Balanced edge-colorings avoiding rainbow cliques of size four

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Abstract

A balanced edge-coloring of the complete graph is an edge-coloring such that every vertex is incident to each color the same number of times. In this short note, we present a construction of a balanced edge-coloring with six colors of the complete graph on $n = 13^k$ vertices, for every positive integer k, with no rainbow K_4. This solves a problem by Erdős and Tuza.

Mathematics Subject Classifications: 05C15, 05C35

1 Introduction and main result

Let H be a graph. An edge-coloring of the complete graph K_n contains a *rainbow copy* of H if it contains a copy of H such that all its edges are assigned different colors. Various conditions on edge-colorings forcing the existence of rainbow copies have been studied [1, 2, 4, 6, 7, 8, 9, 10]. Here, we look at a condition where the colors are well distributed. A *balanced* edge-coloring of the complete graph is an edge-coloring such that every vertex is incident to each color the same number of times.

Question 1 (Erdős and Tuza, Problem 1 in [6])**.** Is the following true for every graph H? For n sufficiently large, every balanced edge-coloring with $|E(H)|$ colors of the complete graph K_n contains a rainbow copy of H.

Recently, this question was answered by Axenovich and Clemen [3] who showed that it is not true for most cliques. In particular, they showed that it is false for all cliques $H = K_q$ with odd number of edges and at least six vertices. They [3] also conjectured Question 1 to be false for every clique with at least four vertices. Here, we answer Question 1 in the negative for $H = K_4$, a subcase which has been given particular attention by Erdős and Tuza.

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Indeed, Erdős and Tuza [6] suggested that the simplest counterexamples to Question 1 might be K_4, C_6 and $2K_3$ and commented "we could not prove or disprove that every t-regular 6-coloring of K_{6t+1}, contains them as rainbow subgraphs (here t has to be even)". In Erdős' list "Some of my favourite problems on cycles and colourings" [5], he further remarked that one of the most interesting unsolved problems in this area is Question 1 for $H = C_6$ and $H = K_4$.

Theorem 2. *For every $k \geqslant 1$ there exists a balanced edge-coloring of K_{13^k} with 6 colors and no rainbow K_4.*

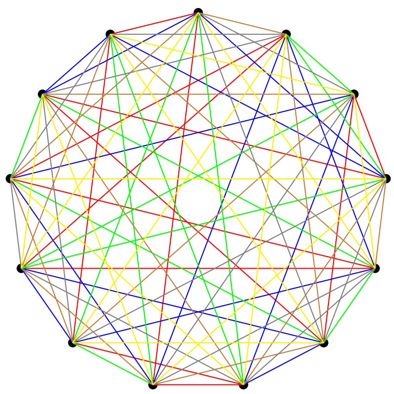
A key idea in the constructions by Axenovich and Clemen [3] is that iterating a balanced coloring with no rainbow copy of some clique K_q maintains those properties:

Lemma 3 (Axenovich, Clemen, Lemma 2.2 in [3])**.** *If there exists a balanced edge-coloring of K_n with ℓ colors and no rainbow K_q, then for every $k \geqslant 1$ there exists a balanced edge-coloring of K_{n^k} with ℓ colors and no rainbow K_q.*

Therefore, it suffices to prove Theorem 2 for $k = 1$. The base constructions used in [3] are the standard examples of 1- and 2-factorizations, i.e. edge-colorings such that every color class is a 1-regular, respectively 2-regular, spanning subgraph. Note that those colorings do not work as constructions for Theorem 2.

$$\begin{pmatrix}
0 & 2 & 5 & 4 & 1 & 3 & 3 & 6 & 4 & 2 & 6 & 5 & 1 \\
2 & 0 & 3 & 6 & 5 & 6 & 4 & 1 & 3 & 1 & 4 & 5 & 2 \\
5 & 3 & 0 & 5 & 4 & 2 & 6 & 3 & 1 & 6 & 2 & 1 & 4 \\
4 & 6 & 5 & 0 & 2 & 4 & 5 & 2 & 1 & 3 & 3 & 1 & 6 \\
1 & 5 & 4 & 2 & 0 & 3 & 1 & 6 & 2 & 5 & 4 & 6 & 3 \\
3 & 6 & 2 & 4 & 3 & 0 & 1 & 4 & 5 & 6 & 5 & 2 & 1 \\
3 & 4 & 6 & 5 & 1 & 1 & 0 & 2 & 5 & 4 & 2 & 6 & 3 \\
6 & 1 & 3 & 2 & 6 & 4 & 2 & 0 & 3 & 5 & 1 & 4 & 5 \\
4 & 3 & 1 & 1 & 2 & 5 & 5 & 3 & 0 & 4 & 6 & 2 & 6 \\
2 & 1 & 6 & 3 & 5 & 6 & 4 & 5 & 4 & 0 & 1 & 3 & 2 \\
6 & 4 & 2 & 3 & 4 & 5 & 2 & 1 & 6 & 1 & 0 & 3 & 5 \\
5 & 5 & 1 & 1 & 6 & 2 & 6 & 4 & 2 & 3 & 3 & 0 & 4 \\
1 & 2 & 4 & 6 & 3 & 1 & 3 & 5 & 6 & 2 & 5 & 4 & 0
\end{pmatrix}$$

(a) Adjacency matrix of the coloring



(b) A drawing of the coloring

Figure 1: The edge-coloring of K_{13} with no rainbow K_4

In Figure 1, we present an edge-coloring of K_{13} with six colors, such that every vertex is incident to every color exactly twice and there is no rainbow K_4. This coloring was found by a computer search, based on a simple local search algorithm. It was greedily attempting to maximize a score function, that measured how close the coloring is to being balanced, and how few rainbow K_4's the coloring contains. The computer search showed that the solution is not unique. While it can be checked quickly that this coloring is indeed balanced, it takes more effort to check by hand that it does not contain a rainbow K_4, simply because there are $\binom{13}{4} = 715$ copies of K_4 in K_{13}.

We remark that our construction differs from the constructions in [3] in the sense that it seemingly does not follow a visible pattern. Indeed, colorclasses 1 and 3 are both a disjoint union of cycles of lengths 5,4 and 4; colorclass 2 is a disjoint union of cycles of lengths 9 and 4; colorclasses 4,5 and 6 are each cycles of length 13.

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