Bounding branch-width

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Abstract

If (X, Y) is a partition of the vertices of a graph $G = (V, E)$ and there are k edges joining vertices in X to vertices in Y, then (X, Y) is an *edge separation* of G of order k. The graph G is (n, k)-edge connected, if whenever (X, Y) is an edge separation of G of order at most k, then either X or Y has at most n elements. We prove that if G is cubic and (n, k)-edge connected, then one can find edges to delete so that the resulting graph is $(6n + 2, k)$-edge connected. We find an explicit bound on the size of a cubic graph that is minimal in the immersion order with respect to having carving-width k. The techniques we use generalise techniques used to prove similar theorems for other structures. In an attempt to develop a unified setting we set up an axiomatic framework to describe certain classes of connectivity functions. We prove a theorem for such classes that gives sufficient conditions to enable a bound on the size of members that are minimal with respect to having branch-width greater than k. As well as proving the above mentioned result for edge connectivity in this setting, we prove (known) bounds on the size of excluded minors for the classes of matroids and graphs of branch-width k. We also bound the size of a connectivity function that has branch-width greater than k and is minimal with respect to an operation known as elision.

Mathematics Subject Classifications: 05B35, 05C40

1 Introduction

Let S be a finite set. A function $\lambda : 2^S \to \mathbb{Z}$ is *normalised* if $\lambda(\emptyset) = 0$, is *symmetric* if $\lambda(X) = \lambda(S - X)$ for all $X \subseteq S$, and is *submodular* if $\lambda(X \cap Y) + \lambda(X \cup Y) \leqslant \lambda(X) + \lambda(Y)$ for all $X, Y \subseteq S$. If λ is normalised, symmetric and submodular, then the pair (λ, S) is a *connectivity function on S* and we say that S is the *ground set* of λ.

Connectivity in a variety of combinatorial structures can be encoded via an associated connectivity function. Examples include connectivity in matroids and polymatroids, vertex connectivity in graphs and edge connectivity in graphs.

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Branch-width was originally defined for graphs by Robertson and Seymour [12]. It is implicit in [12] that branch-width is well defined for any connectivity function and hence, for any structure with an associated connectivity function. This is made explicit in [3] where branch-width for matroids is studied. Readers unfamiliar with branch-width can find the standard definitions at the end of this section. For vertex connectivity in graphs branch-width is qualitatively equivalent to tree width [12]. For edge connectivity in graphs it is known as *carving-width*; see for example [14]. In many situations branch-width is a powerful controller of the complexity of structures. Hard problems can have polynomial-time algorithms for certain classes of structures of bounded branch-width. In such classes it is natural to find, or at least bound the size of, the objects that are minimal obstructions to having branch-width k for some fixed k. But what one means by "minimal obstruction" depends on the class. In [4] a bound is given on the size of a matroid that is minor-minimal with respect to having branch-width greater than k for any fixed k. Analogous techniques are used in the thesis of Jowett [6]. There the class of all connectivity functions is considered and obstructions are considered that are minimal with respect to an operation known as elision.

In this paper we continue the theme by considering edge connectivity in cubic graphs. The natural order on graphs when one considers edge connectivity is the immersion order (see for example [13]) although for cubic graphs this order is, in essence, the topological-minor order. With respect to this order we obtain, for any fixed k, an explicit bound on the size of a cubic graph that is minimal with respect to having branch-width greater than k.

The general strategy for such results is as follows. First one defines an appropriately parameterised notion of connectivity. Then one shows that it is possible to find an element to remove from the structure that does not damage the connectivity too much. After that it is simply a matter of adapting the strategy used in [4] and [6] to the new situation.

Rather than simply mining the same techniques in a somewhat different context it seemed worthwhile to prove a more general theorem that could potentially be used in this, and possibly future situations, and that is the approach we have taken in this paper. Finding the right mathematical umbrella to cover all the cases required an exercise in axiomatics. This exercise may or may not be interesting in its own right.

Our main result on edge connectivity is proved in Section 6, the last section of the paper. This section can be read independently from the rest of the paper. For positive integers m and k, a graph $G = (V, E)$ is (m, k)-edge connected if, whenever a partition (X, Y) of V induces an edge separation in G of order at most k, then either X or Y has size at most m. It follows from Theorem 29 that a cubic (m, k)-edge connected graph G has an edge e such that $G\backslash e$ is $(6m + 2, k)$-edge connected.

The remainder of the paper is structured as follows. Section 2 recalls standard material on branch-width. Section 3 introduces the notion of a monotone order for a class of connectivity functions. Examples are given coming from matroids, vertex connectivity in graphs, edge connectivity in graphs, and from the class of all connectivity functions. In Section 4 a theorem is proved that gives sufficient conditions to obtain an explicit bound on the size of minimal obstacles for branch-width k in a monotone order. Section 5 applies the results of Section 4 to matroids, to vertex connectivity in graphs, to the class of all connectivity functions and finally to edge connectivity in cubic graphs.

2 Branch-width

In this section we review the basic definitions of branch-width and review some standard classes of structures with associated connectivity functions.

Branch-width

A tree is *cubic* if every vertex has either degree 1 or degree 3. We call a vertex with degree 1 a *leaf*. A *partial branch-decomposition* of a connectivity function (λ, S) is a cubic tree T together with a function ϕ from S to the set of leaves of T. For a leaf l, the set $\{s \in S : \phi(s) = l\}$ is the set of *labels* of l. This set may be empty. A *branch-decomposition* is a partial branch-decomposition in which no leaf of T is labelled by more than one element of E. If T is a branch-decomposition and T' is a subgraph of T whose labelled leaves are labelled by exactly $X \subseteq E$, then we say that T' displays X. The *width* of an edge, e, of T is $\lambda(X)$ where X is one of the components displayed by $T \setminus \{e\}$. Note that this is well-defined as λ is symmetric. The *width* of T is the maximum of the widths of the edges. The *branch-width* of a connectivity function λ, denoted $\mathrm{bw}(\lambda)$, is the minimum of the widths of all possible branch-decompositions of λ. While we allow branch-decompositions to have unlabelled leaves we note that such a branch-decomposition can always be modified, without altering the branch-width, so that all leaves are labelled.

Matroids

Few readers of this paper would be unfamiliar with matroids. But, given that very little knowledge of matroid theory is required it is worthwhile to review some basics. While there are many ways to define a matroid, if our interest is in connectivity functions, then the best way to proceed is definitely via the rank function. Let S be a finite set. Recall that a *matroid M on S* is a pair $M = (r_M, S)$, where $r_M : 2^S \to \mathbb{N}$ is a function satisfying the following.

(**R1**) If $X \subseteq S$, then $0 \leqslant r(X) \leqslant |X|$.

(**R2**) If $X \subseteq Y \subseteq S$, then $r(X) \leqslant r(Y)$.

(**R3**) If X and Y are subsets of S, then

$$r(X \cup Y) + r(X \cap Y) \leqslant r(X) + r(Y).$$

We say that r_M is the *rank function* of M. The *connectivity function* $\lambda_M : 2^S \to \mathbb{N}$ of the matroid M on S is defined, for all subsets X of S by $\lambda_M(X) = r_M(X)+r_M(S-X)-r_M(S)$. It is an easy exercise to prove that submodularity of the connectivity function is inherited from the submodularity of the rank function of M. In other words, we have

Lemma 1. *Let M be a matroid on S. Then λ_M is a connectivity function.*

The *branch-width* of a matroid M, denoted $\mathrm{bw}(M)$ is defined by $\mathrm{bw}(M) = \mathrm{bw}(\lambda_M)+1$. This awkward offset occurs for historical reasons where it was felt desirable to match connectivity in a graph with that in its cycle matroid.

There are two natural connectivity functions associated with a graph; one captures vertex connectivity and the other captures edge connectivity. Somewhat confusingly vertex connectivity is captured by a set function on the *edges* of the graph, while edge connectivity is captured by a function on the *vertices* of the graph. Both are interesting.

Vertex Connectivity

Let $G = (V, E)$ be a graph. For a subset X of E let $V(X)$ denote the set of vertices incident with edges in X. Let $\nu_G : 2^E \to \mathbb{N}$ be defined by $\nu_G(X) = |V(X)|+|V(E-X)|-|V(E)|$ for all $X \subseteq V$. It follows from an elementary counting argument that (ν_G, E) is a connectivity function and we say that ν_G is the *vertex-connectivity function* of G. It is well known that, modulo eliminating trivialities, the function ν_G captures vertex connectivity in G.

The *branch-width* of G, denoted $\mathrm{bw}(G)$, is defined to be the branch-width of ν_G. Let $M(G)$ denote the cycle matroid of the graph G. Note that ν_G and $\lambda_{M(G)}$ are certainly different functions; for example consider values on singletons. Nonetheless, it is known that for graphs with a cycle of size at least two, the branch-width of a graph is equal to that of its cycle matroid [5, 10].

Edge Connectivity

Again, let $G = (V, E)$ be a graph. For a set X let $E(X)$ denote the set of edges incident with at least one vertex in X. Define $\varepsilon_G : 2^V \to \mathbb{N}$ to be the set function on V defined by $\varepsilon_G(X) = |E(X)| + |E(V - X)| - |E(V)|$ for all $X \subseteq V$. Again, it is well known and easily seen that (ε_G, V) is a connectivity function. We say that ε_G is the *edge-connectivity* function of G. The *carving-width* of G is defined to be the branch-width of ε_G.

3 Monotone Orders

The problem of finding the "obstacles" to branch-width k for a given class depends very much on the class since the natural notion of substructure for a connectivity function will depend on the class we are interested in, and not just the connectivity function itself. The purpose of this section is to develop an axiomatic framework in which these notions can be unified.

We begin by recalling some straightforward facts about connectivity functions. Let (λ, S) be a connectivity function. If $X \subseteq S$, then $2\lambda(X) = \lambda(X) + \lambda(S - X) \geqslant \lambda(\emptyset) + \lambda(S) = 0$ so that $\lambda(X) \geqslant 0$. We say that λ is *connected* if $\lambda(X) > 0$ for all proper nonempty subsets of S. The case when λ is not connected leads to a familiar decomposition. We omit the elementary proof of the next lemma.

Lemma 2. *Let* (λ_1, S_1) *and* (λ_2, S_2) *be connectivity functions on disjoint sets* S_1 *and* S_2*. Define* $(\lambda_1 \oplus \lambda_2, S_1 \cup S_2)$ *by* $(\lambda_1 \oplus \lambda_2)(X) = \lambda_1(X \cap S_1) + \lambda_2(X \cap S_2)$*. Then* $\lambda_1 \oplus \lambda_2$ *is a connectivity function.*

We say that $\lambda_1 \oplus \lambda_2$ is the *direct sum* of λ_1 and λ_2. Consider the converse. Note that simply restricting a connectivity function to a subset T of the ground set does not usually give a connectivity function as we may lose symmetry. But in the special case that $\lambda(T) = 0$ no problems arise. The straightforward proof of Lemma 3 is given in [6].

Lemma 3. *Let* (λ, S) *be a connectivity function and let* (S_1, S_2) *be a partition of* S *with* $\lambda(S_1) = 0$*. For* $i \in \{1, 2\}$*, define* (λ_i, S_i) *by* $\lambda_i(X) = \lambda(X)$ *for all* $X \subseteq S_i$*. Then* λ_i *is a connectivity function and* $(\lambda, S) = (\lambda_1, S_1) \oplus (\lambda_2, S_2)$*.*

Monotone Orders

Let (λ, S) be a connectivity function. For disjoint sets $X, Y \subseteq S$ we define $\kappa_\lambda(X, Y)$ by $\kappa_\lambda(X, Y) = \min\{\lambda(Z) : X \subseteq Z \subseteq E - Y\}$.

Let $\mathcal{C}$ be a class of connectivity functions and let $\preceq$ be a partial order on $\mathcal{C}$. We say that (λ, S) *covers* (μ, T) if $(\lambda, S) \succ (\mu, T)$, and there is no member (ν, U) of $\mathcal{C}$ such that $(\lambda, S) \succ (\nu, U) \succ (\mu, T)$. We say that the pair $(\mathcal{C}, \preceq)$ is a *monotone order* if the following hold for all $(\mu, T), (\lambda, S) \in \mathcal{C}$.

C1 If $(\mu, T) \preceq (\lambda, S)$, then $T \subseteq S$.

C2 If (λ, S) covers (μ, T), then $|T| \geqslant |S| - 1$.

C3 If $s \in S$, then there exists a pair $(\nu, S - \{s\})$ such that $(\nu, S - \{s\}) \preceq (\lambda, S)$.

C4 If $(\lambda, S) = (\lambda_1, S_1) \oplus (\lambda_2, S_2)$ for some pair of connectivity functions λ_1 and λ_2, then $(\lambda_1, S_1) \preceq (\lambda, S)$.

C5 If $(\mu, T) \preceq (\lambda, S)$, and $X \subseteq T$, then $\mu(X) \leqslant \kappa_\lambda(X, T - X)$.

Properties (C1), (C2), (C3) and (C4) are essentially non-triviality conditions that are easily seen to hold in any natural situations that we can think of. Saying that $\mu \preceq \lambda$ is meant to express that μ is some sort of substructure of λ. One should never expect to gain information in moving to a substructure; and that is what (C5) is attempting to express. Let $(\mathcal{C}, \preceq)$ be a monotone order. If $(\mu, T) \preceq (\lambda, S)$, then we say that μ is a $\mathcal{C}$-*minor* of λ, or simply *minor* if no ambiguity threatens. If λ covers μ in $(\mathcal{C}, \preceq)$, then we say that μ is an *immediate minor* of λ. If μ is a minor of λ and $\mu \neq \lambda$, then we say that μ is a *proper* minor of λ. It is possible for a proper minor of λ to have the same ground set as λ. In particular, if μ is an immediate minor of λ, then either $T = S$, or $T = S - \{s\}$ for some $s \in S$.

It is not clear whether monotone orders are sufficiently interesting to be worthy of investigation in their own right. Nonetheless, they do give a general setting that serves us well for this paper.

Each of the classes of connectivity functions considered in the previous section leads to natural monotone orders. We begin with matroids. Let $\mathcal{C}_M$ denote the class of connectivity functions of matroids. Thus $\lambda \in \mathcal{C}_M$ if and only if there exists a matroid $M \in \mathcal{M}$ such that $\lambda = \lambda_M$. If $\mu, \lambda \in \mathcal{C}_M$, then we say that $\mu \preceq \lambda$ if there exists a matroid M and a minor N of M such that $\mu = \lambda_M$ and $\nu = \lambda_N$. Then $(\mathcal{C}_M, \preceq)$ is a monotone order. Property (C1), (C2), (C3) and (C4) are clear, while property (C5) is well known and easily verified.

Let $\mathcal{C}_\nu$ denote the class of vertex connectivity functions of graphs. For $\mu, \lambda \in \mathcal{C}_\nu$, we say that $\mu \preceq \lambda$ if there exists a graph G and a minor H of G such that $\mu = \nu_H$ and $\lambda = \nu_G$. Again it follows from easily established well-known facts that $(\mathcal{C}_\nu, \preceq)$ is a monotone class of connectivity functions.

As noted in the introduction, if one is interested in edge connectivity for graphs, the natural associated order on graphs is the immersion order. It is straightforward to translate this order to a monotone order on edge connectivity functions. This is particularly straightforward for cubic graphs as the operation of vertex splitting does not arise except in an essentially trivial way. We postpone more detailed discussion of this until later in the paper.

Finally, let $\mathcal{U}$ denote the class of all connectivity functions. It seems that there is just one natural order that makes sense on this all embracing class. Let (λ, S) be a connectivity function and let R be a subset of S. Recall that, for a partition (X, Y) of $S - R$, we define $\kappa_\lambda(X, Y)$ by

$$\kappa_\lambda(X, Y) = \min\{\lambda(Z) : X \subseteq Z \subseteq X \cup R\}.$$

We define the function $\lambda \downarrow R : 2^{S-R} \to \mathbb{N}$ by $\lambda \downarrow R(X) = \kappa_\lambda(X, S - (X \cup R))$ for all $X \subseteq S - R$. In this case we say that $\lambda \downarrow R$ is obtained from λ by *elision*. In the case that $|R| = 1$, elision has a particularly simple description. Say $s \in S$, and $X \subseteq S - \{s\}$. Then $\lambda\downarrow s(X) = \min\{\lambda(X), \lambda(X \cup \{s\})\}$. Now, for connectivity functions μ and λ, define $\mu \preceq \lambda$ if $\mu = \lambda\downarrow R$ for some subset R of the ground set of λ. It is an almost immediate consequence of the definitions that $(\mathcal{U}, \preceq)$ is a monotone order.

4 Bounding branch-width

Let $(\mathcal{C}, \preceq)$ be a monotone order of connectivity functions. A subclass $\mathcal{D}$ of $\mathcal{C}$ is *minor-closed* if every $\mathcal{C}$-minor of a member of $\mathcal{D}$ also belongs to $\mathcal{D}$. If $\mathcal{D}$ is minor-closed and $\lambda \in \mathcal{C}$ has the property that $\lambda \notin \mathcal{D}$, but all proper minors of λ are in $\mathcal{D}$, then λ is an *excluded minor* for $\mathcal{D}$.

Lemma 4. *Let $(\mathcal{C}, \preceq)$ be a monotone order of connectivity functions. For any integer $k \geqslant 0$, the class consisting of those members of $\mathcal{C}$ of branch-width at most k is minor-closed.*

Proof. Assume that (λ, S) has branch-width at most k and that (μ, R) is a minor of (λ, S). Let T be a branch-decomposition of λ. Let T' be the labelled tree obtained by removing the labels in $S - R$ from T. Let e be an edge of T (and hence also of T'). Let

(A, B) and (A', B') be the partitions displayed in T and T' respectively by e. By (C5), $\mu(A') \leqslant \lambda(A) \leqslant k$. Hence T' is a branch-decomposition of μ of width at most k. □

Let (λ, S) be a connectivity function and let (A, B) be a partition of S. A *branching* of B is a partial branch-decomposition of λ in which there is a leaf displaying A and no other leaf is multiply labelled. We say that B is *k-branched* if there is a branching, T, of B with branch-width at most k. The following lemma is proved in [4]. In the lemma we allow members of a partition to be empty.

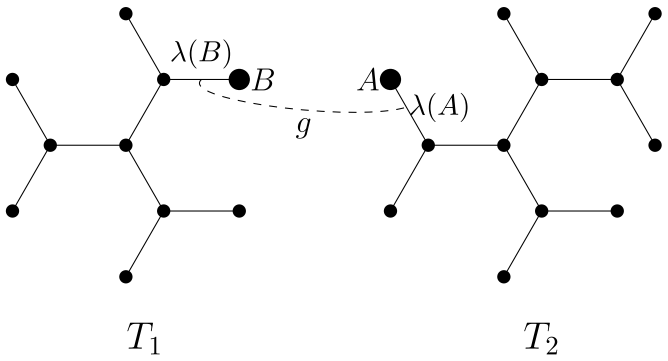
Lemma 5. *Let (λ, S) be a connectivity function. Suppose λ has branch-width at most k. Let (A, B) be a partition of S such that $\lambda(A) \leqslant k$. If B is not k-branched, then there is a partition (A_1, A_2, A_3) of A such that $\lambda(A_i) < \lambda(A)$ for all $i \in \{1, 2, 3\}$.*

Lemma 6. *Let $(\mathcal{C}, \preceq)$ be a monotone order of connectivity functions. Let $k \geqslant 0$ be an integer and let (λ, S) be an excluded minor for the members of $\mathcal{C}$ of branch-width at most k. Then λ is connected.*

Proof. Suppose that λ is not connected. Then there exists a partition (S_1, S_2) of S into nonempty parts such that $\lambda(S_1) = 0$. By Lemma 3, there are connectivity functions (λ_1, S_1) and (λ_2, S_2) such that $\lambda = \lambda_1 \oplus \lambda_2$. Since the ground sets of λ_1 and λ_2 are properly contained in S neither λ_1 nor λ_2 is equal to λ. By (C4) λ_1 and λ_2 are proper minors of λ. Let T_1 be a branch-decomposition of λ_1 with branch-width at most k and T_2 be a branch-decomposition of λ_2 with branch-width at most k. Consider the graph obtained by subdividing an edge of T_1 and subdividing an edge of T_2 and joining the two new vertices with a new edge, e. Call this new tree T. Clearly e has weight 0 in T, and T is a branch-decomposition of λ with branch-width at most k. □

Lemma 7. *Let $(\mathcal{C}, \preceq)$ be a monotone class of connectivity functions. Let $k \geqslant 0$ be an integer, let (λ, S) be a member of $\mathcal{C}$, and let (A, B) be a partition of E. If both A and B are k-branched then* $\mathrm{bw}(\lambda) \leqslant k$. *In particular λ is not an excluded minor for the members of $\mathcal{C}$ of branch-width at most k.*

Proof. Assume that A and B are k-branched. Let T_1 be a branching of A of width at most k, and let T_2 be a branching of B of width at most k. There is a vertex in T_1 labelled by B. Let e be the edge incident with this vertex. The width of e is equal to $\lambda(B) = \lambda(A)$. Similarly in T_2 the edge, f, incident with the vertex labeled by A has width $\lambda(A) = \lambda(B)$. Let v_1 be the internal vertex of T_1 incident with e, and let v_2 be the internal vertex of T_2 incident with f. The graph obtained by joining $T_1 \setminus \{e\}$ to $T_2 \setminus \{f\}$ via a new edge g, that is incident with vertices v_1 and v_2 and has weight $\lambda(A)$, gives a branch-decomposition of λ that has width at most k.



□

Let $(\mathcal{C}, \preceq)$ be a monotone order of connectivity functions and let (μ, R) be a minor of (λ, S) in $\mathcal{C}$. A set $X \subseteq R$ is (μ, λ)-*unperturbed* if $\mu(Z) = \lambda(Z)$ for all $Z \subseteq X$. If the function λ is clear from context, we simply say that X is *unperturbed in* μ.

For a non-negative real n and non-negative integer k, we say that a connectivity function (λ, S) is (n, k)-*connected* if whenever $X \subseteq S$ has $\lambda(X) \leqslant k$, then either $|X| \leqslant n$ or $|S - X| \leqslant n$. Let $[k]$ denote the set $\{0, 1, \ldots, k\}$ and let $\mathbb{N}$ denote the set of non-negative integers. Let $f : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be a function. We say that λ is $(f, [k])$-*connected* if λ is $(f(i), i)$-connected for all $i \in [k]$.

Let $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}/3$ be a function. We say that the monotone order $(\mathcal{C}, \preceq)$ is h-*strong* if the following holds. If $(\lambda, S) \in \mathcal{C}$ is (n, k)-connected, (A, B) is a partition of S such that $|A|, |B| \geqslant h(n, k)$ and $\lambda(A) = k + 1$, then there exists a minor (λ_A, S_A) of λ such that the following hold.

S1 Either $S_A = S$ or $S_A = S - \{a\}$ for some $a \in A$.

S2 λ_A is $(h(n, k), k)$-connected.

S3 If $|A| > 3h(n, k)$ then we can, in addition, choose λ_A such that B is unperturbed in λ_A.

Lemma 8. *Let* $\mathbb{N}/3 = \{n/3 : n \in \mathbb{N}\}$. *Let* $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}/3$ *be a function such that* $\mathcal{C}$ *is* h*-strong. For* $k \in \mathbb{N}$ *define the function* $g : [k + 1] \to \mathbb{N}$ *by* $g(0) = 0$ *and* $g(i + 1) = 3h(g(i), i) + 1$ *for all* $i \in \{1, 2, \ldots, k\}$. *If* λ *is an excluded minor for the members of* $\mathcal{C}$ *of branch-width at most* k, *then* λ *is* $(g, [k + 1])$*-connected.*

Proof. By Lemma 6, λ is connected, so that λ is $(g, [0])$-connected. Suppose that $0 \leqslant t \leqslant k$ and that λ is $(g, [t])$-connected. Assume for a contradiction that λ has a separation (A, B) of order $t + 1$ such that $|A|, |B| \geqslant g(t + 1) = 3h(g(t), t) + 1$. Since λ has branch-width greater than k we may assume that B is not k-branched. As $\mathcal{C}$ is h-strong, there is a minor (λ_A, R) of λ such that the following hold.

(i) Either $R = S$ or $R = S - \{a\}$ for some $a \in A$.

(ii) λ_A is $(g(t + 1), t)$-connected, and

(iii) B is unperturbed in λ_A.

Now λ is an excluded minor for $\mathcal{C}_k$ so that λ_A has branch-width at most k. Let $A' = A \cap R$. Since B is unperturbed in λ_A, we have $\lambda_A(A') = t+1$. Consider any partition (A_1, A_2, A_3) of A'. We have $|A_i| \geqslant h(g(t), t) + 1$ for some $i \in \{1, 2, 3\}$. Hence $\lambda_A(A_i) \geqslant t + 1 = \lambda_A(B)$. Thus A' is not k-branched in λ_A, so by Lemma 7, B is k-branched in λ_A. But B is unperturbed in λ_A, so B is k-branched in λ. We deduce from this contradiction that λ is $(g(t+1), t)$-connected and hence that λ is $(g, [t+1])$-connected. □

The following lemma is well-known; see for example [11, Lemma 14.2.2].

Lemma 9. *If T is a tree with at least one edge, then T has an edge, e, such that each of the two components of $T \setminus e$ contains at least one-third of the leaves of T.*

We say that the monotone order $(\mathcal{C}, \preceq)$ is *smooth* if whenever (μ, R) is an immediate minor of (λ, S), and (X, Y) is a partition of R, then $\mu(X) \geqslant \kappa_\lambda(X, Y) - 1$. Said more prosaically we have $\mu(X) \geqslant \lambda(X) - 1$ if $S = T$ and $\mu(X) \geqslant \min\{\lambda(X) - 1, \lambda(Y) - 1\}$ if $T = S - \{z\}$ for some $z \in S$.

Theorem 10. *Let $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}/3$ be a function and let $(\mathcal{C}, \preceq)$ be a smooth, h-strong monotone order of connectivity functions. Define the function $g : \mathbb{N} \to \mathbb{N}$ by $g(0) = 0$ and $g(i+1) = 3h(g(i), i) + 1$ for all $i \in \mathbb{N}$. If (λ, S) is an excluded minor for $\mathcal{C}_k$, then $|S| \leqslant g(k+2)$.*

Proof. It is easily seen that we may assume that $|S| \geqslant 3$, and that $k \geqslant 1$. Let μ be an immediate minor of λ. Assume that μ is not $(h(g(k+1), k+1), k+1)$-connected. Then, as $\mathcal{C}$ is smooth, λ has a $(k+1)$-separation, both sides of which have size at least $h(g(k+1), k+1)$. As $\mathcal{C}$ is h-strong, we deduce that λ has an immediate minor which is, indeed, $(h(g(k+1), k+1), k+1)$-connected. Say that ν is such a minor.

As λ is an excluded minor and $\mathcal{C}$ is smooth, ν has branch-width k. Let T be a width-k branch-decomposition of ν. Since $|S| \geqslant 3$, T has at least two labelled leaves. By Lemma 9, T has an edge e such that the sets X_1 and X_2 labelled by the components of $T \backslash e$ each have at least $(|S| - 1)/3$ elements. Assume that $|X_1| \leqslant |X_2|$. Then, since ν is $(h(g(k+1), k+1), k+1)$-connected, we have $|X_1| \leqslant h(g(k+1), k+1)$. Thus $|S| \leqslant 3h(g(k+1), k+1) + 1 = g(k+2)$, as required. □

5 Examples

We now consider examples. We begin with matroids.

Matroidal Connectivity Functions

Recall that $(\mathcal{C}_M, \preceq)$ denotes the class of matroidal connectivity functions, where, for $\mu, \lambda \in \mathcal{C}_M$, we say that $\mu \preceq \lambda$ if there exist matroids N and M such that $\mu = \lambda_N$, $\lambda = \lambda_M$, and N is a minor of M. The next theorem is [4, Theorem 1.1].

Theorem 11. *If M is an excluded minor for the class of matroids of branch-width at most k, and $k \geqslant 2$, then $|E(M)| \leqslant (6^k - 1)/5$.*

The techniques of this paper are, in essence, a generalisation of the techniques used in [4]. Nonetheless it is of some interest to derive the theorem from the results of this paper and we do this now.

Note that, in Lemma 13, the function $h(n, k)$ is independent of k. This is the case in all our examples. However we foresee applications to edge connectivity in graphs with arbitrary vertex degrees where the more general version is required. We first note an easy lemma, the proof of which is omitted.

Lemma 12. *Let x be an element of the matroid M on S, and let $Z \subseteq S - \{x\}$. If $\lambda_{M/x}(Z) = \lambda_M(Z)$, then Z is unperturbed in $\lambda_{M/x}$.*

Lemma 13. *Let $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $h(n, k) = 2n$. Then $\mathcal{C}_M$ is h-strong.*

Proof. Say that $(\lambda, S) \in \mathcal{C}_M$ is (m, k)-connected. Let M be a matroid such that $\lambda = \lambda_M$. Note that λ is $(f, [k])$-connected where $f : \mathbb{N} \to \mathbb{N}$ is defined by $f(i) = m$ for all $i \in \mathbb{N}$. It now follows from [4, Lemma 3.1] that either $\lambda_{M\backslash x}$ or $\lambda_{M/x}$ is $(2m, k)$-connected for all $x \in S$.

Assume that (A, B) is a partition of S such that $|A|, |B| \geqslant 2n$ and $\lambda_M(A) = k + 1$. Say $a \in A$. Using the argument of the previous paragraph and duality we may assume that $\lambda_{M/a}$ is $(2m, k)$-connected. Thus (S1) and (S2) hold.

Assume that $|A| \geqslant 3h(m, k) = 6m$. Then $\lambda_{M/a}(B) = \lambda_M(B)$, as otherwise $\lambda_{M/a}$ is not $(2m, k)$-connected. Now, by Lemma 12, B is unperturbed in M/a. $\square$

Theorem 14. *Let (λ, S) be an excluded minor for members of $(\mathcal{C}_M, \preceq)$ of branch-width at most k, where $k \in \mathbb{N}$. Then $|S| \leqslant (6^{k+1} - 1)/5$.*

Proof. It is evident that $(\mathcal{C}_M, \preceq)$ is a smooth monotone order. By Lemma 13, $(\mathcal{C}_M, \preceq)$ is h-strong where $h(n, k) = 2n$. Define the function g by $g(0) = 0$, and otherwise $g(i+1) = 3h(g(i), i) + 1 = 6g(i) + 1$. Then $g(k) = (6^{k-1} - 1)/5$ for all k. By Theorem 10, $|S| \leqslant g(k + 2) = (6^{k+1} - 1)/5$. $\square$

Theorem 11 follows by recalling that a matroid M has branch-width k if and only if λ_M has branch-width $k - 1$.

Vertex Connectivity in Graphs

It is shown in [5, 10], that for graphs with a cycle of size at least two, the branch-width of a graph is equal to that of its cycle matroid [5, 10]. Using this fact and Theorem 11, we obtain a bound on the size of excluded minors for vertex connectivity functions of graphs. Specifically, if λ_G is an excluded minor for vertex connectivity functions of graphs of branch-width at most k, then G has at most $(6^k - 1)/5$ edges. Alternatively, one can derive this fact from the results of this paper. Rather than pursuing that routine exercise, we take the liberty of discussing a topic that we find interesting.

An *(integral) polymatroid* is a pair $P = (S, r)$, where E is a finite set, and $r : 2^S \to \mathbb{N}$ is a function satisfying the following.

(P1) $r(\emptyset) = 0$.

(P2) If $X \subseteq Y \subseteq S$, then $r(X) \leqslant r(Y)$.

(P3) If X and Y are subsets of S, then

$$r(X \cup Y) + r(X \cap Y) \leqslant r(X) + r(Y).$$

In other words, we have the same axioms as for matroids but we omit the requirement that singletons have rank at most one. A *k-polymatroid* is one where singletons have rank at most k. The *connectivity function* (λ_P, S) of a polymatroid is defined just as for matroids, so that $\lambda_P(X) = r(X) + r(S - X) - r(S)$. We define the *branch-width* of the polymatroid P to be the branch-width of λ_P.

The *Dilworth truncation* of the collection of lines of a matroid was first defined by Mason [9]. The operation itself was first used by Dilworth [2] in his proof that every lattice can be embedded in a geometric lattice. Lovász [7] generalised the operation to polymatroids, indeed to more general submodular functions. The Dilworth truncation of a 2-polymatroid is a matroid.

One can obtain a 2-polymatroid $P_G = (r, E)$ from a graph $G = (V, E)$, by letting $r(X) = |V(X)|$ for all $X \subseteq E$. Then the vertex connectivity function ν_G of G is precisely the connectivity function of P_G. The fact that the branch-width of a graph and its cycle matroid are the same can then be stated as a result about the branch-width of a class of polymatroids and their Dilworth Truncations.

To enable a more unified statement we remove the pesky $+1$ from the definition of matroid branch-width. We conjecture that the result for connectivity functions of graphs and their cycle matroids holds more generally.

Conjecture 15. Let P be a 2-polymatroid and $D(P)$ be its Dilworth truncation. If $\text{bw}(P) \geqslant 3$, then $\text{bw}(D(P)) = \text{bw}(P) - 1$.

We further believe that the condition that P is a 2-polymatroid is redundant.

Conjecture 16. Let P be a k-polymatroid and $D(P)$ be its Dilworth truncation. If $\text{bw}(P) \geqslant k + 1$, then $\text{bw}(D(P)) = \text{bw}(P) - 1$.

The condition that $\text{bw}(P) \geqslant 3$ in Conjecture 15 is just to eliminate low branch-width counterexamples caused, in the 2-polymatroid case, essentially by trees. We see similar tree-like problematic structures for more general k-polymatroids. Hence the requirement that $\text{bw}(P) \geqslant k + 1$ in Conjecture 16. These requirements could possibly be refined somewhat.

Elision in Connectivity Functions

Let $(\mathcal{U}, \preceq)$ denote the class of all connectivity function ordered by elision. In this world, (μ, R) is a minor of (λ, S) if there is a subset T of S such that $\mu = \lambda{\downarrow} T$.

Theorem 17. *Let (λ, S) be an excluded minor for the members of $(\mathcal{U}, \preceq)$ of branch-width at most k. Then $|S| \leqslant (3^{k+2} - 1)/2$.*

We first establish some elementary facts.

Lemma 18. *If (λ, S) is an (n, k)-connected connectivity function, then $\lambda\!\downarrow s$ is (n, k)-connected for all $s \in S$.*

Proof. Say that (X, Y) is a partition of $S - \{s\}$ and $\lambda\!\downarrow s(Z) \leqslant k$. Then either $\lambda(X)$ or $\lambda(X \cup \{s\}) \leqslant k$. Either case implies that one of X or Y has at most n elements. □

Lemma 19. *Let (λ, S) be a connectivity function, let X be a subset of S and y an element of $S - X$. If $\lambda(X \cup \{y\}) \geqslant \lambda(X)$, then $\lambda(X' \cup \{y\}) \geqslant \lambda(X')$ for all $X' \subseteq X$.*

Proof. By submodularity $\lambda(X' \cup \{y\}) + \lambda(X) \geqslant \lambda(X') + \lambda(X \cup \{y\})$. The lemma follows easily from this observation. □

The next lemma is an immediate consequence of Lemma 19.

Lemma 20. *Let (λ, S) be a connectivity function and $s \in S$. If $Z \subseteq S - \{s\}$ and $\lambda\!\downarrow s(Z) = \lambda(Z)$, then Z is unperturbed in $\lambda\!\downarrow s$.*

Lemma 21. *Let (λ, S) be a connectivity function and (X, Y) be a partition of S with $\lambda(X) = k$. If $|X| > k$, then there exists $x \in X$ such that Y is unperturbed in $\lambda\!\downarrow x$.*

Proof. Say $X = \{x_1, x_2, \dots, x_t\}$. As $t > k$, there exists an $i \in \{1, 2, \dots, t\}$ such that $\lambda(Y \cup \{x_1, x_2, \dots, x_{i-1}\}) = \lambda(Y \cup \{x_1, x_2, \dots, x_i\}$. By Lemma 20 $Y \cup \{x_1, x_2, \dots, x_{i-1}\}$ is unperturbed in $\lambda\!\downarrow x_i$; in particular Y is unperturbed in $\lambda\!\downarrow x_i$. □

The next lemma is an immediate consequence of Lemmas 18 and 21.

Lemma 22. *Let $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}/3$ be defined by $h(n, k) = n/3$. Then $(\mathcal{U}, \preceq)$ is h-strong.*

We are now able to prove Theorem 17.

Proof of Theorem 17. Let $h(n, k)$ be defined as in Lemma 22 and let $g : \mathbb{N} \to N$ be defined as in Theorem 10. Thus $g(0) = 0$, and, for $i > 1$, we have $g(i + 1) = 3h(g(i), i) + 1 = 3g(i) + 1$. Observe that $g(k) = (3^k - 1)/2$. By Theorem 10, if (λ, S) is an excluded minor for the members of $(\mathcal{U}, \preceq)$ of branch width at most k, then $|S| \leqslant g(k + 2)$, that is $|S| \leqslant (3^{k+2} - 1)/2$, as required. □

Edge Connectivity in Subcubic Graphs

Recall that the *edge connectivity function* ε_G of a graph $G = (V, E)$ is defined by $\varepsilon_G(X) = |E(X)| + |E(V - X)| - |E(V)|$ for all $X \subseteq V$. The branch-width of ε_G is the *carving-width* of G.

For edge connectivity in graphs the natural order to consider is the *immersion order*. Let $G = (V, E)$ be a graph and let xy, yz be edges of G with a common neighbour y. The graph G' is said to be obtained by *splitting off* xy, yz *at* y, if G' is obtained by deleting the edges xy, yz and adding an edge xz. We say that a graph H is *immersed in* G if H can be obtained by a sequence of splittings and edge and vertex deletions. Immersion leads to a natural monotone order on edge connectivity functions of graphs. Note that, in this order, one has proper minors with the same ground set, as neither splitting vertices nor deleting edges changes the ground set of the edge connectivity function.

Vertex splitting is not of particular interest for us as we focus on cubic graphs. Nonetheless we believe that it is of interest to extend the results we obtain for cubic graphs to more general graphs where vertex splitting will necessarily play a role.

A graph is *subcubic* if all of its vertices have degree at most 3. Let $G = (V, E)$ be a subcubic graph and let H be a graph whose vertex set is $V - \{v\}$. In this class, the graph H is an immersion minor of G if and only if H is a topological minor of G. In other words, H can be obtained from G by a sequence of edge deletions, suppression of degree-2 vertices (not incident with a loop) and deletion of isolated vertices. Let $\mathcal{C}^3_\varepsilon$ denote the following class of connectivity functions: $\lambda \in \mathcal{C}^3_\varepsilon$ if and only if there is a subcubic graph G such that $\lambda = \varepsilon_G$. Define $\preceq$ in $\mathcal{C}^3_\varepsilon$ by $\mu \preceq \lambda$ if there exist subcubic graphs H and G such that $\mu = \varepsilon_H$ and $\lambda = \varepsilon_G$, and H is a topological minor of G. It is easily seen that $(\mathcal{C}^3_\varepsilon, \preceq)$ is a monotone order.

More formally, for $\mu, \lambda \in \mathcal{C}^3_\varepsilon$, we say that $\mu \preceq \lambda$ if there exist graphs H and G such that $\mu = \varepsilon_H$, $\lambda = \varepsilon_G$, and H can be obtained from G via a sequence of the following operations:

I1 deleting an edge;

I2 suppressing a degree-2 vertex unless it is adjacent to a loop; and

I3 deleting a vertex of degree at most one.

If e is an edge of the graph $G = (V, E)$, then λ_G and $\lambda_{G\backslash e}$ have the same ground set; namely V. It follows that in $(\mathcal{C}^3_\varepsilon, \preceq)$ we have proper minors that preserve ground sets.

Finally we can state our main theorem.

Theorem 23. *If (λ, S) is an excluded minor for the members of $(\mathcal{C}^3_\varepsilon, \preceq)$ of branch-width at most k, then $|S| \leqslant 7(18^{k+2} - 1)/17$.*

In another language we have.

Corollary 24. *If $G = (V, E)$ is an excluded immersion minor for the class of subcubic graphs of carving-width at most k, then $|V| \leqslant 7(18^{k+2} - 1)/17$.*

Theorem 23 will follow from some straightforward lemmas that put us in a position to apply the theorems of Section 4 and 6. We omit the easy proof of the next lemma.

Lemma 25. *The monotone class $(\mathcal{C}^3_\varepsilon, \preceq)$ is smooth.*

Loops have no effect on the edge connectivity function, so may safely be ignored. The next lemma enables us to reduce to the simple cubic case. We omit the straightforward proof.

Lemma 26. *Let $G = (V, E)$ be a subcubic graph with at least three vertices. Assume that G' is obtained from G by one of the following operations: suppressing a degree-2 vertex, deleting a degree-1 vertex, or deleting an edge that is parallel to another edge. Then* $\mathrm{bw}(\varepsilon_{G'}) = \mathrm{bw}(\varepsilon_G)$.

Let G_1, G_2 and G_3 denote graphs having a 2-element vertex set and, respectively, one, two and three edges joining those vertices. It is easily checked that ε_{G_1}, ε_{G_2} and ε_{G_3} are, respectively, the unique excluded minors for the edge connectivity functions of cubic graphs of branch-width 0, 1 and 2 respectively. The next result follows easily from Lemma 26.

Corollary 27. *Let G be an excluded minor for the class of subcubic graphs of branch-width k, where $k \geqslant 3$. Then G is simple and cubic.*

The next lemma is routinely seen to hold for subcubic graphs, but we only need it for cubic graphs.

Lemma 28. *Let $h : \mathbb{N} \to \mathbb{N}$ be defined by $h(m, k) = 6m + 2$. Say $(\lambda, S) \in (\mathcal{C}^3_\varepsilon, \preceq)$ and $\lambda = \varepsilon_G$ for a simple cubic graph G. Then λ is h-strong.*

Proof. Assume that ε_G is (m, k)-connected. Then the graph G is (m, k)-edge connected. Let (A, B) be a partition of S such that $|A|, |B| \geqslant 6n + 2$, and $\varepsilon(A) = k + 1$. By Theorem 30, there exists an edge $e \in E(A)$ such that $G\backslash e$ is $(6n + 2)$-connected, so that $\lambda_{G\backslash e}$ is $(6n+2)$-connected. As $e \in E(A)$, it is easily seen that e is not in the edge boundary of any subset of B. Thus $\varepsilon_{G\backslash e}(Z) = \varepsilon_G(Z)$ for any $Z \subseteq B$, so that B is unperturbed in $\varepsilon_{G\backslash e}$. It now follows that ε_G is h-strong. □

Proof of Theorem 23. Let (λ, S) be an excluded minor for the members of $(\mathcal{C}^3_\varepsilon, \preceq)$ of branch-width at most k. Let $g : \mathbb{N} \to \mathbb{N}$ be defined by $g(0) = 0$, and otherwise $g(i + 1) = 18g(i) + 7$. It follows from Lemma 28 and Theorem 10 that $|S| \leqslant g(k + 2)$, that is, $|S| \leqslant 7(18^{k+2} - 1)/17$. □

Finally we note that Corollary 24 follows easily from Theorem 23.

6 Edge Connectivity in Cubic Graphs

In this section we prove two theorems on edge connectivity in cubic graphs. In essence we prove that if G is a graph with a certain type of edge connectivity then one can always find an edge e whose deletion does not erode the connectivity of G by an arbitrary amount. It is possible that a reader may be interested in the results of this section without being interested in the other results of this paper. To facilitate this we repeat some definitions.

Let $G = (V, E)$ be a graph. For a set $X \subseteq V$ we let $E(X)$ denote the set of edges of G incident with at least one vertex in X. We define the *edge connectivity* function $\varepsilon_G : 2^V \to \mathbb{N}$ of G by $\varepsilon_G(X) = |E(X)| + |E(V - X)| - |E(V)|$ for all $X \subseteq V$. If G is clear from context we abbreviate ε_G to ε. It is well known that the edge connectivity function of a graph is submodular, and it follows easily that it is a connectivity function in the sense of this paper. Note that $\varepsilon(X)$ counts the number of edges joining vertices in X to vertices in $V - X$.

At times it is useful to use the language of separations. We say that a partition (X, Y) of V is an *edge separation* of G of *order* $\varepsilon(X)$. We say that (X, Y) is a *k-separation* if $\varepsilon(X) \leqslant k$. We say that X is *k-separating* if $\varepsilon(X) \leqslant k$ and is *exactly* k-separating if $\varepsilon(X) = k$.

One can use the edge-connectivity function to define various notions of edge connectivity. In particular G is *k-edge connected* if $\varepsilon(X) \geqslant k$ for all proper nonempty subsets of V. In general k-edge connectivity is a restrictive property. A less restrictive notion is to control the size of sets (or the size of their complement) whose connectivity is low.

The notion that will prove useful here is the following. For positive integers k and m, we say that G is (m, k)-edge connected if whenever (X, Y) is a k-separation of G, then either $|X| \leqslant m$ or $|Y| \leqslant m$. We can now state the two main results of this section. Our interest in this section is solely in edge connectivity, we will often abbreviate "(m, k)-edge connected" to "(m, k)-connected".

Theorem 29. *Let G be a cubic, (m, k)-edge connected graph with at least two edges. Then there exist at least two edges, x and y, such that $G\backslash x$ and $G\backslash y$ are both $(6m + 2, k)$-edge connected.*

While Theorem 29 is simple to state, the more useful theorem for us will be the following.

Theorem 30. *Let G be a simple cubic (m, k)-edge connected graph, and let (X, Y) be a $(k + 1)$-separation of G such that $|X|, |Y| > 3m + 2$. Then there exist edges $x \in E(X)$ and $y \in E(Y)$ such that $G\backslash x$ and $G\backslash y$ are both $(6m + 2, k)$-edge connected.*

The strategy for proving the above theorems is as follows. Let v be a vertex of the (m, k)-connected cubic graph G. One might hope that at least one of the edges incident with v could be deleted to give a graph that is $(6m + 2, k)$-connected. Sadly this is not always possible, but when it is not, we can deduce something about the structure of the graph relative to the vertex v. We then show that, if we look carefully, we can find vertices of G that are not compatible with the presence of such a structure. Before diving into the details we note a useful lemma. This lemma appears as an exercise in [8] and follows from a straightforward counting argument.

Lemma 31 (3-Way Submodularity). *Let A, B, and C be subsets of vertices of a graph G. Then*

$$\begin{aligned}&\varepsilon(A) + \varepsilon(B) + \varepsilon(C)\\ \geqslant&\varepsilon(A \cap B \cap C) + \varepsilon(A - (B \cup C)) + \varepsilon(B - (A \cup C)) + \varepsilon(C - (A \cup B)).\end{aligned}$$

The property of 3-way submodularity seems to be an interesting one that is particular to edge connectivity; it is easily seen not to hold for vertex connectivity in graphs. It is perhaps worthwhile to investigate other structures whose associated connectivity function is 3-way submodular. Can one say anything interesting about the classes of matroids or polymatroids whose connectivity functions are 3-way submodular?

And one more definition. Let e be an edge of the graph G and (X, Y) a separation of $G\backslash e$. We say that (X, Y) is *induced* in G if $\varepsilon_G(X) = \varepsilon_{G\backslash e}(X)$.

Tripods

Let $G = (V, E)$ be an (m, k)-connected graph and let v be a degree-3 vertex of G whose incident edges are e, f and g. Let e', f' and g' be the other vertices incident with e, f and g respectively. A *tripod for* v is a partition (A_e, A_f, A_g, C) of V such that the following hold:

- $v \in C$, $e' \in A_e$, $f' \in A_f$, and $g' \in A_g$;
- $\varepsilon(A_e) = \varepsilon(A_f) = \varepsilon(A_g) = k + 1$;
- $|A_e|, |A_f|, |A_g| > 4m + 2$.

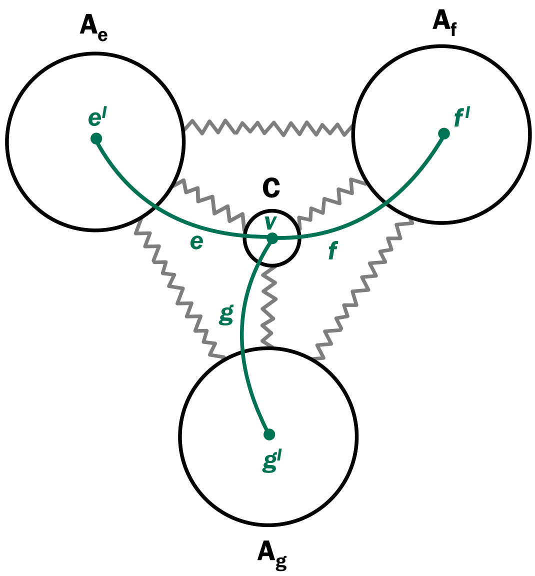


Figure 1: A Tripod for v

The goal is to show that tripods are the obstacles to being able to remove an edge incident with a degree-3 vertex while maintaining the desired connectivity. In particular we prove

Lemma 32. *Let v be a vertex of degree 3 in a simple, connected, (m, k)-connected graph G. If $G\backslash x$ is not $(6m+2, k)$-connected for any edge x incident with v, then G has a tripod for v.*

Proof. Assume that v is incident with edges $e = ve'$, $f = vf'$ and $g = vg'$, and assume that none of $G\backslash e$, $G\backslash f$ and $G\backslash g$ is $(6m+2)$-connected. By assumption, there exist k-separations (E_1, E_2), (F_1, F_2) and (G_1, G_2) of $G\backslash e$, $G\backslash f$ and $G\backslash g$ respectively such that $|E_1|, |E_2|, |F_1|, |F_2|, |G_1|, |G_2| > 6m + 2$. Suppose that $e' \in E_2, f' \in F_2$ and $g' \in G_2$.

32.1. $\{v, f', g'\} \subseteq E_1, \{v, e', g'\} \subseteq F_1$ and $\{v, e', f'\} \subseteq G_1$.

Proof. Consider (E_1, E_2). As $e' \in E_2$, we must have $v \in E_1$, as otherwise (E_1, E_2) is induced in G and we contradict the fact that G is (m, k)-connected. It follows that $v \in E_1 \cap F_1 \cap G_1$.

Assume that $f' \in E_2$. Suppose $g' \in E_1$. Then the separation $(E_1 - \{v\}, E_2 \cup \{v\})$ is also a k-separation of $G\backslash e$ and as $|E_1|, |E_2| > 6m+2$ then $|E_1 - \{v\}|, |E_2 \cup \{v\}| > m$. But $(E_1 - \{v\}, E_2 \cup \{v\})$ is induced in G, contradicting the fact that G is (m, k)-connected.

For the other case, suppose $g' \in E_2$. Then $(E_1 - \{v\}, E_2 \cup \{v\})$ is also a k-separation of $G\backslash e$, and as $|E_1|, |E_2| > 6m+2$, we have $|E_1 - \{v\}|, |E_2 \cup \{v\}| > m$. But $(E_1 - \{v\}, E_2 \cup \{v\})$ is induced in G, again contradicting the fact that G is (m, k)–connected.

It follows $\{v, f', g'\} \subseteq E_1$, and, by symmetry we also have $\{v, e', g'\} \subseteq F_1$ and $\{v, e', f'\} \subseteq G_1$. □

We now consider how the separations (F_1, F_2) and (G_1, G_2) interact with one another. We have $v \in F_1 \cap G_1$, which needs to be remembered as this is not obvious from the diagrams.

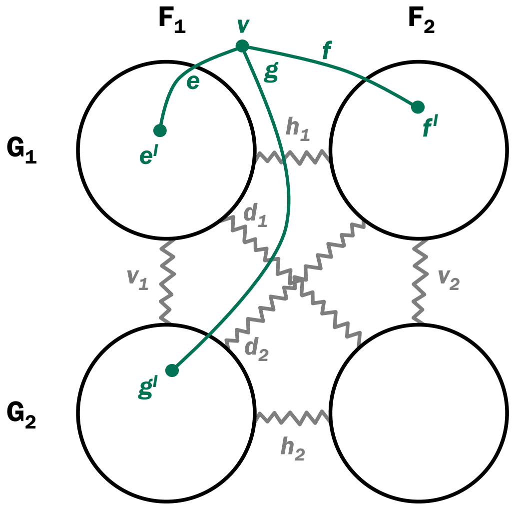


Figure 2: Crossing Separations

In Figure 2 v_1 represents the number of edges of $E - \{e, f, g\}$ joining vertices in $F_1 \cap G_2$ to vertices in $F_1 \cap G_1$. The numbers v_2, d_1, d_2, h_1 and h_2 are defined in a similar obvious way. Recall that $\varepsilon_G(F_1) = \varepsilon_G(G_1) = k + 1$. We therefore have

32.2. $h_1 + d_1 + d_2 + h_2 = k$ and $v_1 + d_1 + d_2 + v_2 = k$.

We now bound the size of either $F_1 \cap G_1$ or $F_2 \cap G_2$, and we do this by bounding connectivity.

32.3. Either $\varepsilon_G(F_2 \cap G_2) \leqslant k$ or $\varepsilon_G((F_1 \cap G_1) - \{v\}) \leqslant k$.

Proof. Suppose the claim does not hold so that $\varepsilon_G(F_2 \cap G_2) \geqslant k+1$ and $\varepsilon_G((F_1 \cap G_1) - \{v\}) \geqslant k+1$. Clearly $\varepsilon_G(F_2 \cap G_2) = h_2 + d_1 + v_2$ and $\varepsilon_G((F_1 \cap G_1) - \{v\}) = h_1 + d_1 + v_1 + 1$. Using this fact and (32.2), we have

$$\begin{aligned} 2k+2 &\leqslant \varepsilon(G_1 \cap F_1 - \{v\}) + \varepsilon(G_2 \cap F_2) \\ &= h_1 + d_1 + v_1 + h_2 + d_1 + v_2 + 1 \\ &< h_1 + d_1 + v_1 + h_2 + d_1 + v_2 + 2 \\ &\leqslant (v_1 + d_1 + d_2 + v_2) + (h_1 + d_1 + d_2 + h_2) + 2 \\ &= 2k+2. \end{aligned}$$

The claim follows from this contradiction. □

As an almost immediate consequence of (32.3) we have

32.4. Either $|F_2 \cap G_2| \leqslant m$ or $|(F_1 \cap G_1) - \{v\}| \leqslant m$.

We now consider $F_1 \cap G_2$ and $F_2 \cap G_1$. By (32.4), and the fact that $|F_1|, |F_2|, |G_1|, |G_2| > 6m+2$, we have

32.5. $|F_1 \cap G_2|, |F_2 \cap G_1| > 5m+2$.

It follows from (32.5) that $\varepsilon_G(F_1 \cap G_2) \geqslant k+1$ and $\varepsilon_G(F_2 \cap G_1) \geqslant k+1$. It turns out that equality holds.

32.6. $\varepsilon(F_1 \cap G_2) = \varepsilon(F_2 \cap G_1) = k+1$.

Proof. Consider $F_1 \cap G_2$. We have

$$\begin{aligned} 2k+2 &= \varepsilon_G(F_1) + \varepsilon_G(G_2) \\ &\geqslant \varepsilon_G(F_1 \cap G_2) + \varepsilon_G(F_1 \cup G_2) \\ &= \varepsilon_G(F_1 \cap G_2) + \varepsilon_G(F_2 \cap G_1) \\ &\geqslant 2k+2. \end{aligned}$$

Hence all inequalities are equalities and the claim follows. □

We are now able to be more specific about the values of $v_1, v_2, d_1, d_2, h_1, h_2$.

32.7. $d_1 = 0, v_1 = h_1$ and $v_2 = h_2$.

Proof. By (32.6), we have $\varepsilon_G(F_2 \cap G_1) = \varepsilon_G(F_1 \cap G_2) = k+1$ and therefore $h_1 + d_2 + v_2 = k$ and $v_1 + d_2 + h_2 = k$ respectively. Therefore $(h_1 + d_2 + v_2) + (v_1 + d_2 + h_2) = 2k$. But $(h_1 + d_1 + d_2 + h_2) + (v_1 + d_1 + d_2 + v_2) = 2k$, and so $d_1 = 0$. As $d_1 = 0$, we know $h_1 + d_1 + d_2 + h_2 = k = h_1 + d_2 + h_2$. As $h_1 + d_2 + v_2 = k$, we must have $h_2 = v_2$.

Similarly, combining $h_1 + d_2 + h_2 = k$ and $v_1 + d_2 + h_2 = k$, we get $h_1 = v_1$ and the claim holds. □

With the information we now have we can refine our diagram to obtain the one illustrated in Figure 3. In the figure $a = h_1 = v_1, d = d_2$ and $b = h_2 = v_2$, and $a + b + d = k$.

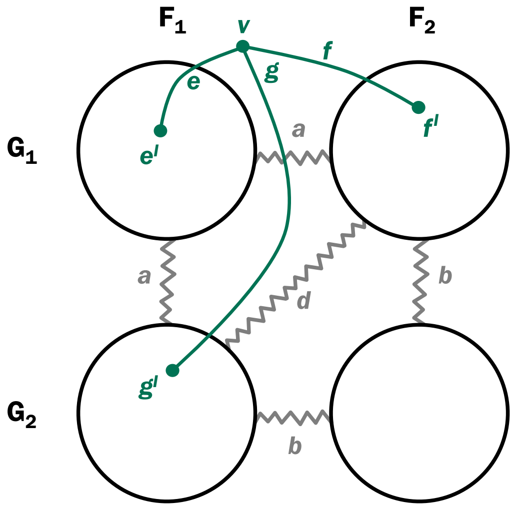


Figure 3: Refined Crossing

Note that everything we have said about how (F_1, F_2) crosses (G_1, G_2) is also true for how (E_1, E_2) crosses (F_1, F_2) and how (E_1, E_2) crosses (G_1, G_2). This symmetry provides additional information, which we now exploit, in order to refine 32.3.

32.8. Up to switching labels $\varepsilon((F_1 \cap G_1) - \{v\}) > k$.

Proof. Suppose for a contradiction that $\varepsilon((F_1 \cap G_1) - \{v\}) \leqslant k$, $\varepsilon((E_1 \cap G_1) - \{v\}) \leqslant k$ and $\varepsilon((E_1 \cap F_1) - \{v\}) \leqslant k$. As the complement of each of these three sets has at least $m + 1$ elements, we have $|F_1 \cap G_1|, |E_1 \cap F_1|, |E_1 \cap G_1| \leqslant m + 1$. As $|E_1| > 6m + 2$, and $|E_1 \cap F_1| + |E_1 \cap G_1| \leqslant 2m + 2$, we deduce that $|E_1 - (F_1 \cup G_1)| > m$. By symmetry, we also have $|F_1 - (E_1 \cup G_1)| > m$, and $|G_1 - (F_1 \cup E_1)| > m$. But the complement of each of these sets has size at least $m + 1$, so we have $\varepsilon_G(E_1 - (F_1 \cup G_1)) \geqslant k + 1$, $\varepsilon_G(F_1 - (E_1 \cup G_1)) \geqslant k + 1$, and $\varepsilon_G(G_1 - (F_1 \cup E_1)) \geqslant k + 1$.

Consider $E_1 \cap F_1 \cap G_1$. Observe that v belongs to this set, but each of e', f' and g' belong to its complement. It follows that $\varepsilon_G(E_1 \cap F_1 \cap G_1) \geqslant 3$. All up we have $\varepsilon_G(E_1 - (F_1 \cup G_1)) + \varepsilon_G(F_1 - (E_1 \cup G_1)) + \varepsilon_G(G_1 - (F_1 \cup E_1)) + \varepsilon_G(E_1 \cap F_1 \cap G_1) \geqslant 3k + 6$. But $\varepsilon_G(E_1) + \varepsilon_G(F_1) + \varepsilon_G(G_1) = 3k + 3$, and we have contradicted 3-way modularity. □

32.9. The following hold.

(i) $\varepsilon(E_2 \cap F_2), \varepsilon(F_2 \cap G_2), \varepsilon(E_2 \cap G_2) \leqslant k$.

(ii) $|E_2 \cap F_2|, |F_2 \cap G_2|, |E_2 \cap G_2| \leqslant m$.

Proof. Note that (ii) follows from (i) and the fact that the corresponding complements of the sets in question have size greater than m. By (32.3) and (32.8), we have $\varepsilon(F_2 \cap G_2) \leqslant k$ and hence $|F_2 \cap G_2| \leqslant m$.

We did not break the symmetry between (E_1, E_2), (F_1, F_2), and (G_1, G_2) until (32.8). Thus, it follows from (32.5) that $|E_1 \cap F_2| > 5m + 2$ and $|E_1 \cap G_2| > 5m + 2$. Since $E_1 \cap F_2 = (E_1 \cap F_2 \cap G_1) \cup (E_1 \cap F_2 \cap G_2)$, and $E_1 \cap F_2 \cap G_2 \subseteq F_2 \cap G_2$, we have $|E_1 \cap F_2 \cap G_1| \geqslant 4m + 2$. But $E_1 \cap F_2 \cap G_1 \subseteq E_1 \cap G_1$, so $|(E_1 \cap G_1) - \{v\}| > m$. The complement of this set also has more than m elements. Hence $\varepsilon((E_1 \cap G_1) - \{v\}) > k$. Therefore, by (32.4) we have $\varepsilon(E_2 \cap F_2) \leqslant k$.

A symmetric argument proves that we also have $\varepsilon(E_2 \cap G_2) \leqslant k$. □

32.10. $|E_2 \cap F_1 \cap G_1|, |E_1 \cap F_2 \cap G_1|, |E_1 \cap F_1 \cap G_2| > 4m + 2$.

Proof. Observe that $E_2 = (E_2 \cap F_1 \cap G_1) \cup (E_2 \cap F_2) \cup (E_2 \cap G_2)$. We have $|E_2| > 6m+2$, and by (32.9), $|E_2 \cap F_2| \leqslant m$ and $|E_2 \cap G_2| \leqslant m$. Hence $|E_2 \cap F_1 \cap G_1| > 4m + 2$. The rest of the claim follows by symmetry. □

32.11. $E_2 \cap F_2 \cap G_2 = \emptyset$.

Proof. By (32.10), $\varepsilon(E_2 \cap F_1 \cap G_1), \varepsilon(E_1 \cap F_2 \cap G_1), \varepsilon(E_1 \cap F_1 \cap G_2) \geqslant k + 1$. But $E_2 \cap F_1 \cap G_1 = E_2 - (F_2 \cup G_2)$, $E_1 \cap F_2 \cap G_1 = F_2 - (E_2 \cup G_2)$, and $E_1 \cap F_1 \cap G_2 = G_2 - (E_2 \cup F_2)$. We are now in a position to utilise 3-way submodularity. We have $\varepsilon(E_2 - (F_2 \cup G_2)) \geqslant k + 1$, $\varepsilon(F_2 - (E_2 \cup G_2)) \geqslant k + 1$, and $\varepsilon(G_2 - (E_2 \cup F_2)) \geqslant k + 1$. But we also have $\varepsilon(E_2) = \varepsilon(F_2) = \varepsilon(G_2) = k + 1$. It follows that $\varepsilon(E_2 \cap F_2 \cap G_2) = 0$, as otherwise we contradict 3-way submodularity. Since G is connected, we conclude that $E_2 \cap F_2 \cap G_2 = \emptyset$. □

32.12. The sets $E_1 \cap G_2$, $E_2 \cap F_1$, $F_2 \cap G_1$ and $E_1 \cap F_1 \cap G_1$ partition V.

Proof. It is easily seen that the sets are disjoint. Say $z \in V$ does not belong to their union. Since $z \notin E_1 \cap F_1 \cap G_1$, we may assume without loss of generality that $z \in E_2$. Then $z \notin E_2 \cap F_1$, so $z \in F_2$. But then $z \notin F_2 \cap G_1$, so $z \in G_2$. Hence $z \in E_2 \cap F_2 \cap G_2$, contradicting the fact that this set is empty. □

By (32.6) and symmetry, we have $\lambda(E_2 \cap F_1) = \lambda(F_2 \cap G_1) = \lambda(E_1 \cap G_2) = k + 1$. Moreover $E_2 \cap F_1 \supseteq E_2 \cap F_1 \cap G_1$. So, by (32.10), $|E_2 \cap F_1| > 4m + 2$, and, of course, we also have $|F_2 \cap G_1| > 4m + 2$ and $|E_1 \cap G_2| > 4m + 2$. Finally, observe that $e' \in E_2 \cap F_1$, $f' \in F_2 \cap G_1$, $g' \in E_1 \cap G_2$ and $v \in E_1 \cap F_1 \cap G_1$.

Altogether we obtain the desired tripod for v by letting $A_e = E_2 \cap F_1$, $A_f = F_2 \cap G_1$, $A_g = E_1 \cap G_2$, and $C = E_1 \cap F_1 \cap G_1$. □

Avoiding a Tripod

Let X be a subset of vertices of a graph G. We say that the vertex $x \in X$ is an *internal* vertex of X if all neighbours of x belong to X. We now prove that internal vertices of sets satisfying certain conditions cannot be contained in a tripod, thus enabling us to find edges to delete.

Lemma 33. *Let $G = (V, E)$ be a simple, connected, cubic (m, k)–connected graph. Assume that a partition (R, B) of V satisfies the following conditions:*

(i) $\varepsilon(R) = k + 1$;

(ii) $|R|, |B| > 3m + 2$;

(iii) *there is no proper subset* $X \subset R$, *such that the separation* $(X, V - X)$ *satisfies the above two conditions.*

Then, if v *is an internal vertex of* R, *then* $G \backslash a$ *is* $(6m + 2, k)$*-connected for at least one edge* a *incident with* v.

Proof. Let v be an internal vertex of R and suppose there is no edge a incident to v such that $G \backslash a$ is $(6m + 2, k)$-connected. Let $e = ve'$, $f = vf'$, and $g = vg'$ denote the edges incident with v. By Lemma 32, G has a tripod (A_e, A_f, A_g, C) for v depicted in Figure 4. In the figure members of R and B are coloured red and blue respectively.

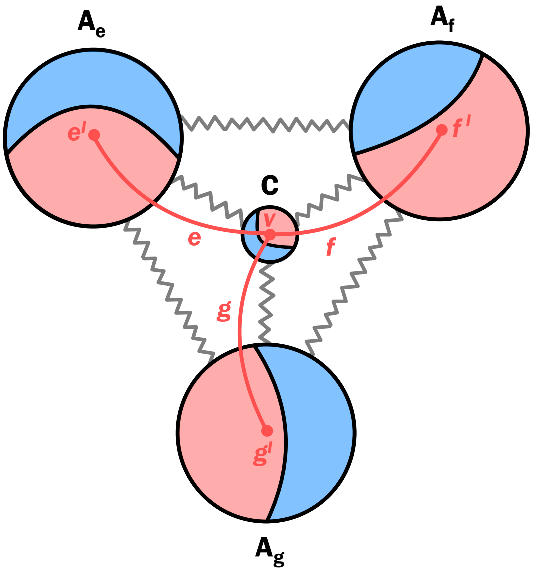


Figure 4: Crossing a Tripod

33.1. $|B \cap A_e|, |B \cap A_f|, |B \cap A_g| > m$.

Proof. Suppose for contradiction that $|B \cap A_e| \leqslant m$. Then as $|B| > 3m + 2$, we have $|B \cap (V - A_e)| > m$. But this set and its complement both have at least $m + 1$ elements. The complement of $B \cap (V - A_e)$ is $R \cup A_e$, so that $\varepsilon(R \cup A_e) \geqslant k + 1$. As $|B \cap A_e| \leqslant m$ and $|A_e| > 4m + 2$, we have $|R \cap A_e| > 3m + 2$ and $\varepsilon(R \cap A_e) \geqslant k + 1$. By submodularity $\varepsilon(R \cap A_e) + \varepsilon(R \cup A_e) \leqslant \varepsilon(R) + \varepsilon(A_e) = 2k + 2$. We deduce that $\varepsilon(R \cap A_e) = k + 1$. But $R \cap A_e \subseteq R$, and $|R \cap A_e| > 3m + 2$, so we have contradicted the minimality of the choice of R, and by symmetry the claim follows. □

33.2. $|R \cap (A_e \cup C)|, |R \cap (A_f \cup C)|, |R \cap (A_g \cup C)| \geqslant m + 2$.

Proof. Suppose for contradiction that $|R \cap (A_e \cup C)| \leqslant m+1$, $|R \cap (A_f \cup C)| \leqslant m+1$ and $|R \cap (A_g \cup C)| \leqslant m+1$. As $v \in C$, we have $|R| \leqslant |R \cap (A_e \cup C)| + |R \cap (A_f \cup C)| + |R \cap (A_g \cup C)| - 2 \leqslant 3m+1$. This contradicts the fact that $|R| > 3m+2$. ☐

We will now use 3-way submodularity to complete the proof. We use the following three sets: R, $A_f \cup \{v\}$ and $A_g \cup \{v\}$. On one side we have $\varepsilon(R) = k+1$, $\varepsilon(A_f \cup \{v\}) = (k+1)+1 = k+2$, and $\varepsilon(A_g \cup \{v\}) = (k+1)+1 = k+2$. Hence

$$\varepsilon(R) + \varepsilon(A_f \cup \{v\}) + \varepsilon(A_g \cup \{v\}) = 3k+5.$$

On the other hand, we first have $\varepsilon(R \cap (A_f \cup \{v\}) \cap (A_g \cup \{v\})) = \varepsilon(\{v\}) = 3$. Note that $R - ((A_f \cup \{v\}) \cup (A_g \cup \{v\})) = (R \cap (A_e \cup C)) - \{v\}$. By (33.2), $|R \cap (A_e \cup C)| \geqslant m+2$. Therefore $|(R \cap (A_e \cup C)) - \{v\}| \geqslant m+1$. As the complement of this set also has size at least $m+1$, we have $\varepsilon((R \cap (A_e \cup C)) - \{v\}) \geqslant k+1$. We also have $(A_f \cup \{v\}) - (R \cup (A_g \cup \{v\})) = B \cap A_f$. Recall that $|B \cap A_f| > m$ so $\varepsilon(A_f \cup \{v\} - (R \cup (A_g \cup \{v\}))) \geqslant k+1$. Similarly $\varepsilon(A_g \cup \{v\} - (R \cup (A_f \cup \{v\}))) \geqslant k+1$. Altogether, we have

$$\begin{aligned}
&\varepsilon(R \cap (A_f \cup \{v\}) \cap (A_g \cup \{v\})) + \varepsilon(R - ((A_f \cup \{v\}) \cup (A_g \cup \{v\})))+\\
&\varepsilon(A_f \cup \{v\} - (R \cup (A_g \cup \{v\}))) + \varepsilon(A_g \cup \{v\} - (R \cup (A_f \cup \{v\})))\\
\geqslant& 3 + 3(k+1) = 3k+6,
\end{aligned}$$

and we have contradicted 3-way submodularity. The lemma follows from this contradiction. ☐

6.1 Proofs of the Main Theorems

We are now in a position to prove our main theorems.

Proof of Theorem 30. Let $R \subseteq X$ be minimal in size with respect to $\varepsilon(R) = k+1$ and $|R| > 3m+2$. Then the separation $(R, V-R)$ satisfies the hypotheses of Lemma 33, so, by that lemma there exists an edge $x \in E(R)$ such that $G \backslash x$ is $(6m+2, k)$-connected. By symmetry, there also exists an edge $y \in Y$ such that $G \backslash y$ is $(6m+2, k)$-connected and the theorem follows. ☐

Proof of Theorem 29. Assume that the theorem fails. Then it is easily seen that G has a vertex v such that none of its incident edges can be deleted to maintain $(6m+2, k)$-connectivity. Therefore by Lemma 32, G must have a tripod (A_e, A_f, A_g, C) for v. We have $|A_e| > 4m+2 > 3m+2$ and $\varepsilon(A_e) = k+1$. By the previous Theorem 30, there exist edges on each side of the separation $(A_e, V - A_e)$ to delete such that the resulting graph is $(6m+2, k)$-connected. ☐

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