

A further extension of Rödl's theorem

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Abstract

Fix $\varepsilon > 0$ and a graph H with at least one vertex. A well-known theorem of Rödl from the 80s says that every graph G with no induced copy of H contains a linear-sized ε -restricted set $S \subseteq V(G)$, which means S induces a subgraph with maximum degree at most $\varepsilon|S|$ in G or its complement. There are two extensions of this result:

- quantitatively, Nikiforov relaxed the condition “no induced copy of H ” to “at most $\kappa|G|^{|H|}$ induced copies of H for some $\kappa > 0$ depending on H and ε ,” and
- qualitatively, Chudnovsky, Scott, Seymour, and Spirkl recently showed that there exists $N > 0$ depending on H and ε such that G is (N, ε) -restricted, which means $V(G)$ has a partition into at most N subsets that are ε -restricted.

A natural common generalization of these two asserts that every graph G with at most $\kappa|G|^{|H|}$ induced copies of H is (N, ε) -restricted for some $\kappa, N > 0$ depending on H and ε . This is unfortunately false; but we prove that for every $\varepsilon > 0$, κ and N still exist so that for every $d \geq 0$, every graph G with at most $\kappa d^{|H|}$ induced copies of H has an (N, ε) -restricted induced subgraph on at least $|G| - d$ vertices. This unifies the two aforementioned theorems, and is optimal up to κ and N for every value of d .

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1 Introduction

Graphs in this paper are finite and simple. For a graph G with vertex set $V(G)$ and edge set $E(G)$, let $|G| := |V(G)|$, and let \overline{G} denote its complement. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S , and let $G \setminus S := G[V(G) \setminus S]$. For a nonnull graph H , a *copy* of H in G is a graph isomorphism from H to $G[S]$ for some $S \subseteq V(G)$. Let $\text{ind}_H(G)$ be the number of copies of H in G ; and say that G is *H -free* if $\text{ind}_H(G) = 0$. Given $\varepsilon > 0$, a subset $S \subseteq V(G)$ is ε -restricted in G if one of $G[S], \overline{G}[S]$ has maximum degree at most $\varepsilon|S|$. The following well-known theorem of Rödl [16] from 1986

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has become a standard tool in the investigation of the Erdős–Hajnal conjecture¹ [8, 7] (see [4] for a survey).

Theorem 1 ([16]). *For every $\varepsilon > 0$ and every graph H , there exists $\delta = \delta(H, \varepsilon) > 0$ such that for every H -free graph G , there is an ε -restricted $S \subseteq V(G)$ in G with $|S| \geq \delta|G|$.*

Since its inception, Theorem 1 has found many extensions. Among these is the following useful quantitative improvement first proved by Nikiforov [14] (see [6, 12, 13] for several applications).

Theorem 2 ([14]). *For every $\varepsilon > 0$ and every graph H , there exist $\delta = \delta_2(H, \varepsilon) > 0$ and $\kappa = \kappa_2(H, \varepsilon) > 0$ such that for every graph G with $\text{ind}_H(G) \leq \kappa|G|^{|H|}$, there is an ε -restricted $S \subseteq V(G)$ in G with $|S| \geq \delta|G|$.*

Rödl’s original proof of Theorem 1 and Nikiforov’s proof of Theorem 2 (we remark that Theorem 2 is already implicit in [16]) both employ the regularity lemma, and so give bounds on δ^{-1} and κ^{-1} which are towers of twos of height polynomial in ε^{-1} with constants depending on H . Fox and Sudakov [10] offered an alternative proof of Theorem 2 showing that both δ and κ can be chosen as $2^{-c \log^2(\varepsilon^{-1})}$ for some constant $c > 0$ depending on H ; and very recently Bucić, Nguyen, Scott, and Seymour [3] improved this to $2^{-c \log^2(\varepsilon^{-1}) / \log \log(\varepsilon^{-1})}$. In [12], it is conjectured that both δ and κ can in fact be taken to be a polynomial of ε in Theorem 2, which would imply the Erdős–Hajnal conjecture itself (see [9, 12, 13] for current progress on this topic).

Recently, Chudnovsky, Scott, Seymour, and Spirkl [5] provided a qualitative refinement of Theorem 1, which says that the vertex set of every H -free graph can even be partitioned into a bounded number of ε -restricted subsets. Formally, for $\varepsilon, N > 0$, a graph G is (N, ε) -restricted if there is a partition of $V(G)$ into at most N subsets that are ε -restricted in G ; thus G is (N, ε) -restricted if and only if \overline{G} is.

Theorem 3 ([5]). *For every $\varepsilon > 0$ and every graph H , there exists $N = N(H, \varepsilon) > 0$ such that every H -free graph is (N, ε) -restricted.*

The *edge density* of a graph G equals $|E(G)| / \binom{|G|}{2}$ if $|G| \geq 2$ and equals 0 if $|G| \leq 1$. For $\varepsilon > 0$, a subset $S \subseteq V(G)$ is *weakly ε -restricted* in G if one of $G[S], \overline{G}[S]$ has edge density at most ε . Thus if S is $\frac{1}{2}\varepsilon$ -restricted in G then it is weakly ε -restricted; and if S is weakly $\frac{1}{4}\varepsilon$ -restricted in G then it has an ε -restricted subset of size $\lceil \frac{1}{2}|S| \rceil$. Hence the strength of Theorems 1 and 2 remain unaffected if “ ε -restricted” is replaced by “weakly ε -restricted.” As discussed in [5], however, Theorem 3 becomes significantly weaker if “ (N, ε) -restricted” is replaced by “*weakly (N, ε) -restricted*,” which means $V(G)$ has a partition into at most N subsets that are weakly ε -restricted in G . Indeed, repeated applications of Theorem 2 yield the following result proved in [14].

Theorem 4 ([14]). *For every $\varepsilon > 0$ and every graph H , there exist $\kappa = \kappa(H, \varepsilon) > 0$ and $N = N(H, \varepsilon) > 0$ such that every graph G with $\text{ind}_H(G) \leq \kappa|G|^{|H|}$ is weakly (N, ε) -restricted.*

¹The very last sentence of [7] was actually the first time Erdős and Hajnal formally stated their well-known conjecture.

(As shown in [14, 10], with more care one can even take the corresponding weakly ε -restricted sets to have size differences at most 1 in this result.) It thus would be natural (and quite tempting) to conjecture the following, which would have unified Theorems 2 and 3 and strengthened Theorem 4 considerably.

Conjecture 5 (false). For every $\varepsilon > 0$ and every graph H , there exist $N = N(H, \varepsilon) > 0$ and $\kappa = \kappa(H, \varepsilon) > 0$ such that every graph G with $\text{ind}_H(G) \leq \kappa|G|^{|H|}$ is (N, ε) -restricted.

Unfortunately, the following proposition² refutes this conjecture in a strong sense.

Proposition 6. *Let $N \geq 1$. Then for all integers m, n with $n \geq m \geq 20N^2$, every $\varepsilon \in (0, \frac{1}{18})$, and every graph H with $h := |H| \geq 2$, there is a graph on n vertices which has at most hmn^{h-1} copies of H and is not (N, ε) -restricted. In particular, for every $\kappa > 0$ and every integer $n \geq 20\kappa^{-1}hN^2$, there is a graph on n vertices which has at most κn^h copies of H and is not (N, ε) -restricted.*

Proof. In what follows, $\Delta(G)$ denotes the maximum degree of a graph G . By taking complements if necessary, we may assume H is connected, and so H has at least one edge as $h \geq 2$.

Let F be a random graph on $m \geq 20N^2$ vertices where each edge appears independently with probability $\frac{1}{2}$. For every $T \subseteq V(F)$ with $|T| \geq \frac{1}{N}m$, since $6\varepsilon < \frac{1}{3}$, Hoeffding's inequality [11] implies that T is weakly 6ε -restricted in F with probability at most $2 \exp(-\frac{1}{72} \binom{|T|}{2}) \leq 2 \exp(-\frac{1}{300N^2}m^2)$; and so, since $2^m \cdot 2 \exp(-\frac{1}{300N^2}m^2) < 1$ (as $m \geq 20N^2$), there is a choice of F with no weakly 6ε -restricted set of size at least $\frac{1}{N}m$. Consequently F has no 3ε -restricted subset of size at least $\frac{1}{N}m$.

Now, fix such an F ; and for every $n \geq m$, let G be a graph obtained from F by adding $n - m$ isolated vertices and making each of them adjacent to every vertex in $V(F)$. Since H has at least one edge, every copy of H in G has at least one image vertex in $V(F)$; thus

$$\text{ind}_H(G) \leq \sum_{i=1}^h \binom{h}{i} m^i (n-m)^{h-i} = n^h - (n-m)^h = m \sum_{i=1}^h n^{i-1} (n-m)^{h-i} \leq hmn^{h-1}.$$

It thus remains to show that G is not (N, ε) -restricted. Suppose not; and let $A_1 \cup \dots \cup A_k$ be a partition of $V(G)$ for some $k \leq N$ such that A_i is ε -restricted for all $i \in \{1, 2, \dots, k\}$. Then $\bigcup_{i=1}^k (A_i \cap V(F))$ is a partition of $V(F)$, and so we may assume $T := A_1 \cap V(F)$ has size at least $\frac{1}{N}m$. Thus T is not 3ε -restricted in F ; hence $S := A_1 \setminus V(F)$ is nonempty. It follows that

$$\begin{aligned} \Delta(G[A_1]) &= |S| + \Delta(F[T]) > \varepsilon|S| + 3\varepsilon|T| > \varepsilon(|S| + |T|) = \varepsilon|A_1|, \\ \Delta(\overline{G}[A_1]) &= \max(|S| - 1, \Delta(\overline{F}[T])) \geq \max(|S| - 1, 3\varepsilon|T|) > \varepsilon(|S| + |T|) = \varepsilon|A_1|. \end{aligned}$$

Therefore A_1 is not ε -restricted in G , a contradiction. This proves Proposition 6. ■

²We remark that Alex Scott (personal communication) independently discovered similar counterexamples.

The graphs constructed in Proposition 6 suggest that an “exceptional” set of vertices should necessarily be removed in order for the remaining vertices to admit a partition into a bounded number of ε -restricted pieces. Our main theorem shows that this is also sufficient.

Theorem 7. *For every $\varepsilon > 0$ and every graph H , there exist $\kappa = \kappa_7(H, \varepsilon) > 0$ and $N = N_7(H, \varepsilon) > 0$ such that for every $d \geq 0$ and every graph G with $\text{ind}_H(G) \leq \kappa d^{|H|}$, there is a set $S \subseteq V(G)$ with $|S| \leq d$ such that $G \setminus S$ is (N, ε) -restricted; equivalently, G can be made (N, ε) -restricted by removing at most $C \cdot \text{ind}_H(G)^{1/|H|}$ vertices where $C = \kappa^{-1/|H|}$.*

We would like to make three remarks. First, Theorem 3 is a special case of Theorem 7 with $d = 0$; and taking $d = \varepsilon|G|$ in Theorem 7 yields Theorem 2. Thus Theorem 7 can be viewed as a remedy for the false Conjecture 5; and the counterexamples in Proposition 6 (with suitable choices of m, n depending on d and more isolated vertices added) show that Theorem 7 is optimal up to κ and N for *any* given value of d .

Second, Theorem 7 is related to the induced removal lemma [1, 17] which also implies Theorem 2. Here, we are dealing with the property of being (N, ε) -restricted which is weaker than H -freeness (by Theorem 3) and not closed under the induced subgraph relation. But the trade-off is worth considering: removing only a handful of vertices instead of adding/deleting edges; and working well for *all* graphs, including those with subquadratic number of edges and only few copies of H .

Third, our proof of Theorem 7 generalizes the proof of Theorem 3 given in [5], demonstrating that the argument there can be extended to graphs with a bounded number of copies of H (at the cost of removing a small number of vertices). The resulting bounds on $\kappa_7^{-1}(H, \varepsilon)$ and $N_7(H, \varepsilon)$, as a result, are better than what the regularity lemma could provide (but still huge functions, namely towers of twos of height depending solely on $|H|$ with ε^{-1} on top). It would be interesting to prove Theorem 7 with bounds on κ^{-1} and N similar to the bounds obtained in [10] or even in [3].

In what follows, for an integer $k \geq 0$, let $[k]$ denote $\{1, 2, \dots, k\}$ if $k \geq 1$ and \emptyset if $k = 0$. The vertex set of H will always be $\{v_1, \dots, v_h\}$ for some $h \geq 1$; and we drop the subscript H from the notation ind_H .

2 A slight digression

This section provides a short and self-contained proof of Theorem 2 without using the regularity lemma, which will be used frequently in the proof of Theorem 7. The presentation here mostly follows [10].

For $\varepsilon > 0$, a graph G , and disjoint subsets A, B of $V(G)$, B is ε -sparse to A in G if every vertex in B is adjacent to fewer than $\varepsilon|A|$ vertices of A in G , and ε -dense to A in G if it is ε -sparse to A in \overline{G} . Say that B is ε -tight to A if it is either ε -sparse or ε -dense to A . The following lemma implicitly appears in [10, Lemma 4.1], which in turn generalizes an old result of Erdős and Hajnal [8, Theorem 1.5]. This result will also be useful later on.

Lemma 8. Let H be a graph, and let $\varepsilon_1, \dots, \varepsilon_{h-1}, \delta_1, \dots, \delta_{h-1} \in (0, 1)$. Let G be a graph, and let D_1, \dots, D_h be disjoint nonempty subsets of $V(G)$ such that for all indices i, j with $1 \leq i < j \leq h$, there do not exist $A \subseteq D_i$ and $B \subseteq D_j$ with $|A| \geq \prod_{t=j}^{h-1} \varepsilon_t \cdot |D_i|$ and $|B| \geq \frac{\delta_{j-1}}{j-1} \prod_{t=j}^{h-1} \varepsilon_t \cdot |D_j|$ satisfying B is ε_j -sparse to A if $v_i v_j \in E(H)$ and ε_j -dense to A if $v_i v_j \notin E(H)$. Then there are at least $\prod_{t=1}^{h-1} (1 - \delta_t) \varepsilon_t^t \cdot \prod_{i=1}^h |D_i|$ copies φ of H in G with $\varphi(v_i) \in D_i$ for all $i \in [h]$.

Proof. Induction on $h \geq 1$. We may assume that $h \geq 2$. For $i \in [h-1]$, let P_i be the set of vertices in D_h with fewer than $\varepsilon_{h-1} |D_i|$ neighbors in D_i if $v_i v_h \in E(H)$ and the set of vertices in D_h with fewer than $\varepsilon_{h-1} |D_i|$ nonneighbors in D_i if $v_i v_h \notin E(H)$. By the hypothesis, $|P_i| \leq \frac{\delta_{h-1}}{h-1} |D_h|$ for all $i \in [h-1]$. Let $D'_h := D_h \setminus (\bigcup_{i \in [h-1]} P_i)$; then $|D'_h| \geq (1 - \delta_{h-1}) |D_h|$.

Now, for each $u \in D'_h$ and $i \in [h-1]$, let D_i^u be the set neighbors of u in D_i if $v_i v_h \in E(H)$ and the set of nonneighbors of u in D_i if $v_i v_h \notin E(H)$; then $|D_i^u| \geq \varepsilon_{h-1} |D_i|$ for all $i \in [h-1]$. Thus for all indices i, j with $1 \leq i < j \leq h-1$, there do not exist $A \subseteq D_i^u$ and $B \subseteq D_j^u$ with $|A| \geq \prod_{t=j}^{h-2} \varepsilon_t \cdot |D_i^u|$ and $|B| \geq \frac{\delta_{j-1}}{j-1} \prod_{t=j}^{h-2} \varepsilon_t \cdot |D_j^u|$ such that B is ε_j -sparse to A if $v_i v_j \in E(H)$ and ε_j -dense to A if $v_i v_j \notin E(H)$. So by induction, there are at least $\prod_{t=1}^{h-2} (1 - \delta_t) \varepsilon_t^t \cdot \prod_{i=1}^{h-1} |D_i^u|$ copies φ_u of $H \setminus v_h$ in $G \setminus D_h$ with $\varphi_u(v_i) \in D_i^u$ for all $i \in [h-1]$. Summing up over all $u \in D'_h$, we deduce that there are at least

$$\begin{aligned} \sum_{u \in D'_h} \left(\prod_{t=1}^{h-2} (1 - \delta_t) \varepsilon_t^t \cdot \prod_{i=1}^{h-1} |D_i^u| \right) &\geq |D'_h| \left(\prod_{t=1}^{h-2} (1 - \delta_t) \varepsilon_t^t \right) \left(\varepsilon_{h-1}^{h-1} \prod_{i=1}^{h-1} |D_i| \right) \\ &\geq \prod_{t=1}^{h-1} (1 - \delta_t) \varepsilon_t^t \cdot \prod_{i=1}^h |D_i| \end{aligned}$$

copies φ of H in G such that $\varphi(v_i) \in D_i$ for all $i \in [h]$. This proves Lemma 8. ■

Corollary 9. Let $\varepsilon \in (0, 1)$, let H be a graph, and let $\kappa = \kappa_9(H, \varepsilon) := (4h)^{-h} \varepsilon^{\binom{h}{2}}$. Then every G with $\text{ind}(G) \leq \kappa |G|^h$ contains disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq (2h)^{-2} \varepsilon^{h-1} |G|$ such that B is ε -tight to A .

Proof. We may assume $|G| \geq h$. Let D_1, \dots, D_h be disjoint subsets of $V(G)$ each of size $\lfloor \frac{1}{h} |G| \rfloor$; then $|D_t| \geq \frac{1}{2h} |G|$ for all $t \in [h]$. It suffices to apply Lemma 8 with $\varepsilon_t = \varepsilon$ and $\delta_t = \frac{1}{2}$ for all $t \in [h]$. ■

For $\varepsilon_1, \varepsilon_2, \kappa > 0$ and a graph H , let $\beta(H, \kappa, \varepsilon_1, \varepsilon_2)$ be the largest constant β with $0 < \beta \leq 1$ such that every graph G with $\text{ind}(G) \leq \kappa |G|^h$ has an induced subgraph with at least $\beta |G|$ vertices and edge density at most ε_1 or at least $1 - \varepsilon_2$; then $\beta(H, \kappa, \varepsilon_1, \varepsilon_2)$ is decreasing in κ and $\beta(H, \kappa, \varepsilon_1, \varepsilon_2) = 1$ for all $\kappa > 0$ whenever $\varepsilon_1 + \varepsilon_2 \geq 1$ (and so whenever $\varepsilon_1 \varepsilon_2 \geq 1$). We need the following lemma.

Lemma 10. Let $\varepsilon_1, \varepsilon_2 > 0$, let H be a graph, and let $\eta := \eta_{10}(H, \varepsilon_1, \varepsilon_2) := \frac{1}{2} (2h)^{-2} (\frac{1}{4} \varepsilon)^{h-1}$ where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then for every κ with $0 < \kappa \leq \kappa_9(H, \frac{1}{4} \varepsilon)$, we have

$$\beta(H, \kappa, \varepsilon_1, \varepsilon_2) \geq \eta \cdot \min(\beta(H, \eta^{-h} \kappa, \frac{3}{2} \varepsilon_1, \varepsilon_2), \beta(H, \eta^{-h} \kappa, \varepsilon_1, \frac{3}{2} \varepsilon_2)).$$

Proof. Let $\beta_1 := \beta(H, \eta^{-h}\kappa, \frac{3}{2}\varepsilon_1, \varepsilon_2)$, $\beta_2 := \beta(H, \eta^{-h}\kappa, \varepsilon_1, \frac{3}{2}\varepsilon_2)$, and $\beta_0 := \eta \cdot \min(\beta_1, \beta_2)$. Let G be a graph with $\text{ind}(G) \leq \kappa|G|^h$; we need to show there is a set $S \subseteq V(G)$ with $|S| \geq \beta_0|G|$ such that $G[S]$ has edge density at most ε_1 or at least $1 - \varepsilon_2$. By Corollary 9, G has disjoint subsets $A, B \subseteq V(G)$ with $|A|, |B| \geq 2\eta|G|$ such that B is $\frac{1}{4}\varepsilon$ -tight to A ; and we may assume B is $\frac{1}{4}\varepsilon$ -sparse to A .

Because $\text{ind}(G[B]) \leq \kappa|G|^h \leq \eta^{-h}\kappa|B|^h$, by the definition of β and by averaging, there exists $B_1 \subseteq B$ with $|B_1| = \lceil \beta_1\eta|G| \rceil \geq \beta_0|G|$ such that $G[B_1]$ has edge density at most $\frac{3}{2}\varepsilon_1$ or at least $1 - \varepsilon_2$. If the latter holds then we are done, so we may assume the former holds.

Let A_0 be the set of vertices in A each with at most $\frac{1}{2}\varepsilon|B_1|$ neighbors in B_1 . Since G has fewer than $\frac{1}{4}\varepsilon|A||B_1|$ edges between A and B_1 , we have $|A_0| \geq \frac{1}{2}|A| \geq \eta|G|$. Thus $\text{ind}(G[A_0]) \leq \eta^{-h}\kappa|A_0|^h$, and so by the definition of β and by averaging, there exists $A_1 \subseteq A_0$ with $|A_1| = \lceil \beta_1\eta|G| \rceil \geq \beta_0|G|$ such that $G[A_1]$ has edge density at most $\frac{3}{2}\varepsilon_1$ or at least $1 - \varepsilon_2$. Again, we may assume the former holds.

Now, let $S := A_1 \cup B_1$; then $|S| = 2|A_1| = 2|B_1| \geq 2\beta_0|G|$. Since $G[A_1], G[B_1]$ each have edge density at most $\frac{3}{2}\varepsilon_1$ and G has at most $\frac{1}{2}\varepsilon|A_1||B_1|$ edges between A_1 and B_1 , we deduce that

$$\begin{aligned} |E(G[S])| &\leq |E(G[A_1])| + |E(G[B_1])| + \frac{1}{2}\varepsilon|A_1||B_1| \\ &\leq \frac{3}{2}\varepsilon_1 \binom{|A_1|}{2} + \frac{3}{2}\varepsilon_1 \binom{|B_1|}{2} + \frac{1}{2}\varepsilon|A_1||B_1| \\ &= 3\varepsilon_1 \binom{|A_1|}{2} + \frac{1}{2}\varepsilon|A_1|^2 \\ &\leq \varepsilon_1 \binom{2|A_1|}{2} \\ &= \varepsilon_1 \binom{|S|}{2}. \end{aligned}$$

Therefore S has the desired property. This proves Lemma 10. ■

We are now give a proof of Theorem 2 in the following equivalent form, which leads to the dependence of $\delta_2(H, \varepsilon)$ and $\kappa_2(H, \varepsilon)$ on ε and h as mentioned in the introduction.

Theorem 11. *For every $\varepsilon > 0$ and every graph H , there exist $\delta = \delta(H, \varepsilon) > 0$ and $\kappa = \kappa(H, \varepsilon) > 0$ such that every graph G with $\text{ind}(G) \leq \kappa|G|^h$ contains a weakly ε -restricted set of size at least $\delta|G|$.*

Proof. Let $s := \lceil \log_{\frac{3}{2}}(\varepsilon^{-2}) \rceil$, $\eta := \eta_{10}(H, \varepsilon, \varepsilon)$, $\delta := \eta^s$, and $\kappa := \eta^{sh} \cdot \kappa_9(H, \frac{1}{4}\varepsilon)$. Note that $\eta_{10}(H, \cdot, \cdot)$ is decreasing in each of the last two components. Thus, since $\beta(H, \cdot, \cdot, \cdot)$ is decreasing in the second component and equals 1 whenever the last two components have product at least 1, applying Lemma 10 for s times yields $\beta(H, \kappa, \varepsilon, \varepsilon) \geq \eta^s = \delta$. This proves Theorem 11. ■

3 Key lemma

This section introduces and proves our key lemma, the following.

Lemma 12. *For all $\varepsilon, \eta, \theta \in (0, \frac{1}{2})$ and every graph H , there are $\kappa = \kappa_{12}(H, \varepsilon, \eta, \theta) > 0$ and $N = N_{12}(H, \varepsilon, \eta, \theta) > 0$ with the following property. For every $d \geq 0$ and every graph G with $\text{ind}(G) \leq \kappa d^h$, there is a set $S \subseteq V(G)$ with $|S| \leq d$ such that $V(G) \setminus S$ can be partitioned into nonempty sets*

$$A_1, \dots, A_m; B_1, \dots, B_m; C_1, \dots, C_n$$

where $m \leq \binom{h}{2}$ and $n \leq N$, such that

- $A_1, \dots, A_m, C_1, \dots, C_n$ are ε -restricted in G ; and
- for every $i \in [m]$, $|B_i| \leq \eta|A_i|$ and B_i is θ -tight to A_i .

This contains [5, Theorem 1.5] as a special case with $d = 0$, and already gives Theorem 2 with $\varepsilon = \eta = \theta$ and $d = \varepsilon|G|$. We shall employ the same approach as in [5, Section 2], and recommend reading the detailed sketch there first. Here we explain the modifications.

We recall some definitions. For $c, \varepsilon > 0$ and a graph G , a pair (A, B) of disjoint nonempty subsets of $V(G)$ is (c, ε) -full in G if for every $A_1 \subseteq A$ and $B_1 \subseteq B$ with $|A_1| \geq c|A|$ and $|B_1| \geq c|B|$, G has at least $\varepsilon|A_1||B_1|$ edges between A_1, B_1 ; and (A, B) is (c, ε) -empty in G if it is (c, ε) -full in \overline{G} . Thus for every $c' > c$ and every $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq c'|A|$ and $|B'| \geq c'|B|$, (A', B') is $(c/c', \varepsilon)$ -full if (A, B) is (c, ε) -full and is $(c/c', \varepsilon)$ -empty if (A, B) is (c, ε) -empty. A collection $\{D_1, \dots, D_h\}$ of disjoint nonempty subsets of $V(G)$ is a (c, ε) -blowup of H if for all distinct $i, j \in [h]$, (D_i, D_j) is (c, ε) -full if $v_i v_j \in E(H)$ and is (c, ε) -empty if $v_i v_j \notin E(H)$.

In proving Lemma 12, we shall be concerned with partitions of $V(G)$ into “rows” of subsets and pairs of subsets as follows:

- *first row:* pairs $(A_1, B_1), \dots, (A_m, B_m)$ for some $m \geq 0$ such that for all $i \in [m]$, A_i is ε -restricted, B_i is very tight to A_i and has size smaller than a tiny fraction of A_i (B_i might be empty);
- *second row:* ε -restricted nonempty sets C_1, \dots, C_n for some $n \geq 0$;
- *third row:* ε' -restricted nonempty sets D_1, \dots, D_t for some t with $0 \leq t \leq h$, such that $\{D_1, \dots, D_t\}$ is a (c, ξ) -blowup of $H[\{v_1, \dots, v_t\}]$ for some appropriately chosen $c, \varepsilon', \xi > 0$; and
- *fourth row:* the set L of “leftover” vertices such that whenever $t > 0$, L has size smaller than a tiny fraction of each D_i .

Such a partition certainly exists, with $m = n = t = 0$ and $L = V(G)$. Starting from $t = 0$ with this partition, we shall attempt to increase t one by one for at most h steps. Let S be the set of vertices in L with the “correct adjacencies” to the collection $\{D_1, \dots, D_t\}$, that is, those having at least a small fraction of neighbors in D_i if $v_{t+1}v_i \in E(H)$ and at least a small fraction of nonneighbors in D_i if $v_{t+1}v_i \notin E(H)$. Then $L \setminus S$ can be partitioned into (possibly empty) sets L_1, \dots, L_t such that L_i is (very) tight to D_i for every $i \in [t]$. As the notation suggests, if $|S| \leq d$ then we stop the iteration and rearrange the sets $A_1, \dots, A_m, B_1, \dots, B_m, C_1, \dots, C_n, D_1, \dots, D_t, L_1, \dots, L_t$ to form a partition of $V(G) \setminus S$ with the desired property (this is not hard, and the bounds on m and n will come up later).

So let us assume $|S| > d$. We can then apply Theorem 2 to find an ε' -restricted subset S_0 of S . Keeping in mind that $\{D_1, \dots, D_t, S_0\}$ now form a “partial” blowup of $H[\{v_1, \dots, v_t, v_{t+1}\}]$, we iteratively construct a nested sequence $S_0 \supseteq S_1 \supseteq \dots \supseteq S_t$ and subsets $P_1 \subseteq D_1, \dots, P_t \subseteq D_t$ such that each pair (S_i, P_i) is reasonably full (if $v_{t+1}v_i \in E(H)$) or reasonably empty (if $v_{t+1}v_i \notin E(H)$); then the collection $\{P_1, \dots, P_t, S_t\}$ will be a sufficiently good blowup of $H[\{v_1, \dots, v_t, v_{t+1}\}]$ while P_1, \dots, P_t, S_t are still ε' -restricted (for suitable c, ε', ξ). To execute this process, we need the following useful theorem of Yuejian, Rödl, and Ruciński [15, Theorem 1.3] which allows us to extract decent fullness/emptiness from moderate denseness/sparseness. (We remark that [15, Theorem 1.3] is stated only for balanced bipartite graphs; but the proof there works equally well for unbalanced ones.)

Lemma 13 ([15]). *Let $c \in (0, 1)$ and $\varepsilon \in (0, \frac{1}{4})$. Then, for $\gamma = \gamma_{13}(c, \varepsilon) := \frac{1}{2}(2\varepsilon)^{12/c} \in (0, \frac{1}{3})$, the following holds. Let G be a graph with $A, B \subseteq V(G)$ disjoint and nonempty such that G has at least $2\varepsilon|A||B|$ edges between A and B . Then there exist $A' \subseteq A$ and $B \subseteq B'$ with $|A'| \geq \gamma|A|$ and $|B'| \geq \gamma|B|$ such that (A', B') is (c, ε) -full.*

Observe that L is nonempty since S is, which implies each D_i is quite large, and so we can take each P_i to have size at least a (small) fraction of D_i yet at most half of D_i simultaneously. Then each L_i is still quite tight to and tiny compared to $D_i \setminus P_i$; and we can move each pair $(D_i \setminus P_i, L_i)$ to the first row.

Now, we want to use Theorem 2 to pull out as many ε -restricted sets as possible from $S \setminus S_t$ (assuming this is nonempty) so that the resulting new “leftover” set L' still has size smaller than a tiny fraction of S_t and of each P_i ; then we can move those new restricted sets to the third row. A potential issue here is that Theorem 2 may not be applicable if $S \setminus S_t$ is not large enough while most of the copies of H in G are “concentrated” on $G[S \setminus S_t]$. This can be avoided, conveniently, by making sure that $|S_0|$ is not too large compared to $|S|$ right in the first place (if $|S| \geq 2$), which will be done by the following simple corollary of Theorem 2 itself (we believe this is well-known, but still include a proof for completeness).

Corollary 14. *For every $\varepsilon > 0$ and every graph H , there exist $\delta = \delta_{14}(H, \varepsilon) \in (0, \frac{1}{4})$ and $\kappa = \kappa_{14}(H, \varepsilon) > 0$ such that for every graph G with $\text{ind}(G) \leq \kappa|G|^h$, G has an ε -restricted set T with $|T| = \lceil \delta|G| \rceil$; in particular $|T| = 1$ if $|G| = 1$ and $|G \setminus T| \geq \frac{1}{2}|G|$ if $|G| \geq 2$.*

Proof. Let $\delta := \frac{1}{4} \cdot \delta_2(H, \frac{1}{8}\varepsilon)$ and $\kappa := \kappa_2(H, \frac{1}{8}\varepsilon)$. By Theorem 2, G has an $\frac{1}{8}\varepsilon$ -restricted set U with $|U| \geq 2\delta|G|$; in particular U is weakly $\frac{1}{4}\varepsilon$ -restricted. By averaging, there is a weakly $\frac{1}{4}\varepsilon$ -restricted subset U' of U such that $|U'| = \lceil 2\delta|G| \rceil$, and so there is $T \subseteq U'$ with $|T| = \lceil \frac{1}{2}|U'| \rceil = \lceil \frac{1}{2}\lceil 2\delta|G| \rceil \rceil = \lceil \delta|G| \rceil$ such that T is ε -restricted in G . In particular, if $|G| \leq 4$ then $|T| = 1$; and if $|G| > 4$ then $|G \setminus T| > |G| - 1 - \delta|G| \geq \frac{3}{4}|G| - \frac{1}{4}|G| = \frac{1}{2}|G|$. This proves Corollary 14. \blacksquare

For $\delta, \eta \in (0, 1)$, let $\phi(\delta, \eta)$ be the least integer $p \geq 1$ with $(1 - \delta)^p \leq \eta$; then $\phi(\delta, \eta) \leq \delta^{-1} \log \eta$. The next corollary of Theorem 2 formalizes the process of repeatedly pulling out ε -restricted sets from $S \setminus S_t$.

Corollary 15. *For every $\varepsilon, \eta \in (0, 1)$, for every graph H , and for $\delta := \delta_2(H, \varepsilon) > 0$, there exists $\kappa = \kappa_{15}(H, \varepsilon, \eta) > 0$ such that for every graph G with $\text{ind}(G) \leq \kappa|G|^h$, there is $T \subseteq V(G)$ with $|T| \leq \eta|G|$ such that $G \setminus T$ is $(\phi(\delta, \eta), \varepsilon)$ -restricted.*

Proof. Let $\kappa := \eta^h \cdot \kappa_2(H, \varepsilon)$. We may assume $|G| \geq 1$. Let $U_0 := V(G)$; and for $i \geq 0$, as long as U_i is defined and $|U_i| > \eta|G|$, let $U_{i+1} \subseteq U_i$ such that $U_i \setminus U_{i+1}$ is ε -restricted and $|U_i \setminus U_{i+1}| \geq \delta|U_i|$, which is possible by Theorem 2 since

$$\text{ind}(G[U_i]) \leq \kappa|G|^h = \kappa_2(H, \varepsilon) \cdot (\eta|G|)^h < \kappa_2(H, \varepsilon) \cdot |U_i|^h.$$

This produces a chain of sets $V(G) = U_0 \supseteq U_1 \supseteq \dots \supseteq U_n$ for some $p \geq 1$ such that $|U_{i+1}| \leq (1 - \delta)|U_i| \leq (1 - \delta)^{i+1}|G|$ and $|U_i| > \eta|G|$ for all $i \in \{0, 1, \dots, p-1\}$. In particular $\eta|G| < |U_{n-1}| \leq (1 - \delta)^{n-1}|G|$; thus $p - 1 < \phi(\delta, \eta)$ and so $p \leq \phi(\delta, \eta)$. Let $T := U_p$; then $\bigcup_{i=1}^p (U_i \setminus U_{i-1})$ is a partition of $V(G) \setminus T$ into p subsets which are ε -restricted in G . This proves Corollary 15. \blacksquare

Now assume we have reached $t = h$ and obtained a decent blowup $\{D_1, \dots, D_h\}$ of H . Observe that to be able to reach $t = h$ means the “exceptional” set S in each step always had size more than d ; so it is not hard to see that each $|D_i|$ is still more than a (tiny) fraction of d . It thus suffices to apply the following, which is a direct corollary of Lemma 8 and is an analogue of the induced counting lemma [2, Lemma 3.2].

Corollary 16. *Let $\varepsilon \in (0, \frac{1}{2})$, let H be a graph, and let G be a graph with an $(\varepsilon^h, \varepsilon)$ -blowup $\{D_1, \dots, D_h\}$ of H . Then there are at least $(1 - \varepsilon)^{h-1} \varepsilon^{\binom{h}{2}} |D_1| \cdots |D_h|$ copies φ of H in G with $\varphi(v_i) \in D_i$ for all $i \in [h]$.*

Proof. This follows from Lemma 8 with $\varepsilon_t := \varepsilon$ and $\delta_t := t \cdot \varepsilon^t$ for all $t \in [h - 1]$; note that $\delta_t \leq t2^{-t+1}\varepsilon \leq \varepsilon$ since $\varepsilon \in (0, \frac{1}{2})$. \blacksquare

We are now ready to prove Lemma 12.

Proof of Lemma 12. Let $\xi := \frac{1}{4}\theta$ and $\varepsilon_h := \min(\varepsilon, \xi^h)$. Let $\Gamma_{t,t} = \lambda_{t,t} := 1$; and for $t = h - 1, h - 2, \dots, 0$ in turn, do the following:

- for $i = t - 1, t - 2, \dots, 0$ in turn, let $\Gamma_{t,i} := \lambda_{t,i+1}\Gamma_{t,i+1}$ and $\lambda_{t,i} := \gamma_{13}(\frac{1}{3}\varepsilon_{t+1}\Gamma_{t,i+1}, \xi)$; and

- let $\varepsilon_t := \varepsilon_{t+1}\lambda_{t,0}$.

Now, define

$$\begin{aligned} \varepsilon' &:= \min_{t \in \{0,1,\dots,h-1\}} \varepsilon_{t+1}\Gamma_{t,0}, & \delta' &:= \delta_{14}(H, \varepsilon'), \\ \eta' &:= \frac{1}{2}\eta\delta' \cdot \min_{t \in \{0,1,\dots,h-1\}} \Gamma_{t,0}, & N &:= \binom{h}{2} + (h-1) \cdot \phi(\delta', \eta'). \end{aligned}$$

Also, for $i = 1, 2, \dots, h$ in turn, do the following:

- let $\Lambda_{i,i} := \delta'\Gamma_{i-1,0}$; and
- for $t = i, i+1, \dots, h-1$ in turn, let $\Lambda_{t+1,i} := \lambda_{t,i}\Lambda_{t,i}$.

Finally, put

$$\kappa := \min \left((1 - \xi)^{h-1} \xi^{\binom{h}{2}} \Lambda_{h,1} \cdots \Lambda_{h,h}, \kappa_{14}(H, \varepsilon'), 2^{-h} \cdot \kappa_{15}(H, \varepsilon, \eta') \right).$$

For integers $m, n, t \geq 0$ with $t \leq h$, an (m, n, t) -partition in G is a partition of $V(G)$ into (not necessarily nonempty) subsets

$$A_1, \dots, A_m; B_1, \dots, B_m; C_1, \dots, C_n; D_1, \dots, D_t; L$$

such that

- $m \leq \binom{t}{2}$ and $n \leq t \cdot \phi(\delta', \eta')$;
- $A_1, \dots, A_m, C_1, \dots, C_n$ are nonempty and ε -restricted;
- for every $i \in [m]$, $|B_i| \leq \eta|A_i|$ and B_i is θ -tight to A_i ;
- $\{D_1, \dots, D_t\}$ is an (ε_t, ξ) -blowup of $H[\{v_1, \dots, v_t\}]$; and
- if $t > 0$, then $|D_i| > \max(\Lambda_{t,i}d, 2\eta^{-1}|L|)$ and D_i is ε_t -restricted for every $i \in [t]$.

For the readers' convenience, let us write such a partition as follows

$$\begin{aligned} &(A_1, B_1), \dots, (A_m, B_m); \\ &C_1, \dots, C_n; \\ &D_1, \dots, D_t; \\ &L. \end{aligned}$$

Observe that $V(G)$ itself is a $(0, 0, 0)$ -partition in G . Thus, there is $t \in \{0, 1, \dots, h\}$ maximal such that there is an (m, n, t) -partition in G . If $t = h$, then $\{D_1, \dots, D_h\}$ would be a (ξ^{h-1}, ξ) -blowup of H ; so by Corollary 16, G would contain at least

$$(1 - \xi)^h \xi^{\binom{h}{2}} |D_1| \cdots |D_h| > (1 - \xi)^h \xi^{\binom{h}{2}} \Lambda_{h,1} \cdots \Lambda_{h,h} d^h \geq \kappa d^h \geq \text{ind}(G)$$

copies of H , a contradiction. Thus $t < h$.

Let S be the set of vertices u in L with the property that for every $i \in [t]$, u has at least $2\xi|D_i|$ neighbors in D_i if $v_{t+1}v_i \in E(H)$ and at least $2\xi|D_i|$ nonneighbors in D_i if $v_{t+1}v_i \notin E(H)$. Then there is a partition $L \setminus S = L_1 \cup \dots \cup L_t$ such that L_i is 2ξ -tight to D_i for all $i \in [t]$. We shall prove that S satisfies the lemma; and the following crucial claim is the key step.

Claim 17. $|S| \leq d$.

Proof. Suppose that $|S| > d$; then $|S| \geq 1$. Since $\delta' = \delta_{14}(H, \varepsilon')$ and

$$\text{ind}(G[S]) \leq \kappa d^h < \kappa_{14}(H, \varepsilon') \cdot |S|^h$$

by the definitions of δ' and κ , Corollary 14 yields an ε' -restricted subset S_0 of S with $|S_0| = \lceil \delta'|G| \rceil$; in particular $|S_0| = 1$ if $|S| = 1$ and $|S \setminus S_0| \geq \frac{1}{2}|S| > \frac{1}{2}d$ if $|S| \geq 2$. We shall define a chain of sets $S_0 \supseteq S_1 \supseteq \dots \supseteq S_t$ together with sets P_1, \dots, P_t of vertices such that for all $i \in [t]$,

- $|S_i| \geq \lambda_{t,i}|S_{i-1}|$;
- $P_i \subseteq D_i$ and $\lambda_{t,i}|D_i| \leq |P_i| \leq \frac{1}{2}|D_i|$; and
- (S_i, P_i) is $(\varepsilon_{t+1}\Gamma_{t,i}, \xi)$ -full if $v_i v_{t+1} \in E(H)$ and is $(\varepsilon_{t+1}\Gamma_{t,i}, \xi)$ -empty if $v_i v_{t+1} \notin E(H)$.

To this end, assume that for $i \in [t]$, S_{i-1} and P_{i-1} have been defined. Assume $v_i v_{t+1} \in E(H)$ without loss of generality; then by the choice of L , every vertex of $S_{i-1} \subseteq S$ has at least $2\xi|D_i|$ neighbors in D_i . Thus Lemma 13 and the definition of $\lambda_{t,i}$ yield $S_i \subseteq S_{i-1}$ and $D'_i \subseteq D_i$ with $|S_i| \geq \lambda_{t,i}|S_{i-1}|$ and $|D'_i| \geq \lambda_{t,i}|D_i|$ so that (S_i, D'_i) is $(\frac{1}{3}\varepsilon_{t+1}\Gamma_{t,i}, \xi)$ -full. Let $P_i \subseteq D'_i$ with $|P_i| = \min(|D'_i|, \lfloor \frac{1}{2}|D_i| \rfloor)$; then since $\lfloor \frac{1}{2}|D_i| \rfloor \geq \frac{1}{3}|D_i| \geq \lambda_{t,i}|D_i|$ (as $|D_i| > 2\eta^{-1}|S| > 1$), $\max(\lambda_{t,i}|D_i|, \frac{1}{3}|D'_i|) \leq |P_i| \leq \frac{1}{2}|D_i|$. In particular, (S_i, P_i) is $(\varepsilon_{t+1}\Gamma_{t,i}, \xi)$ -full. This defines S_i and P_i in the case $v_i v_{t+1} \in E(H)$; and similar arguments with $(\frac{1}{3}\varepsilon_{t+1}\Gamma_{t,i}, \xi)$ -full replaced by $(\frac{1}{3}\varepsilon_{t+1}\Gamma_{t,i}, \xi)$ -empty also define S_i and P_i in the case $v_i v_{t+1} \notin E(H)$.

From the above construction, we see that $|S_t| = 1$ if $|S| = 1$ and $|S \setminus S_t| \geq |S \setminus S_0| > \frac{1}{2}d$ if $|S| \geq 2$. Let $L' := \emptyset$ if the former holds; and if the latter holds, then since

$$\text{ind}(G[S \setminus S_t]) \leq \kappa d^h < \kappa_{15}(H, \varepsilon, \eta') \cdot |S \setminus S_t|^h$$

by the definition of κ , Corollary 15 yields $L' \subseteq S \setminus S_t$ with $|L'| \leq \eta'|S \setminus S_t|$ such that $S \setminus (S_t \cup L')$ is $(\phi(\delta', \eta'), \varepsilon)$ -restricted. Thus there is always a subset $L' \subseteq S \setminus S_t$ with $|L'| \leq \eta'|S \setminus S_t|$ such that $S \setminus (S_t \cup L')$ has a partition into nonempty ε -restricted subsets Q_1, \dots, Q_s for some $s \leq \phi(\delta', \eta')$.

Now, let $P_{t+1} := S_t$; we shall prove that the following partition of $V(G)$

$$\begin{aligned} &(A_1, B_1), \dots, (A_m, B_m), (D_1 \setminus P_1, L_1), \dots, (D_t \setminus P_t, L_t); \\ &C_1, \dots, C_n, Q_1, \dots, Q_s; \\ &P_1, \dots, P_t, P_{t+1}; \\ &L' \end{aligned}$$

is an $(m + t, n + s, t + 1)$ -partition in G , which contradicts the maximality of t . To this end, observe the following.

- Since $m \leq \binom{t}{2}$ and $n \leq t \cdot \phi(\delta', \eta')$, we have $m + t \leq \binom{t+1}{2}$ and $n + s \leq (t + 1) \cdot \phi(\delta', \eta')$.
- $A_1, \dots, A_m, C_1, \dots, C_n, Q_1, \dots, Q_s$ are nonempty and ε -restricted by definition; and for every $i \in [t]$, $|D_i \setminus P_i| \geq \frac{1}{2}|D_i| > 0$ from the construction of P_i , in particular $D_i \setminus P_i$ is ε -restricted since D_i is ε_t -restricted and $2\varepsilon_t \leq \varepsilon$.
- For every $i \in [m]$, $|B_i| \leq \eta|A_i|$ and B_i is θ -tight to A_i by definition; and for every $i \in [t]$, $|L_i| \leq |L| < \frac{1}{2}\eta|D_i| \leq \eta|D_i \setminus P_i|$ and L_i is θ -tight to $D_i \setminus P_i$ since it is 2ξ -tight to D_i while $\xi = \frac{1}{4}\theta$.
- $\{P_1, \dots, P_t\}$ is an (ε_{t+1}, ξ) -blowup of $H[\{v_1, \dots, v_t\}]$ since $|P_i| \geq \lambda_{t,i}|D_i| \geq \varepsilon_t \varepsilon_{t+1}^{-1}|D_i|$ for all $i \in [t]$ and $\{D_1, \dots, D_t\}$ is an (ε_t, ξ) -blowup of $H[\{v_1, \dots, v_t\}]$. Also, from the above construction, for all $i \in \{0, 1, \dots, t\}$, we have

$$|P_{t+1}| = |S_t| \geq (\lambda_{t,t} \lambda_{t,t-1} \cdots \lambda_{t,i+1})|S_i| = \Gamma_{t,i}|S_i|;$$

in particular (P_i, P_{t+1}) is (ε_{t+1}, ξ) -full if $v_i v_{t+1} \in E(H)$ and is (ε_{t+1}, ξ) -empty if $v_i v_{t+1} \notin E(H)$. Thus $\{P_1, \dots, P_t, P_{t+1}\}$ is an (ε_{t+1}, ξ) -blowup of $H[\{v_1, \dots, v_t, v_{t+1}\}]$.

- For every $i \in [t]$, since $\Lambda_{t+1,i} = \lambda_{t,i} \Lambda_{t,i}$, $|L'| \leq \eta'|S \setminus S_t| < \eta'|S| \leq \eta'|L|$ as $|S_t| > 0$, and $\eta' \leq \Gamma_{t,0} < \lambda_{t,i}$ by the definition of η' , we have

$$\begin{aligned} |P_i| \geq \lambda_{t,i}|D_i| &> \lambda_{t,i} \max(\Lambda_{t,i}d, 2\eta^{-1}|L|) \\ &\geq \max(\Lambda_{t+1,i}d, 2\eta^{-1}\lambda_{t,i}(\eta')^{-1}|L'|) \geq \max(\Lambda_{t+1,i}d, 2\eta^{-1}|L'|); \end{aligned}$$

and since $\Lambda_{t+1,t+1} = \delta'\Gamma_{t,0}$ and $\eta' \leq \frac{1}{2}\eta\delta'\Gamma_{t,0}$ by definition, we deduce that

$$\begin{aligned} |P_{t+1}| = |S_t| \geq \Gamma_{t,0}|S_0| &\geq \Gamma_{t,0}\delta'|S| > \max(\Lambda_{t+1,t+1}d, \Gamma_{t,0}\delta'(\eta')^{-1}|L'|) \\ &\geq \max(\Lambda_{t+1,t+1}d, 2\eta^{-1}|L'|). \end{aligned}$$

Also, for every $i \in [t]$, P_i is ε_{t+1} -restricted since D_i is ε_t -restricted; and $P_{t+1} = S_t$ is ε_{t+1} -restricted since S_0 is ε' -restricted and $\varepsilon' \leq \varepsilon_{t+1}\Gamma_{t,0}$ by the definition of ε' .

This proves Claim 17. □

Now, recall that $V(G) \setminus S$ is partitioned into (possibly empty) subsets

$$A_1, \dots, A_m; B_1, \dots, B_m; C_1, \dots, C_n; D_1, \dots, D_t; L_1, \dots, L_t$$

such that

- $m \leq \binom{t}{2}$ and $n \leq t \cdot \phi(\delta', \eta')$;
- $A_1, \dots, A_m, C_1, \dots, C_n$ are nonempty and ε -restricted;

- for every $i \in [m]$, $|B_i| \leq \eta|A_i|$ and B_i is θ -tight to A_i ; and
- if $t > 0$, then for every $i \in [t]$, $|D_i| > 2\eta^{-1}|L_i| \geq \eta^{-1}|L_i|$, D_i is ε -restricted (since it is ε_t -restricted), and L_i is θ -tight to D_i .

By renumbering the above sets if necessary, we may assume there exist $q \in \{0, 1, \dots, m\}$ and $r \in \{0, 1, \dots, t\}$ such that $B_i \neq \emptyset$ for all $i \in [q]$ and $B_i = \emptyset$ for all $i \in [m] \setminus [q]$, and $L_i \neq \emptyset$ for all $i \in [r]$ and $L_i = \emptyset$ for all $i \in [t] \setminus [r]$. Then the following partition of $V(G)$

$$A_1, \dots, A_q, D_1, \dots, D_r; B_1, \dots, B_q, L_1, \dots, L_r; A_{q+1}, \dots, A_m, D_{r+1}, \dots, D_t, C_1, \dots, C_n$$

has the desired property, because $q + r \leq m + t \leq \binom{t}{2} + t = \binom{t+1}{2} \leq \binom{h}{2}$, and

$$(m - q) + (t - r) + n \leq \binom{t}{2} + t + t \cdot \phi(\delta', \eta') \leq \binom{h}{2} + (h - 1) \cdot \phi(\delta', \eta') = N.$$

This proves Lemma 12. ■

We remark that, according to [15, Theorem 1.4], the choice $\gamma_{13}(c, \varepsilon) = \frac{1}{2}(2\varepsilon)^{12/c}$ is optimal up to a constant factor in the exponent. This choice leads to bounds on $\kappa_{12}^{-1}(H, \varepsilon, \eta, \theta)$ and $N_{12}(H, \varepsilon, \eta, \theta)$ which are towers of twos of height depending on h with $(\varepsilon\eta\theta)^{-1}$ on top, which results in the tower-type dependence of $\kappa_7^{-1}(H, \varepsilon)$ and $N_7(H, \varepsilon)$ on ε^{-1} and h mentioned in the introduction.

4 Finishing the proof

With Lemma 12 in hand, it now suffices to make obvious changes to [5, Section 3] to complete the proof of Theorem 7. For $\varepsilon > 0$, an integer $k \geq 0$, and a graph G , a (k, ε) -*path-partition* of G is a sequence (W_0, W_1, \dots, W_k) of disjoint nonempty sets with union $V(G)$ such that for every $i \in \{0, 1, \dots, k - 1\}$,

- W_i is ε -restricted in G ;
- $|W_i| \geq 12|W_k|$; and
- $W_{i+1} \cup \dots \cup W_k$ is $\frac{1}{12}\varepsilon$ -tight to W_i .

We need the following result from [5, Theorem 3.3].

Lemma 18 ([5]). *For all $\varepsilon \in (0, \frac{1}{3})$, every graph with a $(\lceil 4\varepsilon^{-1} \rceil, \frac{1}{4}\varepsilon)$ -path-partition is $(2400\varepsilon^{-2}, \varepsilon)$ -restricted.*

The following lemma allows us to “lengthen” a given path-partition of length less than $\lceil 4\varepsilon^{-1} \rceil$ by one, at the cost of removing a small number of vertices.

Lemma 19. Let $\varepsilon \in (0, \frac{1}{3})$ and $K := \lceil 4\varepsilon^{-1} \rceil$. Let H be a graph with $h := |H| \geq 2$. Let

$$\begin{aligned} \varepsilon' &:= h^{-2K}\varepsilon, & \eta &:= h^{-2}, & \theta &:= \frac{1}{12}h^{-2K}\varepsilon, \\ \kappa &= \kappa_{19}(H, \varepsilon) := h^{-2Kh} \cdot \kappa_{12}(H, \varepsilon', \eta, \theta), & N &= N_{19}(H, \varepsilon) := N_{12}(H, \varepsilon', \eta, \theta). \end{aligned}$$

Let k be an integer with $0 \leq k \leq K$. Let $d \geq 0$, and let G be a graph with $\text{ind}(G) \leq \kappa d^h$ such that G has a $(k, h^{2(k-K)}\varepsilon)$ -path-partition (W_0, W_1, \dots, W_k) . Then there is a set $S \subseteq V(G)$ with $|S| \leq h^{-2k}d$ such that $G \setminus S$ is $(h^{2(K-k)}(2400\varepsilon^{-2} + N) - N, \varepsilon)$ -restricted.

Proof. We proceed by backward induction on k . If $k = K$ then the conclusion follows by Lemma 18. We may assume that $k < K$ and that the lemma holds for $k + 1$. Since

$$\text{ind}(G[W_k]) \leq \kappa d^h \leq \kappa_{12}(H, \varepsilon', \eta, \theta) \cdot (h^{-2(k+1)}d)^h,$$

by Lemma 12 applied to $G[W_k]$ with d replaced by $h^{-2(k+1)}d$, there is $T \subseteq W_k$ with $|T| \leq h^{-2(k+1)}d$ such that $W_k \setminus T$ can be partitioned into nonempty sets

$$A_1, \dots, A_m, B_1, \dots, B_m, C_1, \dots, C_n$$

where $m \leq \binom{h}{2}$ and $n \leq N$, such that

- $A_1, \dots, A_m, C_1, \dots, C_n$ are ε' -restricted in G ; and
- for every $i \in [m]$, $|B_i| \leq \eta|A_i|$ and B_i is θ -tight to A_i .

If $m = 0$ then $G \setminus T$ is $(k + N, \varepsilon)$ -restricted and we are done (note that $h \geq 2$ and $k \leq 4\varepsilon^{-2}$); thus we may assume $m \geq 1$. It follows that $|W_k| \geq |A_1| \geq \eta^{-1}|B_1| \geq h^2 \geq 2m$, and so $|W_i| \geq 12|W_k| \geq 24m$ for all $i \in \{0, 1, \dots, k-1\}$. Thus for each such i , W_i has a partition $W_i^1 \cup \dots \cup W_i^m$ with $|W_i^j| \geq \lfloor \frac{1}{m}|W_i| \rfloor \geq \frac{1}{2m}|W_i| \geq h^{-2}|W_i|$ for all $j \in [m]$. Let $U_j := \bigcup_{i=0}^{k-1} W_i^j \cup (A_j \cup B_j)$ for every $j \in [m]$.

Claim 20. For all $j \in [m]$, $(W_0^j, W_1^j, \dots, W_{k-1}^j, A_j, B_j)$ is a $(k + 1, h^{2(k+1-K)}\varepsilon)$ -path-partition of $G[U_j]$.

Proof. It suffices to observe the following.

- A_j is $h^{2(k+1-K)}\varepsilon$ -restricted since it is ε' -restricted and $\varepsilon' = h^{-2K}\varepsilon$; and also, for each $i \in \{0, 1, \dots, k-1\}$, W_i^j is $h^{2(k+1-K)}\varepsilon$ -restricted since W_i is $h^{2(k-K)}\varepsilon$ -restricted and $|W_i^j| \geq h^{-2}|W_i|$.
- For every $i \in \{0, 1, \dots, k-1\}$, since $12|W_k| \leq |W_i| \leq h^2|W_i^j|$, we have

$$12|B_j| \leq 12\eta|A_j| \leq \min(12h^{-2}|W_k|, |A_j|) \leq \min(|W_i^j|, |A_j|).$$

- B_j is $\frac{1}{12}h^{2(k+1-K)}\varepsilon$ -tight to A_j by the definition of θ ; and also, for every $i \in \{0, 1, \dots, k-1\}$, $(W_{i+1}^j \cup \dots \cup W_{k-1}^j) \cup (A_j \cup B_j)$ is $\frac{1}{12}h^{2(k+1-K)}\varepsilon$ -tight to W_i^j since $W_{i+1} \cup \dots \cup W_k$ is $\frac{1}{12}h^{2(k-K)}\varepsilon$ -tight to W_i and $|W_i^j| \geq h^{-2}|W_i|$.

This proves Claim 20. □

By Claim 20 and induction, for each $j \in [m]$, there is a set $S_j \subseteq U_j$ with $|S_j| \leq h^{-2(k+1)}d$ such that $G[U_j \setminus S_j]$ is $(h^{2(K-k-1)}(2400\varepsilon^{-2} + N) - N, \varepsilon)$ -restricted. Put $S := \bigcup_{j \in [m]} S_j \cup T$; then $|S| \leq (m+1)h^{-2(k+1)}d \leq h^{-2k}d$ as $h \geq 2$, and since

$$\begin{aligned} m \cdot (h^{2(K-k-1)}(2400\varepsilon^{-2} + N) - N) + n &\leq h^2 \cdot (h^{2(K-k-1)}(2400\varepsilon^{-2} + N) - N) + N \\ &\leq h^{2(K-k)}(2400\varepsilon^{-2} + N) - N, \end{aligned}$$

we see that $G \setminus S$ is $(h^{2(K-k)}(2400\varepsilon^{-2} + N) - N, \varepsilon)$ -restricted. This proves Lemma 19. ■

We are now ready to finish the proof of Theorem 7, which we restate here for the reader's convenience.

Theorem 21. *For every $\varepsilon > 0$ and every graph H , there exist $\kappa = \kappa(H, \varepsilon) > 0$ and $N = N(H, \varepsilon) > 0$ such that for every $d \geq 0$ and every graph G with $\text{ind}(G) \leq \kappa d^h$, there is a set $S \subseteq V(G)$ with $|S| \leq d$ such that $G \setminus S$ is (N, ε) -restricted.*

Proof. We may assume $h \geq 2$. Let $\kappa := \kappa_{19}(H, \varepsilon)$ and $N := h^{2K}(2400\varepsilon^{-2} + N_{19}(H, \varepsilon))$. Then Theorem 21 follows from Lemma 19 applied to the $(0, h^{-2K}\varepsilon)$ -partition $V(G)$ of G . ■

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References

- [1] N. Alon, R. A. Duke, H. Lefmann, V. Rödl, and R. Yuster. The algorithmic aspects of the regularity lemma. *J. Algorithms*, 16(1):80–109, 1994.
- [2] N. Alon, E. Fischer, M. Krivelevich, and M. Szegedy. Efficient testing of large graphs. *Combinatorica*, 20(4):451–476, 2000.
- [3] M. Bucić, T. Nguyen, A. Scott, and P. Seymour. Induced subgraph density. I. A loglog step towards Erdős–Hajnal, *preprint*, [arXiv:2301.10147](https://arxiv.org/abs/2301.10147), 2023.
- [4] M. Chudnovsky. The Erdős–Hajnal conjecture—a survey. *J. Graph Theory*, 75(2):178–190, 2014.
- [5] M. Chudnovsky, A. Scott, P. Seymour, and S. Spirkl. Strengthening Rödl's theorem, *preprint*, [arXiv:2105.07370](https://arxiv.org/abs/2105.07370), 2021.
- [6] M. Chudnovsky, A. Scott, P. Seymour, and S. Spirkl. Erdős–Hajnal for graphs with no 5-hole. *Proc. Lond. Math. Soc. (3)*, 126(3):997–1014, 2023.

- [7] P. Erdős and A. Hajnal. On spanned subgraphs of graphs. In *Contributions to graph theory and its applications (Internat. Colloq., Oberhof, 1977) (German)*, pages 80–96. Tech. Hochschule Ilmenau, Ilmenau, 1977.
- [8] P. Erdős and A. Hajnal. Ramsey-type theorems. volume 25, pages 37–52. 1989. *Combinatorics and complexity* (Chicago, IL, 1987).
- [9] J. Fox, T. Nguyen, A. Scott, and P. Seymour. Induced subgraph density. II. Sparse and dense sets in cographs, *preprint*, [arXiv:2307.00801](https://arxiv.org/abs/2307.00801), 2023.
- [10] J. Fox and B. Sudakov. Induced Ramsey-type theorems. *Adv. Math.*, 219(6):1771–1800, 2008.
- [11] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963.
- [12] T. Nguyen, A. Scott, and P. Seymour. Induced subgraph density. III. The pentagon and the bull, *preprint*, [arXiv:2307.06379](https://arxiv.org/abs/2307.06379), 2023.
- [13] T. Nguyen, A. Scott, and P. Seymour. Induced subgraph density. IV. New graphs with the Erdős–Hajnal property, *preprint*, [arXiv:2307.06455](https://arxiv.org/abs/2307.06455), 2023.
- [14] V. Nikiforov. Edge distribution of graphs with few copies of a given graph. *Combin. Probab. Comput.*, 15(6):895–902, 2006.
- [15] Y. Peng, V. Rödl, and A. Ruciński. Holes in graphs. *Electron. J. Combin.*, 9(1):#R1, 18, 2002.
- [16] V. Rödl. On universality of graphs with uniformly distributed edges. *Discrete Math.*, 59(1-2):125–134, 1986.
- [17] S. Sapir and A. Shapira. The induced removal lemma in sparse graphs. *Combin. Probab. Comput.*, 29(1):153–162, 2020.