Inversion sequences avoiding a triple of patterns of 3 letters

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Abstract

An *inversion sequence* of length n is a sequence of integers $e = e_1 \cdots e_n$ which satisfies for each $i \in [n] = \{1, 2, \ldots, n\}$ the inequality $0 \leqslant e_i < i$. For a set of patterns P, we let $\mathbf{I}_n(P)$ denote the set of inversion sequences of length n that avoid all the patterns from P. We say that two sets of patterns P and Q are *I-Wilf-equivalent* if $|\mathbf{I}_n(P)| = |\mathbf{I}_n(Q)|$ for every n. In this paper, we show that the number of I-Wilf-equivalence classes among triples of length-3 patterns is 137, 138 or 139. In particular, to show that this number is exactly 137, it remains to prove $\{101, 102, 110\} \overset{\mathbf{I}}{\sim} \{021, 100, 101\}$ and $\{100, 110, 201\} \overset{\mathbf{I}}{\sim} \{100, 120, 210\}$.

Mathematics Subject Classifications: 05A05, 05A15

1 Introduction

An *inversion sequence* [7, 17] of length n is a sequence of integers $e = e_1 \cdots e_n$ which satisfies for each $i \in [n] = \{1, 2, \ldots, n\}$ the inequality $0 \leqslant e_i < i$. The set of inversion sequences of length n is denoted $\mathbf{I}_n$. Note that $|\mathbf{I}_n| = n!$, and there is a simple bijection between $\mathbf{I}_n$ and the set of all the permutations of the set $[n]$: an inversion sequence $e = e_1 \cdots e_n \in \mathbf{I}_n$ corresponds to the unique permutation $\pi = \pi_1 \cdots \pi_n$ with the property that for each $i \in [n]$, e_i is equal to the number of elements in the set $\{\pi_1, \pi_2, \ldots, \pi_{i-1}\}$ which are larger than π_i.

Let $[k]_0$ denote the set $\{0, 1, 2, \ldots, k\} = \{0\} \cup [k]$. For a set S, we let S^n denote the set of words of length n over the alphabet S, i.e., all the n-tuples $w = w_1 w_2 \cdots w_n$ with $w_i \in S$. In all the words we consider in this paper, the alphabet is a subset of $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The *height* of a word $w = w_1 w_2 \cdots w_n$, denoted $\mathrm{ht}(w)$, is the largest

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number that appears as a symbol in w, or in other words

$$\mathrm{ht}(w) = \max\{w_1, \ldots, w_n\}.$$

We say that a word $x = x_1 \cdots x_n$ is *order-isomorphic* to a word $y = y_1 \cdots y_n$ if for every pair of indices $i, j \in [n]$, we have $x_i < x_j$ if and only if $y_i < y_j$; notice that this also implies that $x_i = x_j$ if and only if $y_i = y_j$. We say that a word $w = w_1 \cdots w_n$ *contains* a word $p = p_1 \cdots p_m$ if w contains a (not necessarily consecutive) subsequence of length m which is order-isomorphic to p. Otherwise, we say that w *avoids* p. A subsequence of w order-isomorphic to p is referred to as *a copy* of p in w.

We say that a word w of height k is *reduced*, if each number from the set $[k]_0$ appears at least once in w. Note that every word is order-isomorphic to a unique reduced word. Furthermore, if p and q are order-isomorphic, then a word w contains p if and only if w contains q. Thus, when dealing with pattern-avoidance in words, we may without loss of generality restrict our attention to reduced patterns. Throughout this paper, we use the term *pattern* as a synonym for reduced word.

For a set B of patterns, we let $\mathbf{I}_n(B)$ denote the set of inversion sequences of length n that avoid all the elements from B, and let $\mathbf{I}(B)$ denote the set $\bigcup_{n=0}^{\infty} \mathbf{I}_n(B)$. To avoid notational clutter, we often omit nested braces and write, e.g, $\mathbf{I}_n(p, q)$ instead of $\mathbf{I}_n(\{p, q\})$. We say that two sets of patterns P and Q are *I-Wilf-equivalent*, denoted $P \overset{\mathbf{I}}{\sim} Q$, if $|\mathbf{I}_n(P)| = |\mathbf{I}_n(Q)|$ for every n.

The systematic study of pattern-avoidance for inversion sequences started around 2015 [7, 17]. Several aspects of pattern-avoidance for inversion sequences have been considered (for example, see [1, 2, 5, 6, 11, 15, 16, 18–21] and references therein). In particular, the results of [3, 5, 20, 21] determined all the I-Wilf-equivalence classes of pairs of length-3 patterns. Note that there are 13 patterns of 3 letters, namely,

$$P_3 = \{000, 001, 010, 011, 012, 021, 100, 101, 102, 110, 120, 201, 210\}.$$

The main result of this paper can be formulated as follows.

Theorem 1. *The number of I-Wilf-equivalence classes of triples of length-3 patterns is* 137*,* 138 *or* 139*.*

To show that there are exactly 137 I-Wilf-equivalence classes, as the computational data seem to suggest, it remains to solve the following two conjectures.

Conjecture 2. We make the following two conjectures:

1. $\{101, 102, 110\} \overset{\mathbf{I}}{\sim} \{021, 100, 101\}$ (see Class 110 in Table 3). Note that in [4] it is shown that $\{021, 100, 101\} \overset{\mathbf{I}}{\sim} \{021, 101, 110\}$, so the conjecture can be equivalently stated as $\{101, 102, 110\} \overset{\mathbf{I}}{\sim} \{021, 101, 110\}$.

2. $\{100, 110, 201\} \overset{\mathbf{I}}{\sim} \{100, 120, 210\}$ (see Class 129 in Table 3). Note that we can show, via generating trees, that $\{100, 110, 201\} \overset{\mathbf{I}}{\sim} \{101, 110, 201\}$ and $\{100, 120, 210\} \overset{\mathbf{I}}{\sim} \{110, 120, 210\}$.

Let L be the set of all triples of patterns in P_3, namely,

$$L = \{X \mid X \subseteq P_3, |X| = 3\}.$$

A *candidate class* is a maximal subset C of L such that for any $B, B' \in C$, $|\mathbf{I}_n(B)| = |\mathbf{I}_n(B')|$ for all $n = 1, 2, \ldots, 9$. Table 7 shows all the 137 candidate classes of L. A candidate class is called *trivial* if it contains exactly one triple, otherwise, it is called *nontrivial*. Clearly, any I-Wilf equivalence class is contained in a candidate class, and Conjecture 2 implies that candidate classes coincide with I-Wilf equivalence classes.

To establish the I-Wilf equivalence of two triples from L, we employ several different approaches. For some triples $T \in L$, we can find a proper subset $T' \subsetneq T$ (i.e., a pair or even a singleton) such that $\mathbf{I}_n(T') = \mathbf{I}_n(T)$ for every n. We can then directly exploit previous results on inversion sequences avoiding smaller sets of patterns to obtain the enumeration of $\mathbf{I}_n(T)$. We develop this approach in Section 2.

Another fruitful approach is based on the concept of generating trees, and uses the recent algorithmic method of Kotsireas, Mansour, and Yıldırım [11]. We outline this method in Section 3. We note that by combining the methods from Sections 3 and 2, we are able to solve several open problems related to the enumeration of inversion sequences avoiding pairs of patterns; see Theorems 10 and 11.

In several cases, neither of the previous two approaches is sufficient for our purposes. We then turn to bijective arguments. Several of our bijections are based on a relationship between inversion sequences and diagram fillings, which we describe in Section 4, and which allows us to tie the previously studied concept of shape-Wilf equivalence with I-Wilf equivalence.

In Section 5, we solve a handful of remaining cases of I-Wilf equivalence by new, direct bijections. Finally, in Section 6, we enumerate several trivial candidate classes, by again using the technique of generating trees.

Our main goal is to focus on the equivalence relations and enumeration results that cannot be deduced by routine applications of known methods. Accordingly, to keep the length of the paper manageable, we omit detailed presentations of repetitive cases which are not meaningfully different from previously solved cases and focus on the 'hard' cases that require novel approaches.

2 Equipotence of pattern sets

We will say that two sets A and B of patterns are *equipotent*, denoted $A \approx B$, if the set $\mathbf{I}(A)$ is equal to the set $\mathbf{I}(B)$. Obviously, equipotent sets are I-Wilf equivalent, and their corresponding generating trees $\mathcal{T}(A)$ and $\mathcal{T}(B)$ are identical.

We are particularly interested in the cases when a set A of three patterns from P_3 is equipotent to a set B of two patterns from P_3 or even a single pattern from P_3. Such a situation will allow us to reduce the enumeration of $\mathbf{I}_n(A)$ to previous results. To deduce all the needed equipotence relations in a uniform way, we now prove several general criteria of equipotence, which cover all the cases of interest.

Observation 3. *The following holds:*

(a) For any three sets of patterns A, B, C, if $A \approx B$ then $A \cup C \approx B \cup C$.

(b) For any set of patterns A, and any reduced pattern p, if p contains at least one pattern from A, then $A \approx A \cup \{p\}$.

For a word $w = w_1 \cdots w_n$ and an integer $k \in \mathbb{N}_0$, let $w + k$ denote the word $w_1 + k, w_2 + k, \ldots, w_n + k$.

Lemma 4. *Let $p = p_1 p_2 \cdots p_k$ be a reduced pattern with $p_1 > 0$. Then every inversion sequence that contains p also contains at least one of the two patterns $0p$ and $00(p+1)$. Consequently,*

$$\{0p, 00(p+1), p\} \approx \{0p, 00(p+1)\}.$$

Proof. Let $e = e_1 \cdots e_n$ be an inversion sequence containing p, and let $s = e_{i(1)} e_{i(2)} \cdots e_{i(k)}$ be a subsequence of e order-isomorphic to p. Since p is reduced, there is an index j such that $p_j = 0$. Let $v = e_{i(j)}$, i.e., v is the smallest value appearing in s. Let q be the prefix of e of length $v+1$, i.e., $q = e_1 e_2 \cdots e_{v+1}$. All the values in q are less than or equal to v, and therefore, they are strictly smaller than $e_{i(1)}$. In particular, the whole subsequence s appears in e to the right of the rightmost entry of q.

We distinguish two possibilities: either q contains an entry equal to v, or it does not. If there is such an entry (necessarily it must be the rightmost entry e_{v+1}), then this entry together with s forms a copy of the pattern $0p$ in e. On the other hand, if there is no such entry, then q only contains the values from the set $[v-1]_0$, and in particular, there is a value $w \in [v-1]_0$ that appears at least twice in q. It follows that e contains the subsequence wws, which is order-isomorphic to $00(p+1)$.

We conclude that any inversion sequence containing p also contains $0p$ or $00(p+1)$, or equivalently, any inversion sequence avoiding both $0p$ and $00(p+1)$ also avoids p. This implies that $\{0p, 00(p+1), p\} \approx \{0p, 00(p+1)\}$. ☐

Example 5. The set $\{001, 010, 120\}$ is equipotent to $\{001, 010\}$. To see this, note that

$$\{0120, 00231\} \approx \{0120, 00231, 120\}$$

by Lemma 4, hence

$$\{0120, 00231\} \cup \{001, 010\} \approx \{0120, 00231, 120\} \cup \{001, 010\}$$

by Observation 3 part (a), and therefore

$$\{001, 010\} \approx \{001, 010, 120\}$$

by Observation 3 part (b).

The argument presented in Example 5 shows the typical way to deduce that a set of three patterns is equipotent to its proper subset. It applies to many analogous situations, and we will from now on omit the details of the arguments, as long as they are analogous to Example 5.

Lemma 6. *Let $p = p_1 p_2 \cdots p_k$ be a reduced pattern with the following properties:*

- *$p_1 = 0$ and $p_2 > 1$,*
- *apart from p_1, no other entry of p is equal to 0.*

Then every inversion sequence containing p also contains at least one of the two patterns p' and p'', where $p' = 0p = 00p_2p_3 \cdots p_k$ and $p'' = 01p_2p_3 \cdots p_k$. Consequently, $\{p', p'', p\} \approx \{p', p''\}$.

Proof. Let $e = e_1 \cdots e_n$ be an inversion sequence containing p, and let $s = e_{i(1)}e_{i(2)} \cdots e_{i(k)}$ be a subsequence of e order-isomorphic to p. Let v be the smallest value appearing in the suffix $e_{i(3)}e_{i(4)} \cdots e_{i(k)}$ of s, i.e., v is the value in s that corresponds to the value 1 in p. By the assumptions on p, we have $e_{i(1)} < v < e_{i(2)}$, and in particular $v > 0$.

Let q be the prefix of e of length $v + 1$, i.e., $q = e_1e_2 \cdots e_{v+1}$. All the values in q are less than or equal to v, and therefore, they are strictly smaller than $e_{i(2)}$, which therefore appears to the right of q.

The rest of the argument is analogous to the proof of Lemma 4: if q contains an entry with value v, then e contains the subsequence $0ve_{i(2)}e_{i(3)} \cdots e_{i(k)}$, which is order-isomorphic to p''. If, on the other hand, q does not contain the value v, then it contains a value $w \in [v-1]_0$ repeated at least twice, and e contains the subsequence $wwe_{i(2)}e_{i(3)} \cdots e_{i(k)}$ order-isomorphic to p'. □

Example 7. The set $\{001, 012, 021\}$ is equipotent to $\{001, 012\}$. This follows by an argument analogous to Example 5, by using Lemma 6 instead of Lemma 4.

Lemma 8. *Let $p = p_1 \cdots p_k$ be a reduced pattern with $p_1 \geqslant 2$, and let e be an inversion sequence containing p. Then e contains the pattern 000 or it contains both the patterns 011 and 012. Consequently, $\{000, 011, p\} \approx \{000, 011\}$ and $\{000, 012, p\} \approx \{000, 012\}$.*

Proof. Let $e = e_1 \cdots e_n$ be an inversion sequence containing p, and let $s = e_{i(1)}e_{i(2)} \cdots e_{i(k)}$ be a subsequence of e order-isomorphic to p. Assume that e avoids the pattern 000; our goal is then to show that e contains both 012 and 011.

Let us first show that e contains the pattern 012. Set $v = e_{i(1)}$. By assumption, we have $v \geqslant p_1 \geqslant 2$. Consider the prefix of e of length v. If this prefix contains a nonzero value w, then e has the subsequence $0wv$ order-isomorphic to 012. Suppose then that the first v values of e are all zeros. Since e avoids 000, this means that $v = 2$. However, this means that $e_{i(1)} = 2$, which implies that all the entries equal to 0, 1, or 2 in p must correspond to entries with the same value in s, and in particular, s contains at least one entry $e_{i(j)}$ equal to 0. This entry, together with the first two entries of e forms a copy of 000, contradicting our assumptions. We conclude that e contains 012.

We now show that e contains 011. Let $v_0 < v_1 < v_2$ be the three smallest values appearing in s, i.e., the values corresponding, respectively, to the values 0, 1, and 2 in p. Let q be the prefix of e of length $v_1 + 1$. Necessarily, all the values in q are smaller than v_2, and so q is completely to the left of s. If any nonzero value w appears in q more than once, then q contains the subsequence $0ww$, and we are done. Also, if q contains the value

v_1, then e contains the subsequence $0v_1v_1$, and we are again done. Finally, if q contains more than two entries equal to 0, then e contains 000 contradicting our assumptions. This leaves us with the case when q contains exactly two entries equal to 0, and $v_1 - 1$ nonzero entries, each equal to a distinct value from the set $[v_1 - 1]$. But now we consider that s contains an entry $e_{i(j)}$ equal to v_0. If $v_0 = 0$, then $e_{i(j)}$ forms a copy of 000 together with the two zero entries of q, and if $v_0 > 0$, we obtain a subsequence $0v_0v_0$ in e, which gives the claimed copy of 011. □

Table 1: Classes involving triples equipotent with their proper subsets. Where no reference for the enumeration result is given, the result follows from the classification of pairs of triples by Yan and Lin [21] or from a reference given therein. For each triple of patterns, we either list an equipotent pair or singleton pattern, or we list the rules of the corresponding generation tree.

Beginning of Table 1			
Class	**Triple**	**Equipotent subset (or gen. tree specification)**	**G.f. formula or reference**
7	001,010,021	001,010	
	001,010,100		
	001,010,101		
	001,010,102		
	001,010,110		
	001,010,120		
	001,010,201		
	001,010,210		
	001,011,021	001,011	
	001,011,101		
	001,011,102		
	001,011,110		
	001,011,201		
	001,011,210		
	001,012,021	001,012	
	001,012,101		
	001,012,102		
	001,012,120		
	001,012,201		
	001,012,210		
	000,001,110	$a_m \rightsquigarrow (00)^{m+1} a_{m+1}$; $a_m = 01 \cdots m$	$\frac{x}{(1-x)^2}$
9	000,001,100	000,001	
	000,001,101		
	000,001,102		
	000,001,201		
	010,011,012	$a_m \rightsquigarrow b_1 \cdots b_m a_{m+1}$, $b_m \rightsquigarrow b_1 \cdots b_{m-1}$; $a_m = 0^m$, $b_m = a_m m$	$\frac{x(1+x)}{1-x-x^2}$
21	000,012,102	000,012	
	000,012,120		
	000,012,201		
	000,012,210		$x + 2x^2 + 4x^3 + 5x^4 + 2x^5 + x^6$
23	001,021,101	001,021	
	001,021,102		
	001,021,201		
	001,021,210		
	001,101,120	001,120	
	001,102,120		
	001,120,201		
	001,120,210		
	001,102,110	001,110	
	001,101,110		
	001,110,201		
	001,110,210		
	000,011,021	$0 \rightsquigarrow 00, 0$, $00 \rightsquigarrow 00, 002$, $002 \rightsquigarrow 002$	
	001,100,210	$a_m \rightsquigarrow (010)^m b_m a_{m+1}$, $b_m \rightsquigarrow (010)^m b_m$; $a_m = 01 \cdots m$, $b_m = a_m m$	
	010,012,021	$a_m \rightsquigarrow (01)^m a_{m+1}$, $01 \rightsquigarrow 01$; $a_m = 0^m$	
	011,012,021		$\frac{x(1-x+x^2)}{(1-x)^3}$
24	001,100,101	001,100	
	001,100,102		
	001,100,201		
	000,011,120	$0 \rightsquigarrow 0, 01$, $01 \rightsquigarrow 0, 012$, $012 \rightsquigarrow 012$	$\frac{x}{(1-x)(1-x-x^2)}$
26	001,102,210	001,210	
	001,201,210		
	001,101,210		
	011,012,210	$a_m \rightsquigarrow b_1 \cdots b_m a_{m+1}$, $b_m \rightsquigarrow (0021)^{m-1} b_m$; $a_m = 0^m$, $b_m = a_m m$	$\frac{x(1-2x+2x^2)}{(1-x)^4}$
29	000,011,100	000,011	
	000,011,101		
	000,011,110		
	000,011,201		
	000,011,210		
	001,101,102	001	
	001,101,201		

Continuation of Table 1			
Class	Triple	Equipotent subset (or gen. tree specification)	G.f. formula or reference
	001,102,201		
	010,012,101 010,012,102 010,012,120 010,012,201	010,012	
	011,012,101 011,012,102 011,012,110 011,012,120	011,012	
	010,011,021	$a_m \rightsquigarrow b_1 \cdots b_m a_{m+1}$, $b_m \rightsquigarrow b_1 \cdots b_m$; $a_m = 0^m$, $b_m = a_m 1$	$\frac{x}{1-2x}$
34	010,011,100 010,011,101 010,011,110	010,011	$\sum_{j\geqslant 1} \frac{x^j}{1-jx}$
42	012,100,102 012,100,120	012,100	Enumeration open
44	012,021,102 012,021,120 012,021,201 012,021,210	012,021	
	012,102,110 012,110,120	012,110	
	012,101,201	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m$, $b_m \rightsquigarrow e b_1 \cdots b_m$, $e \rightsquigarrow e$; $a_m = 0^m$, $b_m = 0^m m$, $e = 010$	
.	011,021,102	$a_m \rightsquigarrow a_{m+1} b_m \cdots b_1$, $b_m \rightsquigarrow e b_m \cdots b_1$, $e \rightsquigarrow e$; $a_m = 0^m$, $b_m = 0^m 1$, $e = 010$	$\frac{x(1-2x+2x^2)}{(1-x)^2(1-2x)}$
46	012,101,102 012,101,120	012,101	$\frac{x(1-x+x^2)}{1-3x+2x^2-x^3}$
49	012,102,201 012,120,201	012,201	
	012,102,210 012,120,210	012,210	$\frac{x(1-4x+7x^2-5x^3+2x^4)}{(1-x)^4(1-2x)}$
53	011,101,102 011,102,110	011,102	
	012,102,120	012	$\frac{x(1-x)}{1-3x+x^2}$
58	000,021,100 000,021,201 000,021,210	000,021	Theorem 10
63	010,021,100 010,021,101 010,021,102 010,021,110 010,021,120 010,021,201 010,021,210	010,021	
	011,021,101 011,021,110 011,021,201 011,021,210	011,021	$C(x) - 1$, [4]
64	011,101,120 011,110,120	011,120	
68	011,100,101 011,100,110	011,100	$(1-x)\sum_{j\geqslant 1} \frac{x^j}{1-(j+1)x}$,
80	011,101,201 011,110,201	011,201	Enumeration open
	011,101,210 011,110,210	011,210	I-Wilf equivalence: [21] See also Example 21
103	021,102,201 021,102,210	021,102	Theorem 11
115	021,120,201 021,120,210	021,120	$\frac{1-4x+\sqrt{-16x^3+20x^2-8x+1}}{2(x-1)(4x-1)}$
117	021,100,201 021,100,210	021,100	
	021,110,201 021,110,210	021,110	Theorem 12
	101,102,210		Theorem 40
120	021,101,201 021,101,210	021,101	A106228 in OEIS
End of Table 1			

3 Inversion sequences and generating trees

To establish a useful connection between generating trees and the avoidance problem in inversion sequences, we recall the generating trees for pattern avoidance in inversion sequences as described in [11]. For a given set of patterns B, let $\mathbf{I}(B) = \cup_{n=0}^{\infty} \mathbf{I}_n(B)$. We will construct a pattern-avoidance tree $\mathcal{T}(B)$ for the class of pattern-avoiding inversion sequences $\mathbf{I}(B)$. The tree $\mathcal{T}(B)$ is understood to be empty if there is no inversion sequence of any length avoiding the set B. Otherwise, the root can always be taken as 0 (inversion sequence with one letter), that is, $0 \in \mathcal{T}(B)$. Starting with this root which stays at level

1, the nodes at level $n+1$ of the tree $\mathcal{T}(B)$ can be constructed from the nodes at level n in such a way that the children of $e = e_1 \cdots e_n \in \mathbf{I}_n(B)$ are $e' = e_1 \cdots e_n j$ with $j = 0, 1, \ldots, n$ such that $e' \in \mathbf{I}_{n+1}(B)$.

Now, we relabel the vertices of the tree $\mathcal{T}(B)$ as follows. Define $\mathcal{T}(B; e)$ to be the subtree consisting of the inversion sequence e as the root and its descendants in $\mathcal{T}(B)$. We say that e is *equivalent* to e', denoted by $e \sim e'$, if and only if $\mathcal{T}(B; e) \cong \mathcal{T}(B; e')$ (in the sense of plane trees). Let $V[B]$ denote the set of all equivalence classes in the quotient set $\mathcal{T}(B)/\sim$. We will represent each equivalence class $[v]$ by the label of the unique node v which appears on the tree $\mathcal{T}(B)$ as the first node (from top to bottom from left to right). Let $\mathcal{T}[B]$ be the same tree $\mathcal{T}(B)$ where we replace each node v by its equivalence class label.

The basic outline of the generating tree method is the following.

(1) We use the main algorithm of [11] for finding the generating tree $\mathcal{T}[B]$ up to a level $4-7$, for all $B \in L$. Those sets B of patterns for which the generating tree is finite ($V[B]$ is a finite set) are called *regular*, while all other sets of patterns are called *non-regular*. In Table 7, we denote each regular class by (r). Thus, in L there are exactly 64 regular classes.

(2) We try to guess the rules of the generating tree $\mathcal{T}[B]$ for any $B \in L$ such that B belongs to a nontrivial candidate class, as described in [11]. Tables 1 and 3 present all the nontrivial candidate classes C, all the triples $B \in C$, and the generating tree $\mathcal{T}[B]$ for any $B \in C$. In the case we fail to guess and prove the generating tree $\mathcal{T}[B]$, we leave the cell in the column of $\mathcal{T}[B]$ empty. Moreover, sometimes we succeed to find the generating tree $\mathcal{T}[B]$ while we fail to find an explicit formula for the generating function $F_B(x) = \sum_{n\geqslant 1} |\mathbf{I}_n(B)| x^n$. In this case, we leave the cell in the column of $F_B(x)$ empty.

(3) For a triple B, we translate the rules of $\mathcal{T}[B]$ (if we succeed to find them) into a system of recurrence relations. Then we solve the system, either by induction or by the kernel method (see [9] and references therein).

Remark 9. For a given triple B, assume that we have guessed the rules R of the generating tree $\mathcal{T}[B]$ (the root is assumed to be the inversion sequence 0). To prove that R are the rules of the generating tree $\mathcal{T}[B]$, we proceed by induction on the length of the labels. For example, let $B = \{001, 100, 210\}$ and our algorithm guessed the following rules

$$a_m \rightsquigarrow c^m a_m \text{ and } b_m \rightsquigarrow c^m a_m b_{m+1},$$

where $a_m = 01 \cdots mm$, $b_m = 01 \cdots m$ and $c = 010$. Note that, for the inversion sequence 010, there are no children (because the children of 010 are 0100, 0101, 0102 and 0103, where each contains either 100 or 001). To show that the rules hold, we have to show that the children of a_m and children of b_m satisfy the same set of rules. The inversion sequence b_m has the children $01 \cdots mj \sim 010$ for all $j = 0, 1, \ldots, m-1$, and a_m, b_{m+1}, which creates the rule $b_m \rightsquigarrow c^m a_m b_{m+1}$. Similarly, the inversion sequence a_m has the children $01\cdots mmj \sim 010$ for all $j = 0, 1, \ldots, m-1$, and $012\cdots mmm \sim a_m$, which creates the rule $a_m \rightsquigarrow c^m a_m$, as required.

We used this method after we guessed the generating tree $\mathcal{T}[B]$ for any possible triple B. From now on, we omit the proofs for the generating trees $\mathcal{T}[B]$.

Table 3: Nontrivial candidate classes not listed in Table 1, generating trees $\mathcal{T}[B]$, and generating functions $F_B(x)$

Begin of Table 3			
Class	B	$\mathcal{T}[B]$	$F_B(x)$
2	000,001,010 000,001,011	$0 \rightsquigarrow 0, 00$	
	001,010,011 001,010,012 001,011,012	$0 \rightsquigarrow 00, 00, \quad 00 \rightsquigarrow 00$	$\frac{x(1+x)}{1-x}$
5	001,011,100(r) 001,012,100(r)	$0 \rightsquigarrow 00, 01, \quad 00 \rightsquigarrow 00, \quad 01 \rightsquigarrow 01, 010$	
	001,011,120(r) 001,012,110(r)	$0 \rightsquigarrow 00, 01, \quad 00 \rightsquigarrow 00, \quad 01 \rightsquigarrow 00, 00$	$\frac{x(1+x+x^2)}{1-x}$
6	000,001,021(r) 000,001,120(r)	$0 \rightsquigarrow 00, 01, \quad 01 \rightsquigarrow 00, 011, 01, \quad 011 \rightsquigarrow 00$	$\frac{x(1+x+x^2+x^3)}{1-x}$
22	000,011,102(r)	$0 \rightsquigarrow 0, 01, \quad 01 \rightsquigarrow 010, 01$	
	001,021,100(r) 001,100,120(r)	$0 \rightsquigarrow 00, 01, \quad 00 \rightsquigarrow 00, \quad 01 \rightsquigarrow 010, 011, 01, \quad 011 \rightsquigarrow 010, 011$	
	001,021,110(r) 001,110,120(r)	$0 \rightsquigarrow 00, 01, \quad 00 \rightsquigarrow 00, \quad 01 \rightsquigarrow 00, 00, 01$	
	001,021,120(r)	$0 \rightsquigarrow 00, 01, \quad 00 \rightsquigarrow 00, \quad 01 \rightsquigarrow 00, 011, 011, \quad 011 \rightsquigarrow 00, 011$	
	001,100,110	$a_m \rightsquigarrow a_{m+1}bc^m, \quad b \rightsquigarrow b$; $a_m = 01\cdots m$, $b = 00$, $c = 010$	$\frac{x(1+x^2)}{(1-x)^2}$
27	010,012,100	$a_m \rightsquigarrow b_1\cdots b_m a_{m+1}, \quad b_m \rightsquigarrow c_2\cdots c_m b_m, \quad c_m \rightsquigarrow c_2\cdots c_{m-1}$; $a_m = 0^m$, $b_m = a_m m$, $c_m = b_m(m-1)$	
	010,012,110	$a_m \rightsquigarrow b_1\cdots b_m a_{m+1}, \quad b_m \rightsquigarrow b_1\cdots b_{m-1}b_1$; $a_m = 0^m$, $b_m = a_m m$	
	011,012,201	$a_m \rightsquigarrow b_1\cdots b_m a_{m+1}, \quad b_m \rightsquigarrow b_1\cdots b_m$; $a_m = 0^m$, $b_m = a_m m$	$\frac{x(1-x+x^3)}{(1-x)^2(1-x-x^2)}$
33	010,011,201 010,011,210	$a_m \rightsquigarrow a_{m+1}b_{m,1}b_{m,1}b_{m,3}\cdots b_{m,m}$, $b_{m,1} \rightsquigarrow b_{m,1}b_{m,1}b_{m,3}\cdots b_{m,m}$, $b_{m,j} \rightsquigarrow b_{m+2-j,1}b_{m+3-j,1}b_{m+4-j,3}\cdots b_{m,j-1}b_{m,j}\cdots b_{m,m}$ with $j = 3, \ldots, m$; $a_m = 0^m$, $b_{m,j} = a_m j$	
35	012,021,100 012,021,101	$a_m \rightsquigarrow a_{m+1}(01)^m, \quad 01 \rightsquigarrow 010, 01, \quad 010 \rightsquigarrow 010$; $a_m = 0^m$	
	012,021,110	$a_m \rightsquigarrow a_{m+1}(01)^m, \quad 01 \rightsquigarrow 011, 01, \quad 011 \rightsquigarrow 011$; $a_m = 0^m$	$\frac{x(1-2x+3x^2-x^3)}{(1-x)^4}$
40	012,100,210	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, \quad b_m \rightsquigarrow c_m e^{m-1}b_m, \quad c_m \rightsquigarrow e^{m-1}c_m$; $a_m = 0^m$, $b_m = 0^m m$, $c_m = 0^m m0$, $e = 0021$	
	012,110,210	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, \quad b_m \rightsquigarrow b_m e^m, \quad e \rightsquigarrow e$; $a_m = 0^m$, $b_m = 0^m m$, $e = 011$	$\frac{x(1-3x+5x^2-3x^3+x^4)}{(1-x)^5}$
41	012,100,201	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, \quad b_m \rightsquigarrow c_1\cdots c_m b_m, \quad c_m \rightsquigarrow dc_2\cdots c_{m-1}$; $a_m = 0^m$, $b_m = a_m m$, $c_m = b_m(m-1)$, $e = 00210$	
	012,110,201	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, \quad b_m \rightsquigarrow b_1c_2\cdots c_m d$, $c_m \rightsquigarrow d^2c_2\cdots c_{m-1}$; $a_m = 0^m$, $b_m = a_m m$, $c_m = b_m(m-1)$, $e = 011$	$\frac{x(1-3x+4x^2-x^3-2x^4)}{(1-x-x^2)(1-x)^4}$
50	011,021,100	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, \quad b_m \rightsquigarrow c_m b_1\cdots b_m, \quad c_m \rightsquigarrow c_1\cdots c_m$, $0103 \rightsquigarrow 0103$; $a_m = 0^m$, $b_m = a_m 1$, $c_m = b_m 0$	
	011,021,120 011,102,210	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m, \quad b_m \rightsquigarrow b_{m+1}c_1\cdots c_m, \quad c_m \rightsquigarrow c_1\cdots c_m$; $a_m = 0^m$, $b_m = a_m 1$, $c_m = b_m 2$	$\frac{x(1-3x+3x^2)}{(1-x)(1-2x)^2}$
62	011,120,201	$a_m \rightsquigarrow a_{m+1}b_{m,1}\cdots b_{m,m}$, $b_{m,j} \rightsquigarrow b_{m+2-j,1}b_{m,j-1}\cdots b_{m+2-j,1}c_{m+1-j,2}\cdots c_{m+1-j,m+2-j}$, $c_{m,j} \rightsquigarrow c_{m,j-1}\cdots c_{m+3-j,2}c_{m+2-j,m+3-j}\cdots c_{m+2-j,2}$; $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m 1j$	
	011,120,210	$a_m \rightsquigarrow a_{m+1}b_{m,1}\cdots b_{m,m}$, $b_{m,j} \rightsquigarrow b_{m+1,j}c_{m,j}\cdots c_{m+2-j,2}c_{m+1-j,m+2-j}\cdots c_{m+1-j,2}$, $c_{m,j} \rightsquigarrow c_{m,j-1}\cdots c_{m+3-j,2}c_{m+2-j,m+3-j}\cdots c_{m+2-j,2}$; $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m 1j$	Theorem 25
65	011,100,201 011,100,210	$a_m \rightsquigarrow a_{m+1}b_{m,1}\cdots b_{m,m}$, $b_{m,j} \rightsquigarrow b_{m+1-j,1}b_{m+2-j,1}\cdots b_{m-1,j-1}b_{m,j}\cdots b_{m,m}$; $a_m = 0^m$, $b_{m,j} = 0^m j$	Example 21
82	010,101,120 010,110,120		Theorem 31
88	000,100,101 000,100,110		Theorem 30
93	000,101,201 000,110,210		Theorem 38

Continuation of Table 3			
Class	B	$\mathcal{T}[B]$	$F_B(x)$
94	010,100,201 010,100,210 010,101,201	$a_m \rightsquigarrow a_{m+1}a_mb_{m,2}\cdots, b_{m,m}$, $b_{m,j} \rightsquigarrow a_{m+2-j}b_{m+3-j,2}\cdots b_{m+1,j}b_{m,j}\cdots b_{m,m}$; $a_m = 0^m$, $b_{m,j} = a_mj$	
	010,101,210		Corollary 36
98	000,100,201 000,100,210		Example 21
105	021,100,120 021,101,120	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m$, $b_m \rightsquigarrow b_{m+1}c_1\cdots c_me$, $c_m \rightsquigarrow c_1\cdots c_{m+1}e$, $e \rightsquigarrow c_1e$; $a_m = 0^m$, $b_m = a_m1$, $c_m = a_m10$, $e = 012$	
	021,110,120	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m$, $b_m \rightsquigarrow eb_1\cdots b_{m+1}$, $e \rightsquigarrow b_1e$; $a_m = 0^m$, $b_m = a_m1$	$\frac{1-4x+2x^2+2x^3-(1-2x)\sqrt{1-4x}}{2x^2(1-x)}$, Theorem 11
110	021,100,101	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m$, $b_m \rightsquigarrow c_mb_1\cdots b_{m+1}$, $c_m \rightsquigarrow c_1\cdots c_{m+1}$; $a_m = 0^m$, $b_m = a_m1$, $c_m = a_m10$	[4]
	021,101,110	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m$, $b_m \rightsquigarrow a_{m+1}c_mb_1\cdots b_m$, $c_m \rightsquigarrow c_1\cdots c_{m+1}e$, $e \rightsquigarrow c_1e$; $a_m = 0^m$, $b_m = a_m1$, $c_m = a_m11$, $e = 0113$	
	101,102,110		OPEN
114	102,110,210 102,120,210		Theorem 39
126	101,120,201 101,120,210		Theorem 27
128	100,120,201 110,120,201		Theorem 29
129	100,110,201 101,110,201		Theorem 22
	100,120,210 110,120,210	$a_m \rightsquigarrow a_{m+1}b_{m,1}\cdots b_{m,m}$, $b_{m,j} \rightsquigarrow a_{m+2-j}b_{m+1,j}\cdots b_{m+2-j,1}b_{m+1-j,1}\cdots b_{m+1-j,m+1-j}$; $a_m = 0^m$, $b_{m,j} = 0^mj$	OPEN
135	100,201,210 101,201,210	$a_m \rightsquigarrow a_{m+1}b_{m,1}\cdots b_{m,m}$, $b_{m,j} \rightsquigarrow (a_{m+2-j})^jb_{m+1,j}\cdots b_{m+1,m+1}$; $a_m = 0^m$, $b_{m,j} = 0^mj$	Remark 24
	110,201,210	$a_m \rightsquigarrow a_{m+1}b_{m,1}\cdots b_{m,m}$, $b_{m,j} \rightsquigarrow (b_{m+2-j,1})^ja_{m+2-j}b_{m+1,j+1}\cdots b_{m+1,m+1}$; $a_m = 0^m$, $b_{m,j} = 0^mj$	
End of Table 3			

As we have said, to find the generating function $F_B(x)$ for a given triple $B \in L$, we translate the rules of $\mathcal{T}[B]$ (if we succeed to find them) into a system of recurrence relations. Then, we solve the system, mostly either by induction or by the kernel method (see [9] and references therein). Since our main focus is on the cases that cannot be handled by routine methods, we do not give the details of finding $F_B(x)$ when $F_B(x)$ is a polynomial or rational generating function.

The enumeration of $\mathbf{I}_n(000, 021)$ has been left as an open problem by Yan and Lin [21]. Our next result settles the problem.

Theorem 10. *Let* $B \in \{\{000, 021, 100\}, \{000, 021, 201\}, \{000, 021, 210\}, \{000, 021\}\}$. *We have*

$$F_B(x) = \frac{3x^3 + x^2 - 3x + 1}{2x^2\sqrt{(1 + x)(1 - 3x)}} + \frac{3x^4 - 4x^3 - 2x^2 + 4x - 1}{2x^2(1 + x)(1 - 3x)}.$$

Proof. Note that the sets $\{000, 021, 100\}$, $\{000, 021, 201\}$, $\{000, 021, 210\}$ and $\{000, 021\}$ are all equipotent, so the choice of B makes no difference for the purposes of the calculation. The generating tree $\mathcal{T}[B]$ is given by a root a_0 and rules

$$\begin{aligned} &a_m \rightsquigarrow b_md_{m+1}\cdots d_1, \quad d_m \rightsquigarrow c_ma_md_m\cdots d_1, \quad c_m \rightsquigarrow b_mc_m\cdots c_1e,\\ &b_m \rightsquigarrow c_{m+1}\cdots c_1e, \quad e \rightsquigarrow b_0e, \end{aligned}$$

where $a_m = d_mm$, $b_m = 0d_mm$, $c_m = 0d_m$, $d_m = 011\cdots(m-1)(m-1)m$, and $e = 002$.

Define $A_m(x)$ (respectively, $B_m(x)$, $C_m(x)$, $D_m(x)$, $E(x)$) to be the generating function for the number of nodes at level $n \geqslant 1$ for the subtree of $\mathcal{T}(L; a_m)$ (respectively, $\mathcal{T}(L; b_m)$, $\mathcal{T}(L; c_m)$, $\mathcal{T}(L; d_m)$, $\mathcal{T}(L; e)$), where its root has level 1. Thus,

$$\begin{aligned}
A_m(x) &= x + xB_m(x) + x(D_{m+1}(x) + \cdots + D_1(x)), \quad m \geqslant 0,\\
B_m(x) &= x + x(C_{m+1}(x) + \cdots + C_1(x)), \quad m \geqslant 0,\\
C_m(x) &= x + xB_m(x) + x(C_m(x) + \cdots + C_1(x)) + xE(x), \quad m \geqslant 1,\\
D_m(x) &= x + xC_m(x) + xA_m(x) + x(D_m(x) + \cdots + D_1(x)), \quad m \geqslant 1,\\
E(x) &= x + xB_0(x) + xE(x).
\end{aligned}$$

Define $G(x, v) = \sum_{m \geqslant s} G_m(x)v^m$ for $G \in \{A, C\}$ and $s = 1$, and for $G \in \{B, D\}$ and $s = 0$. Then this system of recurrence relations can be written as

$$\begin{aligned}
A(x, v) &= \frac{xv}{1-v} + xC(x, v) + x(D(x, v) - D(x, 0)) + \frac{x}{1-v}A(x, v),\\
B(x, v) &= \frac{x}{1-v} + \frac{x}{1-v}E(x) + \frac{x}{v(1-v)}C(x, v),\\
C(x, v) &= \frac{xv}{1-v} + x(B(x, v) - B(x, 0)) + \frac{x}{1-v}C(x, v) + \frac{xv}{1-v}E(x),\\
D(x, v) &= \frac{x}{1-v} + xB(x, v) + \frac{x}{v(1-v)}A(x, v),\\
E(x) &= x + xB(x, 0) + xE(x).
\end{aligned}$$

By using the kernel method for a linear system of equations [9], we obtain, in particular, an explicit formula for the generating function $A(x, 0)$, as required. □

Our next result addresses the enumeration of $\mathbf{I}_n(021, 102)$, solving another open case of Yan and Lin [21].

Theorem 11. *We have that*

$$\begin{aligned}
F_{\{021,102,201\}}(x) = F_{\{021,102,210\}}(x) = F_{\{021,102\}}(x) = f,\\
F_{\{021,100,120\}}(x) = F_{\{021,101,120\}}(x) = F_{\{021,110,120\}}(x) = g,
\end{aligned}$$

where

$$\begin{aligned}
f &= \frac{1 - 7x + 19x^2 - 27x^3 + 24x^4 - 12x^5 + 4x^6}{2x(1-x)^4(1-2x)} - \frac{\sqrt{1-4x}}{2x(1-x)},\\
g &= \frac{1 - 4x + 2x^2 + 2x^3 - (1-2x)\sqrt{1-4x}}{2x^2(1-x)}.
\end{aligned}$$

Proof. Due to the similarity of Classes 103(1-2) and 105(1-3), we present only the proof for the Class 105(1-2). Let $A_m(x)$ (respectively, $B_m(x)$, $C_m(x)$, and $D(x)$) be the generating function for the number of nodes at level $n \geqslant 1$ for the subtree of $\mathcal{T}(L; a_m)$ (respectively,

$\mathcal{T}(L;b_m)$, $\mathcal{T}(L;c_m)$ and $\mathcal{T}(L;e)$), see Class 105(1-2) in Table 3. By translating the rules of the generating trees, we obtain

$$\begin{aligned}A_m(x) &= x + xA_{m+1}(x) + x(B_1(x) + \cdots + B_m(x)),\\ B_m(x) &= x + xB_{m+1}(x) + x(C_1(x) + \cdots + C_m(x)) + xD(x),\\ C_m(x) &= x + x(C_1(x) + \cdots + C_{m+1}(x)) + xD(x),\\ D(x) &= x + xC_1(x) + xD(x).\end{aligned}$$

Define $G(x,v) = \sum_{m\geqslant 1} G_m(x)v^{m-1}$, for all $G \in \{A,B,C\}$. Then

$$\begin{aligned}A(x,v) &= \frac{x}{1-v} + \frac{x}{v}(A(x,v) - A(x,0)) + \frac{x}{1-v}B(x,v),\\ B(x,v) &= \frac{x}{1-v} + \frac{x}{v}(B(x,v) - B(x,0)) + \frac{x}{1-v}C(x,v) + \frac{x}{1-v}D(x),\\ C(x,v) &= \frac{x}{1-v} + \frac{x}{1-v}C(x,v) + \frac{x}{v}(C(x,v) - C(x,0)) + \frac{x}{1-v}D(x),\\ D(x) &= \frac{x(1+C_1(x))}{1-x}.\end{aligned}$$

By solving this system by [9], we complete the proof. □

Theorem 12. *For a set of patterns* $B \in \{\{021,100,201\},\{021,100,210\},\{021,110,201\},$ $\{021,110,210\},\{021,100\},\{021,110\}\}$*, we have*

$$F_B(x) = f,$$

where

$$f = \frac{(1-3x)^2}{2x^2\sqrt{1-4x}} - \frac{(1-x)(1-3x)}{2x^2}.$$

Proof. We know that $\{021,100,201\} \approx \{021,100,210\} \approx \{021,100\}$ and $\{021,110,201\} \approx \{021,110,210\} \approx \{021,110\}$. Moreover, Yan and Lin [21, Theorem 8.1] show that $\{021,100\}$ and $\{021,110\}$ are I-Wilf equivalent. It follows that the value of $F_B(x)$ does not depend on the choice of B. By translating the rules of the generating trees (see Class 117(1-4) in Table 1), we obtain

$$\begin{aligned}A_m(x) &= x + xA_{m+1}(x) + x(B_1(x) + \cdots + B_m(x)),\\ B_m(x) &= x + xC_m(x) + x(B_1(x) + \cdots + B_{m+1}(x)),\\ C_m(x) &= x + x(C_1(x) + \cdots + C_{m+1}(x)) + xD(x),\\ D(x) &= x + xC_1(x) + xD(x).\end{aligned}$$

Define $G(x,v) = \sum_{m\geqslant 1} G_m(x)v^{m-1}$, for all $G \in \{A,B,C\}$. Then

$$\begin{aligned}A(x,v) &= \frac{x}{1-v} + \frac{x}{v}(A(x,v) - A(x,0)) + \frac{x}{1-v}B(x,v),\\ B(x,v) &= \frac{x}{1-v} + xC(x,v) + \frac{x}{1-v}B(x,v) + \frac{x}{v}(B(x,v) - B(x,0)),\\ C(x,v) &= \frac{x}{1-v} + \frac{x}{1-v}C(x,v) + \frac{x}{v}(C(x,v) - C(x,0)) + \frac{x^2(1+C(x,0))}{(1-x)(1-v)}.\end{aligned}$$

By solving this system by [9], we complete the proof. □

4 Inversion sequences and diagram fillings

Some known results on pattern-avoiding fillings of Ferrers diagrams can directly be translated into results on pattern-avoiding inversion sequences.

Recall that for a word $w = w_1 \cdots w_n$ and an integer $k \in \mathbb{N}_0$, we let $w + k$ denote the word $w_1 + k, w_2 + k, \ldots, w_n + k$. Let x and y be two words with $\mathrm{ht}(x) = k$ and $\mathrm{ht}(y) = \ell$. Their *direct sum*, denoted $x \oplus y$, is the word obtained by concatenating x with $y + (k+1)$, while their *skew sum*, denoted $x \ominus y$, is the concatenation of $x + (\ell+1)$ and y. Notice that if x and y are reduced, then so is $x \oplus y$ and $x \ominus y$. If x and y are words and X and Y are sets of words, we use the shorthand $x \oplus Y$ for $\{x \oplus y;\ y \in Y\}$, $X \oplus y$ for $\{x \oplus y;\ x \in X\}$, $X \oplus Y$ for $\{x \oplus y;\ x \in X \wedge y \in Y\}$, and similarly for $\ominus$.

A *diagram* (or *polyomino*) is a finite collection of unit boxes in the Cartesian plane whose vertices have integer coordinates. *Box* (i, j) refers to the box in the i-th column and j-th row of the diagram. We assume that columns are numbered left to right and rows are numbered bottom to top. We adopt the convention that the leftmost nonempty column of D is column number 1, while the bottommost nonempty row is row 0.

A diagram D is *convex* if, whenever D contains two boxes b and b' in the same row or column, it also contains all the boxes lying between b and b'. A diagram is *bottom-justified* if the bottommost boxes in all the nonempty columns lie in the same row (by convention, this is row 0); right-justified, top-justified, and left-justified diagrams are defined analogously. A diagram is bottom-right justified if it is both bottom-justified and right-justified. A *Ferrers diagram* is a convex, bottom-right justified diagram[1].

For the purposes of this paper, a *filling* of a diagram D is a mapping that assigns to every box of D the value of 0 or 1, in such a way that every column of D contains exactly one box with value 1.

A word $w = w_1 w_2 \cdots w_n$ of height k and length n can be naturally represented as a filling of a rectangular diagram with $k + 1$ rows and n columns: the filling has value 1 in each box (i, w_i) for $i = 1, \ldots, n$, and value 0 elsewhere. We let $F(w)$ denote this filling.

Let P be a filling of a rectangular diagram with n columns and $k + 1$ rows. Let F be any filling of a diagram. We say that F *contains* P, if F has n distinct columns $c_1 < c_2 < \cdots < c_n$ and $k + 1$ distinct rows $r_0 < r_1 < \cdots < r_k$ with the following two properties:

- For every $i \in [n]$ and $j \in [k]_0$, the column c_i intersects the row r_j inside F, i.e., the diagram of F actually contains the box (c_i, r_j).
- For every $i \in [n]$ and $j \in [k]_0$, if the box (i, j) is a 1-entry of P, then the box (c_i, r_j) is a 1-entry of F.

Note that since we only consider fillings that have exactly one 1-entry in each column, the properties above actually imply that for every 0-entry (i, j) in P, the entry (c_i, r_j) in

[1] In the literature, it is usual to define Ferrers diagrams as either top-left or bottom-left justified shapes, but our convention will be more practical for our applications.

F is a 0-entry as well. In particular, the two properties informally state that the rows $r_0, \dots, r_k$ and columns $c_1, \dots, c_n$ induce in F a rectangular subdiagram equal to P.

Observe that a word x contains a reduced word y if and only if the filling $F(x)$ contains the filling $F(y)$.

Let T_n denote the Ferrers diagram with n columns whose i-th column contains exactly i boxes. Observe that to an inversion sequence $e = e_1e_2 \cdots e_n \in \mathbf{I}_n$ we may associate a filling of T_n whose 1-entries are precisely the boxes (i, e_i) for $i = 1, \dots, n$. This filling will be denoted $T_n(e)$. Note that the mapping $e \mapsto T_n(e)$ is a bijection between $\mathbf{I}_n$ and the set of fillings of T_n.

Let p be a reduced word and let $e \in \mathbf{I}_n$ be an inversion sequence. Note that if the filling $T_n(e)$ contains the rectangular filling $F(p)$, then e contains p; however, the converse does not necessarily hold: for example, the inversion sequence 011 contains the pattern 01, but the triangular filling $T_3(011)$ does not contain the 2×2 filling $F(01)$.

To get an equivalence between the containment of words and the containment of inversion sequences, we need to put a restriction on the pattern p. We say that a word $p = p_1 \cdots p_m$ is a *top-first pattern* if p is reduced and the first symbol of p is its maximum; that is, $p_1 = \mathrm{ht}(p)$.

Lemma 13. *Let p be a top-first pattern. Then an inversion sequence $e \in \mathbf{I}_n$ contains p if and only if the filling $T_n(e)$ contains $F(p)$.*

Proof. $\Longrightarrow$: Let k be the height of p and m its length. Suppose e contains p. Let $i_1 < i_2 < \cdots < i_m$ be the indices inducing a copy of p in e, that is, the subsequence $e_{i_1}e_{i_2} \cdots e_{i_m}$ is order-isomorphic to p. Let $j_0 < j_1 < \cdots < j_k$ be the $k+1$ values that appear in the subsequence $e_{i_1}e_{i_2} \cdots e_{i_m}$. Since p is a top-first pattern, we know that $e_{i_1} = j_k$. To show that $T_n(e)$ contains $F(p)$, we consider the columns $i_1 < \cdots < i_m$ and rows $j_0 < \cdots < j_k$. Note that $e_{i_1} = j_k$, so the box (i_1, j_k) is a 1-entry, and in particular the box lies inside T_n. It follows that each of the columns $i_1, \dots, i_m$ intersects any of the rows $j_0, \dots, j_k$ inside T_n, and by construction, these columns and rows induce in $T_n(e)$ a copy of $F(p)$. Hence $T_n(e)$ contains $F(p)$.

$\Longleftarrow$: If $T_n(e)$ contains a copy of $F(p)$ in columns $i_1 < i_2 < \cdots < i_m$, then the subsequence $e_{i_1} \cdots e_{i_m}$ of e is order-isomorphic to p, hence e contains p. ☐

Lemma 13 allows us to exploit known results on fillings of diagrams to obtain results on pattern-avoiding inversion sequences.

We say that two fillings P and Q are *shape-Wilf-equivalent*, denoted $P \overset{s}{\sim} Q$, if for every Ferrers diagram D, the number of P-avoiding fillings of D is the same as the number of Q-avoiding fillings of D. We extend the notion of shape-Wilf equivalence to sets of patterns in an obvious way. To avoid clutter in our notation, we will identify a word w with its corresponding filling $F(w)$, and we will say, e.g., that two words x and y are shape-Wilf-equivalent when $F(x)$ and $F(y)$ are shape-Wilf-equivalent.

Lemma 14. *If X and Y are shape-Wilf-equivalent sets of top-first patterns, then X and Y are also I-Wilf-equivalent.*

Proof. If X and Y are shape-Wilf-equivalent, then for every n, the number of X-avoiding fillings of T_n is the same as the number of its Y-avoiding fillings. By Lemma 13, this means that $|\mathbf{I}_n(X)| = |\mathbf{I}_n(Y)|$, and hence $X \overset{\mathbf{I}}{\sim} Y$. □

In a word $w = w_1 \cdots w_n$, an element w_i is a *weak LR maximum* if $w_j \leqslant w_i$ for each $j < i$, and it is a *strict LR maximum* if $w_j < w_i$ for each $j < i$. Similarly, in a filling F, we say that a 1-entry (c, r) is a weak LR maximum if all the 1-entries in columns $1, \ldots, c-1$ only appear in rows $0, \ldots, r$, and (c, r) is a strict LR maximum if all the 1-entries in columns $1, \ldots, c-1$ only appear in rows $0, \ldots, r-1$.

Shape-Wilf equivalence has a long history, and the paper by Guo et al. [8] gives a summary of known results to date. We now summarize here the known facts that are relevant to us.

Fact 15.

- *For any $k \geqslant 0$, we have $012 \cdots (k-1)k \overset{s}{\sim} k(k-1) \cdots 210$. This equivalence is witnessed by a bijection that preserves the number of 1-entries in each row. See Krattenthaler [12].*
- $\{021, 011\} \overset{s}{\sim} \{102, 101\}$. *See Guo et al. [8, Theorem 12].*
- $\{021, 010\} \overset{s}{\sim} \{102, 001\}$. *See Guo et al. [8, Theorem 13].*
- *If the reduced words x and y are shape-Wilf-equivalent, and z is any nonempty reduced word, then $z \ominus x$ and $z \ominus y$ are also shape-Wilf-equivalent. More generally, if X and Y are shape-Wilf-equivalent sets of reduced words and z is a nonempty reduced word, then $z \ominus X$ and $z \ominus Y$ are also shape-Wilf-equivalent. Additionally, the bijection witnessing $z \ominus X \overset{s}{\sim} z \ominus Y$ preserves the positions and values of weak LR maxima, and if the bijection witnessing $X \overset{s}{\sim} Y$ preserves row-sums, then the bijection witnessing $z \ominus X \overset{s}{\sim} z \ominus Y$ preserves them too. See Jelínek and Mansour [10, Lemma 14].*

Combining Fact 15 with Lemma 14 yields the following examples of I-Wilf-equivalent patterns or sets of patterns.

Corollary 16. *For any top-first pattern p, the following holds:*

- *for an integer $k \geqslant 1$, we have $p \ominus 012 \cdots (k-1)k \overset{\mathbf{I}}{\sim} p \ominus k(k-1) \cdots 210$, via a bijection that preserves the number of occurrences of each symbol and also preserves the positions and values of the weak LR maxima,*
- $p \ominus \{021, 011\} \overset{\mathbf{I}}{\sim} p \ominus \{102, 101\}$, *and*
- $p \ominus \{021, 010\} \overset{\mathbf{I}}{\sim} p \ominus \{102, 001\}$.

Remark 17. The previous corollary can in fact be restated in a slightly more general form, where instead of the single pattern p we consider a set P of top-first patterns. While this was not explicitly mentioned in any of the previous papers, it can be proven by the same arguments.

Example 18. Here are the I-Wilf equivalences between single patterns of small size that follow from Corollary 16:

- $201 \overset{\mathbf{I}}{\sim} 210$,
- $3012 \overset{\mathbf{I}}{\sim} 3210 \overset{\mathbf{I}}{\sim} 3201$, $2201 \overset{\mathbf{I}}{\sim} 2210$,
- $40123 \overset{\mathbf{I}}{\sim} 43210 \overset{\mathbf{I}}{\sim} 43012 \overset{\mathbf{I}}{\sim} 43201$, $42301 \overset{\mathbf{I}}{\sim} 42310$, $33012 \overset{\mathbf{I}}{\sim} 33210 \overset{\mathbf{I}}{\sim} 33201$, $32201 \overset{\mathbf{I}}{\sim} 32210$, $32301 \overset{\mathbf{I}}{\sim} 32310$, $22201 \overset{\mathbf{I}}{\sim} 22210$.

Note that for some patterns p, we may determine whether an inversion sequence $e \in \mathbf{I}_n$ contains p merely by looking at the total number of occurrences of each symbol in e and at the number of times each symbol occurs as a weak LR maximum of e. We say that such a pattern p is conservative. Formally, a pattern p (or a set of patterns P) is *conservative*, if for every n and every two sequences $e, e' \in \mathbf{I}_n$ such that

- e and e' have the same number of occurrences of each symbol, and
- in e and e', each symbol appears the same number of times as a weak LR maximum,

the sequence e avoids p (or P) if and only if e' avoids p (or P, respectively).

Observe that if a set of patterns P contains only conservative patterns, then P is itself conservative. However, a set of patterns P may be conservative even when its individual patterns are not, as we will see in the next observation. We use the short-hand notation a^m for the word $aa \cdots a$ of length m.

Observation 19. *For any $m \in \mathbb{N}$, the following patterns and sets are conservative:*

- *The pattern 0^m: indeed $\mathbf{I}_n(0^m)$ contains precisely those inversion sequences in which each symbol appears at most $m-1$ times.*
- *The pattern 01^m: $\mathbf{I}_n(01^m)$ contains precisely those inversion sequences in which each symbol other than 0 appears at most $m-1$ times.*
- *The pattern 10^m: $\mathbf{I}_n(10^m)$ contains precisely those inversion sequences in which each symbol has at most $m-1$ occurrences that are not a weak LR maximum.*
- *The pattern 021^m: $\mathbf{I}_n(021^m)$ contains precisely those inversion sequences in which each symbol other than 0 has at most $m-1$ occurrences that are not a weak LR maximum.*
- *The set $\{10^{m+1}, 010^m\}$: $\mathbf{I}_n(10^{m+1}, 010^m)$ contains precisely those inversion sequences in which each symbol has at most m occurrences that are not weak LR maxima, and moreover, each symbol that appears as a weak LR maximum has at most $m-1$ occurrences that are not weak LR maxima.*

Combining the first item of Corollary 16 with Observation 19, we reach the following conclusion.

Corollary 20. *For any conservative set C of patterns, any top-first pattern p, and any $k \geqslant 1$,*

$$C \cup \{p \ominus 012 \cdots (k-1)k\} \overset{\mathbf{I}}{\sim} C \cup \{p \ominus k(k-1) \cdots 210\}.$$

Example 21. By Corollary 20, we have

- $\{011, 100, 201\} \overset{\mathbf{I}}{\sim} \{011, 100, 210\}$,
- $\{011, 101, 201\} \overset{\mathbf{I}}{\sim} \{011, 110, 201\} \overset{\mathbf{I}}{\sim} \{011, 201\} \overset{\mathbf{I}}{\sim} \{011, 210\} \overset{\mathbf{I}}{\sim} \{011, 101, 210\} \overset{\mathbf{I}}{\sim} \{011, 110, 210\}$,
- $\{000, 100, 201\} \overset{\mathbf{I}}{\sim} \{000, 100, 210\}$,
- $\{010, 100, 201\} \overset{\mathbf{I}}{\sim} \{010, 100, 210\}$.

5 Bijections

In this section, we will present bijective proofs for several **I**-Wilf equivalence relations that do not follow from the general methods we described in the previous sections. We remark that although we confine all our results to the setting of inversion sequences, many of these bijections (specifically, those described in Theorems 22, 23, 27, 29, 30, 31, 33, 37 and 38) can in fact be applied to arbitrary words, yielding bijections between pattern-avoiding sets of words with arbitrarily prescribed strict LR maxima.

Theorem 22. *We have*

$$\{100, 110, 201\} \overset{\mathbf{I}}{\sim} \{101, 110, 201\}.$$

Proof. Given an inversion sequence $e = (e_1, \ldots, e_n) \in \mathbf{I}_n$, an index $k \in [n]$ and a pattern $p = p_1p_2p_3 \in P_3$, we say that *e has a copy of p ending at position k*, if there are indices i and j such that $i < j < k$ and $e_ie_je_k$ is order-isomorphic to p. For the purposes of this proof, we will say that $e \in \mathbf{I}_n$ is a *k-hybrid inversion sequence* if it satisfies the following properties:

- e avoids 110 and 201,
- for every $\ell \leqslant k$, e has no copy of 101 ending at position ℓ, and
- for every $\ell > k$, e has no copy of 100 ending at position ℓ.

We let $\mathbf{I}_n^k(110, 201)$ denote the set of k-hybrid inversion sequences. Note that $\mathbf{I}_n^0(110, 201)$ is precisely the set $\mathbf{I}_n(100, 110, 201)$, while $\mathbf{I}_n^n(110, 201)$ is the set $\mathbf{I}_n(101, 110, 201)$. To prove the theorem, we will establish the stronger statement that all the sets $\mathbf{I}_n^k(110, 201)$ for $k = 0, \ldots, n$ have the same size. To this end, we will describe, for a fixed $k \in [n]$, a bijection $\psi\colon \mathbf{I}_n^{k-1}(110, 201) \to \mathbf{I}_n^k(110, 201)$.

Fix $e = (e_1, \dots, e_n) \in \mathbf{I}_n^{k-1}(110, 201)$. If e has no copy of 101 ending at position k, then e is also in $\mathbf{I}_n^k(110, 201)$, and we set $\psi(e) = e$. Suppose now that e has a copy of 101 ending at position k, and fix $i < j < k$ such that $e_i e_j e_k$ is order-isomorphic to 101. In addition, choose i and j in such a way that the value e_j is as small as possible. We now define a sequence $e' = (e'_1, \dots, e'_n) = \psi(e)$ as follows: the entry e'_k is equal to e_j, and every other entry of e' is equal to the corresponding entry of e. Informally speaking, ψ replaces the value of e_k with a smaller value, so that a copy of 101 ending at position k in e turns into a copy of 100; if there are more possible values achieving this, the smallest one is chosen. We now check that e' belongs to $\mathbf{I}_n^k(110, 201)$:

- **e' avoids 110:** suppose $e'_a e'_b e'_c$ forms a copy of 110 in e', for some $a < b < c$. Clearly $k \in \{a, b, c\}$ otherwise e would contain 110 as well. If $k = a$ or $k = b$, then $e_i e_k e_c$ forms a 110 in e, which is impossible. This leaves $k = c$. If $b < j$, then $e_a e_b e_j$ forms a 110 in e, and $j = b$ is impossible, since $e'_j = e'_k$ while $e'_b > e'_k$. Thus, $j < b$. Now if $e_b > e_i$, then $e_a e_b e_k$ forms 110 in e, while if $e_b \leqslant e_i$, then $e_i e_j e_b$ forms either a 201 or a 101 ending at position $b < k$, which are both impossible.

- **e' avoids 201:** suppose $e'_a e'_b e'_c$ forms a copy of 201 in e', for some $a < b < c$. Again, k is one of a, b, c. If $k = a$, then $e_a e_b e_c$ is a 201 in e. Suppose $k = b$. Now if $e_c > e_k$, then $e_a e_b e_c$ is a 201 in e, if $e_c = e_k$, then we get a 100 ending at position $c > k$ in e, and if $e_c < e_k$, then $e_i e_j e_c$ is a 201 in e, all of which is impossible.

- **e' avoids 101 ending at positions $0, 1, \dots, k$:** there can be no 101 ending at $\ell < k$ in e', because e would contain it as well. Suppose there is a copy $e'_a e'_b e'_k$ of 101 ending at k in e'. Now if $b < j$, then $e_a e_b e_j$ is a copy of 101 ending at $j < k$ in e, which is impossible, and if $b > j$, then b should have been chosen instead of j in the choice of i and j above, since $e_i e_b e_k$ is a copy of 101 with $e_b < e_j$, contradicting the minimality of e_j.

- **e' avoids 100 ending at positions $k+1, \dots, n$:** suppose $e'_a e'_b e'_c$ is a copy of 100 with $c > k$. If $k = b$, then $e_i e_j e_c$ forms a 100 in e, while if $k \neq b$, then $e_a e_b e_c$ is a copy of 100 in e.

Having verified that $\psi(e)$ is in $\mathbf{I}_n^k(110, 201)$, we now show that the mapping ψ can be inverted by defining a function $\psi^*\colon \mathbf{I}_n^k(110, 201) \to \mathbf{I}_n^{k-1}(110, 201)$ and showing that it is the inverse of ψ. Choose $e' \in \mathbf{I}_n^k(110, 201)$. We will find a sequence $\psi^*(e') = e \in \mathbf{I}_n^{k-1}(110, 201)$ as follows. If e' has no 100 ending in k, then it belongs to $\mathbf{I}_n^{k-1}(110, 201)$ and we set $\psi^*(e') = e'$.

Suppose there is a copy $e'_i e'_j e'_k$ of 100 in e', and choose i and j so that e'_i is as large as possible. Define $\psi^*(e') = e = (e_1, \dots, e_n)$ to be the sequence with $e_k = e'_i$, and all the other entries of e are the same as the corresponding entries of e'. Informally, we increase the value of e'_k to turn a copy of 100 ending in k into a copy of 101, and choose the largest possible value to achieve this. In particular, $e_i e_j e_k$ is now a copy of 101 ending at position k in e.

Let us check that e is in $\mathbf{I}_n^{k-1}(110, 201)$:

- ***e* avoids 110:** suppose $e_a e_b e_c$ is a copy of 110 in e. If $c = k$, then $e'_a e'_b e'_c$ is a copy of 110 in e', and if $k = a$, then $e'_i e'_b e'_c$ is a copy of 110 in e'. Suppose $k = b$. If $e_c > e'_j$, then $e'_i e'_j e'_c$ is a copy of 201, if $e_c = e'_j$, then $e'_i e'_j e'_c$ is a copy of 100 ending at $c > k$, and if $e_c < e'_j$, then $e'_j e'_k e'_c$ is a copy of 110.

- ***e* avoids 201:** suppose $e_a e_b e_c$ is a copy of 201 in e. If $k = a$, then $e'_i e'_b e'_c$ is a copy of 201 in e', and if $k = b$, then $e'_a e'_b e'_c$ is a copy of 201 in e'. Suppose $k = c$. If $b > j$, then $e'_i e'_j e'_b$ forms a 201 in e', hence $b \leqslant j$. Then $e'_a e'_j e'_k$ forms a 100, and since $e'_a > e'_i$, we should have chosen a instead of i before.

- ***e* avoids 101 ending at positions $\mathbf{0, 1, \ldots, \boldsymbol{k} - 1}$:** this is clear since the first $k - 1$ positions of e have the same values as the corresponding positions of e'.

- ***e* avoids 100 ending at positions $\boldsymbol{k}, \ldots, \boldsymbol{n}$:** suppose $e_a e_b e_c$ is such a copy of 100 with $c \geqslant k$. If $k = a$, then $e'_i e'_b e'_c$ is a copy of 100 in e' ending in $c > k$. If $k = b$, then $e'_a e_b e'_c$ is a copy of 201 in e'. Finally, if $k = c$, then either $b > j$, and $e'_i e'_j e'_b$ forms a 101 in e' ending at $b < k$, or $b < j$ and $e'_a e'_j e'_k$ forms a copy of 100 with $e'_a > e'_i$, contradicting again the choice of i.

Hence $\psi^*(e')$ is in $\mathbf{I}_n^{k-1}(110, 201)$.

We now check that for any $e \in \mathbf{I}_n^{k-1}(110, 201)$, $\psi^*(\psi(e)) = e$. This is clear when e has no copy of 101 ending at k as then e belongs to $\mathbf{I}_n^{k-1}(110, 201) \cap \mathbf{I}_n^k(110, 201)$ and both ψ and ψ^* maps e to e. If e has a copy of 101 ending at k, then ψ chooses such a copy $e_i e_j e_k$ with e_j smallest possible, then changes it into a copy of 100 by decreasing the k-th element appropriately, resulting in a sequence $e' = \psi(e)$. To show that ψ^* reverses this operation, we need to argue that e' has no subsequence $e'_a e'_b e'_k$ forming a copy of 100, with $e'_a > e'_i$. This holds, because if such a subsequence existed, then $e_a e_b e_k$ would have been a copy of 201 in e, which is impossible. Hence $\psi^*(\psi(e)) = e$.

Finally, we check that $\psi(\psi^*(e')) = e'$ for any $e' \in \mathbf{I}_n^k(110, 201)$. Again, the case when e' has no copy of 100 ending at k is trivial. Suppose $e'_i e'_j e'_k$ is a copy of 100, with e'_i as large as possible, so ψ^* changes e' into a sequence e whose k-th element is equal to e'_i. We need to show that there are no a and b such that $a < b < k$, $e_a e_b e_k$ is a copy of 101, and $e_b < e_j$. If such a and b existed, then either $b < j$, and $e_a e_b e_j$ forms a copy of 201 in e, or $b > j$, and $e'_j e'_b e'_k$ would form a copy of 101 ending at k in e', contradicting $e' \in \mathbf{I}_n^k(110, 201)$. ☐

Theorem 23. *We have*

$$\{101, 201, 210\} \overset{\mathbf{I}}{\sim} \{110, 201, 210\}.$$

Proof. Let $e \in \mathbf{I}_n$ be an inversion sequence that avoids 201 and 210, and let $\in [n]_0$ be an integer. We will say that e *contains 110 at height k* if e contains the subsequence $kk\ell$ for some $\ell < k$, or in other words, e contains a copy of 110 in which the two symbols '1' correspond to the value k. Similarly, we say that e *contains 101 at height k* if it contains the subsequence $k\ell k$ for some $\ell < k$.

For $m \in [n]_0$, we will say that a sequence $e \in \mathbf{I}_n$ is an *m-hybrid sequence* if it satisfies the following properties:

- e avoids 201 and 210,
- for every $k < m$, e avoids 110 at height k, and
- for every $k \geqslant m$, e avoids 101 at height k.

Note that 0-hybrid sequences are precisely the sequences from $\mathbf{I}_n(101, 201, 210)$, while n-hybrid sequences are precisely the elements of $\mathbf{I}_n(110, 201, 210)$ (recall that in a sequence $e \in \mathbf{I}_n$, all the elements have value at most $n-1$). We will show, for every $m \in [n-1]_0$, that there is a bijection ψ between m-hybrids and $(m+1)$-hybrids.

Fix an m-hybrid sequence $e = (e_1, \ldots, e_n)$. By definition, it must avoid 101 at height m. If the sequence also avoids 110 at height m, then it is an $(m+1)$-hybrid, and we set $\psi(e) = e$.

Suppose then that e contains 110 at height m. Let e_i be the leftmost occurrence of m in e. We say that an element e_j is *m-low* if $j > i$ and $e_j < m$. Note that e must contain at least one m-low element (since e contains 110 at height m), and that all the m-low elements have the same value (otherwise e would contain 201 or 210). Let ℓ be the value of the m-low elements.

We also say that an element e_j is an *m-repeat* if $j > i$ and $e_j = m$. The sequence must contain at least one m-repeat, since it contains 110 at height m, and all the m-repeats must appear to the left of any m-low element, since e avoids 101 at height m. Note also that any element larger than m in e must appear to the right of any m-repeat, otherwise e would contain 210.

We now construct a sequence $e' = \psi(e)$ as follows: for any j, if e_j is an m-low element, we define $e'_j = m$, if e_j is an m-repeat, we define $e'_j = \ell$, and in all other cases we define $e'_j = e_j$. Informally, ψ changes m-low elements into m-repeats and vice versa. Thus, in e', all the m-repeats are to the right of all the m-low elements, and any copy of 110 at height m in e is transformed into a copy of 101 at height m in e'.

We claim that e' is an $(m+1)$-hybrid. It is clear that e' is an inversion sequence (since it has the same positions and values of strict LR maxima as e) and that it avoids 110 at height m (since all its m-repeats are to the right of all the m-low elements). It is also straightforward to check that e' avoids both 210 and 201. Furthermore, for any $k > m$ any copy of 101 at height k in e' implies that the same three positions form a copy of 101 at height k in e, which is impossible. Finally, for $k < m$ if e' contained a copy $e'_a e'_b e'_c$ of 110 at height k, then necessarily $b < i$ and either $c < i$ as well or e'_c is an m-low element. In any case, e would contain 110 at height k as well, which is impossible.

Hence, e' is an $(m+1)$-hybrid. Conversely, any $(m+1)$-hybrid sequence is either an m-hybrid already (if it avoids 110 at height m), or is transformed into an m-hybrid sequence by exchanging the m-low and m-repeat elements, inverting the operation ψ defined above. Therefore, the number of m-hybrid sequences is independent of the choice of $m \in [n]_0$, implying the theorem. □

Remark 24. Note that by combining Theorem 23 with the identity $\mathcal{T}[\{100, 201, 210\}] = \mathcal{T}[\{101, 201, 210\}]$ of generating trees (see Class 135 in Table 3), we obtain

$$\{100, 201, 210\} \overset{\mathbf{I}}{\sim} \{101, 201, 210\} \overset{\mathbf{I}}{\sim} \{110, 201, 210\}.$$

Theorem 25. *We have the equivalence*

$$\{011, 120, 201\} \overset{\mathbf{I}}{\sim} \{011, 120, 210\},$$

and the equivalence is witnessed by a bijection from $\mathbf{I}_n(011, 120, 201)$ *to* $\mathbf{I}_n(011, 120, 210)$ *that preserves the positions and values of strict LR maxima, the positions and values of weak LR maxima, and the number of occurrences of each symbol.*

Proof. We will describe a bijection between the sets

$$A := \mathbf{I}_n(011, 120, 201) \text{ and } B := \mathbf{I}_n(011, 120, 210),$$

but first we will analyze the structure of the inversion sequences in the two sets A and B.

An inversion sequence $e = (e_1, \dots, e_n) \in \mathbf{I}_n$ that has k strict LR maxima can be uniquely decomposed into a concatenation $e = B_1 B_2 \cdots B_k$, where B_i is the subword of e that begins with the i-th strict LR maximum and ends with the element immediately preceding the $(i+1)$-th strict LR maximum. For example, with $e = (0, 0, 0, 2, 0, 1, 3, 3, 5, 3, 4)$, we have (after omitting redundant punctuation) $B_1 = 000$, $B_2 = 201$, $B_3 = 33$, and $B_4 = 534$. We will call B_i the *i-th LR block of e*.

Note that e avoids 011 if and only if each value greater than 0 appears at most once in e. Note further that if e avoids 120, then for any two LR blocks B_i and B_j with $i < j$, the smallest value in B_j is at least as large as the largest value of B_i; in other words $\max B_i \leqslant \min B_j$. Furthermore, the previous inequality is strict, except perhaps when $j = i + 1$.

It follows that for any $e \in \mathbf{I}_n(011, 120)$, any copy of the pattern 210 must appear within a single LR block of e, and also any copy of the pattern 201 must appear within a single LR block of e.

Suppose that an inversion sequence e avoids 011, and let $B_i = b_1 b_2 \cdots b_m$ be its i-th LR block. Note that B_i avoids 210 if and only if $b_2 b_3 \cdots b_m$ is a weakly increasing sequence, and B_i avoids 201 if and only if $b_2 b_3 \cdots b_m$ is weakly decreasing – here we use the fact that due to 011-avoidance, either all the elements of B_i are zeros, or b_1 is the unique maximum of B_i.

For a sequence $B_i = b_1 b_2 \cdots b_m$, let B_i^* denote the sequence $b_1 b_m b_{m-1} \cdots b_2$, i.e., the sequence obtained from B_i by reversing the order of all the elements after the first one. We now describe an involution ψ on $\mathbf{I}_n$ which, when restricted to the set A, yields the required bijection between A and B. Fix $e \in \mathbf{I}_n$, and decompose it into LR blocks as $e = B_1 B_2 \cdots B_k$. Define $\psi(e)$ as the concatenation $B_1^* B_2^* \cdots B_k^*$. Observe that $\psi(e)$ is again an inversion sequence, B_i^* is its i-th LR block, and $\psi(\psi(e)) = e$. Moreover, e belongs to A if and only if $\psi(e)$ belongs to B. Thus, the restriction of ψ to the set A provides the required bijection.

By construction, ψ preserves the positions and values of strict LR maxima and the number of occurrences of each element. Moreover, when restricted to 011-avoiding sequences, ψ also preserves the positions and values of weak LR maxima, since in a 011-avoiding sequence, the only weak LR maxima that are not strict LR maxima appear in the first LR block, which is unchanged by ψ. □

Since the bijection used to prove Theorem 25 preserves the number of occurrences of each element as well as the number of occurrences of each element as weak LR maximum, we know that the bijection preserves the avoidance of any conservative set of patterns. Thus, by the same argument as in Corollary 20, we get the following consequence.

Corollary 26. *For any conservative set of patterns C, we have $C \cup \{011, 120, 201\} \stackrel{\mathbf{I}}{\sim} C \cup \{011, 120, 210\}$. For example, taking $C = \{000\}$, we get $\{000, 011, 120, 201\} \stackrel{\mathbf{I}}{\sim} \{000, 011, 120, 210\}$.*

Theorem 27. *We have*

$$\{101, 120, 201\} \stackrel{\mathbf{I}}{\sim} \{101, 120, 210\}.$$

Proof. Our argument is very similar to the proof of Theorem 25. We consider again the decomposition of an inversion sequence $e \in \mathbf{I}_n$ into LR blocks $B_1, \dots, B_k$. Again, if e avoids 120, then for any two LR blocks B_i and B_j with $i < j$, we have $\max B_i \leqslant \min B_j$, with equality only possible when $j = i + 1$. It follows that any copy of 210 or 201 in e must be confined to a single LR block.

If in addition to 120 the sequence e also avoids 101, then the equality $\max B_i = \min B_{i+1}$ can only occur when all the elements of B_i are equal to $\max B_i$. Moreover, in an inversion sequence that avoids 101, in every LR block B_i, all the elements equal to $\max B_i$ appear consecutively at the beginning of B_i. We will say that the elements of B_i that are equal to $\max B_i$ form the *head* of B_i, and the remaining elements form the *tail* of B_i; note that the tail may be empty. Note also that B_i avoids 210 if and only if its tail is a weakly increasing sequence, and it avoids 201 if and only if its tail is weakly decreasing.

For an LR block B_i, let B_i^* denote the sequence obtained by keeping the head of B_i the same, and reversing the order of elements in the tail of B_i. For a sequence $e \in \mathbf{I}_n$ with LR block decomposition $B_1 B_2 \cdots B_k$, define $\psi(e)$ as $\psi(e) = B_1^* B_2^* \cdots B_k^*$. We observe that ψ is an involution on $\mathbf{I}_n$, which restricts to a bijection between $\mathbf{I}_n(101, 120, 210)$ and $\mathbf{I}_n(101, 120, 201)$. □

The argument we used to deduce Corollary 26 from Theorem 25 can be used here as well, since the bijection we used to prove Theorem 27 has all the required statistic-preserving properties.

Corollary 28. *For any conservative set of patterns C, we have $C \cup \{101, 120, 201\} \stackrel{\mathbf{I}}{\sim} C \cup \{101, 120, 210\}$. For example, taking $C = \{000\}$, we get $\{000, 101, 120, 201\} \stackrel{\mathbf{I}}{\sim} \{000, 101, 120, 210\}$.*

Theorem 29. *We have*

$$\{100, 120, 201\} \stackrel{\mathbf{I}}{\sim} \{110, 120, 201\}.$$

Proof. We again apply the decomposition if $e \in \mathbf{I}_n$ into LR blocks $B_1, \dots, B_k$, as in Theorems 25 and 27. If e avoids 120, this implies that for any $i \in [k]$, the maximum of B_i

cannot be larger than the minimum of B_{i+1}. This implies that in a 120-avoiding sequence, any copy of any of the patterns 100, 110 or 201 must be confined to a single LR block B_i. We shall therefore investigate the structure of individual blocks imposed by avoidance of these patterns.

The avoidance of 201 implies that in each LR block $B = b_1 b_2 \cdots b_m$, the elements smaller than the maximum b_1 must form a weakly decreasing sequence. If we further impose 110-avoidance, this means that every value other than the maximum or the minimum must appear at most once, and moreover, if there are any further occurrences of the maximum value b_1, these must appear after all the other values. In particular, for $e = \mathbf{I}_n(110, 120, 201)$, each LR block B of e has the structure $B = b_1 b_2 \cdots b_{q-1} b_q^a b_1^b$, where $b_1 > b_2 > \cdots > b_q$, $a \geqslant 1$ and $b \geqslant 0$. Conversely, we routinely verify that if $e \in \mathbf{I}_n$ is an inversion sequence whose every block has this structure, and additionally the maximum of B_i is not larger than the minimum of B_{i+1}, then e belongs to $\mathbf{I}_n(110, 120, 201)$.

Assume now that e is from $\mathbf{I}_n(100, 120, 201)$, and let $B = b_1 \cdots b_m$ be an LR block of e. Avoidance of 100 means that each value in B smaller than the maximum b_1 can only appear once. Avoidance of 120 further means that any occurrence of the maximum value b_1 can only appear either before all the smaller values or after them. Thus, the block has the form $B = b_1^a b_2 \cdots b_{q-1} b_q b_1^b$, where $b_1 > b_2 > \cdots > b_q$, $a \geqslant 1$ and $b \geqslant 0$. Conversely, if $e \in \mathbf{I}_n$ is an inversion sequence whose every block has this structure, and additionally the maximum of B_i is not larger than the minimum of B_{i+1}, then e belongs to $\mathbf{I}_n(100, 120, 201)$.

It is now clear how to transform bijectively a sequence $e \in \mathbf{I}_n(110, 120, 201)$ into a sequence $e^* \in \mathbf{I}_n(100, 120, 201)$: we partition e into LR blocks and then transform each LR block of e, which as we know has the form $b_1 b_2 \cdots b_{q-1} b_q^a b_1^b$, into the sequence $b_1^a b_2 \cdots b_{q-1} b_q b_1^b$. This changes e into a sequence e^*, which has the same strict LR maxima as e, and belongs to $\mathbf{I}_n(100, 120, 201)$. □

Theorem 30. *We have*

$$\{000, 100, 101\} \overset{\mathbf{I}}{\sim} \{000, 100, 110\}.$$

Proof. Let $e \in \mathbf{I}_n$ be an inversion sequence that avoids 000 and 100, and let $k \in [n]_0$ be an integer. We will say that e contains the pattern 101 *at height* k if it contains a copy of the pattern 101 in which the symbol 1 of the pattern is represented by the symbol k in e; in other words, e contains 101 at height k if it contains a subsequence of the form $k\ell k$ for some $\ell < k$. Similarly, we say that e contains 110 at height k, if it contains a subsequence $kk\ell$ for some $\ell < k$.

We will say that an inversion sequence $e \in \mathbf{I}_n$ is an *m-hybrid* sequence, if it satisfies the following properties:

- e avoids 000 and 100,
- for every $k < m$, e avoids 101 at height k, and
- for every $k \geqslant m$, e avoids 110 at height k.

Note that the 0-hybrid sequences are precisely the sequences avoiding $\{000, 100, 110\}$, while the n-hybrid ones are precisely the avoiders of $\{000, 100, 101\}$. Thus, the theorem is equivalent to the statement that 0-hybrid sequences are equinumerous with the n-hybrid ones. To prove this, we will in fact show that the number of m-hybrid sequences does not depend on m. To this end, we now describe, for any $m \in [n-1]_0$, a bijection ψ_m between the m-hybrid and the $(m+1)$-hybrid sequences.

Fix an m-hybrid sequence $e = e_1 \cdots e_n$. If e has at most one occurrence of the symbol m, then it contains neither 101 nor 110 at height m, and therefore it is also an $(m+1)$-hybrid sequence. In such case, we define $\psi_m(e) = e$.

Suppose now that e has at least two occurrences of m. Since e avoids 000, it follows that e in fact has exactly two occurrences of m. Let these occurrences be e_a and e_b, with $a < b$. Note that e_a is a strict LR maximum otherwise we would have a copy on 100 in e. Note also that all the elements of e after e_b are larger than m, otherwise e would contain a copy of 000, or a copy of 110 at height m, which is impossible in an m-hybrid sequence.

Let us say that an element e_i of e is *crucial* if $a < i \leqslant b$ and $e_i \leqslant m$. In particular, e_b is a crucial element, and the remaining crucial elements (if any) all form a copy of the pattern 101 at height m with e_a and e_b. Let $i_1 < i_2 < \cdots < i_c = b$ be the indices of all the crucial elements, in left-to-right order. We now define a new sequence $\psi_m(e) = e^* = e^*_1 \dots e^*_n$ as follows:

- If e_i is not a crucial element, then $e^*_i = e_i$.
- If e_i is the leftmost crucial element (i.e., $i = i_1$), we set $e^*_i = e_b = m$.
- If e_i is a crucial element, but not the leftmost one (i.e., $i = i_q$ for some $q > 1$), we let e^*_i be equal to the immediately preceding crucial element of e, i.e., $e^*_i = e_{i_{q-1}}$.

Intuitively speaking, we obtain e^* from e by performing a cyclic shift of the crucial elements, with the rightmost crucial element being moved to the position of the leftmost one, and any other crucial element being moved one step to the right in the subsequence of crucial elements.

We claim that e^* is an $(m+1)$-hybrid sequence. First note that the strict LR maxima of e coincide with those of e^*, which implies that e^* is indeed an inversion sequence. Note also that every symbol has the same number of occurrences in e^* as in e, and in particular e^* avoids 000. Also, in e^* as in e, each symbol has at most one occurrence that is not a strict LR maximum, implying e^* avoids 100. It remains to analyze the copies of 101 and 110 at various heights. Note that the symbols smaller than m form the same subsequence in e as in e^* (although not necessarily at the same positions), and in particular, e^* avoids 101 at height k for each $k < m$, because e avoided it. Moreover, e^* also avoids 101 at height m, since in e^*, the two occurrences of m (namely e^*_a and $e^*_{i_1}$) have no element smaller than m between them. It remains to check that e^* avoids 110 at every height $k > m$: to see this, note that the elements larger than m are identical in e^* as in e, and for any $i \in [n]$, we have $e^*_i \leqslant m \iff e_i \leqslant m$. Thus, any copy of 110 at height $k > m$ in e^* would imply that the same positions in e also have a copy of 110 at the same height, which is impossible. We conclude that e^* is an $(m+1)$-hybrid sequence.

To show that the mapping ψ_m is a bijection, we describe its inverse ψ_m^{-1}. Suppose e is an $(m+1)$-hybrid sequence. If it has at most one occurrence of m, we put $\psi_m^{-1}(e) = e$, otherwise e has two occurrences of m, say e_a and e_b. Note that there are no elements smaller than m between e_a and e_b, since e avoids 101 at height m. We say that an element e_i is *crucial*, if $i \geqslant b$ and $e_i \leqslant m$. To define $e^* = \psi_m^{-1}(e)$, we rearrange the crucial elements by moving the leftmost crucial element (namely e_b) to the position of the rightmost one and moving every other crucial element to the position of the immediately preceding one. We easily check, with a similar argument as in the preceding paragraph, that e^* is an m-hybrid sequence, and that the mapping we now described is the inverse to the mapping ψ_m defined above. □

Theorem 31. *We have*

$$\{010, 120, 101\} \overset{\mathbf{I}}{\sim} \{010, 120, 110\}.$$

Proof. We use a similar argument, and analogous terminology, as in the proof of Theorem 30.

Let $e \in \mathbf{I}_n$ be an inversion sequence that avoids 010 and 120, and let $k \in [n]_0$ be an integer. We will again say that e contains the pattern 101 *at height* k if it contains a subsequence of the form $k\ell k$ for some $\ell < k$, and we say that e contains 110 at height k, if it contains a subsequence $kk\ell$ for some $\ell < k$.

We will say that an inversion sequence $e \in \mathbf{I}_n$ is an *m-hybrid* sequence, if it satisfies the following properties:

- e avoids 010 and 120,
- for every $k < m$, e avoids 101 at height k, and
- for every $k \geqslant m$, e avoids 110 at height k.

We again want to show that 0-hybrid sequences are equinumerous with the n-hybrid ones, and we again do this by proving that the number of m-hybrid sequences does not depend on m. Hence we again describe a bijection ψ_m between the m-hybrid and the $(m+1)$-hybrid sequences.

Fix an m-hybrid sequence $e = e_1 \cdots e_n$. Let q be the number of occurrences of the symbol m in e. If $q \leqslant 1$, then e contains neither 101 nor 110 at height m, and therefore it is also an $(m+1)$-hybrid sequence. In such case, we define $\psi_m(e) = e$.

Suppose now that $q > 1$. Let e_a be the leftmost occurrence of m in e. Let us say that an element e_i of e is *crucial* if $a < i$ and $e_i \leqslant m$. In particular, all the occurrences of m other than e_a are crucial. Note that the crucial elements form a consecutive block of e starting immediately to the right of e_a; in other words, any non-crucial element of e is either one of $e_1, e_2, \ldots, e_a$, or it appears to the right of the rightmost crucial element. If not, then e would contain 010 or 120 (in the latter case, using e_a in the place of '1').

Note also, that since e avoids 110 at height m, all the $q-1$ crucial elements equal to m appear to the right of any crucial element smaller than m; in particular, both the crucial elements equal to m and those smaller than m form consecutive blocks. We now create a sequence $e^* = \psi_m(e)$, by exchanging the order of these two blocks, that is, by shifting the crucial elements equal to m to the beginning of the sequence of crucial elements, and by shifting every other crucial element by $q-1$ steps to the right. To describe ψ_m more formally, let p be the number of crucial elements of e smaller than m; as we know, these elements are $e_{a+1}, e_{a+2}, \dots, e_{a+p}$, and they are followed by $q-1$ elements $e_{a+p+1}, \dots, e_{a+p+q-1}$ all equal to m. Then e^* is defined as follows:

- If e_i is not crucial, then $e^*_i = e_i$.
- For $i \in \{a+1, a+2, \dots, a+q-1\}$, we have $e^*_i = m$.
- For $i \in \{a+q, a+q+1, \dots, a+p+q-1\}$, we have $e^*_i = e_{i-q+1}$.

We easily observe that e^* is an inversion sequence, that it avoids 010, that it avoids 101 at all heights $k \leqslant m$, and that it avoids 110 at all heights $k > m$, using the same ideas as in the proof of Theorem 30. To see that e^* is an $(m+1)$-hybrid, we need to check that it avoids 120. Suppose for contradiction that a triple $e^*_i e^*_j e^*_k$ with $i < j < k$ forms a copy of 120 in e^*. Necessarily at least one of the three elements must belong to the block of crucial elements, i.e., at least one of the three indices i, j, k must belong to the set $\{a+1, a+2, \dots, a+p+q-1\}$. Since e^*_k is the smallest and rightmost of the three, it must belong to this crucial block, i.e., $a < k < a+p+q$. If e^*_k is the only such element, i.e., if $j \leqslant a$, then e also contains 120 formed by the same three values $e^*_i e^*_j e^*_k$ (although the last value may be at a different position than in e^*). If, on the other hand, we have $i \leqslant a$ and $a < j < k < a+p+q$, then necessarily $i < a$ (otherwise $e^*_i = e^*_a = m$, which cannot be smaller than e^*_j), and we may replace e^*_j with $e^*_a = e_a = m$ to transform the situation to the previous case. Finally, if all three elements are in the crucial block, i.e., $a < i < j < k < a+p+q$, then the three values $e^*_i e^*_j e^*_k$ are all smaller than m (otherwise $e^*_j = m$, but no crucial element equal to m has a smaller crucial element to its left in e^*). But that means that the three elements $e_{i-q+1}, e_{j-q+1}, e_{k-q+1}$ form a copy of 120 in e, a contradiction. This shows that e^* is indeed an $(m+1)$-hybrid.

We easily observe that ψ_m is a bijection between the m-hybrids and $(m+1)$-hybrids. □

We may observe that the bijection in the proof of Theorem 31 preserves the number of occurrences of each element, as well as the number of its occurrences as weak LR maximum. This leads to the usual conclusion.

Corollary 32. *For any conservative set of patterns C, we have $C \cup \{010, 120, 101\} \overset{\mathbf{I}}{\sim} C \cup \{010, 120, 110\}$.*

Theorem 33. *We have*

$$\{010, 210, 100\} \overset{\mathbf{I}}{\sim} \{010, 210, 101\}.$$

Proof. For a sequence $e = e_1e_2 \cdots e_n$, we say that an element e_i is a *repeat* if the value e_i already appears among the elements $e_1, \ldots, e_{i-1}$. Consider now a sequence $e = e_1 \cdots e_n \in \mathbf{I}_n(010, 210)$. Observe that such a sequence avoids 100 if and only if every repeat e_i is equal to the largest element among $e_1, \ldots, e_{i-1}$, while the sequence avoids 101 if and only if every repeat e_i is equal to e_{i-1}.

We construct a bijection $\psi\colon \mathbf{I}_n(010, 210, 100) \to \mathbf{I}_n(010, 210, 101)$ as follows: from a sequence $e \in \mathbf{I}_n(010, 210, 100)$, we obtain a sequence $e^* = \psi(e)$ by replacing, in left-to-right order, the value of every repeat e_i in e with the value $e^*_i := e_{i-1}$. Note that e^* has the same strict LR maxima as e and in particular, it is again an inversion sequence. Note also that e^* has repeats at the same positions as e. In particular, ψ is injective, and its inverse $\psi^{-1}(e^*)$ is obtained by replacing every repeat e^*_i of e^* by the value $\max\{e^*_1, \ldots, e^*_{i-1}\}$.

We may routinely check that neither $\psi(e)$ nor $\psi^{-1}(e)$ can contain any copy of 010 or 210, as long as e avoids both these patterns. It follows that ψ is a bijection witnessing that $\{010, 210, 100\} \overset{\mathbf{I}}{\sim} \{010, 210, 101\}$. □

Theorem 34. *We have*

$$\{010, 101, 210\} \overset{\mathbf{I}}{\sim} \{010, 101, 201\}.$$

Proof. As in the proof of Theorem 33, we call an element e_i of a sequence $e = e_1 \cdots e_n$ a *repeat* if it is equal to some of the previous elements. Notice that a sequence of $e = e_1 \cdots e_n$ avoids the two patterns 010 and 101 if an only if every repeat e_i is equal to the immediately preceding element e_{i-1}. In other words, in a sequence avoiding 010 and 101, all the occurrences of a given value v appear in a single consecutive block.

We have seen in Fact 15, that the patterns $210 = 0\ominus10$ and $201 = 0\ominus01$ are shape-Wilf equivalent. Since the two patterns are top-first, this implies by Lemma 13 that they are also $\overset{\mathbf{I}}{\sim}$-equivalent. However, the bijection witnessing this equivalence does not preserve avoidance of 010 and 101, so we cannot use it directly. Instead, we combine the bijection with a 'compression' step, which removes repeats from the sequence.

Fix $e = e_1 \cdots e_n \in \mathbf{I}_n(010, 101)$. As we have seen, for each value v appearing in e, the occurrences of v will form a consecutive block of elements. The *compression* $c(e)$ of e is the sequence $c_1c_2 \ldots c_k$ obtained from e by replacing, for each $v \in \{e_1, \ldots, e_n\}$, all the (necessarily consecutive) occurrences of v in e by a single occurrence. For example, with $e = 0001444435$, we have $c(e) = 01435$. Note that $c(e)$ is not necessarily an inversion sequence. Observe that all the elements of $c(e)$ are distinct and that $c(e)$ avoids 210 if and only if e avoids 210, and likewise for the pattern 201. For $j = 1, \ldots, k$, let m_j denote the number of occurrences of the value c_j in e. For instance, with the above example of $e = 0001444435$ and $c(e) = 01435$, we have $m_1 = 3$, $m_2 = 1$, $m_3 = 4$, $m_4 = 1$ and $m_5 = 1$.

Suppose now that the inversion sequence e additionally avoids the pattern 210. Then $c(e)$ avoids 210 as well, and by Fact 15, there is a bijection transforming the 210-avoiding rectangular filling $F(c(e))$ into a 201-avoiding rectangular filling $F(c^*)$, for some 201-avoiding sequence $c^* = c^*_1 \cdots c^*_k$. Additionally, we know that c^* has the same positions and values of weak LR maxima as $c(e)$, and the same number of occurrences of each symbol as $c(e)$. In particular, the elements of c^* are pairwise distinct. We now use the values $m_1, \ldots, m_k$, defined above, to transform c^* into a sequence e^*, obtained from c^* by replacing each element c_i^* by a sequence of m_i consecutive copies of c_i^*. Note that e^* has the same positions and values of weak LR maxima as e, and in particular, e^* is an inversion sequence. By construction, e^* belongs to $\mathbf{I}_n(010, 101, 201)$. All the steps of the transform from e to e^* can be inverted, and therefore the transform yields a bijection between $\mathbf{I}_n(010, 101, 210)$ and $\mathbf{I}_n(010, 101, 201)$. $\square$

Note that the bijection used in the preceding proof does not necessarily preserve the number of occurrences of each symbol, and therefore it does not allow us to add any conservative set to the list of forbidden patterns. However, we may directly observe that the bijection preserves 0^m-avoidance for any $m \geqslant 1$. We state this as a corollary.

Corollary 35. *For any $m \geqslant 1$, we have*

$$\{0^m, 010, 101, 210\} \overset{\mathbf{I}}{\sim} \{0^m, 010, 101, 201\} \text{ and } \{10^m, 010, 101, 210\} \overset{\mathbf{I}}{\sim} \{10^m, 010, 101, 201\}.$$

Corollary 36. *The four sets of patterns $A = \{010, 100, 201\}$, $B = \{010, 100, 210\}$, $C = \{010, 101, 201\}$ and $D = \{010, 101, 210\}$ are all I-Wilf-equivalent.*

Proof. We know that $A \overset{\mathbf{I}}{\sim} B$ by Corollary 20 (see Example 21), $B \overset{\mathbf{I}}{\sim} D$ by Theorem 33, and $C \overset{\mathbf{I}}{\sim} D$ by Theorem 34. $\square$

Theorem 37. *We have*

$$\{000, 010, 201\} \overset{\mathbf{I}}{\sim} \{000, 010, 210\}.$$

Proof. Avoidance of 000 means that each value can appear at most twice. As in the proof of Theorem 34, we will use the shape-Wilf equivalence of 201 and 210. But we again need to take care of repeated elements.

For a sequence $e \in \mathbf{I}_n$, we say that e_i is a *low repeat*, if e_i is a repeat (i.e., $e_i \in \{e_1, \ldots, e_{i-1}\}$), and moreover, e_i is not a weak LR maximum.

We claim that if the sequence e avoids 000 and 010, and moreover avoids at least one of the two patterns 201 and 210, then every low repeat e_i satisfies $e_i = e_{i-1}$. To see this, suppose that e_i is a low repeat such that $e_i = e_j$ for some $j < i - 1$. Since each value appears at most twice in e, we know that all the values between e_j and e_i are different from e_i. If at least one of these values is larger than e_i, we obtain a copy of 010. If all these values are smaller than e_i, then e_j is not a weak LR maximum (recall that e_i is not a weak LR maximum since it is a low repeat), hence there is a $k < j$ such that $e_k > e_j = e_i$. For any ℓ strictly between j and i, we further have $e_\ell < e_j = e_i$. Thus, $e_k e_j e_\ell$ is a copy of 210, while $e_k e_\ell e_i$ is a copy of 201, contradicting our assumptions.

We will now use the same compression argument as in the proof of Theorem 34, except now we will only compress low repeats. Fix $e \in \mathbf{I}_n(000, 010, 201)$. Let $c(e) = c_1 \cdots c_k$ be the sequence obtained from e by erasing all the low repeats. Note that $c(e)$ has the same values of weak LR maxima as e (although not necessarily at the same positions). Note also that any value c_i that is not a weak LR maximum is distinct from all the other values in $c(e)$. Let us say that an element c_j is *compressed* if c_j is not a weak LR maximum in $c(e)$ and e has two occurrences of the value c_j (hence one of them is necessarily a low repeat).

Since $c(e)$ avoids 201, we may apply the bijection from Fact 15 to transform it into a 210-avoiding sequence $c^* = c_1^* \cdots c_k^*$ with the same weak LR maxima and the same multiplicities of elements as $c(e)$. In particular, if c_j is compressed in $c(e)$, then c_j^* is not a weak LR maximum (because c_j isn't). We claim that c_j^* is distinct from all the other elements of c^*; indeed, if the value c_j^* occurred more than once in c^*, then it would also occur more than once in $c(e)$, hence all its occurrences in $c(e)$ would have to be weak LR maxima, but since c^* has the same weak LR maxima and at the same positions as $c(e)$, this would mean that in c^*, the value c_j^* has more occurrences than in $c(e)$, which contradicts the properties of the bijection.

We then 'decompress' c^* into a sequence e^* in an obvious way: whenever an element c_j is compressed in $c(e)$, modify c^* by replacing c_j^* by two consecutive copies of c_j^*. It follows from the discussion in the previous paragraph that this cannot create a copy of the pattern 000, and it is easy to see that this cannot create a copy of 010 or 210 either. We see that e^* has the same weak LR maxima as e, and therefore it is an inversion sequence, hence $e^* \in \mathbf{I}_n(000, 010, 210)$. The mapping $e \mapsto e^*$ can be inverted in an obvious way and is the required bijection. □

Theorem 38. *We have*

$$\{000, 101, 201\} \overset{\mathbf{I}}{\sim} \{000, 110, 210\}.$$

Proof. Note that a sequence avoids the pattern 000 if and only if each symbol appears at most twice in it. We will prove the theorem by showing that there is a bijection ψ between $\mathbf{I}_n(101, 201)$ and $\mathbf{I}_n(110, 210)$ which additionally preserves the number of occurrences of each symbol. In fact, ψ will also preserve the positions and values of strict LR maxima.

To describe the bijection, we consider an arbitrary inversion sequence $e = e_1 \cdots e_n \in \mathbf{I}_n$. First, we describe a procedure that encodes e into a particular filling of a Ferrers diagram.

For $i \in [n]$, define $h_i = \max\{e_1, e_2, \ldots, e_i\}$. Note that $h_1 \leqslant h_2 \leqslant \cdots \leqslant h_n$, and that h_n is the height $\mathrm{ht}(e)$ of e. Consider the filling $F(e)$, and recall that this filling has $h_n + 1$ rows and n columns. We will now restrict the filling $F(e)$ to a filling of a Ferrers diagram, by removing from $F(e)$ every box (i, j) such that $j > h_i$. Let D be the resulting filling. Note that all the boxes we removed from $F(e)$ were 0-cells, and that D is a filling of a Ferrers diagram. In fact, the underlying diagram of D is the smallest Ferrers subdiagram of $F(e)$ which contains all the 1-cells of $F(e)$. Note also that the shape of D only depends on the values of $h_1, \ldots, h_n$, and therefore it only depends on the positions and values of the strict LR maxima of e.

As the next step, we transform the filling D into its subfilling D^- be removing from D all the columns corresponding to the strict LR maxima of e (that is, if e_i is a strict LR maximum of e, we remove from D its i-column). After removing a column, we shift the columns to its right by one step to the left, to fill the gap. D^- is again a filling of a

Ferrers diagram. The key observation is that e belongs to $\mathbf{I}_n(110, 210)$ if and only if D^- avoids the pattern 10 (i.e., it avoids the 2×2 subdiagram $F(10) = {}^{1\,0}_{0\,1}$), while e belongs to $\mathbf{I}_n(101, 201)$ if and only if D^- avoids the pattern 01.

We now use the fact that 10 and 01 are shape-Wilf equivalent to describe the required bijection ψ. Start with an inversion sequence $e \in \mathbf{I}_n(101, 201)$. Construct successively the fillings D and D^- as described above. As we observed, D^- avoids 01. By Fact 15, D^- can be bijectively transformed to a filling $\tilde{D}^-$ which avoids 10 and has the same row-sums. We then enlarge $\tilde{D}^-$ into a filling $\tilde{D}$, by reinserting the columns that were removed from D to obtain D^-. These columns will have the same position and filling in $\tilde{D}$ as in D. We now use $\tilde{D}$ to define a sequence $\tilde{e} = \tilde{e}_1, \ldots, \tilde{e}_n$, where $\tilde{e}_i = j$ if and only if $\tilde{D}$ has the 1-cell (i, j). Note that $\tilde{e}$ has the same strict LR maxima as e, and in particular, it is an inversion sequence. Since $\tilde{D}^-$ avoided 10, we easily deduce that $\tilde{e}$ avoids 101 and 201. Defining $\psi(e) := \tilde{e}$, we have obtained the required bijection. □

Theorem 39. *We have*

$$\{102, 110, 210\} \overset{\mathbf{I}}{\sim} \{102, 120, 210\}.$$

Proof. Let us say that an element e_i of an inversion sequence $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n$ is *high* if there is an element e_j such that $i < j$ and $e_i > e_j$. An element is *low* if it is not high. Observe that the low elements must form a weakly increasing subsequence of e. Moreover, e avoids 210 if and only if its high elements form a weakly increasing subsequence. We will now characterize the two classes of inversion sequences of interest, and state the characterization as a pair of claims.

Claim A. An inversion sequence e belongs to $\mathbf{I}_n(102, 110, 210)$ if and only if it satisfies the following properties:

- For any high element e_i and any low element e_j, we have $e_i \geqslant e_j$.
- The high elements form a strictly increasing subsequence.
- The high elements all appear consecutively, that is, there are no three indices $i < j < k$ such that e_i and e_k are high while e_j is low.

To prove Claim A, we first easily observe that an inversion sequence containing a copy of 102, 110, or 210 must violate at least one of these three conditions. Conversely, suppose e violates at least one of the three conditions of the claim. If it violates the first one, it contains a low element e_j and a high element e_i with $e_j > e_i$. Then necessarily e_j is to the right of e_i otherwise e_j would be high; moreover, since e_i is high, there is a smaller element e_k to the right of it. But e_k cannot be to the right of e_j, since e_j is low. Thus, we get $i < k < j$, and e_i, e_j and e_k form a forbidden copy of 102. If e violates the second condition, then it contains 110 or 210, which are both forbidden. Finally, if e satisfies the first two conditions but violates the third one, then it contains 102. This proves Claim A.

Claim B. An inversion sequence e belongs to $\mathbf{I}_n(102, 120, 210)$ if and only if it satisfies these two conditions:

- For any high element e_i and any low element e_j, we have $e_i \geqslant e_j$.
- The high elements all have the same value.

We again easily observe that if e contains 102, 120, or 210, then it violates one of the two conditions. To see the converse, suppose e violates one of the two conditions. If it violates the first one, then it contains 102, by the same argument as in the previous claim. Suppose that e violates the second condition, i.e., it contains two high elements e_i and e_j, with $e_i > e_j$. If $i < j$, then e contains 210, while if $i > j$, then e contains 102 or 120. This proves Claim B.

It is now easy to describe a bijection $\psi\colon \mathbf{I}_n(102,110,210) \to \mathbf{I}_n(102,120,210)$. Fix $e = e_1 \cdots e_n \in \mathbf{I}_n(102,110,210)$. If e has no high element, then it avoids 10 and therefore it is a weakly increasing sequence. In such case, we define $\psi(e) = e$. Suppose that e has at least one high element. Let e_k be the leftmost high element, and let $m \geqslant 0$ be the number of high elements other than e_k. We know from Claim A that these remaining high elements are $e_{k+1}, e_{k+2}, \ldots, e_{k+m}$, and that $e_k < e_{k+1} < \cdots < e_{k+m}$. For $i = 0, 1, \ldots, m$, define $d_i = e_{k+i} - e_k$, so that we have $0 = d_0 < d_1 < d_2 < \cdots < d_m$. Note that, by the definition of inversion sequence, we have

$$k + m > e_{k+m} = e_k + d_m. \tag{1}$$

We now transform the sequence e into a sequence $e' = \psi(e) \in \mathbf{I}_n(102,120,210)$ by the following two steps:

1. Delete from e all the high elements $e_k, e_{k+1}, \ldots, e_{k+m}$, leaving only the weakly increasing sequence of length $n - m - 1$ formed by the low elements of e.
2. Into the obtained sequence, insert $m + 1$ new symbols, all of them equal to e_k, so that the newly inserted symbols will appear at positions $k + m - d_m, k + m - d_{m-1}, \ldots, k + m - d_0 = k + m$. Call the resulting sequence e'.

We claim that e' has exactly $m + 1$ high elements, and these correspond precisely to the elements inserted in step 2 above. To see that the inserted elements are high in e', it is enough to note that e_k was high in e, and therefore e has an element e_j smaller than e_k which appears to the right of all the high elements $e_k, e_{k+1}, \ldots, e_{k+m}$. Since the positions to the right of e_{k+m} are not modified by the mapping $e \mapsto e'$, we have $e_j = e'_j$, and this element e'_j guarantees that all the symbols inserted in the second step are high. By construction, there can be no other high symbols in e'.

Let us verify that e' is an inversion sequence. For this, it is enough to check that each symbol inserted in the second step is smaller than its index. Since all the inserted symbols have the same value e_k, it is enough to verify this inequality for the leftmost inserted symbol, i.e., to verify $e_k = e'_{k+m-d_m} < k + m - d_m$. However, this follows from (1). We conclude that e' is an inversion sequence, and from Claim B, it follows that e' belongs to $\mathbf{I}_n(102,120,210)$.

To see that the mapping ψ is a bijection, let us describe a transformation ψ', which will turn out to be its inverse. Fix $f = f_1 \cdots f_n \in \mathbf{I}_n(102,120,210)$. If f has no high elements, then it is weakly increasing and we put $\psi'(f) = f$. Suppose f has at least one high element, and let f_k be the rightmost high element of f. Let $m \geqslant 0$ be the number of high elements to the left of f_k. Fix a sequence $0 = d_0 < d_1 < d_2 < \cdots < d_m$ so that the high elements of f (in right to left order) are precisely at positions $k - d_0, k - d_1, \ldots, k - d_m$. Recall from Claim B that all the $m + 1$ high elements are equal to f_k. Note that the definition of inversion sequence implies that

$$f_k = f_{k-d_m} < k - d_m. \tag{2}$$

We then transform f into a sequence f' as follows

1. Delete from f all the high elements, leaving only the weakly increasing sequence of length $n - m - 1$ formed by the low elements of f.

2. Into the obtained sequence, insert $m+1$ new symbols forming an increasing sequence $f_k, f_k + d_1, f_k + d_2, \ldots, f_k + d_m$; the symbols are inserted at positions $k - m, k - m + 1, \ldots, k$. Call the resulting sequence f'.

We may easily verify that the high elements of f' are precisely the $m+1$ elements inserted in the second step. With the help of (2), we can verify that f' is an inversion sequence. With the help of Claim A, we may then confirm that f' belongs to $\mathbf{I}_n(102, 110, 210)$. We can then define $\psi'(f) = f'$, and check that ψ' is the inverse of ψ. This shows that both ψ and ψ' are injective, and therefore bijections witnessing that $\{102, 110, 210\} \overset{\mathbf{I}}{\sim} \{102, 120, 210\}$. $\square$

Our next goal is to establish the I-Wilf equivalence of $\{021, 100\}$ and $\{101, 102, 210\}$. Recall from Section 2 that the set $\{021, 100\}$ is equipotent to $\{021, 100, 201\}$ as well as to $\{021, 100, 210\}$ (and to $\{021, 100, 201, 210\}$ as well). It follows that all these sets of patterns are I-Wilf equivalent.

Theorem 40. *We have*

$$\{021, 100\} \overset{\mathbf{I}}{\sim} \{101, 102, 210\},$$

and therefore also $\{021, 100, 210\} \overset{\mathbf{I}}{\sim} \{101, 102, 210\}$.

Proof. Let us say that an element e_i in an inversion sequence e is *covered* if there is a $j < i$ such that $e_j > e_i$. In other words, an element is covered if and only if it is not a weak LR maximum.

Claim A. An inversion sequence $e = e_1 \cdots e_n$ avoids the two patterns 021 and 100 if and only if it has at most one covered element, and this element (if it exists) is equal to 0.

To prove the claim, notice that an inversion sequence contains 021 if and only if it contains a covered element larger than zero. Moreover, a 021-avoiding inversion sequence has at most one covered element if and only if it avoids 100. The claim follows.

Recall from the proof of Theorem 39 that an element e_i of an inversion sequence is high if there is a $j > i$ such that $e_i > e_j$, otherwise e_i is low. The low elements necessarily form a weakly increasing sequence.

Claim B. An inversion sequence $e = e_1 \cdots e_n$ avoids the three patterns 101, 102 and 210 if and only if it satisfies the following conditions:

- Any high element is strictly larger than all the low elements.
- The high elements form a weakly increasing subsequence of e.
- The high elements all appear consecutively, that is, there are no three indices $i < j < k$ such that e_i and e_k are high while e_j is low.

It is straightforward to check that the copy of any of the three patterns 101, 102 and 210 in e implies that e violates at least one of the three conditions of Claim B. Suppose conversely that e violates one of the conditions. If the first condition is violated, then e contains a high element e_i and a low element e_j such that $e_i \leqslant e_j$. Since e_i is high, there is also an element e_k such that $i < k$ and $e_i > e_k$. Since e_j is low, it must be to the right of e_k, otherwise, it would be high due to e_k. Thus, we have $i < k < j$ and $e_k < e_i \leqslant e_j$, which means that the three elements form a copy of 101 or 102. If the first condition holds but the second does not, then e contains 210. Finally, if the first two conditions hold but the third does not, we again obtain a copy of 101 or 102. This proves Claim B.

We now describe a bijection $\psi\colon \mathbf{I}_n(021, 100) \to \mathbf{I}_n(101, 102, 210)$. To describe the bijection, it is convenient to encode inversion sequences from $\mathbf{I}_n$ as lattice paths of a special form connecting the point $(0,0)$ to the point (n,n). To a sequence $e \in \mathbf{I}_n$, we associate a path $P(e)$ defined as follows.

- For each $i \in [n]$, the path $P(e)$ contains a horizontal segment connecting the points $(i-1, e_i)$ and (i, e_i).
- For each $i \in [n-1]$, the path $P(e)$ contains a (possibly trivial) vertical segment connecting (i, e_i) to (i, e_{i+1}).
- The path $P(e)$ contains the vertical segment from (n, e_n) to (n, n).

Notice that if we orient $P(e)$ from $(0,0)$ towards (n,n), then it can be decomposed into a sequence of unit-length steps of three types: *right-steps* going from a point (i,j) to $(i+1, j)$, *up-steps* from (i,j) to $(i, j+1)$, and *down-steps* from (i,j) to $(i, j-1)$. Notice also that $P(e)$ is wholly inside the closed triangle with vertices $(0,0)$, $(n,0)$ and (n,n). Conversely, any lattice path inside this triangle composed of steps of the above three types encodes a unique inversion sequence.

Fix $e \in \mathbf{I}_n(021, 100)$. If e has no covered element, it means that e is a weakly increasing sequence with no high elements, and we define $\psi(e) = e$. Consider therefore that e has a covered element e_i. As we know from Claim A, e_i is equal to 0 and it is the only covered element of e, while the remaining $n-1$ elements of e form a weakly increasing sequence. Define $h = e_{i-1}$. Since e_i is covered, we know that $h > 0$. Furthermore, let d be the number of occurrences of the value h in the subsequence $e_1 e_2 \cdots e_{i-1}$. By monotonicity, we know that $h = e_{i-1} = e_{i-2} = \cdots = e_{i-d} > e_{i-d-1}$. Since e is an inversion sequence, we know that $h = e_{i-d} < i - d$, and therefore $h + d < i$.

The bijection $\psi\colon \mathbf{I}_n(021,100) \to \mathbf{I}_n(101,102,210)$ in this case will be described in terms of a geometric manipulation with the lattice path $P(e)$. We begin by identifying four auxiliary points W, X, Y, Z on the path $P(e)$:

- W is the point $(i-d-1, h)$
- X is the point $(i-1, h)$
- Y is the point (i, h)
- Z is the leftmost intersection of $P(e)$ with the horizontal line $y = h+d$. Note that such an intersection point exists since, as we have pointed out, $h + d < i$ so $P(e)$ must cross the line $y = h + d$ at least once.

These four points partition $P(e)$ into five subpaths, denoted $P_0, P_1, \dots, P_4$ in their left-to-right order. Note that P_1 is a horizontal segment of length d, while P_2 consists of two vertical segments of length h separated by a single right-step.

We now transform $P(e)$ into a path P' via the mapping that sends a point (x, y) to $(n-y, n-x)$. Note that this mapping is the mirror reflection through the line passing through the two points $(n, 0)$ and $(0, n)$. Let W', X', Y', Z' be the respective images of W, X, Y, Z under this mapping, and let P'_i denote the image P_i for $i = 0, \dots, 4$.

We now obtain a path P'' from P' by this sequence of steps:

- Delete P'_1, P'_2, and all the vertical steps of P'_0 that belong to the vertical line $x = n$.
- Take the subpath P'_0 (which connects W' to (n, n)) and move it d steps to the left and d steps down, and call the resulting path P''_0. Notice that the leftmost point of P''_0 is at the same vertical line as the point Z', and that all the points of P''_0 are strictly above the horizontal line $y = n - i$, while $P'_3 \cup P'_4$ has its topmost point on this line. Note also that P''_0 has exactly h horizontal steps.
- Take P'_3, and move it h steps to the right, calling the resulting path P''_3. Note that the leftmost point of P''_3 is at the same vertical line as the rightmost point of P''_0, while the rightmost point of P''_3 is on the vertical line $x = n$. Note that P''_3 has exactly d horizontal steps.
- Insert three vertical segments, connecting, respectively, Z' to the leftmost point of P''_0, the rightmost point of P''_0 to the leftmost point of P''_3, and the rightmost point of P''_3 to (n, n). This yields a path P''.

Note that for every $i \in [n]$, P'' has a unique right-step of the form $(i-1, j)$ to (i, j) for some j. In particular, there is a unique inversion sequence $e' \in \mathbf{I}_n$ such that $P'' = P(e')$; recall that $e'_i = j$ if and only if P'' has a unique right-step of the form $(i-1, j)$ to (i, j). We define the image of e under ψ to be the sequence e'.

With the help of Claim B, we check that e' avoids the patterns 101, 102 and 210. Note that the high elements of e' correspond precisely to the horizontal steps of P''_0, and these steps are all higher than any of the other horizontal steps. Thus, e' satisfies the first condition of Claim B. It follows from the construction, that e' also satisfies the other two conditions, hence e' is in $\mathbf{I}_n(101, 102, 210)$.

To show that the mapping ψ is injective, all we need to do is show that from the sequence $f \in \mathbf{I}_n(101, 102, 210)$, we can uniquely reconstruct the preimage under ψ. If f has no high elements, then f is weakly increasing, and we have $\psi^{-1}(f) = f$. Suppose that f has a high element. Let h be the number of high elements (which, as we know, form a consecutive subsequence in f), and let d be the number of elements that follow the rightmost high element. We may now define $P'' = P(f)$, let P''_3 be its subpath induced by the d rightmost horizontal steps, and P''_0 the subpath induced by the h horizontal steps preceding P''_3. With the knowledge of P''_0, P''_3, d and h, we can reverse the mapping $P \mapsto P''$ described above, and obtain the path P encoding the sequence $e = \psi^{-1}(e')$. By construction, e contains a unique covered element, which is equal to 0, and therefore e is in $\mathbf{I}_n(021, 100)$. □

6 Further results: Trivial Classes

The main goal of our paper was to show there are at least 137 and at most 139 I-Wilf-equivalences for inversion sequences avoiding triples of patterns of length three, see Theorem 1. The main tool that we used to achieve this goal was the concept of generating trees. We remark that by applying the same tool, we can also enumerate several trivial classes. The results we obtained are summarized in Table 5. The proofs for the classes in this table are omitted because of their similarity to the analytical proofs presented in Section 3.

Table 5: Several trivial classes, generating trees $\mathcal{T}[B]$, and generating functions $F_B(x)$

Begin of Table 5			
Class	B	$\mathcal{T}[B]$	$F_B(x)$
1	000,001,012(r)	$0 \rightsquigarrow 00, 01,\quad 01 \rightsquigarrow 00, 011,\quad 011 \rightsquigarrow 00$	$x + 2x^2 + 2x^3 + x^4$
3	000,011,012(r)	$0 \rightsquigarrow 00, 01,\quad 00 \rightsquigarrow 001, 01,\quad 01 \rightsquigarrow 001$	$x + 2x^2 + 3x^3 + x^4$
4	000,010,012(r)	$0 \rightsquigarrow 00, 01,\quad 00 \rightsquigarrow 01, 002,$ $002 \rightsquigarrow 01, 0022,\quad 01 \rightsquigarrow 011,\quad 022 \rightsquigarrow 01,$	$x + 2x^2 + 3x^3 + 3x^4 + 2x^5 + x^6$
8	000,001,210	$a_m \rightsquigarrow (00)^m b_m a_{m+1},\quad b_m \rightsquigarrow (00)^m;\quad a_m = 01\cdots m,\ b_m = a_m m$	$\frac{x(1+x^3)}{(1-x)^2}$
10	000,010,011(r)	$0 \rightsquigarrow 00, 01,\quad 00 \rightsquigarrow 00, 00,\quad 01 \rightsquigarrow 01$	$\frac{x(1-x-x^2)}{(1-x)(1-2x)}$
14	000,010,100	$a_{mj} \rightsquigarrow (a_{m(j-1)})^{j-m} b_{mj} a_{mj} \cdots a_{m(2m)},$ $b_{mj} \rightsquigarrow (b_{m(j-1)})^{j-m} a_{(m+1)j} \cdots a_{(m+1)(2m+2)};$ $a_{mj} = 00\cdots(m-1)(m-1)j$ and $b_{mj} = a_{mj}j,\ j = m, m+1, \ldots, 2m$	
15	000,010,101	$a_{mj} \rightsquigarrow b_m a_{mm} \cdots a_{m(2m)},$ $b_{mj} \rightsquigarrow a_{(m+1)(m+1)} \cdots a_{(m+1)(2m+2)};\ a_{mj} = 00\cdots(m-1)(m-1)j,$ $b_m = 00\cdots mm,\ j = m, m+1, \ldots, 2m$	$\sum_{j\geqslant 0} \frac{j!x^{2j}}{(1-x)\cdots(1-(j+1)x)}$
17	000,012,021(r)	$0 \rightsquigarrow 00, 00,\quad 00 \rightsquigarrow 001, 001,\quad 001 \rightsquigarrow 0011$	$x + 2x^2 + 4x^3 + 4x^4$
18	000,012,110(r)	$0 \rightsquigarrow 00, 01,\quad 00 \rightsquigarrow 001, 01,\quad 01 \rightsquigarrow 001, 011,\quad 001 \rightsquigarrow 011$	$x + 2x^2 + 4x^3 + 4x^4 + x^5$
19	000,012,101(r)	$0 \rightsquigarrow 00, 01,\quad 00 \rightsquigarrow 001, 002,\quad 01 \rightsquigarrow 010, 001,\quad 001 \rightsquigarrow 010,$ $002 \rightsquigarrow 001, 0022,\quad 0022 \rightsquigarrow 001$	$x + 2x^2 + 4x^3 + 4x^4 + 2x^5 + x^6$
20	000,012,100(r)	$0 \rightsquigarrow 00, 01,\quad 00 \rightsquigarrow 001, 002,\quad 01 \rightsquigarrow 001, 001,\quad 001 \rightsquigarrow 0011,$ $002 \rightsquigarrow 0011, 001$	$x + 2x^2 + 4x^3 + 5x^4 + x^5$
25	011,012,100	$a_m \rightsquigarrow b_1 \cdots b_m a_{m+1},\quad b_m \rightsquigarrow b_1 \cdots b_{m-1} c_m,$ $c_m \rightsquigarrow c_1 \cdots c_{m-1};\ a_m = 0^m,\ b_m = a_m m,\ c_m = b_m 0$	$\frac{x(1-x^2-x^3)}{(1-x-x^2)^2}$
28	010,012,210	$a_m \rightsquigarrow b_1 \cdots b_m a_{m+1},\quad b_m \rightsquigarrow b_1^m b_m;\ a_m = 0^m,\ b_m = a_m m$	$\frac{x(1-3x+4x^2-2x^3+x^4)}{(1-x)^5}$
31	000,010,021	$a_m \rightsquigarrow a_0 \cdots a_m b_m,\quad b_m \rightsquigarrow a_0 \cdots a_{m+1};$ $a_m = 0011\cdots(m-1)(m-1)m,\ b_m = a_m m$	$\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2} - 1$
38	012,101,110	$a_m \rightsquigarrow b_1 \cdots b_m a_{m+1},\quad b_m \rightsquigarrow b_1 \cdots b_{m-1} c_m d,$ $c_m \rightsquigarrow b_1 \cdots b_{m-1} c_m,\quad d \rightsquigarrow d;\ a_m = 0^m,\ b_m = a_m m,\ c_m = b_m 0,$ $d = 010$	$\frac{x(1-x+x^2)}{(1-x)(1-2x)}$

Continuation of Table 5			
Class	B	$\mathcal{T}[B]$	$F_B(x)$
39	000,021,102	$a_0 \rightsquigarrow b_0c_1$, $a_m \rightsquigarrow b_ma_1\cdots a_me$, $e \rightsquigarrow b_0e$, $b_m \rightsquigarrow a_{m+1}\cdots a_1e$, $c_1 \rightsquigarrow gd_1h$, $c_m \rightsquigarrow fd_mc_m\cdots c_2e$, $d_m \rightsquigarrow fc_{m+1}\cdots c_2h$, $g \rightsquigarrow f$, $h \rightsquigarrow fd_1h$; $a_m = 00\cdots(m-1)(m-1)m$, $b_m = a_mm$, $c_m = 011\cdots(m-1)(m-1)m$, $d_m = c_mm$, $e = 002$, $f = 0101$, $g = 010$, $h = 012$	$\frac{1-x-x^2-x^3-(1+x^2)\sqrt{1-2x-3x^2}}{2x^2} - x^2 + x^4$
43	012,101,210	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m$, $e \rightsquigarrow e$, $b_m \rightsquigarrow c_me^{m-1}b_m$, $c_m \rightsquigarrow c_me^{m-1}$; $a_m = 0^m$, $e = 0101$, $b_m = a_mm$ $c_m = b_m0$	$\frac{x(1-x+x^2)(1-3x+4x^2-x^3)}{(1-x)^6}$
47	000,021,120	$a_0 \rightsquigarrow b_0(01)$, $01 \rightsquigarrow a_1a_1(002)$, $a_m \rightsquigarrow b_ma_1\cdots a_m(002)$, $b_m \rightsquigarrow a_{m+1}\cdots a_1(002)$, $002 \rightsquigarrow b_0(002)$; $a_m = 00\cdots(m-1)(m-1)m$, $b_m = a_mm$	$\frac{3-6x-4x^2+3x^3}{2x^2} - \frac{(3-3x-x^2)\sqrt{1-2x-3x^2}}{2x^2}$
51	011,102,201	$a_m \rightsquigarrow a_{m+1}b_{m,1}\cdots b_{m,m}$, $b_{m,1} \rightsquigarrow db_{m,1}\cdots b_{m,m}$, $b_{m,j} \rightsquigarrow d^2c_3\cdots c_jb_{m,j}\cdots b_{m,m}$, $d \rightsquigarrow d$; $a_m = 0^m$, $b_{m,j} = a_mj$, $c_m = a_mm(m-1)$, $d = 010$	$\frac{x(1-2x+x^2+x^3)}{(1-x)^2(1-2x-x^2)}$
52	000,021,110	$0 \rightsquigarrow a_0e$, $e \rightsquigarrow b_1a_0e$, $a_m \rightsquigarrow b_{m+1}\cdots b_1f$, $b_m \rightsquigarrow a_mb_m\cdots b_1f$, $f \rightsquigarrow a_0f$; $a_m = 00\cdots mm$, $b_m = a_{m-1}m$, $e = 01$, $f = 002$	$\frac{2-4x-3x^2+3x^3}{2x^2(1-x)} + \frac{(x^2+2x-2)\sqrt{(1+x)(1-3x)}}{2x^2(1-x)}$
55	000,021,101	$a_0 \rightsquigarrow b_0c_1$, $a_m \rightsquigarrow b_ma_1\cdots a_me$, $b_m \rightsquigarrow a_1\cdots a_{m+1}e$, $c_m \rightsquigarrow b_{m-1}d_mc_1\cdots c_m$, $d_m \rightsquigarrow b_mc_1\cdots c_{m+1}$, $e \rightsquigarrow b_0e$; $a_m = 00\cdots(m-1)(m-1)m$, $b_m = a_mm$, $c_m = 011\cdots(m-1)(m-1)m$, $d_m = c_mm$, $e = 002$	$\frac{2x}{3x-1+\sqrt{1-2x-3x^2}} - 1$, [4]
71	010,102,120	$a_m \rightsquigarrow a_{m+1}a_mb_{m,2}\cdots, b_{m,m}$, $b_{m,j} \rightsquigarrow c_{j,1}\cdots c_{j,j-1}b_{m+1,j}b_{m+1-j,2}\cdots b_{m+1-j,m+1-j}$, $c_{m,j} \rightsquigarrow c_{j,1}\cdots c_{j,j-1}c_{m,j}c_{m-j,1}\cdots c_{m-j,m-1-j}d_{m+1-j}$, $d_m \rightsquigarrow c_{m-1,1}\cdots c_{m-1,m-2}d_m$; $a_m = 0^m$, $b_{m,j} = a_mj$, $c_{m,j} = 0^mmj$, $d_m = 0^mm1m$	
83	010,100,110	$a_m \rightsquigarrow a_{m+1}a_mb_{m,2}\cdots, b_{m,m}$, $b_{m,j} \rightsquigarrow (b_{m,j-1})^{j-1}a_{m+2-j}b_{m,j}\cdots b_{m,m}$; $a_m = 0^m$, $b_{m,j} = a_mj$	
84	010,101,110	$a_m \rightsquigarrow a_{m+1}a_mb_{m,2}\cdots, b_{m,m}$, $b_{m,j} \rightsquigarrow a_ma_{m+2-j}b_{m,2}\cdots b_{m,m}$; $a_m = 0^m$, $b_{m,j} = a_mj$	
87	010,100,101	$a_m \rightsquigarrow a_{m+1}a_mb_{m,2}\cdots, b_{m,m}$, $b_{m,j} \rightsquigarrow (c_{m,j})^{j-1}b_{m+1,j}b_{m,j}\cdots b_{m,m}$, $c_{m,2} \rightsquigarrow a_mb_{m,2}\cdots b_{m,m}$, $c_{m,j} \rightsquigarrow (c_{m,j-1})^{j-2}b_{m,j-1}\cdots b_{m,m}$; $a_m = 0^m$, $b_{m,j} = a_mj$, $c_{m,j} = 0^mj1$	
99	021,102,110	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m$, $b_m \rightsquigarrow c_md_1\cdots d_me$, $c_m \rightsquigarrow c_1\cdots c_{m+1}f$, $d_m \rightsquigarrow d_1\cdots d_mc_mg$, $e \rightsquigarrow eg$, $f \rightsquigarrow c_1f$, $g \rightsquigarrow g$; $a_m = 0^m$, $b_m = a_m1$, $c_m = a_m11$, $d_m = a_m12$, $e = 010$, $f = 0113$, $g = 0101$	$\frac{1-8x+26x^2-44x^3+43x^4-22x^5+2x^6}{2x(1-x)^4(1-2x)} - \frac{\sqrt{1-4x}}{2x}$
100	021,102,120	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m$, $b_m \rightsquigarrow b_{m+1}c_1\cdots c_md$, $c_m \rightsquigarrow c_1\cdots c_{m+1}$, $d \rightsquigarrow d^2$; $a_m = 0^m$, $b_m = a_m1$, $c_m = a_m12$, $d = 010$	$\frac{1-7x+19x^2-25x^3+18x^4-4x^5}{2x(1-x)^3(1-2x)} - \frac{\sqrt{1-4x}}{2x}$
101	021,100,102	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m$, $b_m \rightsquigarrow b_{m+1}c_1\cdots c_md$, $c_m \rightsquigarrow c_1\cdots c_{m+1}e$, $d \rightsquigarrow d$; $a_m = 0^m$, $b_m = a_m1$, $c_m = a_m12$, $d = 010$, $e = 0120$	$\frac{1-5x+8x^2-4x^3-x^4+3x^5}{2x(1-x)^4} - \frac{(1+x)\sqrt{1-4x}}{2x}$
102	021,101,102	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m$, $b_m \rightsquigarrow b_1\cdots b_{m+1}c$, $c \rightsquigarrow c$; $a_m = 0^m$, $b_m = a_m1$, $c = 010$	$\frac{1-3x+2x^2-2x^3-(1-x)\sqrt{1-4x}}{2x(1-x)^2}$
104	021,100,110	$a_m \rightsquigarrow a_{m+1}b_1\cdots b_m$, $b_m \rightsquigarrow c_m^2b_1\cdots b_m$, $c_m \rightsquigarrow c_1\cdots c_{m+1}e$, $e \rightsquigarrow c_1e$; $a_m = 0^m$, $b_m = a_m1$, $c_m = a_m10$, $e = 0103$	$\frac{(1-x)(1-3x)-(1-2x)\sqrt{1-4x}}{x(1-2x)}$
119	101,102,201		[4]
127	021,201,210	$a_m \rightsquigarrow a_{m+1}a_{m+1}\cdots a_2$; $a_m = 0^m$,	$\mathbf{I}_n(021,201,210) = \mathbf{I}_n(021)$
End of Table 5			

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7 Appendix A

Table 7: Inversion sequences avoiding a set $B \subset P_3$ with $|B| = 3$

Begin of Table 7

Class	B	$\{\lvert \mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
1	000,001,012(r)	1,2,2,1,0,0,0,0,0
2	000,001,010(r)	
	000,001,011(r)	
	001,010,011(r)	
	001,010,012(r)	
	001,011,012(r)	1,2,2,2,2,2,2,2,2
3	000,011,012(r)	1,2,3,1,0,0,0,0,0
4	000,010,012(r)	1,2,3,3,2,1,0,0,0
5	001,011,100(r)	
	001,011,120(r)	
	001,012,100(r)	
	001,012,110(r)	1,2,3,3,3,3,3,3,3
6	000,001,021(r)	
	000,001,120(r)	1,2,3,4,4,4,4,4,4
7	000,001,110	
	001,010,021(r)	
	001,010,100(r)	
	001,010,101(r)	
	001,010,102(r)	
	001,010,110(r)	
	001,010,120(r)	
	001,010,201(r)	
	001,010,210(r)	
	001,011,021(r)	
	001,011,101(r)	
	001,011,102(r)	
	001,011,110(r)	
	001,011,201(r)	
	001,011,210(r)	
	001,012,021(r)	
	001,012,101(r)	
	001,012,102(r)	
	001,012,120(r)	
	001,012,201(r)	
	001,012,210(r)	1,2,3,4,5,6,7,8,9
8	000,001,210	1,2,3,5,7,9,11,13,15
9	000,001,100	
	000,001,101	
	000,001,102	
	000,001,201	
	010,011,012	1,2,3,5,8,13,21,34,55
10	000,010,011(r)	1,2,3,5,9,17,33,65,129
11	000,010,102	1,2,4,10,27,73,204,587,1716
12	000,010,120	1,2,4,10,28,85,279,979,3624
13	000,010,110	1,2,4,10,28,86,284,1003,3762
14	000,010,100	1,2,4,10,28,87,297,1099,4373
15	000,010,101	1,2,4,10,28,88,304,1144,4648
16	000,010,201	
	000,010,210	1,2,4,10,29,95,343,1341,5599
17	000,012,021(r)	1,2,4,4,0,0,0,0,0
18	000,012,110(r)	1,2,4,4,1,0,0,0,0
19	000,012,101(r)	1,2,4,4,2,1,0,0,0
20	000,012,100(r)	1,2,4,5,1,0,0,0,0
21	000,012,102(r)	
	000,012,120(r)	
	000,012,201(r)	
	000,012,210(r)	1,2,4,5,2,1,0,0,0
22	000,011,102(r)	
	001,021,100(r)	
	001,021,110(r)	
	001,021,120(r)	
	001,100,110	
	001,100,120(r)	
	001,110,120(r)	1,2,4,6,8,10,12,14,16
23	000,011,021(r)	
	001,021,101(r)	
	001,021,102(r)	
	001,021,201(r)	

Class	B	$\{\lvert \mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
47	000,021,120	1,2,5,13,32,81,207,537,1409
48	000,102,120	1,2,5,13,32,85,223,599,1617
49	012,102,201	
	012,102,210	
	012,120,201	
	012,120,210	1,2,5,13,33,80,185,411,885
50	011,021,100	
	011,021,120	
	011,102,210	1,2,5,13,33,81,193,449,1025
51	011,102,201	1,2,5,13,33,82,201,489,1185
52	000,021,110	1,2,5,13,33,84,215,556,1453
53	011,101,102	
	011,102,110	
	012,102,120	1,2,5,13,34,89,233,610,1597
54	000,102,110	1,2,5,13,34,91,246,672,1850
55	000,021,101	1,2,5,13,35,96,267,750,2123
56	011,100,120	1,2,5,13,36,103,306,935,2933
57	000,101,102	1,2,5,13,37,108,327,1010,3180
58	000,021,100	
	000,021,201	
	000,021,210	1,2,5,14,39,111,317,911,2627
59	000,102,210	1,2,5,14,39,113,325,945,2747
60	000,100,102	1,2,5,14,39,115,347,1069,3351
61	000,102,201	1,2,5,14,39,116,345,1060,3289
62	011,120,201	
	011,120,210	1,2,5,14,41,123,375,1156,3590
63	010,021,100	
	010,021,101	
	010,021,102	
	010,021,110	
	010,021,120	
	010,021,201	
	010,021,210	
	011,021,101	
	011,021,110	
	011,021,201	
	011,021,210	1,2,5,14,42,132,429,1430,4862
64	011,101,120	
	011,110,120	1,2,5,14,42,132,431,1452,5026
65	011,100,201	
	011,100,210	1,2,5,14,42,133,441,1521,5425
66	000,101,120	1,2,5,14,43,143,505,1874,7258
67	000,101,110	1,2,5,14,43,143,509,1922,7651
68	011,100,101	
	011,100,110	1,2,5,14,43,144,523,2048,8597
69	000,110,120	1,2,5,14,45,156,581,2289,9468
70	000,100,120	1,2,5,15,49,176,670,2679,11159
71	010,102,120	1,2,5,15,50,175,627,2277,8347
72	010,102,110	1,2,5,15,50,175,628,2289,8436
73	011,201,210	1,2,5,15,50,176,638,2354,8789
74	010,100,102	1,2,5,15,50,177,650,2449,9410
75	010,101,102	1,2,5,15,50,178,662,2540,9977
76	000,120,201	1,2,5,15,50,183,713,2924,12480
77	000,120,210	1,2,5,15,50,183,715,2944,12642
78	010,102,210	1,2,5,15,51,185,692,2629,10076
79	010,102,201	1,2,5,15,51,185,693,2648,10277
80	011,101,201	
	011,101,210	
	011,110,201	
	011,110,210	1,2,5,15,51,189,746,3091,13311
81	010,100,120	1,2,5,15,51,190,758,3192,14045
82	010,101,120	
	010,110,120	1,2,5,15,51,190,759,3206,14180
83	010,100,110	1,2,5,15,51,190,761,3238,14515
84	010,101,110	1,2,5,15,51,190,762,3256,14722
85	000,110,201	1,2,5,15,51,191,769,3273,14552
86	000,101,210	1,2,5,15,51,191,773,3336,15200
87	010,100,101	1,2,5,15,51,192,789,3505,16706

Continuation of Table 7

Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$	Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
	001,021,210(r)		88	000,100,101	
	001,100,210			000,100,110	1,2,5,15,51,193,797,3548,16866
	001,101,110		89	010,110,201	1,2,5,15,52,200,829,3636,16672
	001,101,120(r)		90	010,120,201	1,2,5,15,52,200,829,3638,16704
	001,102,110		91	010,120,210	1,2,5,15,52,200,830,3654,16869
	001,102,120(r)		92	010,110,210	1,2,5,15,52,200,830,3655,16893
	001,110,201		93	000,101,201	
	001,110,210			000,110,210	1,2,5,15,52,201,849,3856,18607
	001,120,201(r)		94	010,100,201	
	001,120,210(r)			010,100,210	
	010,012,021			010,101,201	
	011,012,021	1,2,4,7,11,16,22,29,37		010,101,210	1,2,5,15,52,202,859,3930,19095
24	000,011,120(r)		95	011,101,110	1,2,5,15,52,203,877,4140,21147
	001,100,101		96	010,201,210	1,2,5,15,53,213,938,4403,21640
	001,100,102		97	000,201,210	1,2,5,16,59,242,1065,4932,23703
	001,100,201	1,2,4,7,12,20,33,54,88	98	000,100,201	
25	011,012,100	1,2,4,7,13,23,41,72,126		000,100,210	1,2,5,16,59,245,1111,5413,27961
26	001,101,210		99	021,102,110	1,2,6,19,57,168,506,1585,5165
	001,102,210		100	021,102,120	1,2,6,19,58,174,528,1649,5328
	001,201,210		101	021,100,102	1,2,6,19,59,183,580,1893,6347
	011,012,210	1,2,4,8,15,26,42,64,93	102	021,101,102	1,2,6,19,60,191,619,2048,6909
27	010,012,100		103	021,102,201	
	010,012,110			021,102,210	1,2,6,20,66,213,683,2211,7291
	011,012,201	1,2,4,8,15,27,47,80,134	104	021,100,110	1,2,6,20,68,232,794,2732,9468
28	010,012,210	1,2,4,8,16,31,57,99,163	105	021,100,120	
29	000,011,100(r)			021,101,120	
	000,011,101(r)			021,110,120	1,2,6,20,68,233,805,2807,9879
	000,011,110(r)		106	100,102,120	1,2,6,20,69,240,842,2979,10628
	000,011,201(r)		107	102,110,120	1,2,6,20,69,242,859,3080,11140
	000,011,210(r)		108	101,102,120	1,2,6,20,69,243,869,3145,11491
	001,101,102		109	100,102,110	1,2,6,20,70,248,891,3236,11866
	001,101,201		110	021,100,101	
	001,102,201			021,101,110	
	010,011,021			101,102,110	1,2,6,20,70,252,924,3432,12870
	010,012,101		111	100,101,102	1,2,6,20,73,280,1116,4572,19140
	010,012,102		112	102,110,201	1,2,6,21,75,267,951,3404,12268
	010,012,120		113	102,120,201	1,2,6,21,76,274,979,3479,12351
	010,012,201		114	102,110,210	
	011,012,101			102,120,210	1,2,6,21,76,276,1002,3641,13261
	011,012,102		115	021,120,201	
	011,012,110			021,120,210	1,2,6,21,77,287,1079,4082,15522
	011,012,120	1,2,4,8,16,32,64,128,256	116	100,102,210	1,2,6,21,78,296,1133,4356,16797
30	010,011,102	1,2,4,9,21,51,126,316,799	117	021,100,201	
31	000,010,021	1,2,4,9,21,51,127,323,835		021,100,210	
32	010,011,120	1,2,4,9,22,58,161,467,1402		021,110,201	
33	010,011,201			021,110,210	
	010,011,210	1,2,4,9,23,65,198,639,2160		101,102,210	1,2,6,21,78,297,1144,4433,17238
34	010,011,100		118	100,102,201	1,2,6,21,78,299,1176,4729,19378
	010,011,101		119	101,102,201	1,2,6,21,79,311,1265,5275,22431
	010,011,110	1,2,4,9,23,66,210,733,2781	120	021,101,201	
35	012,021,100			021,101,210	1,2,6,21,80,322,1347,5798,25512
	012,021,101		121	100,101,120	1,2,6,21,81,333,1439,6466,29985
	012,021,110	1,2,5,11,21,36,57,85,121	122	101,110,120	1,2,6,21,81,335,1463,6676,31596
36	012,100,110	1,2,5,11,22,39,66,108,175	123	100,101,110	1,2,6,21,81,337,1491,6945,33827
37	012,100,101	1,2,5,11,23,45,85,156,281	124	100,110,120	1,2,6,21,83,354,1601,7573,37125
38	012,101,110	1,2,5,11,23,47,95,191,383	125	102,201,210	1,2,6,22,85,328,1253,4754,17994
39	000,021,102	1,2,5,12,25,60,148,374,962	126	101,120,201	
40	012,100,210			101,120,210	1,2,6,22,89,384,1743,8239,40215
	012,110,210	1,2,5,12,26,51,92,155,247	127	021,201,210	1,2,6,22,90,394,1806,8558,41586
41	012,100,201		128	100,120,201	
	012,110,201	1,2,5,12,26,51,93,161,269		110,120,201	1,2,6,22,91,408,1939,9623,49371
42	012,100,102		129	100,110,201	
	012,100,120	1,2,5,12,27,56,110,207,378		100,120,210	
43	012,101,210	1,2,5,12,27,57,113,211,373		101,110,201	
44	011,021,102			110,120,210	1,2,6,22,91,409,1953,9763,50583
	012,021,102		130	100,101,210	1,2,6,22,91,409,1955,9803,51085
	012,021,120		131	101,110,210	1,2,6,22,91,410,1973,10012,53094
	012,021,201		132	100,110,210	1,2,6,22,92,422,2074,10754,58202
	012,021,210		133	100,101,201	1,2,6,22,92,424,2106,11102,61436
	012,101,201		134	120,201,210	1,2,6,23,101,484,2468,13166,72630
	012,102,110		135	100,201,210	
	012,110,120	1,2,5,12,27,58,121,248,503		101,201,210	
45	011,102,120	1,2,5,12,28,64,144,320,704		110,201,210	1,2,6,23,102,495,2549,13682,75714
46	012,101,102		136	012,201,210	1,2,5,13,32,73,156,318,629
	012,101,120	1,2,5,12,28,65,151,351,816	137	011,100,102	1,2,5,12,30,75,190,483,1235

End of Table 7