{ins}_{3,3}((1,4,2,3)) = (1,4+1,2,3,3+1) = (1,5,2,3,4).$$

For $\sigma = (1,3,2,4)$ we have $p = 1$. Computing $\psi(\sigma,q)$ for all $\sigma \in \Sigma_{(4)}$ and $q \in \{0,1,2\}$, we obtain the following table. By Equation (4.12), it lists all elements of $\Sigma_{(5)}$.

$\psi(\sigma,q)$	0	1	2
$(1,4,2,3)$	$(1,5,3,2,4)$	$(1,5,2,3,4)$	$(1,5,2,4,3)$
$(1,3,2,4)$	$(1,3,4,2,5)$	$(1,4,3,2,5)$	$(1,4,2,3,5)$

Corollary 53. *Let $n \in \mathbb{N}$. Then*

$$\left|\Sigma_{(n)}\right| = \begin{cases} 1 & \text{if } n \leqslant 2 \\ 2 \cdot 3^{\lfloor \frac{n-3}{2} \rfloor} & \text{if } n \geqslant 3. \end{cases}$$

Proof. Let $x_n := |\Sigma_{(n)}|$ for $n \geqslant 1$, $y_1 := y_2 := 1$ and $y_n := 2 \cdot 3^{\left\lfloor \frac{n-3}{2} \right\rfloor}$ for $n \geqslant 3$. We show that both sequences have the same initial values and recurrence relations. First note that

$$(x_1,x_2,x_3) = (1,1,2) = (y_1,y_2,y_3).$$

where we obtain the x_i from Table 1. Now let $n \geqslant 4$. By Corollary 51 we have to show that $y_n = y_{n-1}$ if n is even and $y_n = 3y_{n-1}$ if n is odd. If n is even, we have

$$\left\lfloor \frac{n-3}{2} \right\rfloor = \left\lfloor \frac{n-4}{2} + \frac{1}{2} \right\rfloor = \frac{n-4}{2} = \left\lfloor \frac{n-1-3}{2} \right\rfloor$$

and thus $y_n = y_{n-1}$. If n is odd, we have

$$\left\lfloor \frac{n-3}{2} \right\rfloor = \frac{n-3}{2} = \frac{n-5}{2} + 1 = \left\lfloor \frac{n-5}{2} + \frac{1}{2} \right\rfloor + 1 = \left\lfloor \frac{n-4}{2} \right\rfloor + 1$$

and hence $y_n = 3y_{n-1}$. □

4.2 Equivalence classes of odd hook type

Let $\alpha = (k, 1^{n-k}) \vDash n$ be a hook. Then α is a maximal composition. Recall that a hook α is called *odd* if k is odd and *even* otherwise. In the main result of this section, Theorem 69, we show for each odd hook α that the elements of Σ_α are characterized by three combinatorial properties which we call the *hook properties* (Definition 63). Two of these properties are generalizations of the concepts of being oscillating and having connected intervals. The third property is a requirement on the orbits on $[n]$ of the permutation in question.

We emphasize that the hook properties are defined for all hooks α but in this section we only show that Σ_α is characterized by them if α is odd. In the next section we prove in Theorem 93 that the characterization also holds if α is an even hook. This will follow from an application of the *inductive product*, which is the topic of that section.

The current section is structured as follows. We first generalize the properties of being oscillating and having connected intervals from n-cycles to all permutations. In Corollary 60 and Lemma 61 we then show that the relation $\approx$ also propagates the generalized properties. After defining the hook properties in Definition 63, the argumentation leading to Theorem 69 is similar to that in the last section. Let α be an odd hook. We show that the element in stair form σ_α has the hook properties (Lemma 65), that $\approx$ preserves the hook properties (Lemma 67) and that for each permutation of cycle type α satisfying the hook properties there is a chain of $\approx$-equivalent permutations that ends up at σ_α (Lemma 68). The latter result is based on a generalization of the algorithm used in Lemma 47 for the case of n-cycles. After having established Theorem 69, we use it for odd $k \geqslant 3$ to obtain a bijection which allows to compute $\Sigma_{(k,1^{n-k})}$ from $\Sigma_{(k)}$ (Corollary 70) and infer a cardinality formula for $\Sigma_{(k,1^{n-k})}$ (Corollary 71).

Lets start with the generalization of being oscillating and having connected intervals. In order to do this, we standardize cycles in the following way. Let $\sigma := (c_1, \ldots, c_k) \in \mathfrak{S}_n$ be a k-cycle. Replace the smallest element among $c_1, \ldots, c_k$ by 1, the second smallest by 2 and so on. The result is a k-cycle with entries $1, 2, \ldots, k$ which can be regarded as an element $\mathfrak{S}_k$. This permutation is called the *cycle standardization* $\operatorname{cst}(\sigma)$ of σ.

Example 54. Consider $\sigma = (3, 11, 4, 10, 5) \in \mathfrak{S}_{11}$. Then $\operatorname{cst}(\sigma) = (1, 5, 2, 4, 3) \in \mathfrak{S}_5$ which is oscillating with connected intervals.

We formally define the cycle standardization as follows.

Definition 55. (1) Given $\sigma \in \mathfrak{S}_n$ and $i \in [n]$, there is a cycle $(c_1, \ldots, c_k)$ of σ containing i. Then we define

$$\rho_\sigma(i) := |\{j \in [k] \mid c_j \leqslant i\}|\,.$$

(2) Let $\sigma = (c_1, \ldots, c_k) \in \mathfrak{S}_n$ be a k-cycle. The *cycle standardization* of σ is the k-cycle of $\mathfrak{S}_k$ given by

$$\operatorname{cst}(\sigma) := (\rho_\sigma(c_1), \rho_\sigma(c_2), \ldots, \rho_\sigma(c_k)).$$

Note that the permutation $\mathrm{cst}(\sigma)$ is independent from the choice of the cycle notation $\sigma = (c_1, c_2, \ldots, c_k)$ in Definition 55.

Remark 56. Let $\sigma = (c_1, c_2, \ldots, c_k) \in \mathfrak{S}_n$ be a k-cycle.

(1) The anti-rank of $i \in [n]$ among the elements in its cycle in σ is $\rho_\sigma(i)$.

(2) For all $i, j \in [k]$ we have $c_i < c_j$ if and only if $\rho_\sigma(c_i) < \rho_\sigma(c_j)$.

(3) Let i be an element appearing in the cycle $(c_1, c_2, \ldots, c_k)$. Then we have

$$\mathrm{cst}(\sigma)(\rho_\sigma(i)) = \rho_\sigma(\sigma(i)).$$

We now generalize the notions of being oscillating and having connected intervals to arbitrary permutations via the cycle decomposition and the cycle standardization. Recall that trivial cycles are those of length 1.

Definition 57. Let $\sigma \in \mathfrak{S}_n$ and write σ as a product $\sigma = \sigma_1 \cdots \sigma_l$ of disjoint cycles including the trivial ones.

(1) We say that σ is *oscillating* if $\mathrm{cst}(\sigma_i)$ is oscillating for each cycle σ_i.

(2) We say that σ has *connected intervals* if $\mathrm{cst}(\sigma_i)$ has connected intervals for each cycle σ_i

Let $(c) \in \mathfrak{S}_n$ be a trivial cycle. Then $\mathrm{cst}((c)) = (1) \in \mathfrak{S}_1$ which is oscillating and has connected intervals. Therefore, in order to show that a permutation σ is oscillating (has connected intervals) it suffices to consider the nontrivial cycles.

Example 58. Let $\alpha = (4, 5, 3, 1) \vDash_e 13$ and

$$\sigma_\alpha = (1, 13, 2, 12)(3, 11, 4, 10, 5)(9, 6, 8)(7).$$

The cycle standardizations of the nontrivial cycles of σ_α are

$$(1, 4, 2, 3), (1, 5, 2, 4, 3) \text{ and } (1, 2, 3).$$

Each of these three permutations is oscillating and has connected intervals (see Table 1). Thus, σ_α is oscillating and has connected intervals.

Assume that $\sigma \in \mathfrak{S}_n$ is an n-cycle. Then σ has only one cycle σ in cycle notation and $\mathrm{cst}(\sigma) = \sigma$. Thus, for n-cycles our new notion of being oscillating (having connected intervals) from Definition 57 is equivalent to the old concept from Definition 30 (Definition 38).

We now prove some general results on oscillating permutations with connected intervals. As in the last section, we are interested in the effect of swapping entries i and $i+1$ in cycle notation (that is, conjugating with s_i). This will in particular be useful to prove our results on odd hooks. We consider the case where i and $i+1$ appear in the same cycle first.

Lemma 59. *Let $\sigma \in \mathfrak{S}_n$ and write σ as a product $\sigma = \sigma_1 \cdots \sigma_l$ of disjoint cycles. Assume that there is an $i \in [n-1]$ and a $k \in [l]$ such that i and $i+1$ both appear in the cycle σ_k. Set $i' := \rho_\sigma(i)$ and $\tau := \mathrm{cst}(\sigma_k)$. Then we have*

(1) $\mathrm{cst}(s_i \sigma_k s_i) = s_{i'} \tau s_{i'}$,

(2) $s_i \sigma s_i \approx \sigma$ *if and only if* $s_{i'} \tau s_{i'} \approx \tau$.

Proof. By the definition of ρ_σ, we have that $\rho_\sigma(j) = \rho_{\sigma_k}(j)$ for all entries j in the cycle σ_k.

(1) We obtain $s_i \sigma_k s_i$ from σ_k by interchanging i and $i+1$ in cycle notation. Since i and $i+1$ appear in σ_k, we have $\rho_{\sigma_k}(i+1) = i'+1$. Thus, we obtain $\mathrm{cst}(s_i \sigma_k s_i)$ from $\tau = \mathrm{cst}(\sigma_k)$ by interchanging i' and $i'+1$ in cycle notation. That is, $\mathrm{cst}(s_i \sigma s_i) = s_{i'} \tau s_{i'}$.

(2) We have $s_i \sigma s_i \approx \sigma$ if and only if $\ell(s_i \sigma s_i) = \ell(\sigma)$. By Lemma 25, this is the case if and only if either $\sigma(i) < \sigma(i+1)$ or $\sigma^{-1}(i) < \sigma^{-1}(i+1)$. From the definition of the cycle standardization we obtain that $\tau(\rho_\sigma(j)) = \rho_\sigma(\sigma(j))$ for each entry j in σ_k (cf. Remark 56). Moreover, by the definition of ρ_σ and the fact that i and $i+1$ appear in the same cycle of σ,

$$\sigma(i) < \sigma(i+1) \iff \rho_\sigma(\sigma(i)) < \rho_\sigma(\sigma(i+1)).$$

Hence,

$$\sigma(i) < \sigma(i+1) \iff \tau(i') < \tau(i'+1).$$

Similarly, one shows that this equivalence is also true for σ^{-1} and τ^{-1}. Therefore, we have $s_i \sigma s_i \approx \sigma$ if and only if either $\tau(i') < \tau(i'+1)$ or $\tau^{-1}(i') < \tau^{-1}(i'+1)$. As for σ, the latter is equivalent to $s_{i'} \tau s_{i'} \approx \tau$. □

We now infer from Lemma 59 that swaps of i and $i+1$ within a cycle that preserve $\approx$ also preserve the properties of being oscillating with connected intervals.

Corollary 60. *Let $\sigma \in \mathfrak{S}_n$ be oscillating with connected intervals, $i \in [n-1]$ such that i and $i+1$ appear in the same cycle of σ and $\sigma' := s_i \sigma s_i$. If $\sigma \approx \sigma'$ then σ' is oscillating with connected intervals.*

Proof. We write σ as a product $\sigma = \sigma_1 \cdots \sigma_l$ of disjoint cycles and choose k such that i and $i+1$ appear in the cycle σ_k. Moreover, we set $\tau := \mathrm{cst}(\sigma_k)$, $\tau' := \mathrm{cst}(s_i \sigma_k s_i)$ and m to be the length of the cycle σ_k.

As i and $i+1$ only appear in σ_k, $\sigma' = \sigma_1 \cdots \sigma_{k-1}(s_i \sigma_k s_i)\sigma_{k+1} \cdots \sigma_l$ is the decomposition of σ' in disjoint cycles. Since σ is oscillating with connected intervals, $\mathrm{cst}(\sigma_j)$ is oscillating with connected intervals for all $j \in [l]$. Therefore, it remains to show that τ' has these properties. Since $\sigma \approx \sigma'$, Lemma 59 yields that $\tau \approx \tau'$. In addition, τ is an oscillating m-cycle with connected intervals and thus $\tau \in \Sigma_{(m)}$ by Theorem 49. Hence, also $\tau' \in \Sigma_{(m)}$, i.e. τ' is oscillating with connected intervals. □

The next result is concerned with the interchange of i and $i+1$ between two cycles. This also preserves the properties of being oscillating and having connected intervals.

Lemma 61. *Let $\sigma \in \mathfrak{S}_n$ be oscillating with connected intervals, $i \in [n-1]$ such that i and $i+1$ appear in different cycles of σ and $\sigma' := s_i \sigma s_i$. Then σ' is oscillating and has connected intervals.*

Proof. We obtain σ' from σ by interchanging i and $i+1$ between two cycles in cycle notation. It is easy to see that this does not affect the cycle standardization of the cycles in question. In addition, all other cycles of σ' appear as cycles of σ. Since σ is oscillating with connected intervals, it follows that the standardization of each cycle of σ' is oscillating with connected intervals. That is, σ' is oscillating with connected intervals. $\square$

Note that from Corollary 60 and Lemma 61 it follows that $\approx$ propagates the properties of being oscillating and having connected intervals also in the general form. For n-cycles we showed this in Lemma 46. We now narrow our scope to hooks.

Example 62. Let $\alpha = (3,1,1) \vDash_e 5$. The elements of Σ_α are

$$(1,5,2),(1,2,5),(1,5,3),(1,3,5),(1,5,4),(1,4,5).$$

Note that 1 and 5 always appear in the cycle of length 3.

Recall that we use *type* as a short form for *cycle type*.

Definition 63. Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be a hook, $\sigma \in \mathfrak{S}_n$ of type α, $m := \frac{k-1}{2}$ if k is odd and $m := \frac{k}{2}$ if k is even. We say that σ satisfies the *hook properties* if

(1) σ is oscillating,

(2) σ has connected intervals,

(3) if $k > 1$ then i and $n-i+1$ appear in the cycle of length k of σ for all $i \in [m]$.

The permutations from Example 62 satisfy the hook properties. The main result of this section is that for an odd hook α, the elements of Σ_α are characterized by the hook properties (Theorem 69). In Theorem 93 of Section 4.3 we will see that the same is true for even hooks.

Example 64. (1) Let $\sigma \in \mathfrak{S}_n$ be of type (1^n). Then $\sigma = \mathrm{id}$ and σ satisfies the hook properties. Moreover, $\Sigma_{(1^n)} = \{\sigma\}$.

(2) Let $\sigma \in \mathfrak{S}_n$ be of type (n). That is, σ is an n-cycle. Then the third hook property is satisfied by σ since all elements of $[n]$ appear in the only cycle of σ. Thus, σ has the hook properties if and only if σ is oscillating with connected intervals. By Theorem 49, this is equivalent to $\sigma \in \Sigma_{(n)}$.

(3) Let $\alpha = (3,1,1) \vDash n$. We want to determine all permutations in $\mathfrak{S}_n$ of type α that satisfy the hook properties. Let $\sigma \in \mathfrak{S}_n$ be of type α, σ_1 be the cycle of length 3 of σ and $\mathcal{O}_1$ be the set of elements in σ_1.

Since σ_1 is the only nontrivial cycle of σ, σ is oscillating and has connected intervals if and only if $\tau := \mathrm{cst}(\sigma_1)$ has these properties. The type of τ is (3). By Theorem 49, the oscillating permutations of type (3) with connected intervals form $\Sigma_{(3)}$. From Table 1 we read $\Sigma_{(3)} = \{(1,3,2),(1,2,3)\}$.

The third hook property is satisfied by σ if and only if $\mathcal{O}_1 \in M$ where we set

$$M = \{\{1,5\} \cup \{j\} \mid j \in [2,4]\} = \{\{1,2,5\},\{1,3,5\},\{1,4,5\}\}.$$

Therefore, σ fulfills the hook properties if and only if there is a $\tau \in \Sigma_{(3)}$ and an $\mathcal{O}_1 \in M$ such that we obtain σ_1 by writing $\mathcal{O}_1$ in a cycle such that the relative order of entries matches that one in τ. For instance, from $\tau = (1,3,2)$ and $\mathcal{O}_1 = \{1,4,5\}$ we obtain $\sigma = (1,5,4)$. Going through all possibilities for τ and $\mathcal{O}_1$ we obtain the desired set of permutations. These are the ones shown in Example 62.

In order to show that Σ_α is characterized by the hook properties when α is an odd hook in Theorem 69, we follow the same strategy as in in the case of compositions with one part from Section 4.1: For any odd hook α we show that σ_α satisfies the hook properties (Lemma 65), $\approx$ is compatible with the hook properties (Lemma 67) and there is an algorithm that computes a sequence of $\approx$-equivalent permutations starting with σ and ending up with σ_α for each permutation σ of type α satisfying the hook properties (Lemma 68).

Lemma 65. *Let $\alpha \vDash_e n$ be an odd hook. Then the element in stair form $\sigma_\alpha \in \mathfrak{S}_n$ satisfies the hook properties.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_l) = (k, 1^{n-k}) \vDash_e n$ be an odd hook. If $k = 1$ then σ_α is the identity which satisfies the hook properties. Assume $k > 1$ and set $m := \frac{k-1}{2}$. By definition, the cycle of length k of σ_α is given by

$$\sigma_{\alpha_1} = (1, n, 2, n-1, \ldots, m, n-m+1, m+1).$$

Hence, σ_α satisfies the third hook property. In order to show that σ_α is oscillating and has connected intervals, it suffices to consider σ_{α_1} because the other cycles of σ_α are trivial. From the description of σ_{α_1} we obtain its cycle standardization

$$\operatorname{cst}(\sigma_{\alpha_1}) = (1, k, 2, k-1, \ldots, m, k-m+1, m+1).$$

That is, $\operatorname{cst}(\sigma_{\alpha_1})$ is the element in stair form $\sigma_{(k)}$ which is oscillating and has connected intervals by Lemma 40. $\square$

Let $\alpha \vDash_e n$ be an odd hook and $\sigma \in \mathfrak{S}_n$ be of type α satisfying the hook properties. In order to show $\sigma_\alpha \approx \sigma$ we will successively interchange elements i and $i+1$ in the cycle notation of σ. The next lemma considers the case where at least one of i and $i+1$ is a fixed point of σ.

Lemma 66. *Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be an odd hook, $m := \frac{k-1}{2}$ and $\sigma \in \mathfrak{S}_n$ of type α satisfying the hook properties. Furthermore, assume that there are $i, i+1 \in [m+1, n-m]$ such that i or $i+1$ is a fixed point of σ. Then $s_i\sigma s_i \approx \sigma$ and $s_i\sigma s_i$ satisfies the hook properties.*

Proof. If both i and $i+1$ are fixed points of σ then $s_i\sigma s_i = \sigma$ and there is nothing to show. Therefore, we assume that either i or $i+1$ is not a fixed point and call this element j. By choice of i and $i+1$, $m < j < n-m+1$. Since σ satisfies the hook properties, the cycle of length k of σ consists of the elements $1, \ldots, m, j, n-m+1, \ldots, n$.

First we show that $s_i\sigma s_i$ satisfies the hook properties. As σ is oscillating with connected intervals and i and $i+1$ appear in different cycles of σ, Lemma 61 yields that $s_i\sigma s_i$ is oscillating with connected intervals too. As we obtain $s_i\sigma s_i$ by interchanging i and $i+1$ in cycle notation of σ and

$$i, i+1 \notin \{1,\dots,m,n-m+1,\dots,n\},$$

$s_i\sigma s_i$ satisfies the third hook property.

In order to show $s_i\sigma s_i \approx \sigma$, we assume that $i+1$ is a fixed point of σ and i is not. The other case is proven analogously. Let $\tau := \mathrm{cst}(\sigma)$ and $i' := \rho_\sigma(i)$. Then $i' = m+1 = \frac{k+1}{2}$ by the description of the cycle of length k from above. Since σ is oscillating, τ is oscillating. Thus, Lemma 35 implies that there is an $\varepsilon \in \{-1,1\}$ such that

$$\tau^{\varepsilon}(i') > m+1 \text{ and } \tau^{-\varepsilon}(i') < m+1.$$

Now we use that $\tau^{\delta}(i') = \rho_\sigma(\sigma^{\delta}(i))$ for $\delta = -1, 1$ and obtain that

$$\sigma^{\varepsilon}(i) \geqslant n-m+1 \text{ and } \sigma^{-\varepsilon}(i) \leqslant m.$$

As $\sigma(i+1) = i+1 \in [m+2, n-m]$, it follows that

$$\sigma^{\varepsilon}(i) > \sigma^{\varepsilon}(i+1) \text{ and } \sigma^{-\varepsilon}(i) < \sigma^{-\varepsilon}(i+1).$$

Hence, Lemma 25 implies $\ell(s_i\sigma s_i) = \ell(\sigma)$. Therefore, $s_i\sigma s_i \approx \sigma$. $\square$

The following lemma shows that $\approx$ preserves the hook properties. It is an analogue to Lemma 46.

Lemma 67. *Given an odd hook $\alpha = (k, 1^{n-k}) \vDash_e n$, $\sigma \in \mathfrak{S}_n$ of type α satisfying the hook properties and $\sigma' := s_i\sigma s_i$ with $\sigma \approx \sigma'$, we have that also σ' satisfies the hook properties.*

Proof. We show that σ' has the hook properties. If $k = 1$ then $\sigma = \sigma' = \mathrm{id}$ so that σ' satisfies the hook properties. Hence, assume $k > 1$. Set $m := \frac{k-1}{2}$, $\tau := \mathrm{cst}(\sigma)$ and $\tau' := \mathrm{cst}(\sigma')$. We deal with three cases.

First, assume that neither i nor $i+1$ is a fixed point of σ. Then i and $i+1$ both appear in the cycle of length k of σ. Since σ satisfies the hook properties, it is oscillating and has connected intervals. Therefore, Corollary 60 yields that also σ' has these properties. The elements $1,\dots,m,n-m+1,\dots m$ all appear in the cycle of length k of σ because σ satisfies the hook properties. Since we interchange two entries in this cycle to obtain σ' from σ, all the elements also appear in the cycle of length k of σ'.

Second, assume that $i+1$ is a fixed point of σ but i is not. Since $\sigma \approx \sigma'$, we have $\ell(\sigma) = \ell(\sigma')$ and by Lemma 25

$$\begin{aligned} &\text{either } \sigma(i) > i+1 \text{ and } \sigma^{-1}(i) < i+1 \\ &\text{or } \sigma(i) < i+1 \text{ and } \sigma^{-1}(i) > i+1 \end{aligned} \tag{4.13}$$

where we used $\sigma(i+1) = i+1$. The elements of the cycle of length k of σ are $1, \dots, m, j, n-m+1, \dots, n$ where $j \in [m+1, n-m]$. We now show that $i, i+1 \in [m+1, n-m]$.

As $i+1$ is a fixed point, we have $i+1 \leqslant n-m$ and it remains to show that $i \geqslant m+1$. Assume that $i \leqslant m$ instead and set $i' := \rho_\sigma(i)$. Then $i' < \frac{k+1}{2}$. Since $\tau \in \mathfrak{S}_k$ is an oscillating k-cycle, Lemma 35 yields that $\tau^{-1}(i'), \tau(i') \geqslant \frac{k+1}{2}$. Because $\rho_\sigma(j) = \frac{k+1}{2}$, it follows that $\sigma^{-1}(i), \sigma(i) \geqslant j$. Moreover, $i+1$ being a fixed point and $i \leqslant m$ imply that $i+1 < j$. Hence, $\sigma^{-1}(i), \sigma(i) > i+1$ which contradicts Equation (4.13).

Since $i, i+1 \in [m+1, n-m]$ and $i+1$ is a fixed point of σ, we can apply Lemma 66 which implies that σ' satisfies the hook properties.

In the same vein, one proves the remaining case where i is a fixed point but $i+1$ is not. $\square$

We now extend Lemma 47 to the case of odd hooks. That is, we consider one step of the algorithm mentioned earlier.

Lemma 68. *Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be an odd hook and $\sigma \in \mathfrak{S}_n$ such that σ is of type α, σ satisfies the hook properties and $\sigma \neq \sigma_\alpha$. Then there exists a minimal integer p such that $1 \leqslant p \leqslant k-1$ and $\sigma^p(1) \neq \sigma_\alpha^p(1)$. Set $a := \sigma^p(1)$, $b := \sigma_\alpha^p(1)$ and*

$$\sigma' := \begin{cases} s_{a-1}\sigma s_{a-1} & \text{if } a > b \\ s_a \sigma s_a & \text{if } a < b. \end{cases}$$

Then $\sigma' \approx \sigma$ and σ' satisfies the hook properties.

Proof. Set $m := \frac{k-1}{2}$. If $\alpha = (1^n)$ then the only permutation of type α is the identity and there is nothing to show. If $\alpha = (n)$ then this is Lemma 47. Therefore, assume $1 < k < n$. Since σ satisfies the hook properties, 1 appears in the cycle of length k of σ. By definition, σ_α has the form

$$\sigma_\alpha = \begin{cases} (1, n, 2, n-1, \dots, m+1)(n-m)(m+2)\cdots(\frac{n+3}{2})(\frac{n+1}{2}) & \text{if } n \text{ is odd} \\ (1, n, 2, n-1, \dots, m+1)(n-m)(m+2)\cdots(\frac{n}{2})(\frac{n}{2}+1) & \text{if } n \text{ is even.} \end{cases}$$

In particular, $[m+2, n-m]$ is the set of fixed points of σ_α and 1 also appears in the cycle of length k of σ_α. Thus, from $\sigma \neq \sigma_\alpha$ it follows that there exists p as claimed. In particular, we can define a, b and σ' as in the theorem.

If n is odd, $k < n$ implies that $\frac{n+1}{2}$ is a fixed point of σ_α and hence $b \neq \frac{n+1}{2}$. If n is even, we have $b \neq \frac{n+1}{2}$ anyway. Let $\tau := \operatorname{cst}(\sigma)$ and note that $\operatorname{cst}(\sigma_\alpha)$ is just the element in stair form $\sigma_{(k)}$. Moreover set $a' := \rho_\sigma(a)$.

Assume $b < \frac{n+1}{2}$. The proof for $b > \frac{n+1}{2}$ is similar and hence omitted. If $b < \frac{n+1}{2}$ then $b \leqslant m+1$ by the description of σ_α from above. The choice of p and $1 < b \leqslant m+1$ imply

$$\sigma^{-1}(a) = \sigma_\alpha^{-1}(b) = n-b+2 > m+1$$

and

$$\{1, 2, \dots, b-1\} \subseteq \{\sigma_\alpha^r(1) \mid r = 0, \dots, p-1\} = \{\sigma^r(1) \mid r = 0, \dots, p-1\}.$$

The last equality and $a \neq b$ imply $b < a$. Thus, we consider $\sigma' := s_{a-1}\sigma s_{a-1}$. From the hook properties, we obtain that the elements in the cycle of length k of σ are $1, \ldots, m, j, n-m+1, \ldots n$ where $j \in [m+1, n-m]$. Thus, $\sigma^{-1}(a) > m+1$ implies $\tau^{-1}(a') > m+1$. But since σ is oscillating, τ is oscillating and therefore Lemma 35 implies $a' \leqslant m+1$. From the description of the elements in the k-cycle of σ, it now follows that $a \leqslant n-m$.

To sum up, we have $b < a \leqslant n-m$ and $\sigma' = s_{a-1}\sigma s_{a-1}$. Now we have two cases depending on $a-1$. If $a-1$ is a fixed point of σ then because of $a \leqslant n-m$, we can apply Lemma 66 and obtain that $\sigma' \approx \sigma$ and σ' satisfies the hook properties.

If $a-1$ is not a fixed point of σ then $\rho_\sigma(a-1) = a'-1$. Moreover, interchanging $a-1$ and a in σ does not affect the third part of the hook property. Therefore, we obtain from Lemma 59 that $\sigma' \approx \sigma$ and σ' satisfies the hook properties if $\tau' := s_{a'-1}\tau s_{a'-1} \approx \tau$ and τ' is oscillating with connected intervals. By Lemma 47, τ' has these properties if $\tau^r(1) = \sigma^r_{(k)}(1)$ for $0 \leqslant r \leqslant p-1$, $\tau^p(1) > \sigma^p_{(k)}(1)$ and $\tau^p(1) = a'$. This is what remains be shown.

As $\sigma^r(1) = \sigma^r_\alpha(1)$ for $0 \leqslant r \leqslant p-1$, we have the following equality of tuples

$$\begin{aligned}(\tau^0(1), \tau^1(1), \ldots, \tau^{p-1}(1)) &= (\rho_\sigma(1), \rho_\sigma(n), \rho_\sigma(2), \rho_\sigma(n-1), \ldots, \rho_\sigma(n-b+2)) \\ &= (1, k, 2, k-1, \ldots, k-b+2) \\ &= (\sigma^0_{(k)}(1), \sigma^1_{(k)}(1), \ldots, \sigma^{p-1}_{(k)}(1)).\end{aligned}$$

Since the cycle of length k of σ contains exactly one element of $[m+1, n-m]$, $a-1$ and a appear in this cycle and $a \leqslant n-m$, we have that $a \leqslant m+1$. Moreover, $1, \ldots, m$ appear in the cycle of length k of σ and σ_α. Since $b < a \leqslant m+1$, this implies

$$\sigma^p_{(k)}(1) = \rho_{\sigma_\alpha}(b) = b \text{ and } \tau^p(1) = \rho_\sigma(a) = a.$$

In particular, $a' = \tau^p(1)$. Moreover, we have $b < a$ so that $\sigma^p_{(k)}(1) < \tau^p(1)$ as desired. $\square$

We now come to the main result of this section.

Theorem 69. *Let $\alpha \vDash_e n$ be an odd hook and $\sigma \in \mathfrak{S}_n$ of type α. Then $\sigma \in \Sigma_\alpha$ if and only if σ satisfies the hook properties.*

Proof. Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be an odd hook and σ_α be the element in stair form. The proof is analogous to the one of Theorem 49. By Lemma 65, σ_α satisfies the hook properties. Let $\sigma \in \mathfrak{S}_n$.

For the direction from left to right assume that $\sigma \in \Sigma_\alpha$. Then $\sigma \approx \sigma_\alpha$. From the definition of $\approx$ and Lemma 67 it follows that $\approx$ transfers the hook properties from σ_α to σ.

For the converse direction, assume that σ satisfies the hook properties. By using Lemma 68 iteratively, we obtain a sequence of $\approx$-equivalent permutations starting with σ and ending in σ_α. Hence $\sigma \in \Sigma_\alpha$. $\square$

We continue with a rule for the construction of $\Sigma_{(k,1^{n-k})}$ from $\Sigma_{(k)}$ in the case where k is odd and $k \geqslant 3$. The rule can be sketched as follows. Given a $\tau \in \Sigma_{(k)}$ we can choose a subset of $[n]$ of size k in accordance with the third hook property. Arranging the elements of this subset in a cycle of length k such that its cycle standardization is τ (and letting the other elements of $[n]$ be fixed points) then results in an element of $\Sigma_{(k,1^{n-k})}$. See Part (3) of Example 64 for an illustration.

Corollary 70. *Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be an odd hook with $k \geqslant 3$. Set $m := \frac{k-1}{2}$. For $\tau \in \Sigma_{(k)}$ and $j \in [m+1, n-m]$ define $\varphi(\tau, j)$ to be the element $\sigma \in \mathfrak{S}_n$ of type α such that $\operatorname{cst}(\sigma) = \tau$ and the entries in the cycle of length k of σ are $1, \ldots, m, j, n-m+1, \ldots, n$. Then*

$$\varphi\colon \Sigma_{(k)} \times [m+1, n-m] \to \Sigma_\alpha, \quad (\tau, j) \mapsto \varphi(\tau, j)$$

is a bijection.

Proof. Given a $\tau \in \Sigma_{(k)}$ and a $j \in [m+1, n-m]$ there is only one way (up to cyclic shift) to write the elements $1, 2, \ldots, m, j, n-m+1, \ldots, n$ in a cycle of length k such that the standardization of the corresponding k-cycle in $\mathfrak{S}_n$ is τ. This k-cycle is $\varphi(\tau, j)$. By construction, $\varphi(\tau, j)$ satisfies the hook properties. Hence, Theorem 69 yields $\varphi(\tau, j) \in \Sigma_\alpha$. That is, φ is well defined.

Let $\sigma \in \Sigma_\alpha$. Then by Theorem 69, σ satisfies the hook properties. The third hook property yields that there is a unique $j \in [m+1, n-m]$ such that the elements in the cycle of length k of σ are $1, 2, \ldots, m, j, n-m+1, \ldots, n$. From the first two hook properties it follows that $\tau := \operatorname{cst}(\sigma)$ is oscillating and has connected intervals. Thus, $\tau \in \Sigma_{(k)}$ by Theorem 49. By definition of φ, the cycles of length k of $\varphi(\tau, j)$ and σ contain the same elements. Moreover, they have the same cycle standardization τ. Consequently, $\varphi(\tau, j) = \sigma$. That is, φ is surjective. Since (τ, j) uniquely depends on σ, φ is also injective. $\square$

In the last result of the section we determine the cardinality of Σ_α for each odd hook α.

Corollary 71. *If $\alpha = (k, 1^{n-k}) \vDash_e n$ is an odd hook then*

$$|\Sigma_\alpha| = \begin{cases} 1 & \text{if } k = 1 \\ 2(n-k+1)3^{\frac{k-3}{2}} & \text{if } k \geqslant 3. \end{cases}$$

Proof. Let $\sigma \in \Sigma_\alpha$. If $k = 1$ then $\Sigma_\alpha = \{1\}$. Now suppose that $k \geqslant 3$ and set $m := \frac{k-1}{2}$. The cardinality of $[m+1, n-m]$ is $n-k+1$. As a consequence, Corollary 70 yields that $|\Sigma_\alpha| = (n-k+1)|\Sigma_{(k)}|$. In addition, we have $|\Sigma_{(k)}| = 2 \cdot 3^{\frac{k-3}{2}}$ from Corollary 53. $\square$

4.3 The inductive product

The inductive product is a binary operator

$$\odot\colon \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \to \mathfrak{S}_n, (\sigma_1, \sigma_2) \mapsto \sigma_1 \odot \sigma_2$$

where $n_1 + n_2 = n$. The main result of this section is that it restricts to a bijection $\odot\colon \Sigma_{(\alpha_1)} \times \Sigma_{(\alpha_2,\ldots,\alpha_l)} \to \Sigma_\alpha$ for all $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$ with α_1 even (Theorem 84).

Thus, we obtain a decomposition rule for Σ_α in this case. Together with Theorem 69 this leads to a description of Σ_α for all maximal compositions α whose odd parts form a hook (Remark 87) and a result on the cardinality of Σ_α (Corollary 88). We then infer in Theorem 93 that Σ_α is characterized by the hook properties if α is an even hook and hence generalize Theorem 69 to all hooks.

Let $\alpha \vDash_e n$ and α' be the maximal composition consisiting of the odd parts of α. Combining Theorem 84 with Theorem 49 reduces the problem of describing Σ_α to $\Sigma_{\alpha'}$. If α' is an odd hook we can do this with Theorem 69. So the problem that remains is to describe Σ_α for all maximal compositions α with only odd parts which are not hooks. A solution to this problem would be interesting but is out of scope of this paper (see Remark 94). In the case where α_1 is odd, Theorem 84 provides a bijection given by $\odot$ only between certain subsets of $\Sigma_{(\alpha_1)} \times \Sigma_{(\alpha_2,\ldots,\alpha_l)}$ and Σ_α.

The main steps towards Theorem 84 are the following. We first define and formalize the inductive product. Then we consider basic properties such as its injectivity (Lemma 79) and the length ℓ of certain images under the inductive product (Lemma 80). We then show in Lemma 83 how the elements in stair form can be decomposed by the inductive product. These results allow us to show that $\odot\colon \Sigma_{(\alpha_1)} \times \Sigma_{(\alpha_2,\ldots,\alpha_l)} \to \Sigma_\alpha$ is surjective for even α_1, and we obtain Theorem 84.

We now begin with the definition of the inductive product. Recall that we write $\gamma \vDash_0 n$ if γ is a weak composition of n, that is, a finite sequence of nonnegative integers that sum up to n.

Definition 72. Let $(n_1, n_2) \vDash_0 n$. The *inductive product* on $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ is the binary operator

$$\begin{aligned} \odot\colon \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} &\to \mathfrak{S}_n \\ (\sigma_1, \sigma_2) &\mapsto \sigma_1 \odot \sigma_2 \end{aligned}$$

where $\sigma_1 \odot \sigma_2$ is the element of $\mathfrak{S}_n$ whose cycles are the cycles of σ_1 and σ_2 altered as follows:

(1) in the cycles of σ_1, add n_2 to each entry $> k$,

(2) in the cycles of σ_2, add k to each entry

where $k := \lceil \frac{n_1}{2} \rceil$.

For two sets $X_1 \subseteq \mathfrak{S}_{n_1}$ and $X_2 \subseteq \mathfrak{S}_{n_2}$ we define

$$X_1 \odot X_2 := \{\sigma_1 \odot \sigma_2 \mid \sigma_1 \in X_1, \sigma_2 \in X_2\}.$$

We will see in Lemma 76 that the inductive product is well-defined.

Example 73. (1) Let $\emptyset \in \mathfrak{S}_0$ be the empty function and $\sigma \in \mathfrak{S}_n$. Then

$$\emptyset \odot \sigma = \sigma \odot \emptyset = \sigma.$$

(2) Consider $n_1 = 6$, $n_2 = 4$, $n = 10$ and the elements in stair form $\sigma_{(6)} \in \mathfrak{S}_{n_1}$ and $\sigma_{(3,1)} \in \mathfrak{S}_{n_2}$. Then $k = 3$ and

$$\begin{aligned}\sigma_{(6)} \odot \sigma_{(3,1)} &= (1, 6, 2, 5, 3, 4) \odot (1, 4, 2)(3)\\ &= (1, 6 + 4, 2, 5 + 4, 3, 4 + 4)(1 + 3, 4 + 3, 2 + 3)(3 + 3)\\ &= (1, 10, 2, 9, 3, 8)(4, 7, 5)(6).\end{aligned}$$

(3) Consider $n_1 = 5$, $n_2 = 4$ and the elements in stair form $\sigma_{(5)} = (1, 5, 2, 4, 3) \in \mathfrak{S}_{n_1}$ and $\sigma_{(3,1)} = (1, 4, 2)(3) \in \mathfrak{S}_{n_2}$. Then $\sigma^{w_0}_{(3,1)} = (1, 3, 4)(2)$ where $w_0 = (1, 4)(2, 3)$ is the longest element of $\mathfrak{S}_4$. We have $k = 3$ and

$$\begin{aligned}\sigma_{(5)} \odot \sigma^{w_0}_{(3,1)} &= (1, 5 + 4, 2, 4 + 4, 3)(1 + 3, 3 + 3, 4 + 3)(2 + 3)\\ &= (1, 9, 2, 8, 3)(7, 4, 6)(5).\end{aligned}$$

Note that in Parts (2) and (3) we obtain the elements in stair form $\sigma_{(6,3,1)}$ and $\sigma_{(5,3,1)}$, respectively.

In order to work with the inductive product, it is convenient to describe it more formally. To this end we introduce the following notation which we will use throughout the section.

Notation 74. *Let $n \geqslant 0$, $(n_1, n_2) \vDash_0 n$, $k := \lceil \frac{n_1}{2} \rceil$,*

$$N_1 := [k] \cup [k + n_2 + 1, n] \quad \textit{and} \quad N_2 := [k + 1, k + n_2].$$

We have that $|N_1| = n_1, |N_2| = n_2$, N_1 and N_2 are disjoint and $N_1 \cup N_2 = [n]$. Note that $[0] = [1, 0] = \emptyset$. Define the bijections $\varphi_1 \colon [n_1] \to N_1$ and $\varphi_2 \colon [n_2] \to N_2$ by

$$\varphi_1(i) := \begin{cases} i & \text{if } i \leqslant k\\ i + n_2 & \text{if } i > k \end{cases} \quad \textit{and} \quad \varphi_2(i) := i + k.$$

The bijections φ_1 and φ_2 formalize the alteration of the cycles of σ_1 and σ_2 in Definition 72, respectively. Their inverses are given by

$$\varphi_1^{-1}(i) := \begin{cases} i & \text{if } i \leqslant k\\ i - n_2 & \text{if } i > k \end{cases} \quad \textit{and} \quad \varphi_2^{-1}(i) := i - k.$$

For $i = 1, 2$ and $\sigma_i \in \mathfrak{S}_{n_i}$, write $\sigma_i^{\varphi_i} := \varphi_i \circ \sigma_i \circ \varphi_i^{-1}$. Then $\sigma_i^{\varphi_i} \in \mathfrak{S}(N_i)$ and $\sigma_i^{\varphi_i}$ can naturally be identified with the element of $\mathfrak{S}_n$ that acts on N_i as $\sigma_i^{\varphi_i}$ and fixes all elements of $[n] \setminus N_i$.

We will see in Lemma 76 that we obtain $\sigma_i^{\varphi_i}$ by applying φ_i on each entry in of σ_i in cycle notation.

Example 75. Let $n_1 = 6$ and $n_2 = 4$ and consider the elements in stair form

$$\sigma_1 := \sigma_{(6)} = (1, 6, 2, 5, 3, 4) \in \mathfrak{S}_6 \quad \text{and} \quad \sigma_2 := \sigma_{(3,1)} = (1, 4, 2)(3) \in \mathfrak{S}_4.$$

Then $k = 3$ and

$$\begin{aligned}\sigma_1^{\varphi_1} &= (1, 6+4, 2, 5+4, 3, 4+4) = (1, 10, 2, 9, 3, 8),\\ \sigma_2^{\varphi_2} &= (1+3, 4+3, 2+3)(3+3) = (4, 7, 5)(6).\end{aligned}$$

Thus, from Example 73 it follows that $\sigma_1 \odot \sigma_2 = \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}$. The next lemma shows that this is true in general.

We now come to the more formal description of the inductive product.

Lemma 76. *Let $\sigma_r \in \mathfrak{S}_{n_r}$ with cycle decomposition $\sigma_r = \sigma_{r,1}\sigma_{r,2}\cdots\sigma_{r,p_r}$ for $r = 1, 2$.*

(1) We have

$$\sigma_1 \odot \sigma_2 = \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}.$$

(2) Let $r \in \{1, 2\}$ and $\sigma_{r,j} = (c_1, \dots, c_t)$ be a cycle of σ_r. Then

$$\sigma_{r,j}^{\varphi_r} = (\varphi_r(c_1), \dots, \varphi_r(c_t)).$$

(3) The decomposition of $\sigma_1 \odot \sigma_2$ in disjoint cycles is given by

$$\sigma_1 \odot \sigma_2 = \sigma_{1,1}^{\varphi_1} \cdots \sigma_{1,p_1}^{\varphi_1} \cdot \sigma_{2,1}^{\varphi_2} \cdots \sigma_{2,p_2}^{\varphi_2}.$$

Proof. Set $\sigma := \sigma_1 \odot \sigma_2$ and $\sigma' := \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}$. It will turn out that $\sigma = \sigma'$.

We first show Part (2). Let $r \in \{1, 2\}$, ξ be a cycle of σ_r and $i \in [n_r]$. Then

$$\xi^{\varphi_r}(\varphi_r(i)) = (\varphi_r \circ \xi \circ \varphi_r^{-1} \circ \varphi_r)(i) = \varphi_r(\xi(i)).$$

Hence, if $\xi = (c_1, \dots, c_t) \in \mathfrak{S}_{n_r}$ then $\xi^{\varphi_r} = (\varphi_r(c_1), \dots, \varphi_r(c_t)) \in \mathfrak{S}(N_r)$.

We continue with showing Part (3) for σ'. For $r = 1, 2$ we have

$$\begin{aligned}\sigma_r^{\varphi_r} &= \varphi_r \circ \sigma_r \circ \varphi_r^{-1}\\ &= \varphi_r \circ \sigma_{r,1} \cdots \sigma_{r,p_r} \circ \varphi_r^{-1}\\ &= (\varphi_r \circ \sigma_{r,1} \circ \varphi_r^{-1}) \cdots (\varphi_r \circ \sigma_{r,p_r} \circ \varphi_r^{-1})\\ &= \sigma_{r,1}^{\varphi_r} \cdots \sigma_{r,p_r}^{\varphi_r}.\end{aligned}$$

Thus,

$$\sigma' = \sigma_{1,1}^{\varphi_1} \cdots \sigma_{1,p_1}^{\varphi_1}\sigma_{2,1}^{\varphi_2} \cdots \sigma_{1,p_2}^{\varphi_2}. \tag{4.14}$$

The cycles in this decomposition are given by Part (1). As φ_1 and φ_2 are bijections with disjoint images, the cycles are disjoint.

Lastly, we show $\sigma = \sigma'$. From Equation (4.14), Part (2) and the definition of φ_1 and φ_2 it follows that we obtain the cycles of σ' by altering the cycles of σ_1 and σ_2 as described in Definition 72. Hence, $\sigma = \sigma'$. □

Corollary 77. *Let $\sigma_1 \in \mathfrak{S}_{n_1}, \sigma_2 \in \mathfrak{S}_{n_2}$ and $\sigma := \sigma_1 \odot \sigma_2$. Then*

$$P(\sigma) = \varphi_1(P(\sigma_1)) \cup \varphi_2(P(\sigma_2)).$$

We continue with basic properties of the inductive product.

Lemma 78. *Let $\sigma_1 \in \mathfrak{S}_{n_1}, \sigma_2 \in \mathfrak{S}_{n_2}$ and $\sigma := \sigma_1 \odot \sigma_2$. Then for all $i \in [n]$*

$$\sigma(i) = \begin{cases} \sigma_1^{\varphi_1}(i) & \textit{if } i \in N_1 \\ \sigma_2^{\varphi_2}(i) & \textit{if } i \in N_2. \end{cases}$$

Proof. By Lemma 76, $\sigma = \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}$. If $n_1 = 0$ or $n_2 = 0$ the claim is trivially true. Thus, suppose $n_1, n_2 \geqslant 1$ and let $i \in [n]$. Consider $\sigma_1^{\varphi_1}$ and $\sigma_2^{\varphi_2}$ as elements of $\mathfrak{S}_n$. Since $\{N_1, N_2\}$ is a partition of $[n]$ there is exactly one $r \in \{1, 2\}$ such that $i \in N_r$. We have that $\sigma_r^{\varphi_r}(N_r) = N_r$ and that $\sigma_{2-r+1}^{\varphi_{2-r+1}}$ fixes each element of N_r. Hence,

$$\sigma(i) = \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}(i) = \sigma_r^{\varphi_r}(i). \qquad \square$$

We now determine the image of the inductive product and show that it is injective.

Lemma 79. *Let $(n_1, n_2) \vDash_0 n$.*

(1) The image of $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ under $\odot$ is given by

$$\mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2} = \{\sigma \in \mathfrak{S}_n \mid \sigma(N_i) = N_i \textit{ for } i = 1, 2\}.$$

(2) The inductive product on $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ is injective.

Proof. (1) Set $Y := \{\sigma \in \mathfrak{S}_n \mid \sigma(N_i) = N_i \text{ for } i = 1, 2\}$.

We show $\mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2} \subseteq Y$ first. Let $\sigma \in \mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2}$. Then there are $\sigma_i \in \mathfrak{S}_{n_i}$ for $i = 1, 2$ such that $\sigma = \sigma_1 \odot \sigma_2$. By Lemma 78 we have $\sigma(N_i) = \sigma^{\varphi_i}(N_i) = N_i$ for $i = 1, 2$. Hence, $\sigma \in Y$.

We now show $Y \subseteq \mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2}$. Let $\sigma \in Y$. For $i = 1, 2$ set $\tilde{\sigma}_i = \sigma|_{N_i}$ (the restriction to N_i). Consider $i \in \{1, 2\}$. Since $\sigma \in Y$, $\tilde{\sigma}_i(N_i) = N_i$ and thus $\tilde{\sigma}_i \in \mathfrak{S}(N_i)$. Therefore, $\sigma_i := \varphi_i^{-1} \circ \tilde{\sigma}_i \circ \varphi_i$ is an element of $\mathfrak{S}_{n_i}$. Moreover, $\sigma_i^{\varphi_i}$ considered as an element of $\mathfrak{S}_n$ leaves each element of N_{2-i+1} fixed. Hence, we have

$$(\sigma_1 \odot \sigma_2)|_{N_i} = \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}|_{N_i} = \sigma_i^{\varphi_i}|_{N_i} = \tilde{\sigma}_i|_{N_i} = \sigma|_{N_i}.$$

Consequently, $\sigma = \sigma_1 \odot \sigma_2$.

(2) Since $|N_i| = n_i$ for $i = 1, 2$, the cardinality of Y is $n_1!n_2!$. This is also the cardinality of $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$. As the image of $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ under $\odot$ is Y, it follows that $\odot$ is injective. $\square$

Recall that for $\alpha \vDash_e n$, each element of Σ_α has the property that its length is maximal in its conjugacy class. We want to use this property to prove our main result.

Consider $\sigma = \sigma_1 \odot \sigma_2$ such that σ_1 has type (n_1). We seek a formula for $\ell(\sigma)$ depending on σ_1 and σ_2. We are particularly interested in the case where the n_1-cycle σ_1 is oscillating.

Given $\sigma \in \mathfrak{S}_n$ let $\mathrm{Inv}(\sigma) := \{(i, j) \mid 1 \leqslant i < j \leqslant n, \sigma(i) > \sigma(j)\}$ be the *set of inversions* of σ. Then $\ell(\sigma) = |\operatorname{Inv}(\sigma)|$ by [1, Proposition 1.5.2].

Lemma 80. *Let $\sigma_1 \in \mathfrak{S}_{n_1}$ be an n_1-cycle, $\sigma_2 \in \mathfrak{S}_{n_2}$, $\sigma := \sigma_1 \odot \sigma_2$,*

$$\begin{aligned} P &:= \{i \in [k] \mid \sigma_1(i) > k\}, \\ Q &:= \{i \in [k+1, n_1] \mid \sigma_1(i) \leqslant k\}, \end{aligned}$$

$p := |P|$ and $q := |Q|$. Then we have

$$\ell(\sigma) = \ell(\sigma_1) + \ell(\sigma_2) + (p+q)n_2.$$

Moreover,

(1) *$p, q \leqslant \lfloor \frac{n_1}{2} \rfloor$,*

(2) *if σ_1 is oscillating, then $p = q = \left\lfloor \frac{n_1}{2} \right\rfloor$.*

Proof. Let $i, j \in [n]$ and $m := \left\lfloor \frac{n_1}{2} \right\rfloor$. We distinguish three types of pairs (i, j) and count the number of inversions of σ type by type.

Type 1. There is an $r \in \{1, 2\}$ such that $i, j \in N_r$. In this case let $t \in \{i, j\}$ and set $t' := \varphi_r^{-1}(t)$. Then $t' \in [n_r]$. From Lemma 78 we obtain

$$\sigma(t) = \varphi_r(\sigma_r(t')).$$

In addition, we have

$$\varphi_r(\sigma_r(i')) > \varphi_r(\sigma_r(j')) \iff \sigma_r(i') > \sigma_r(j')$$

since φ_r is a stricly increasing function. As φ_r^{-1} is stricly increasing as well, we also have that

$$i < j \iff i' < j'.$$

Hence,

$$\begin{aligned} (i, j) \in \operatorname{Inv}(\sigma) &\iff i < j \text{ and } \sigma(i) > \sigma(j) \\ &\iff i' < j' \text{ and } \varphi_r(\sigma_r(i')) > \varphi_r(\sigma_r(j')) \\ &\iff i' < j' \text{ and } \sigma_r(i') > \sigma_r(j') \\ &\iff (i', j') \in \operatorname{Inv}(\sigma_r). \end{aligned}$$

Thus, the number of inversions of Type 1 is

$$|\operatorname{Inv}(\sigma_1)| + |\operatorname{Inv}(\sigma_2)| = \ell(\sigma_1) + \ell(\sigma_2).$$

Type 2. We have $i \in N_1$, $j \in N_2$ and $i < j$. Assume that (i, j) is of this type and recall that $N_1 = [k] \cup [k+n_2+1, n]$ and $N_2 = [k+1, k+n_2]$ where $k = \lceil \frac{n}{2} \rceil$. Since $i < j$, we have $i \leqslant k$ which in particular means that $\varphi_1^{-1}(i) = i$. As $\sigma(j) \in N_2$, $k+1 \leqslant \sigma(j) \leqslant k+n_2$. Moreover, $\sigma(i) = \sigma_1^{\varphi_1}(i)$ by Lemma 78. Consequently,

$$\sigma(i) = \sigma_1^{\varphi_1}(i) = \varphi_1(\sigma_1(i)) = \begin{cases} \sigma_1(i) < \sigma(j) & \text{if } \sigma_1(i) \leqslant k \\ \sigma_1(i) + n_2 > \sigma(j) & \text{if } \sigma_1(i) > k. \end{cases}$$

Therefore,

$$(i, j) \in \mathrm{Inv}(\sigma) \iff \sigma_1(i) > k.$$

Hence, the number of inversions of Type 2 is the cardinality of the set $P \times N_2$. Thus, we have pn_2 inversions of Type 2.

Type 3. We have $i \in N_2$, $j \in N_1$ and $i < j$. Let (i, j) be of Type 3. Then from $i < j$ we obtain $j \geqslant k + n_2 + 1$. In particular, this type can only occur if $n_1 > 1$ because otherwise $n = 1 + n_2 < j$.

Since $i \in N_2$, also $\sigma(i) \in N_2$. That is, $k + 1 \leqslant \sigma(i) \leqslant k + n_2$. Moreover, from $i < j$ and $i \in N_2$ it follows that $j \geqslant k + n_2 + 1$. Thus,

$$j' := \varphi_1^{-1}(j) = j - n_2$$

and $j' \in [k + 1, n_1]$. Hence,

$$\sigma(j) = \sigma_1^{\varphi_1}(j) = \varphi_1(\sigma_1(j')) = \begin{cases} \sigma_1(j') < \sigma(i) & \text{if } \sigma_1(j') \leqslant k \\ \sigma_1(j') + n_2 > \sigma(i) & \text{if } \sigma_1(j') > k. \end{cases}$$

That is,

$$(i, j) \in \mathrm{Inv}(\sigma) \iff \sigma_1(j') \leqslant k \iff j' \in Q \iff j \in \varphi_1(Q)$$

where we use that $j' \in [k + 1, n_1]$ for the second equivalence. Consequently, the set of inversion of Type 3 is the set $N_2 \times \varphi_1(Q)$. Since φ_1 is a bijection, it follows that there are exactly qn_2 inversions of this type.

Summing up the number of inversions of each type, we obtain the formula for the length of σ.

We now prove (1) and (2).

(1) By definition, $\sigma_1(P) \subseteq [k+1, n_1]$ and $Q \subseteq [k+1, n_1]$. The cardinality of $[k+1, n_1]$ is $\left\lfloor \frac{n_1}{2} \right\rfloor$. Therefore, $p, q \leqslant \left\lfloor \frac{n_1}{2} \right\rfloor$.

(2) Assume that σ_1 is oscillating. Suppose first that n is even. Then $k = \frac{n_1}{2}$. Because σ_1 is oscillating, we obtain that

$$\sigma_1([k]) = [k + 1, n_1] \quad \text{and} \quad \sigma_1([k + 1, n_1]) = [k]$$

from Definition 30 and Lemma 33. Hence, $p = q = k = \lfloor \frac{n_1}{2} \rfloor$.

Suppose now that n is odd. Then $k = \frac{n_1+1}{2}$. Since σ_1 is oscillating, Definition 30 and Lemma 33 yield that there is an $m \in \{k - 1, k\}$ such that

$$\sigma_1([m]) = [n_1 - m + 1, n_1] \quad \text{and} \quad \sigma_1([m + 1, n_1]) = [n_1 - m].$$

It is not hard to see that this implies $p = q = k - 1 = \lfloor \frac{n_1}{2} \rfloor$. $\square$

We have seen in Example 73 that the elements in stair form $\sigma_{(5,3)}$ and $\sigma_{(6,3)}$ can be decomposed as

$$\sigma_{(5,3)} = \sigma_{(5)} \odot \sigma_{(3)}^{w_0} \quad \text{and} \quad \sigma_{(6,3)} = \sigma_{(6)} \odot \sigma_{(3)}$$

where w_0 is the longest element of $\mathfrak{S}_3$. We want to show that these are special cases of a general rule for decomposing the element in stair form σ_α. Before we state the rule in Lemma 83, we compare the sequences used to define the element in stair form in Definition 14 for compositions of n, n_1 and n_2.

Lemma 81. *For $m \in \mathbb{N}_0$ let $x^{(m)}$ be the sequence $(x_1^{(m)}, \dots, x_m^{(m)})$ given by $x_{2i-1}^{(m)} = i$ and $x_{2i}^{(m)} = m - i + 1$. Set $x := x^{(n)}$, $y := x^{(n_1)}$ and $z := x^{(n_2)}$.*

(1) We have $\varphi_1(y_i) = x_i$ for all $i \in [n_1]$.

(2) If n_1 is even then $\varphi_2(z_i) = x_{i+n_1}$ for all $i \in [n_2]$.

(3) If n_1 is odd then $\varphi_2(w_0(z_i)) = x_{i+n_1}$ for all $i \in [n_2]$ where w_0 is the longest element of $\mathfrak{S}_{n_2}$.

Proof. Recall that $k = \left\lceil \frac{n_1}{2} \right\rceil$ and $(n_1, n_2) \vDash_0 n$ by Notation 74. Let $i \in \mathbb{N}$. We mainly do straight forward calculations.

(1) Assume $2i - 1 \in [n_1]$. Then $i \leqslant k$ and thus $\varphi_1(i) = i$. Consequently,

$$\varphi_1(y_{2i-1}) = \varphi_1(i) = i = x_{2i-1}.$$

Now, assume $2i \in [n_1]$. Then

$$\begin{aligned} n_1 - i + 1 = \lceil n_1 - i + 1 \rceil &\geqslant \left\lceil n_1 - \frac{n_1}{2} + 1 \right\rceil \\ &= \left\lceil \frac{n_1}{2} + 1 \right\rceil = \left\lceil \frac{n_1}{2} \right\rceil + 1 = k + 1, \end{aligned}$$

i.e. $\varphi_1(n_1 - i + 1) = n_1 + n_2 - i + 1$. Therefore,

$$\varphi_1(y_{2i}) = \varphi_1(n_1 - i + 1) = n_1 + n_2 - i + 1 = n - i + 1 = x_{2i}.$$

(2) Assume that n_1 is even. Then $n_1 = 2k$. If $2i - 1 \in [n_2]$ then we have

$$2(k + i) - 1 = n_1 + 2i - 1 \leqslant n_1 + n_2 = n.$$

Thus,

$$\varphi_2(z_{2i-1}) = \varphi_2(i) = k + i = x_{2(k+i)-1} = x_{2i-1+n_1}.$$

Suppose $2i \in [n_2]$. Then $2(k + i) = n_1 + 2i \leqslant n$ and

$$\begin{aligned} \varphi_2(z_{2i}) = k + n_2 - i + 1 &= (n - 2k - n_2) + k + n_2 - i + 1 \\ &= n - k - i + 1 \\ &= x_{2(k+i)} = x_{2i+n_1}. \end{aligned}$$

(3) Assume that n_1 is odd. In this case $n_1 = 2k-1$. Let w_0 be the longest element of $\mathfrak{S}_{n_2}$. We have $w_0(j) = n_2 - j + 1$ for all $j \in [n_2]$. If $2i-1 \in [n_2]$ then $2i-1+n_1 \in [n]$ and

$$\begin{aligned}
\varphi_2(w_0(z_{2i-1})) &= \varphi_2(w_0(i)) \\
&= \varphi_2(n_2 - i + 1) \\
&= n_2 + k - i + 1 \\
&= (n - 2k + 1 - n_2) + n_2 + k - i + 1 \\
&= n - (k + i - 1) + 1 \\
&= x_{2(i+k-1)} \\
&= x_{2i-1+2k-1} = x_{2i-1+n_1}.
\end{aligned}$$

If $2i \in [n_2]$ then $2i + n_1 \in [n]$ and

$$\varphi_2(w_0(z_{2i})) = \varphi_2(w_0(n_2 - i + 1)) = \varphi_2(i) = i + k = x_{2(i+k)-1} = x_{2i+n_1}. \qquad \square$$

Example 82. Consider $n = 9$, $n_1 = 6$ and $n_2 = 3$. Then $k = 3$. Using the notation from Lemma 81 we obtain

$$\begin{aligned}
x &= (1, 9, 2, 8, 3, 7, 4, 6, 5), \\
y &= (1, 6, 2, 5, 3, 4), \\
z &= (1, 3, 2).
\end{aligned}$$

Then $x = (\varphi_1(y_1), \ldots, \varphi_1(y_6), \varphi_2(z_1), \varphi_2(z_2), \varphi_2(z_3))$ as predicted by Lemma 81. Moreover, x, y and z are the sequences used to define the elements in stair form $\sigma_{(6,3)}$, $\sigma_{(6)}$ and $\sigma_{(3)}$, respectively. Therefore,

$$\sigma_{(6,3)} = (\varphi_1(y_1), \ldots, \varphi_1(y_6))(\varphi_2(z_1), \varphi_2(z_2), \varphi_2(z_3)) = \sigma_{(6)}^{\varphi_1} \sigma_{(3)}^{\varphi_2} = \sigma_{(6)} \odot \sigma_{(3)}.$$

This also illustrates the idea of the proof of the next lemma on the decomposition of σ_α.

Lemma 83. *Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$ with $l \geqslant 1$. Then we have the following.*

(*1*) *If α_1 is even then $\sigma_\alpha = \sigma_{(\alpha_1)} \odot \sigma_{(\alpha_2,\ldots,\alpha_l)}$.*

(*2*) *If α_1 is odd then $\sigma_\alpha = \sigma_{(\alpha_1)} \odot \left(\sigma_{(\alpha_2,\ldots,\alpha_l)}\right)^{w_0}$ where w_0 is the longest element of $\mathfrak{S}_{\alpha_2+\cdots+\alpha_l}$.*

Proof. Set $n_1 := \alpha_1$ and $n_2 := \alpha_2 + \cdots + \alpha_l$. As in Lemma 81, let $x^{(m)}$ be the sequence $(x_1^{(m)}, \ldots, x_m^{(m)})$ given by $x_{2i-1}^{(m)} = i$ and $x_{2i}^{(m)} = m - i + 1$ for $m \in \mathbb{N}_0$ and set $x := x^{(n)}$, $y := x^{(n_1)}$ and $z := x^{(n_2)}$. We have that

(1) σ_α has the cycles

$$\sigma_{\alpha_i} = \left(x_{\alpha_1+\cdots+\alpha_{i-1}+1}, x_{\alpha_1+\cdots+\alpha_{i-1}+2}, \ldots, x_{\alpha_1+\cdots+\alpha_{i-1}+\alpha_i}\right)$$

for $i = 1, \ldots, l$,

(2) $\sigma_{(\alpha_1)} = (y_1, y_2, \ldots, y_{n_1})$ and
(3) $\sigma_{(\alpha_2,\ldots,\alpha_l)}$ has the cycles

$$\tilde{\sigma}_{\alpha_i} = \Big(z_{\alpha_2+\cdots+\alpha_{i-1}+1}, z_{\alpha_2+\cdots+\alpha_{i-1}+2}, \ldots, z_{\alpha_2+\cdots+\alpha_{i-1}+\alpha_i}\Big)$$

for $i = 2, \ldots, l$.

Assume that α_1 is even and set $\sigma := \sigma_{(\alpha_1)} \odot \sigma_{(\alpha_2,\ldots,\alpha_l)}$. From Lemma 76 we obtain that σ has the cycles $(\sigma_{(\alpha_1)})^{\varphi_1}$ and $(\tilde{\sigma}_{(\alpha_i)})^{\varphi_2}$ for $i = 2, \ldots, l$. By Lemma 81, $\varphi_1(y_j) = x_j$ for $j \in [n_1]$ and $\varphi_2(z_j) = x_{\alpha_1+j}$ for $j \in [n_2]$. As a consequence,

$$(\sigma_{(\alpha_1)})^{\varphi_1} = (\varphi_1(y_1), \ldots, \varphi_1(y_{\alpha_1})) = (x_1, \ldots, x_{\alpha_1}) = \sigma_{\alpha_1}$$

and

$$\begin{aligned}(\tilde{\sigma}_{\alpha_i})^{\varphi_2} &= \Big(\varphi_2(z_{\alpha_2+\cdots+\alpha_{i-1}+1}), \ldots, \varphi_2(z_{\alpha_2+\cdots+\alpha_{i-1}+\alpha_i})\Big)\\ &= \Big(x_{\alpha_1+\cdots+\alpha_{i-1}+1}, \ldots, x_{\alpha_1+\cdots+\alpha_{i-1}+\alpha_i}\Big)\\ &= \sigma_{\alpha_i}\end{aligned}$$

for $i = 2, \ldots, l$. Hence, $\sigma = \sigma_\alpha$.

Now let α_1 be odd. Set $\sigma := \sigma_{(\alpha_1)} \odot \Big(\sigma_{(\alpha_2,\ldots,\alpha_l)}\Big)^{w_0}$ where w_0 is the longest element of $\mathfrak{S}_{\alpha_2+\cdots+\alpha_l}$. Then σ has the cycles $(\sigma_{(\alpha_1)})^{\varphi_1}$ and $((\tilde{\sigma}_{(\alpha_i)})^{w_0})^{\varphi_2}$ for $i = 2, \ldots, l$. Moreover, from Lemma 81 we have that $\varphi_2(w_0(z_j)) = x_{\alpha_1+i}$ for $j \in [n_2]$. Thus,

$$\begin{aligned}(\tilde{\sigma}_{(\alpha_i)})^{w_0})^{\varphi_2} &= \Big(\varphi_2(w_0(z_{\alpha_2+\cdots+\alpha_{i-1}+1})), \ldots, \varphi_2(w_0(z_{\alpha_2+\cdots+\alpha_{i-1}+\alpha_i})\Big))\\ &= \Big(x_{\alpha_1+\cdots+\alpha_{i-1}+1}, \ldots, x_{\alpha_1+\cdots+\alpha_{i-1}+\alpha_i}\Big)\\ &= \sigma_{\alpha_i}\end{aligned}$$

for $i = 2, \ldots, l$. As we have already shown that $(\sigma_{(\alpha_1)})^{\varphi_1} = \sigma_{\alpha_1}$, it follows that $\sigma = \sigma_\alpha$. $\square$

We now come to the main result of the section. It enables us to decompose Σ_α if α_1 is even. Before we can state the result, we need to introduce some more notation. For $\alpha \vDash_e n$ we define

$$\Sigma_\alpha^\times := \{\sigma \in \Sigma_\alpha \mid P(\sigma) = P(\sigma_\alpha)\}.$$

Below the set $(\Sigma_\alpha^\times)^{w_0}$ appears where w_0 the longest element of $\mathfrak{S}_n$. Let $\sigma \in \Sigma_\alpha$. Then by Corollary 13, $\sigma^{w_0} \in \Sigma_\alpha$. Since $P(\sigma^{w_0}) = w_0(P(\sigma))$, we have

$$\sigma \in \Big(\Sigma_\alpha^\times\Big)^{w_0} \iff P(\sigma^{w_0}) = P(\sigma_\alpha) \iff P(\sigma) = P(\sigma_\alpha^{w_0}). \tag{4.15}$$

Theorem 84. *Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$ with $l \geqslant 1$.*

(1) Suppose that α_1 is even. Then the map

$$\Sigma_{(\alpha_1)} \times \Sigma_{(\alpha_2,\ldots,\alpha_l)} \to \Sigma_\alpha, \quad (\sigma_1, \sigma_2) \mapsto \sigma_1 \odot \sigma_2$$

is a bijection. In particular, we have the decomposition

$$\Sigma_\alpha = \Sigma_{(\alpha_1)} \odot \Sigma_{(\alpha_2,\ldots,\alpha_l)}.$$

(2) *Suppose that α_1 is odd and let w_0 bet the longest element of $\mathfrak{S}_{\alpha_2+\cdots+\alpha_l}$. Then the map*

$$\Sigma^\times_{(\alpha_1)} \times \left(\Sigma^\times_{(\alpha_2,\ldots,\alpha_l)}\right)^{w_0} \to \Sigma^\times_\alpha, \quad (\sigma_1,\sigma_2) \mapsto \sigma_1 \odot \sigma_2$$

is a bijection. In particular, we have the decomposition

$$\Sigma^\times_\alpha = \Sigma^\times_{(\alpha_1)} \odot \left(\Sigma^\times_{(\alpha_2,\ldots,\alpha_l)}\right)^{w_0}.$$

Proof. Let $\alpha^{(1)} := (\alpha_1)$, $\alpha^{(2)} := (\alpha_2,\ldots,\alpha_l)$, $n_1 := |\alpha^{(1)}|$, $n_2 := |\alpha^{(2)}|$ and w_0 be the longest element of $\mathfrak{S}_{n_2}$. We use the inductive product on $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ and the related notation. By Lemma 79 the two maps from the theorem are injective so that it remains to prove the surjectivity. That is, we have to show the following.

(1) If α_1 is even then $\Sigma_\alpha = \Sigma_{\alpha^{(1)}} \odot \Sigma_{\alpha^{(2)}}$.

(2) If α_1 is odd then $\Sigma^\times_\alpha = \Sigma^\times_{\alpha^{(1)}} \odot \left(\Sigma^\times_{\alpha^{(2)}}\right)^{w_0}$.

The proofs of (1) and (2) have a lot in common. Hence, we do them simultaneously as much as possible and separate the cases α_1 even and α_1 odd only when necessary.

If $l = 1$ then $\alpha = \alpha^{(1)}$, $\alpha^{(2)} = \emptyset$ and thus

$$\Sigma_{\alpha^{(1)}} \odot \Sigma_{\alpha^{(2)}} = \Sigma_\alpha \odot \mathfrak{S}_0 = \Sigma_\alpha.$$

Moreover, $\Sigma^\times_{(\alpha_1)} = \Sigma_{(\alpha_1)}$ and $\left(\Sigma^\times_\emptyset\right)^{w_0} = \Sigma_\emptyset$. Thus we have (1) and (2) in this case.

Now suppose $l \geqslant 2$. Let $\sigma := \sigma_\alpha$, $\sigma_1 := \sigma_{\alpha^{(1)}}$ and $\sigma_2 := \sigma_{\alpha^{(2)}}$ if α_1 is even and $\sigma_2 = \sigma^{w_0}_{\alpha^{(2)}}$ if α_1 is odd. From Lemma 83 we have $\sigma = \sigma_1 \odot \sigma_2$. By Theorem 18, $\sigma_{\alpha^{(i)}} \in \Sigma_{\alpha^{(i)}}$ for $i = 1, 2$. In addition, Corollary 13 then yields that $\sigma^{w_0}_{\alpha^{(2)}} \in \Sigma_{\alpha^{(2)}}$. Thus, $\sigma_i \in \Sigma_{\alpha^{(i)}}$ for $i = 1, 2$.

We begin with the inclusions "$\subseteq$". Let $\tau \in \Sigma_\alpha$ with $P(\tau) = P(\sigma)$ if α_1 is odd. First we show $\tau \in \mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2}$. By Lemma 79, we have to show $\tau(N_i) = N_i$ for $i = 1, 2$. Since $\{N_1, N_2\}$ is a set partition of $[n]$, it suffices to show $\tau(N_1) = N_1$. As $\sigma_1 \in \mathfrak{S}_{n_1}$ is an n_1-cycle, $P(\sigma_1) = \{[n_1]\}$. Moreover, Corollary 77 yields $P(\sigma) = \varphi_1(P(\sigma_1)) \cup \varphi_2(P(\sigma_2))$. Thus,

$$N_1 = \varphi_1([n_1]) \in \varphi_1(P(\sigma_1)) \subseteq P(\sigma).$$

If α_1 is even then $N_1 \in P_e(\sigma)$. Moreover, Proposition 28 yields $P_e(\tau) = P_e(\sigma)$. Thus, $N_1 \in P(\tau)$ which means that $\tau(N_1) = N_1$. If α_1 is odd then $P(\tau) = P(\sigma)$ by assumption. Hence, $N_1 \in P(\sigma) = P(\tau)$ and thus $\tau(N_1) = N_1$.

Because $\tau \in \mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2}$, there are $\tau_1 \in \mathfrak{S}_{n_1}$ and $\tau_2 \in \mathfrak{S}_{n_2}$ such that $\tau = \tau_1 \odot \tau_2$. Let $i \in \{1, 2\}$. We want to show $\tau_i \in \Sigma_{\alpha^{(i)}}$. Recall that $\sigma_i \in \Sigma_{\alpha^{(i)}}$. Thus, from Proposition 28 it follows that $\tau_i \in \Sigma_{\alpha^{(i)}}$ if and only if

(i) σ_i and τ_i are conjugate in $\mathfrak{S}_{n_i}$,

(ii) $\ell(\sigma_i) = \ell(\tau_i)$ and

(iii) $P_e(\sigma_i) = P_e(\tau_i)$.

Therefore, we show that τ_i satisfies (i) – (iii). Let i be arbitrary again.

(i) For a permutation ξ, let $C(\xi)$ be the multiset of cycle lengths of ξ. Assume $\xi = \xi_1 \odot \xi_2$ for $\xi_i \in \mathfrak{S}_{n_i}$ and $i = 1, 2$. From Lemma 76 it follows that

$$C(\xi) = C(\xi_1) \cup C(\xi_2). \tag{4.16}$$

Since $\tau = \tau_1 \odot \tau_2$, Corollary 77 implies $P(\tau) = \varphi_1(P(\tau_1)) \cup \varphi_2(P(\tau_2))$. Therefore, from $N_1 \in P(\tau)$ it follows that $P(\tau_1) = \{[n_1]\}$. That is, τ_1 is an n_1-cycle of $\mathfrak{S}_{n_1}$. By definition, σ_1 is an n_1-cycle of $\mathfrak{S}_{n_1}$ too. Thus, $C(\tau_1) = C(\sigma_1)$. Since $\tau \in \Sigma_\alpha$, τ and σ are conjugate so that $C(\tau) = C(\sigma)$. Because of Equation (4.16) and $C(\tau_1) = C(\sigma_1)$, it follows that also $C(\tau_2) = C(\sigma_2)$. In other words, τ_i and σ_i are conjugate for $i = 1, 2$.

(ii) Let $m := \left\lfloor \frac{n_1}{2} \right\rfloor$. By Lemma 80, there are $p, q \leqslant m$ such that

$$\ell(\tau) = \ell(\tau_1) + \ell(\tau_2) + (p+q)n_2.$$

Moreover, we have $\ell(\tau_i) \leqslant \ell(\sigma_i)$ for $i = 1, 2$ because τ_i and σ_i are conjugate and $\sigma_i \in \Sigma_{\alpha^{(i)}}$. On the other hand, σ_1 is oscillating by Theorem 49 and hence Lemma 80 yields

$$\ell(\sigma) = \ell(\sigma_1) + \ell(\sigma_2) + 2mn_2.$$

Since $\tau \in \Sigma_\alpha$, we have $\ell(\tau) = \ell(\sigma)$. Therefore, we obtain from the equalities for $\ell(\tau)$ and $\ell(\sigma)$ and the inequalities for $\ell(\tau_1), \ell(\tau_2), p$ and q that $\ell(\tau_1) = \ell(\sigma_1)$ and $\ell(\tau_2) = \ell(\sigma_2)$.

(iii) Corollary 77 states that

$$P(\xi) = \varphi_1(P(\xi_1)) \cup \varphi(P(\xi_2)) \tag{4.17}$$

for $\xi = \sigma, \tau$. This equality remains valid if we replace P by P_e. From $\tau \in \Sigma_\alpha$ and Proposition 28 it follows that $P_e(\tau) = P_e(\sigma)$. Hence,

$$\varphi_1(P_e(\tau_1)) \cup \varphi_2(P_e(\tau_2)) = \varphi_1(P_e(\sigma_1)) \cup \varphi_2(P_e(\sigma_2)).$$

Since φ_1 and φ_2 are bijections and the images of φ_1 and φ_2 are disjoint, it follows that $P_e(\tau_i) = P_e(\sigma_i)$ for $i = 1, 2$. This finishes the proof of $\tau \in \Sigma_{\alpha^{(1)}} \odot \Sigma_{\alpha^{(2)}}$.

It remains to show that $\tau_1 \in \Sigma^\times_{\alpha^{(1)}}$ and $\tau_2 \in \left(\Sigma^\times_{\alpha^{(2)}}\right)^{w_0}$ if α_1 is odd. Thus, assume that α_1 is odd. We have already seen that $P(\tau_1) = P(\sigma_1)$. Hence, $\tau_1 \in \Sigma^\times_{\alpha^{(1)}}$. Since α_1 is odd, $P(\tau) = P(\sigma)$ by assumption and therefore we deduce from Equation (4.17) as above that $P(\tau_2) = P(\sigma_2)$. Now we can use that $\sigma_2 = \sigma^{w_0}_{\alpha^{(2)}}$ and obtain $\tau_2 \in \left(\Sigma^\times_{\alpha^{(2)}}\right)^{w_0}$ from Equation (4.15).

We continue with the inclusions "$\supseteq$". Let $\tau_i \in \Sigma_{\alpha^{(i)}}$ for $i = 1, 2$ and $\tau := \tau_1 \odot \tau_2$. If α_1 is odd, assume that in addition $\tau_1 \in \Sigma^\times_{\alpha^{(1)}}$ and $\tau_2 \in \left(\Sigma^\times_{\alpha^{(2)}}\right)^{w_0}$ which by Equation (4.15) is equivalent to $P(\tau_i) = P(\sigma_i)$ for $i = 1, 2$.

We want to show that $\tau \in \Sigma_\alpha$ and again use Proposition 28 to do this. That is, we show the properties (i) – (iii) for τ and σ.

(i) For $i \in \{1, 2\}$ we have $C(\tau_i) = C(\sigma_i)$ since $\tau_i \in \Sigma_{\alpha^{(i)}}$. Hence, from Equation (4.16) it follows that $C(\tau) = C(\sigma)$, i.e. τ and σ are conjugate.

(ii) Since $\tau_1, \sigma_1 \in \Sigma_{\alpha^{(1)}}$, they are oscillating n_1-cycles by Theorem 49. Therefore, Lemma 80 yields

$$\ell(\xi) = \ell(\xi_1) + \ell(\xi_2) + 2mn_2$$

for $\xi = \sigma, \tau$ and $m = \lfloor \frac{n_1}{2} \rfloor$. Moreover, as $\sigma_i, \tau_i \in \Sigma_{\alpha^{(i)}}$, $\ell(\tau_i) = \ell(\sigma_i)$ for $i = 1, 2$. As a consequence, $\ell(\tau) = \ell(\sigma)$.

(iii) Since $\xi = \xi_1 \odot \xi_2$ for $\xi = \sigma, \tau$, Equation Equation (4.17) holds. This equation remains true if we substitute P by P_e. In addition, from Proposition 28 we obtain that $P_e(\tau_i) = P_e(\sigma_i)$ for $i = 1, 2$. Thus, $P_e(\tau) = P_e(\sigma)$.
Because of (i) – (iii) we can now apply Proposition 28 and obtain that $\tau \in \Sigma_\alpha$. In the case where α_1 is odd, it remains to show $P(\tau) = P(\sigma)$. But this is merely a consequence of $P(\tau_i) = P(\sigma_i)$ for $i = 1, 2$ and Equation (4.17). □

Recall that, given a maximal composition $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$, there exists $0 \leqslant j \leqslant l$ such that $\alpha_1, \ldots, \alpha_j$ are even and $\alpha_{j+1} \geqslant \ldots \geqslant \alpha_l$ are odd. Using Part (1) of Theorem 84 iteratively, we obtain the following decomposition of the elements of Σ_α.

Corollary 85. *Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$, $\sigma \in \mathfrak{S}_n$ of type α and $0 \leqslant j \leqslant l$ be such that $\alpha' := (\alpha_{j+1}, \ldots, \alpha_l)$ are the odd parts of α. Then $\sigma \in \Sigma_\alpha$ if and only if there are $\sigma_i \in \Sigma_{(\alpha_i)}$ for $i = 1, \ldots, j$ and $\tau \in \Sigma_{\alpha'}$ such that*

$$\sigma = \sigma_1 \odot \sigma_2 \odot \cdots \odot \sigma_j \odot \tau$$

where the product is evaluated from right to left.

Example 86. Consider $\alpha = (2, 4, 3, 1, 1) \vDash_e 11$. From Table 1 and Example 62 we obtain

$$\begin{aligned}
\Sigma_{(2)} &= \{(1,2)\}\,, \\
\Sigma_{(4)} &= \{(1,4,2,3), (1,3,2,4)\}\,, \\
\Sigma_{(3,1,1)} &= \{(1,5,2), (1,2,5), (1,5,3), (1,3,5), (1,5,4), (1,4,5)\}\,.
\end{aligned}$$

By Corollary 85, Σ_α consists of all elements $(1,2) \odot (\sigma \odot \tau)$ with $\sigma \in \Sigma_{(4)}$ and $\tau \in \Sigma_{(3,1,1)}$. Thus, $|\Sigma_\alpha| = 12$. For instance,

$$\begin{aligned}
(1,2) \odot ((1,3,2,4) \odot (1,3,5)) &= (1,2) \odot (1,8,2,9)(3,5,7) \\
&= (1,11)(2,9,3,10)(4,6,8)
\end{aligned}$$

is an element of Σ_α.

Remark 87. For compositions with one part $\alpha = (n)$, Theorem 49 provides a combinatorial characterization of $\Sigma_{(n)}$. Therefore, Corollary 85 reduces the problem of describing Σ_α for each maximal composition α to the case where α has only odd parts. These α are the partitions consisting of odds parts.

If α is an odd hook, then Theorem 69 yields that the hook properties characterize the elements of Σ_α. That is, we have a description of Σ_α for all maximal compositions α whose odd parts form a hook.

Let $\alpha \vDash_e n$ and α' be the composition formed by the odd parts of α. We infer from Corollary 85 a formula that expresses $|\Sigma_\alpha|$ as a product of $|\Sigma_{\alpha'}|$ and a factor that only depends on the even parts of α. In the case where α' is an odd hook, we can determine $|\Sigma_{\alpha'}|$ explicitly and thus obtain a closed formula.

Corollary 88. *Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$, $0 \leqslant j \leqslant l$ be such that $(\alpha_1, \ldots, \alpha_j)$ are the even and $\alpha' := (\alpha_{j+1}, \ldots, \alpha_l)$ are the odd parts of α, $n' := |\alpha'|$, $P := \{i \in [j] \mid \alpha_i \geqslant 4\}$, $p := |P|$ and $q := -2p + \frac{1}{2}\sum_{i\in P} \alpha_i$. Then*

$$|\Sigma_\alpha| = 2^p 3^q |\Sigma_{\alpha'}|.$$

Moreover, if α' is a hook $(r, 1^{n'-r})$ then

$$|\Sigma_\alpha| = \begin{cases} 2^p 3^q & \text{if } r \leqslant 1 \\ (n'-r+1)2^{p'}3^{q'} & \text{if } r \geqslant 3 \end{cases}$$

where $p' := p+1$ and $q' := q + \frac{r-3}{2}$.

Proof. Since $\alpha_1, \ldots, \alpha_j$ are the even parts of α, Corollary 85 implies that

$$|\Sigma_\alpha| = |\Sigma_{\alpha'}| \prod_{i=1}^{j} |\Sigma_{(\alpha_i)}|. \tag{4.18}$$

For the same reason, Corollary 53 yields

$$|\Sigma_{(\alpha_i)}| = \begin{cases} 1 & \text{if } n \leqslant 2 \\ 2 \cdot 3^{\frac{\alpha_i - 4}{2}} & \text{if } n \geqslant 4. \end{cases}$$

for $i = 1, \ldots, j$. Therefore,

$$\prod_{i=1}^{j} |\Sigma_{(\alpha_i)}| = \prod_{i\in P} 2 \cdot 3^{\frac{\alpha_i - 4}{2}} = 2^p 3^{-2p + \frac{1}{2}\sum_{i\in P}\alpha_i} = 2^p 3^q.$$

and with Equation (4.18) we get the first statement.

For the second part, assume that α' is a hook. Then, by the choice of j, α' is an odd hook. It remains to compute $|\Sigma_{\alpha'}|$. If $\alpha' = \emptyset$ or $\alpha' = (1^{n'})$ we have $|\Sigma'_\alpha| = 1$. If $\alpha' = (r, 1^{n'-r})$ with $r \geqslant 3$ then Corollary 71 provides the formula

$$|\Sigma_{\alpha'}| = 2(n' - r + 1)3^{\frac{r-3}{2}}.$$

□

Example 89. Consider $\alpha = (2, 8, 4, 5, 1, 1, 1) \vDash_e 22$. Then $\alpha' = (5, 1, 1, 1) \vDash_e 8$ is a hook, $P = \{2, 3\}$, $p' = 2 + 1$ and $q' = -2 \cdot 2 + \frac{1}{2}(8 + 4) + \frac{5-3}{2} = 3$. Thus, Corollary 88 yields $|\Sigma_\alpha| = (8 - 5 + 1)2^3 3^3 = 864$.

Let $\alpha = (l, 1^{n-l}) \vDash_e n$ be a hook. From Corollary 70 we know how to construct Σ_α from $\Sigma_{(l)}$ if k is odd. If l is even, we obtain Σ_α in the following way.

Corollary 90. *Let $\alpha = (l, 1^{n-l}) \vDash_e n$ be an even hook and* $\mathrm{id} \in \mathfrak{S}_{n-k}$*. Then the map*

$$\Sigma_{(l)} \to \Sigma_\alpha, \quad \sigma \mapsto \sigma \odot \mathrm{id}$$

is a bijection.

Proof. Recall that $\Sigma_{(1^{n-l})} = \{\mathrm{id}\}$. Then Theorem 84 yields that the map from the claim is a bijection. □

Example 91. Consider $\alpha = (4, 1, 1)$ and $\mathrm{id} \in \mathfrak{S}_2$. From Table 1 we read

$$\Sigma_{(4)} = \{(1, 4, 2, 3), (1, 3, 2, 4)\}$$

Hence, Corollary 90 yields

$$\sigma_\alpha = \left\{\sigma \odot \mathrm{id} \mid \sigma \in \Sigma_{(4)}\right\} = \{(1, 6, 2, 5), (1, 5, 2, 6)\}.$$

In Theorem 69 we showed that Σ_α is characterized by the hook properties if α is an odd hook. In the remainder of the section we want to prove that the same is true for even hooks. We first show that $\odot$ is compatible with the concepts of being oscillating and having connected intervals.

Lemma 92. *Let $\sigma_1 \in \mathfrak{S}_{n_1}$, $\sigma_2 \in \mathfrak{S}_{n_2}$ and $\sigma := \sigma_1 \odot \sigma_2$. Then σ is oscillating (has connected intervals) if and only if σ_1 and σ_2 are oscillating (have connected intervals).*

Proof. Let $\sigma_r = \sigma_{r,1}\sigma_{r,2} \cdots \sigma_{r,p_r}$ be a decomposition in disjoint cycles for $r = 1, 2$. Fix an $r \in \{1, 2\}$ and a cycle $(c_1, \dots c_t) = \sigma_{r,j}$ of σ_r. Then by Lemma 76 we have that

$$\sigma_{r,j}^{\varphi_r} = (\varphi_r(c_1), \dots, \varphi_r(c_t)).$$

As φ_r is strictly increasing, it preserves the relative order of the cycle elements so that

$$\mathrm{cst}(\sigma_{r,j}) = \mathrm{cst}(\sigma_{r,j}^{\varphi_r}).$$

In addition, Lemma 76 provides the cycle decomposition

$$\sigma = \sigma_{1,1}^{\varphi_1} \cdots \sigma_{1,p_1}^{\varphi_1} \cdot \sigma_{2,1}^{\varphi_2} \cdots \sigma_{2,p_2}^{\varphi_2}.$$

of σ. Hence, σ is oscillating if and only σ_1 and σ_2 are oscillating. For the same reason, σ has connected intervals if and only if σ_1 and σ_2 have connected intervals. □

We now generalize Theorem 69 to all hooks. The hook properties can be looked up in Definition 63.

Theorem 93. *Let $\alpha \vDash_e n$ be a hook and $\sigma \in \mathfrak{S}_n$ of type α. Then $\sigma \in \Sigma_\alpha$ if and only if σ satisfies the hook properties.*

Proof. Let $\alpha = (l, 1^{n-l}) \vDash_e n$ and $\sigma \in \mathfrak{S}_n$ be of type α. The case where l is odd was done in Theorem 69. Therefore, assume that l is even. If $l = n$ then the third hook property is satisfied and therefore the n-cycle $\sigma \in \mathfrak{S}_n$ has the hook properties if and only if it is oscillating and has connected intervals. By Theorem 49 this is equivalent to $\sigma \in \Sigma_{(n)}$. Therefore we now assume $l < n$. Write $\sigma = (d_1, \dots, d_l)$ omitting the trivial cycles. We consider the inductive product on $\mathfrak{S}_l \times \mathfrak{S}_{n-l}$ and $\mathrm{id} \in \mathfrak{S}_{n-l}$. Following Notation 74 we then have that

$$N_1 = \varphi_1([l]) = \left[\frac{l}{2}\right] \cup \left[n - \frac{l}{2} + 1, n\right].$$

Note that σ satisfies the third hook property if and only if $\{d_1, \dots, d_l\} = N_1$.

We begin with the implication form left to right. Assume that $\sigma \in \Sigma_{(l,1^{n-l})}$. By Corollary 90 there is $\tau \in \Sigma_{(l)}$ such that $\sigma = \tau \odot \mathrm{id}$. Certainly id is oscillating and has connected intervals. Moreover, τ has these properties by Theorem 49. Therefore, σ is oscillating with connected intervals by Lemma 92. Because $\sigma = \tau \odot \mathrm{id}$, Lemma 76 implies that we can write $\tau = (c_1, \dots, c_l)$ such that $d_i = \varphi_1(c_i)$ for $i = 1, \dots, l$. Therefore,

$$\{d_1, \dots, d_l\} = \varphi_1(\{c_1, \dots c_l\}) = \varphi_1([l]) = N_1$$

which means that σ satisfies the third hook property.

We now show the implication from right to left. Assume that σ fulfills the hook properties. Then the third hook property yields that $\{d_1, \dots, d_l\} = N_1$ which implies that $\sigma(N_1) = N_1$. Therefore, $\sigma \in \mathfrak{S}_l \odot \mathfrak{S}_{n-l}$ by Lemma 79, i.e. there are $\sigma_1 \in \mathfrak{S}_l$ and $\sigma_2 \in \mathfrak{S}_{n-l}$ such that $\sigma = \sigma_1 \odot \sigma_2$. From Lemma 78 we obtain that $\sigma|_{N_1} = \sigma_1^{\varphi_1}$ so that we can write σ_1 as $\sigma_1 = (c_1, \dots, c_l)$ with $c_i = \varphi_1^{-1}(d_i)$ for $i = 1, \dots, l$. It follows that σ_1 is an l-cycle of $\mathfrak{S}_l$. Since σ fixes each element of N_2, it follows from Lemma 78 that $\sigma_2 = \mathrm{id}$. As σ is oscillating with connected intervals, Lemma 92 implies that σ_1 has these properties as well. Thus, $\sigma_1 \in \Sigma_{(l)}$ by Theorem 49. Hence, we can apply Corollary 90 and obtain that

$$\sigma = \sigma_1 \odot \mathrm{id} \in \Sigma_{(l,1^{n-l})}.$$

□

Remark 94. In Remark 87 we reduced the problem of describing Σ_α for all maximal compositions α to the partitions with only odd parts. As we have such a description for odd hooks, it remains to find a combinatorial description of Σ_α in the case where α is a partition of odd parts which is not a hook. Then Σ_α consists of all permutations of type α of maximal length. Unfortunately, the situation is a lot more complex. One reason for this is the following. For any subset Σ of $\mathfrak{S}_n$ define

$$P(\Sigma) := \{P(\sigma) \mid \sigma \in \Sigma\}.$$

In general, $P(\sigma_\alpha)$ is not the only element of $P(\Sigma_\alpha)$ and there seems to be no obvious way to describe $P(\Sigma_\alpha)$. Moreover, the number of $\sigma \in \Sigma_\alpha$ whose orbits yield the same set partition of $[n]$ depends on this very set partition. For example, $\Sigma_{(3,3)}$ consists of the following elements where elements with the same orbit partition occur in the same row.

$$\begin{array}{llll}
(1,6,2)(3,4,5) & (1,2,6)(3,4,5) & (1,6,2)(3,5,4) & (1,2,6)(3,5,4) \\
(1,6,3)(2,4,5) & (1,6,3)(2,5,4) & (1,3,6)(2,4,5) & (1,3,6)(2,5,4) \\
(1,4,5)(2,6,3) & (1,5,4)(2,3,6) & (1,5,4)(2,6,3) & (1,4,5)(2,3,6) \\
(1,6,4)(2,3,5) & (1,4,6)(2,3,5) & (1,6,4)(2,5,3) & (1,4,6)(2,5,3) \\
(1,6,5)(2,3,4) & (1,5,6)(2,3,4) & (1,5,6)(2,4,3) & (1,6,5)(2,4,3) \\
(1,5,3)(2,4,6) & (1,3,5)(2,6,4) & &
\end{array}$$

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