On the Limit of the Positive ℓ-Degree Turán Problem

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Abstract

The *minimum positive ℓ-degree* $\delta^+_\ell(G)$ of a non-empty k-graph G is the maximum m such that every ℓ-subset of $V(G)$ is contained in either none or at least m edges of G; let $\delta^+_\ell(G) := 0$ if G has no edges. For a family $\mathcal{F}$ of k-graphs, let $\mathrm{co}^+\mathrm{ex}_\ell(n,\mathcal{F})$ be the maximum of $\delta^+_\ell(G)$ over all $\mathcal{F}$-free k-graphs G on n vertices. We prove that the ratio $\mathrm{co}^+\mathrm{ex}_\ell(n,\mathcal{F})/\binom{n-\ell}{k-\ell}$ tends to limit as $n\to\infty$, answering a question of Halfpap, Lemons and Palmer. Also, we show that the limit can be obtained as the value of a natural optimisation problem for k-hypergraphons; in fact, we give an alternative description of the set of possible accumulation points of almost extremal k-graphs.

Mathematics Subject Classifications: 05D05, 05C65

1 Introduction

A *k-graph* G is a pair $(V(G),E(G))$, where $V(G)$ is the *vertex set* of G and $E(G)$ is a collection of k-subsets of $V(G)$, called *edges*. We call G *non-empty* if $E(G)\neq\emptyset$.

Fix an integer ℓ with $0\leqslant\ell\leqslant k-1$. The *minimum positive ℓ-degree* of a non-empty k-graph G, denoted by $\delta^+_\ell(G)$, is the maximum m such that every ℓ-subset L of $V(G)$ is contained in either none or at least m edges of G. If G has no edges then we define $\delta^+_\ell(G) := 0$. For a family $\mathcal{F}$ of k-graphs, the corresponding *positive ℓ-degree Turán problem* is to determine $\mathrm{co}^+\mathrm{ex}_\ell(n,\mathcal{F})$, the maximum of $\delta^+_\ell(G)$ over all $\mathcal{F}$-free k-graphs G on n vertices. Let

$$\gamma^+_\ell(\mathcal{F}) := \limsup_{n\to\infty}\frac{\mathrm{co}^+\mathrm{ex}_\ell(n,\mathcal{F})}{\binom{n-\ell}{k-\ell}}. \tag{1}$$

If $\ell=0$, then $\mathrm{co}^+\mathrm{ex}_0(n,\mathcal{F})$ is the usual *Turán function* $\mathrm{ex}(n,\mathcal{F})$, the maximum number of edges in an n-vertex $\mathcal{F}$-free k-graph and thus $\gamma^+_0(\mathcal{F})$ is the *Turán density* $\pi(\mathcal{F}) := \lim_{n\to\infty}\mathrm{ex}(n,\mathcal{F})/\binom{n}{k}$, where the existence of the limit was established by Katona, Nemetz and Simonovits [9] by an easy averaging argument. Also, one can show that $\gamma^+_1(\mathcal{F}) = \pi(\mathcal{F})$. (For example, this easily follows from Proposition 1 which states that the ratio in

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the right-hand side of (1) tends to a limit.) For surveys of the hypergraph Turán problem, we refer the reader to Sidorenko [16] and Keevash [10].

Another relative is the *ℓ-degree Turán fuction* $\text{co-ex}_\ell(n,\mathcal{F})$, the maximum m such that there is an $\mathcal{F}$-free n-vertex k-graph G such that for every ℓ-subset L of $V(G)$ we have $\deg_L(G) \geqslant m$, where the *degree* $\deg_L(G)$ of a set $L \subseteq V(G)$ is the number of edges of G that contain L. Trivially, we always have $\text{co}^+\text{ex}_\ell(n,\mathcal{F}) \geqslant \text{co-ex}_\ell(n,\mathcal{F})$. A systematic study of the latter function for $\ell = k-1$ was started by Mubayi and Zhao [14, Proposition 1.2] who in particular proved that the limit

$$\gamma_\ell(\mathcal{F}) := \lim_{n\to\infty} \frac{\text{co-ex}_\ell(n,\mathcal{F})}{\binom{n-\ell}{k-\ell}} \tag{2}$$

exists for $\ell = k-1$. The existence of the limit in (2) for every ℓ was proved by Lo and Markström [12, Proposition 1.5] (see also Keevash [10, Page 118]). Balogh, Clemen and Lidicky [2] present a survey of these (and some related) Turán-type problems for 3-graphs.

The problem of determining $\text{co}^+\text{ex}(n,\mathcal{F})$ for $\ell = k-1$ was introduced and studied by Halfpap, Lemons and Palmer [7], motivated by an earlier paper of Balogh, Lemons and Palmer [3] who studied positive degree in the context of intersection families. Note that our definition of the minimum positive ℓ-degree of G deviates from the one in [3] when G has no edges: in this case the authors of [3] leave $\delta_\ell^+(G)$ undefined while we set $\delta_\ell^+(G) := 0$ (with this leading to slightly cleaner statements of some of our results).

Halfpap, Lemons and Palmer [7] computed $\gamma_2(\{F\})$ (and, in some cases, the exact value of $\text{co}^+\text{ex}_2(n,\{F\})$) for a few natural 3-graphs F; see also Wu [18] for some further such results. The value of $\gamma_2^+(\{F\})$ is in general different from $\gamma_2(\{F\})$ (as well as from $\pi(\{F\})$).

Halfpap, Lemons and Palmer [7] asked whether, for every fixed k-graph F, the ratio $\text{co}^+\text{ex}_{k-1}(n,\{F\})/n$ tends to a limit as $n \to \infty$ and proved this ([7, Proposition 21]) in a special case when every $(k-1)$-set of vertices of F is covered by an edge.

Here we answer this question for every k-graph family $\mathcal{F}$ (and an arbitrary integer ℓ).

Proposition 1. *For every (possibly infinite) k-graph family $\mathcal{F}$ and every integer ℓ with $0 \leqslant \ell \leqslant k-1$, the ratio $\text{co}^+\text{ex}_\ell(n,\mathcal{F})/\binom{n-\ell}{k-\ell}$ tends to a limit as $n \to \infty$.*

Our second result, Theorem 5, describes the set of possible k-hypergraphons that are the limits of sequences of almost optimal constructions for $\text{co}^+\text{ex}_\ell(n,\mathcal{F})$ as $n \to \infty$, in particuar giving an optimisation problem for k-hypergraphons that produces $\gamma_\ell^+(\mathcal{F})$. This can be viewed as the natural limit version of the positive ℓ-degree Turán problem.

Since the statement of Theorem 5 is technical and requires quite a few definitions (for a non-expert), we state it only after presenting the (purely combinatorial) proof of Proposition 1.

2 Proof of Proposition 1

The main proof idea (to take an n-vertex sample from a larger nearly optimal $\mathcal{F}$-free k-graph) is the same as in the proofs of Mubayi and Zhao [14, Proposition 1.2] and Lo and Markström [12, Proposition 1.5] that the limit in (2) exists. Here, we face some new (minor) technicalities due to the fact that ℓ-sets of zero degree have to be treated differently.

We need the following auxiliary lemma which states, roughly speaking, that a k-graph with $n \to \infty$ vertices and positive ℓ-degree $\Omega(n^{k-\ell})$ must have $\Omega(n^k)$ edges.

Lemma 2. *Let $0 \leqslant \ell < k$ be integers and $\gamma \geqslant 0$ be a real number. If a k-graph G with m vertices and $e > 0$ edges satisfies $\delta_\ell^+(G) \geqslant \gamma m^{k-\ell}/(k-\ell)!$ then $e \geqslant \gamma^{k/(k-\ell)} m^k/k!$.*

Proof. Assume that $\gamma > 0$ as otherwise there is nothing to prove. Let $\lambda > 0$ be the number of ℓ-sets covered by at least one edge of G. Let x be the real number at least k that satisfies $e = \binom{x}{k}$, where we define

$$\binom{y}{k} := \frac{y(y-1)\cdots(y-k+1)}{k!}, \quad \text{for } y \in \mathbb{R}.$$

Note that the function $\binom{y}{k}$ is strictly increasing and continuous for $y \geqslant k$, so x exists and is unique. By a version of the Kruskal-Katona Theorem [8, 11] that is due to Lovász [13, Exercise 13.31(b)], we have that $\lambda \geqslant \binom{x}{\ell}$.

Thus the number of pairs (K, L), where $K \in E(G)$ and L is an ℓ-subset of K is at least $\lambda \cdot \gamma m^{k-\ell}/(k-\ell)!$ on one hand and is exactly $e \cdot \binom{k}{\ell}$ on the other hand. Putting these two estimates together, we get that

$$\binom{x}{k}\binom{k}{\ell} \geqslant \binom{x}{\ell} \frac{\gamma m^{k-\ell}}{(k-\ell)!}.$$

By canceling the same (positive) factors and rearranging, we get that $(x-\ell)\cdots(x-k+1) \geqslant \gamma m^{k-\ell}$. Now, the required inequality follows:

$$k!\, e = \prod_{i=1}^{k}(x-i+1) \geqslant \left(\prod_{i=\ell+1}^{k}(x-i+1)\right)^{k/(k-\ell)} \geqslant \gamma^{k/(k-\ell)} m^k,$$

where the first inequality can be proved by observing that, after raising it to power $k-\ell$ and cancelling identical terms, we are left with two products of $\ell(k-\ell)$ factors, with each factor at least $x-\ell+1$ on the left-hand side and at most $x-\ell$ on the right-hand side. $\square$

Proof of Proposition 1. Let $\gamma := \limsup_{n\to\infty} \frac{\mathrm{co}^+\mathrm{ex}_\ell(n,\mathcal{F})}{\binom{n-\ell}{k-\ell}}$ be the limit superior in of the stated ratios. If $\gamma = 0$, then by the non-negativity of each term, the limit exists and is 0. So suppose that $\gamma > 0$. Take any $\varepsilon \in (0, \gamma)$.

Let n be sufficiently large. Pick any $\mathcal{F}$-free k-graph G with $N \geqslant n$ vertices such that $\delta_\ell^+(G) \geqslant (\gamma - \varepsilon/2)\binom{N-\ell}{k-\ell}$.

Take a uniformly random n-subset of $V(G)$ and a uniformly random enumeration $v_1, \ldots, v_n$ of its vertices. Equivalently, for $i = 1, \ldots, n$, let v_i be a random element of $V(G) \setminus \{v_1, \ldots, v_{i-1}\}$ with all $N - i + 1$ choices being equally likely.

Let H be the k-graph on $[n]$ where a k-set $\{i_1, \ldots, i_k\} \subseteq [n]$ is an edge if and only if $\{v_{i_1}, \ldots, v_{i_k}\}$ is an edge of G. Thus, up to relabelling of its vertices, H is the k-graph induced in G by a uniformly random set of n vertices. Clearly, H is $\mathcal{F}$-free.

Claim 3. *Let* $g := \lfloor (\gamma-\varepsilon)\binom{n-\ell}{k-\ell} \rfloor$*. Then the probability that there is an* ℓ*-set* $L \subseteq [n]$ *with* $0 < \deg_L(H) \leqslant g$ *is less than* $1/2$*.*

Proof of Claim. It is enough to show by the Union Bound that, for every ℓ-set $L \subseteq [n]$, the probability over the random choices of $v_1, \ldots, v_n$ that $\deg_L(H) \in [g]$ is less than $\frac{1}{2}\binom{n}{\ell}^{-1}$. Fix any $L \in \binom{[n]}{k}$. By symmetry between the vertices of H, we can assume for notational convenience that $L = [\ell]$. It is enough to prove the stated bound when we condition on $A := \{v_1, \ldots, v_\ell\}$. The conditional distribution can be obtained by picking $v_{\ell+1}, \ldots, v_n$ one by one, each being uniform in the remaining subset of $V(G) \setminus A$. If $\deg_A(G) = 0$, then $\deg_L(H) = 0$ deterministically and the stated event cannot occur. So suppose that $\deg_A(G) > 0$ and thus it is at least $\delta_\ell^+(G) \geqslant (\gamma - \varepsilon/2)\binom{N-\ell}{k-\ell}$. For $i = 0, \ldots, n-\ell$, let X_i be the expectation of $\deg_L(H)$ after we have exposed $v_{\ell+1}, \ldots, v_{\ell+i}$. In other words, $(X_0, \ldots, X_{n-\ell})$ is the vertex-exposure martingale for $\deg_L(H)$ conditioned on A. Note that X_0 is constant and its value is

$$X_0 = \mathbb{E}[X_{n-\ell}] = \deg_A(G)\binom{n-\ell}{k-\ell}\Big/\binom{N-\ell}{k-\ell} \geqslant (\gamma-\varepsilon/2)\binom{n-\ell}{k-\ell}, \tag{3}$$

while $X_{n-\ell} = \deg_L(H)$.

Let us show that $|X_i - X_{i-1}| \leqslant \binom{n-\ell-1}{k-\ell-1}$ for every $i \in [n-\ell]$. It is enough to prove this inequality, conditioned on every choice of $v_{\ell+1}, \ldots, v_{\ell+i-1}$. For every two different choices u and u' for the vertex $v_{\ell+i}$ there is a natural coupling of the follow-up processes so that $|B \triangle B'| \leqslant 2$ always holds for the current unordered sets $B, B' \subseteq V(H)$ of the selected vertices: namely, run the process for u and let the process for u' choose the same vertex at each step, except if we see a vertex on which the current sets B and B' differ then we make these two sets equal from this step onwards. Thus the final unordered sets $\{v_{\ell+1}, \ldots, v_n\}$ differ in at most two places and the respective degrees of L differ by at most $\binom{n-\ell-1}{k-\ell-1}$, giving the claimed inequality.

Thus Azuma's inequality (see e.g. [1, Theorem 7.2.1]) gives that the probability of $X_{n-\ell} \leqslant X_0 - (\varepsilon/2)\binom{n-\ell}{k-\ell}$ is at most

$$\mathrm{e}^{-\left(\frac{\varepsilon}{2}\binom{n-\ell}{k-\ell}\right)^2\Big/\left(2(n-\ell)\binom{n-\ell-1}{k-\ell-1}^2\right)} < \mathrm{e}^{-\varepsilon^2 n/(9k^2)} < \frac{1}{2}\binom{n}{\ell}^{-1}.$$

Recalling that $X_{n-\ell} = \deg_L(H)$ while the constant X_0 satisfies (3), we obtain that the probability of $\deg_H(L) \leqslant g$ is less than $\frac{1}{2}\binom{n}{\ell}^{-1}$, giving the claim. ∎

Claim 4. *The probability that* H *is empty is less than* $1/2$*.*

Proof of Claim. Since n is large, the edge density of G is by Lemma 2 at least, for example, $\beta := (\gamma-\varepsilon)^{k/(k-\ell)} > 0$. Thus the expected size of H is at least $\beta\binom{n}{k}$. Consider the vertex-exposure martingale $(Y_0, \ldots, Y_n)$ for $|E(H)|$. Similarly as before, one can show that $|Y_i - Y_{i-1}| \leqslant \binom{n-1}{k-1}$ for each $i \in [n]$. Thus if H has no edges then $Y_n = |E(H)| = 0$ is at least $\beta\binom{n}{k}$ away from its mean Y_0 and, again by Azuma's inequality, the probability of this is at most

$$\mathrm{e}^{-\left(\beta\binom{n}{k}\right)^2\Big/\left(2n\binom{n-1}{k-1}^2\right)} < \mathrm{e}^{-\beta^2 n/(3k^2)} < \frac{1}{2},$$

as desired. ▎

By Claims 3 and 4 the random $\mathcal{F}$-free k-graph H on $[n]$ is non-empty and satisfies $\delta_\ell^+(H) > (\gamma-\varepsilon)\binom{n-\ell}{k-\ell}$ with positive probability. So at least one such choice for H exists and $\mathrm{co}^+\mathrm{ex}(n,\mathcal{F}) > (\gamma-\varepsilon)\binom{n-\ell}{k-\ell}$. Since $\varepsilon > 0$ was arbitrary and this inequality holds for all sufficiently large n, we conclude that the ratio $\mathrm{co}^+\mathrm{ex}(n,\mathcal{F})/\binom{n-\ell}{k-\ell}$ tends to γ, finishing the proof of the proposition. □

3 Positive degree via hypergraph limits

In order to state our main result of this paper (Theorem 5) we need to give various definitions related to the limit theory of hypergraphs. We generally follow the notation from [19].

For a finite set A and an integer $m \geqslant 1$, let

$$r(A,m) := \{X \subseteq A : 0 < |X| \leqslant m\},$$

consist of all non-empty subsets of A with at most m elements. Also, let

$$r(A) := r(A,|A|) = \{X : \emptyset \neq X \subseteq A\}$$

denote the set of all non-empty subsets of A and

$$r_<(A) := r(A,|A|-1) = \{X : \emptyset \neq X \subsetneq A\}$$

denote the set of all *proper* subsets of A. If A is $[m] = \{1,\ldots,m\}$, then we may abbreviate $r([m])$ and $r_<([m])$ to $r[m]$ and $r_<[m]$ respectively.

For a family $\mathcal{A}$ of sets, let $\mathbf{x}_\mathcal{A} \in \mathbb{R}^\mathcal{A}$ denotes the vector of reals $(x_A)_{A\in\mathcal{A}}$ indexed by $\mathcal{A}$. When $\mathbf{x}$ has already been specified, we use this notation as follows. If every index in $\mathbf{x}$ appears in $\mathcal{A}$ then by $\mathbf{x}_\mathcal{A}$ we mean an *extension* of $\mathbf{x}$, that is, any vector indexed by $\mathcal{A}$ which coincides with $\mathbf{x}$ on the common set of indices. If every element of $\mathcal{A}$ appears as an index in $\mathbf{x}$ then $\mathbf{x}_\mathcal{A}$ means the *restriction* of $\mathbf{x}$ to $\mathcal{A}$, that is, the vector indexed by $\mathcal{A}$ whose A-coordinate is the same as the A-coordinate of $\mathbf{x}$ for every $A \in \mathcal{A}$. We assume that the sets in $\mathcal{A}$ come in some fixed and consistently used order, which is preserved when we pass to subfamilies. In all concrete examples for $\mathcal{A} \subseteq r[n]$ that we give, we first order the sets increasingly by their size and then use the lexicographic order to break ties on sets of equal size.

The *symmetric group* Sym_k (consisting of all permutations of $[k]$) acts naturally on $[0,1]^{r_<[k]}$. A function $W : [0,1]^{r_<[k]} \to \mathbb{R}$ is called *symmetric* if its values do not change under the action of Sym_k on $[0,1]^{r_<[k]}$. For example, for $r=2$, this means that $W(x_1,x_2) = W(x_2,x_1)$ for all $(x_1,x_2)\in[0,1]^2$, and for $r=3$ this means that

$$W(x_1,x_2,x_3,x_{12},x_{13},x_{23}) = W(\mathbf{x}^\sigma), \quad \text{for all } \mathbf{x}\in[0,1]^{r_<[3]} \text{ and } \sigma\in S_3, \tag{4}$$

where we abbreviate $x_{\{a_1,\ldots,a_s\}}$ to $x_{a_1\ldots a_s}$ and denote

$$\mathbf{x}^\sigma := (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(1)\,\sigma(2)}, x_{\sigma(1)\,\sigma(3)}, x_{\sigma(2)\,\sigma(3)}).$$

A *k-hypergraphon* is a symmetric (Lebesgue) measurable function $W:[0,1]^{r_<[k]}\to[0,1]$.

Elek and Szegedy [4, Theorem 7] (see also Zhao [19, Theorem 1.5] for a different proof) showed that k-hypergraphons can serve as limit objects for k-graphs as follows. The *(homomorphism) density* of a k-graph F in W is defined as

$$t(F,W) := \int_{[0,1]^{r(V(F),k-1)}} \prod_{A\in E(F)} W(\mathbf{x}_{r_<(A)})\,\mathrm{d}\mathbf{x}. \tag{5}$$

A k-graph G with vertices enumerated as $v_1,\ldots,v_n$ corresponds to the k-hypergraphon W^G constructed as follows: partition $[0,1]$ into n intervals $I_1,\ldots,I_n$ of length $1/n$ each, and let

$$W^G(\mathbf{x}) := \begin{cases} 1, & \text{if } \{v_{i_1},\ldots,v_{i_k}\}\in E(G), \text{ where } i_j \text{ is the index with } I_{i_j}\ni x_j \text{ for } j\in[k],\\ 0, & \text{otherwise.}\end{cases} \tag{6}$$

Thus W^G is a $\{0,1\}$-valued function on $[0,1]^{r_<[k]}$ that depends on the 1-dimensional coordinates only, naturally encoding the edge set of G. Note that

$$t(F,W^G) = t(F,G), \quad \text{for every } k\text{-graph } F, \tag{7}$$

where $t(F,G)$ is the usual *(homomorphism) density* of F in G, which is the probability that a random function $f:V(F)\to V(G)$, with all $|V(G)|^{|V(F)|}$ choices being equally likely, sends every edge of F to an edge of G. Call a sequence $(G_n)_{n=1}^\infty$ of k-graphs *convergent* if, for every k-graph F, the densities $t(F,G_n)$ converge to a limit as $n\to\infty$. One of the main results of Elek and Szegedy [4, Theorem 7] is that, for every convergent sequence $(G_n)_{i=1}^\infty$ of k-graphs, there is a k-hypergraphon W, called the *limit* of $(G_n)_{i=1}^\infty$, such that

$$t(F,W) = \lim_{n\to\infty} t(F,G_n), \quad \text{for every } k\text{-graph } F. \tag{8}$$

Note that, even though the right-hand side of (8) involves hypergraphons depending on the 1-dimensional coordinates only (when we replace $t(F,G_n)$ by $t(F,W^{G_n})$ using (7)), the resulting limit may in general depend on the extra (more than 1-dimensional) coordinates. Informally speaking, we may need to account for the limits of hypergraph constructions where the density of edges of $E(G)$ depends not only on the locations of vertices in a regularity partition of G but also on the higher-dimensional cylinder structure, that is, the "colours" of subsets of sizes between 2 and $k-1$. One such example is the *directed cycle*

3-*graph construction* $C(T)$ where one takes a quasi-random tournament T and defines the edge set of $C(T)$ to consist of all triples spanning a directed cycle; this construction is known to be asymptotically optimal for some extermal 3-graph problems, see e.g. [5, 6, 15]. A possible corresponding 3-hypergraphon can be defined to be 0 except it is 1 on $\mathbf{x}^\sigma$ for all $\sigma \in S_3$ and $\mathbf{x}$ such that $x_1 < x_2 < x_3$, $x_{12}, x_{23} \in [0, 1/2]$ and $x_{13} \in (1/2, 1]$.

Accordingly, when we extend the definition of (positive) ℓ-degree to a k-hypergraphon W, we need to take into account not only the 1-dimensional coordinates (ℓ of them) but also all higher-dimensional ones (within the corresponding ℓ-set). Formally, for $0 \leqslant \ell < k$, the *degree* of $\mathbf{x} \in [0,1]^{r[\ell]}$ in W is defined as

$$\deg_W(\mathbf{x}) := \int_{[0,1]^{r_<[k] \setminus r[\ell]}} W(\mathbf{x}_{r_<[k]}) \,\mathrm{d}\mathbf{x}_{r_<[k] \setminus r[\ell]}, \tag{9}$$

that is, we take the average of W over all extensions $\mathbf{x}_{r_<[k]} \in [0,1]^{r_<[k]}$ of $\mathbf{x} \in [0,1]^{r[\ell]}$, where the new coordinates (those indexed by $r_<[k] \setminus r[\ell]$) are independent and uniformly distributed in $[0, 1]$. Note that by Fubini-Tonelli's theorem (see e.g. [17, Theorem 2.3.2]), the integral in (9) is well-defined for a.e. choice of $\mathbf{x} \in [0,1]^{r[\ell]}$. For those $\mathbf{x} \in [0,1]^{r[\ell]}$ for which the integral is undefined, we set $\deg_W(\mathbf{x}) := 0$ for definiteness.

For example, if $k = 3$ and $\ell = 2$, then the degree of $\mathbf{x} = (x_1, x_2, x_{12})$ is

$$\deg_W(\mathbf{x}) := \int_{[0,1]^3} W(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}) \,\mathrm{d}x_3 \,\mathrm{d}x_{13} \,\mathrm{d}x_{23}.$$

If $W = W^G$ for a k-graph G on $\{v_1, \ldots, v_n\}$ then, with i_j being the unique index such that $x_j \in I_{i_j}$ for $j \in [\ell]$ (as in (6)), we have that

$$\deg_{W^G}(\mathbf{x}) = \begin{cases} \deg_G(\{v_{i_1}, \ldots, v_{i_\ell}\}) \, \frac{(k-\ell)!}{n^{k-\ell}}, & \text{if } i_1, \ldots, i_\ell \text{ are pairwise distinct,} \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Indeed, every $(k - \ell)$-subset $\{v_{i_{\ell+1}}, \ldots, v_{i_k}\}$ that makes an edge of G with $\{v_{i_1}, \ldots, v_{i_\ell}\}$ corresponds to $(k - \ell)!$ products (over all possible permutations of $\{i_{\ell+1}, \ldots, i_k\}$) of the corresponding $k - \ell$ distinct intervals I_v, with each product having measure $1/n^{k-\ell}$.

Call a k-hypergraphon W *non-zero* if the measure of $\mathbf{x} \in [0,1]^{r_<[k]}$ with $W(\mathbf{x}) > 0$ is positive. Let the *minimum positive ℓ-degree* of a non-zero k-hypergraphon W be defined as

$$\delta_\ell^+(W) := \operatorname*{ess-inf}_{\substack{\mathbf{x} \in [0,1]^{r[\ell]} \\ \deg_W(\mathbf{x}) > 0}} \deg_W(\mathbf{x}) = \sup_{\substack{A \subseteq [0,1]^{r[\ell]} \\ \mu(A) = 0}} \inf \left\{ \deg_W(\mathbf{x}) : \mathbf{x} \in [0,1]^{r[\ell]} \setminus A,\ \deg_W(\mathbf{x}) > 0 \right\}, \tag{11}$$

the *essential infimum* (that is, the infimum after ignoring a set A of $\mathbf{x}$ measure 0) of the degrees $\deg_W(\mathbf{x})$ which are positive. If W is zero, then we define $\delta_\ell^+(W) := 0$. Note that, for every k-graph G, we have by (10) that

$$\delta_\ell^+(W^G) = \frac{(k-\ell)!}{|V(G)|^{k-\ell}} \, \delta_\ell^+(G). \tag{12}$$

For $0 \leqslant \ell < k$ and a k-graph family $\mathcal{F}$, let $\mathcal{W}_\ell^+(\mathcal{F})$ consist those k-hypergraphons W which are the limits of some sequence of *almost extremal* k-graphs, that is, a sequence $(G_n)_{n=1}^\infty$ of $\mathcal{F}$-free k-graphs such that, as $n \to \infty$, we have $|V(G_n)| \to \infty$ and $\delta_\ell^+(G_n) = (\gamma_\ell^+(\mathcal{F}) + o(1))\binom{|V(G_n)|-\ell}{k-\ell}$. Also, a k-hypergraphon W is called *$\mathcal{F}$-free* if $t(F,W) = 0$ for every $F \in \mathcal{F}$. With this preparation, we can now formulate our main result which expresses the limit in Proposition 1 as the value of an optimisation problem involving k-hypergraphons; in fact, we also give an alternative description of the set $\mathcal{W}_\ell^+(\mathcal{F})$.

Theorem 5. *Take any integers $0 \leqslant \ell < k$ and a (possibly infinite) family $\mathcal{F}$ of k-graphs. Define $\gamma := \gamma_\ell^+(\mathcal{F})$. Then the following statements hold.*

1. *The value of γ is the supremum (in fact, maximum) of $\delta_\ell^+(W)$ over all $\mathcal{F}$-free k-hypergraphons W.*

2. *A k-hypergraphon W belongs to $\mathcal{W}_\ell^+(\mathcal{F})$ if and only if it is $\mathcal{F}$-free and satisfies $\delta_\ell^+(W) = \gamma$.*

Theorem 5 will be a direct consequence of the following two lemmas. In order to state them, we need a few more definitions. The *n-sample* of W is the distribution $\mathbb{G}(n,W)$ on (vertex-labelled) k-graphs on $[n]$ where we sample $G \sim \mathbb{G}(n,W)$ in the following two steps. First, we sample a uniform $\mathbf{x} \in [0,1]^{r([n],k-1)}$ (i.e. each x_A is uniform in $[0,1]$ and the choices over all different $A \subseteq [n]$ are mutually independent). Second, every k-subset $A = \{i_1,\ldots,i_k\}$ of $[n]$ is included into $E(G)$ with probability $W(\mathbf{x}_{r_<(A)}) \in [0,1]$, with all $\binom{n}{k}$ choices being mutually independent. (Recall that $\mathbf{x}_{r_<(A)}$ denotes the sub-vector of $\mathbf{x} \in [0,1]^{r([n],k-1)}$ where we take all x_B with $\emptyset \neq B \subsetneq A$.) For example, if $k=3$ then $\{u,v,w\} \subseteq [n]$ is made an edge with probability $W(x_u,x_v,x_w,x_{uv},x_{uw},x_{vw})$. One relation between $G \sim \mathbb{G}(n,W)$ and the densities in W is that, for every k-graph F on $[n]$, we have

$$t(F,W) = \mathbb{P}\left[E(F) \subseteq E(G)\right], \tag{13}$$

that is, the $t(F,W)$ is the probability that every edge of F is an edge of G.

Now, we are ready to state the two key lemmas and show how they imply Theorem 5.

Lemma 6. *Let $k > \ell \geqslant 0$. Let $(G_n)_{n=1}^\infty$ be an arbitrary sequence of k-graphs convergent to a k-hypergraphon W such that $|V(G_n)| \to \infty$ as $n \to \infty$. Then*

$$\delta_\ell^+(W) \geqslant \limsup_{n\to\infty} \frac{\delta_\ell^+(G_n)}{\binom{|V(G_n)|-\ell}{k-\ell}}. \tag{14}$$

This lemma states that, informally speaking, the (normalised) minimum positive ℓ-degree does not decrease when we pass to the limit. If $\ell \geqslant 1$ then we have only one-sided inequality here because there may be $o(1)$-fraction of "outlier" ℓ-tuples in G_n whose degree is positive but strictly smaller than $(\delta_\ell^+(W) + o(1))\binom{|V(G_n)|}{k-\ell}$; these ℓ-tuples bring the positive ℓ-degree of G_n down but leave no trace in the limit W.

Lemma 7. *Let $k > \ell \geqslant 0$ and let W be any k-hypergraphon. For every $\varepsilon > 0$, there is n_0 such that for all $n \geqslant n_0$, if $G \sim \mathbb{G}(n, W)$ then the probability that $|\delta_\ell^+(G) - \delta_\ell^+(W)\binom{n-\ell}{k-\ell}| > \varepsilon\binom{n-\ell}{k-\ell}$ is at most ε.*

This lemma states that minimum positive ℓ-degree of a non-zero hypergraphon is inherited within additive error $o(1)$ by a typical n-sample as $n \to \infty$.

Proof of Theorem 5. First, suppose that $\gamma > 0$. Take any sequence $(G_n)_{n=1}^\infty$ of almost extremal k-graphs. By the standard diagonalization argument run over all (countably many) non-isomorphic k-graphs F, pass to a convergent subsequence (where $t(F, G_n)$ tends to a limit as $n \to \infty$ for each F). By the result of Elek and Szegedy [4, Theorem 7] (see also [19, Theorem 1.5]) there is a k-hypergraphon W such that $\lim_{n\to\infty} t(F, G_n) = t(F, W)$ for every k-graph F. Of course, $t(F, W) = \lim_{n\to\infty} t(F, G_n) = 0$ for every $F \in \mathcal{F}$, that is, W is $\mathcal{F}$-free. By Lemma 6, it holds that $\delta_\ell^+(W) \geqslant \gamma$.

Let us show that for every $\mathcal{F}$-free k-hypergraphon U, we have

$$\delta_\ell^+(U) \leqslant \gamma. \tag{15}$$

By Lemma 7, a typical n-vertex sample H_n of U for all large n has minimum positive degree at least $(\delta_\ell^+(U) + o(1))\binom{n-\ell}{k-\ell}$. (In fact, we only need this one-sided estimate from Lemma 7.) Also, for every $F \in \mathcal{F}$ the probability that H_n contains F as a subgraph is at most $n!\, t(F, U) = 0$ by (13). It follows that $(\delta_\ell^+(U) + o(1))\binom{n-\ell}{k-\ell} \leqslant \mathrm{co}^+\mathrm{ex}(n, \mathcal{F})$. We conclude that (15) holds.

By applying (15) to W, the limit of almost extremal k-graphs G_n, we conclude that $\delta_\ell^+(W) = \gamma$. This proves the first part and the forward implication in the second part (also reproving Proposition 1).

Let us show the converse implication in the second part. Let an $\mathcal{F}$-free k-hypergraphon U satisfy $\delta_\ell^+(U) = \gamma$. The sequence of random independent samples $(H_n)_{n=1}^\infty$, $H_n \sim \mathbb{G}(n, U)$, converges to U with probability 1 by [4, Theorem 12]. Furthermore, by Lemma 7, for every $m \geqslant 1$ we can find N_m such that, for example, $N_m \geqslant m$ and the probability of $\delta_\ell^+(H_{N_m}) < (\gamma - 1/m)\binom{N_m-\ell}{k-\ell}$ is at most 2^{-m-1}. Clearly, H_{N_m} is $\mathcal{F}$-free with probability 1. Thus, with probability at least $1 - \sum_{m=1}^\infty 2^{-m-1} > 0$, $(H_{N_m})_{m=1}^\infty$ is a sequence of almost extremal k-graphs convergent to U, that is, $U \in \mathcal{W}_\ell^+(\mathcal{F})$, proving the second part.

Finally, suppose that $\gamma = 0$. The only non-trivial claim that we have to establish is that no $\mathcal{F}$-free k-hypergraph U can satisfy $\delta_\ell^+(U) > 0$ and this follows as above by taking random samples from U and applying Lemma 7. $\square$

4 Proofs of Lemma 6 and 7

In order to present the proofs of Lemma 6 and 7 we need some further notation.

The following definition of a partially vertex-labelled hypergraph will suffice for the purposes of this paper. Namely, for $k > \ell \geqslant 0$, an *ℓ-labelled* k-graph is a triple $F = (V, E, \ell)$ where (V, E) is a k-graph and $V \supseteq [\ell]$. We view $1, \ldots, \ell \in V$ as labelled vertices and call them the *roots*.

For a k-hypergraphon W, a vector $\mathbf{x} \in [0,1]^{r[\ell]}$ and an ℓ-labelled k-graph $F = (V, E, \ell)$, the *($\mathbf{x}$-rooted) density* of F in W is

$$t_{\mathbf{x}}(F, W) := \int_{[0,1]^{r(V,k-1)\setminus r[\ell]}} \prod_{A \in E} W(\mathbf{x}_{r_<(A)})\, \mathrm{d}\mathbf{x}_{r(V,k-1)\setminus r[\ell]}. \tag{16}$$

By Fubini-Tonelli's theorem, this is defined for a.e. $\mathbf{x} \in [0,1]^{r[\ell]}$; for all other $\mathbf{x}$ we set $t_{\mathbf{x}}(F, W) := 0$ for definiteness. For example, we have that the definition in (16) for

$$\mathrm{Edge}_{k,\ell} := ([k], \{[k]\}, \ell),$$

the ℓ-labelled single k-edge, becomes exactly the one in (9), and thus

$$t_{\mathbf{x}}(\mathrm{Edge}_{k,\ell}, W) = \deg_W(\mathbf{x}), \quad \text{for every } \mathbf{x} \in [0,1]^{r[\ell]},$$

The densities of an ℓ-labelled k-graph $F = (V, E, \ell)$ and its unlabelled version $[\![F]\!] := (V, E)$, where we just forget the labelling, satisfy by Fubini-Tonelli's theorem the following relation:

$$t([\![F]\!], W) = \int_{[0,1]^{r[\ell]}} t_{\mathbf{x}}(F, W)\, \mathrm{d}\mathbf{x}. \tag{17}$$

This can be informally interpreted as that the density of the unlabelled k-graph $[\![F]\!]$ is the average over the uniform choice of an "ℓ-tuple" $\mathbf{x} \in [0,1]^{r[\ell]}$ of the $\mathbf{x}$-rooted density of F.

The *product* FF' of any two ℓ-labelled k-graphs $F = (V, E, \ell)$ and $F' = (V', E', \ell)$ is obtained by replacing F' by an isomorphic ℓ-labelled k-graph with $V \cap V' = [\ell]$ (that is, making F and F' vertex-disjoint except for the roots) and then taking the union of the vertex and edge sets:

$$FF' := (V \cup V', E \cup E, \ell).$$

The name "product" comes from the relation

$$t_{\mathbf{x}}(FF', W) = t_{\mathbf{x}}(F, W)\, t_{\mathbf{x}}(F', W), \quad \text{for every } \mathbf{x} \in [0,1]^{r[\ell]}, \tag{18}$$

which holds since the integral for $t_{\mathbf{x}}(FF', W)$ can be written by Fubini-Tonelli's theorem as the product of two integrals by partioning its variables into two groups: namely x_A with $A \cap (V \setminus [\ell]) \neq \emptyset$ and $x_{A'}$ with $A' \cap (V' \setminus [\ell]) \neq \emptyset$. (Recall that $\ell < k$ so the roots do not span any edges.) It follows that, with $\mathrm{Edge}_{k,\ell}^m$ denoting the m-fold product of the ℓ-labelled single k-edge with itself, we have

$$t_{\mathbf{x}}(\mathrm{Edge}_{k,\ell}^m, W) = \left(t_{\mathbf{x}}(\mathrm{Edge}_{k,\ell}, W)\right)^m, \quad \text{for every } \mathbf{x} \in [0,1]^{r[\ell]}. \tag{19}$$

Now we are ready to prove the lemmas.

Proof of Lemma 6. Suppose that a sequence $(G_n)_{n=1}^{\infty}$ convergent to some W gives a counterexample to the lemma. By passing to a subsequence, we can assume that the ratios $\delta_\ell^+(G_n)/\binom{|V(G_n)|-\ell}{k-\ell}$ converge to some δ, which is strictly larger that $\delta_\ell^+(W)$. By

Lemma 2, the edge density of G_n is at least $\delta^{k/(k-\ell)} + o(1) > 0$ as $n \to \infty$, so their limit W is non-zero. Thus the set

$$X := \{\mathbf{x} \in [0,1]^{r[\ell]} : \deg_W(\mathbf{x}) \in (0,\delta)\}$$

has positive measure.

By the countable additivity of measure, there is $\varepsilon > 0$ such that the set

$$X_\varepsilon := \left\{\mathbf{x} \in [0,1]^{r[\ell]} : \deg_W(\mathbf{x}) \in (\varepsilon, \delta-\varepsilon)\right\},$$

has measure at least ε. Indeed, X is the countable union $\cup_{m=1}^{\infty} X_{1/m}$, so $X_{1/m}$ has positive measure for some m and we can take $\varepsilon := \min\{1/m, \mu(X_{1/m})\}$.

Since $X_\varepsilon \neq \emptyset$, we have $\varepsilon < \delta/2$. Fix any $\beta \in (0, \varepsilon/2)$. Let $L(x) : [0,1] \to \mathbb{R}$ be the piecewise linear function whose graph in $[0,1] \times \mathbb{R}$ consists of the linear segments connecting the points $(0,\beta)$, $(\varepsilon, 1+\beta)$, $(\delta-\varepsilon, 1+\beta)$, $(\delta-\varepsilon/2, \beta)$, and $(1,\beta)$ in this order, see Figure 1. In particular, L is constant $1+\beta$ on $[\varepsilon, \delta-\varepsilon]$ and constant β on $[\delta-\varepsilon/2, 1]$. Informally, the "penalty" function L penalises values strictly between 0 and δ.

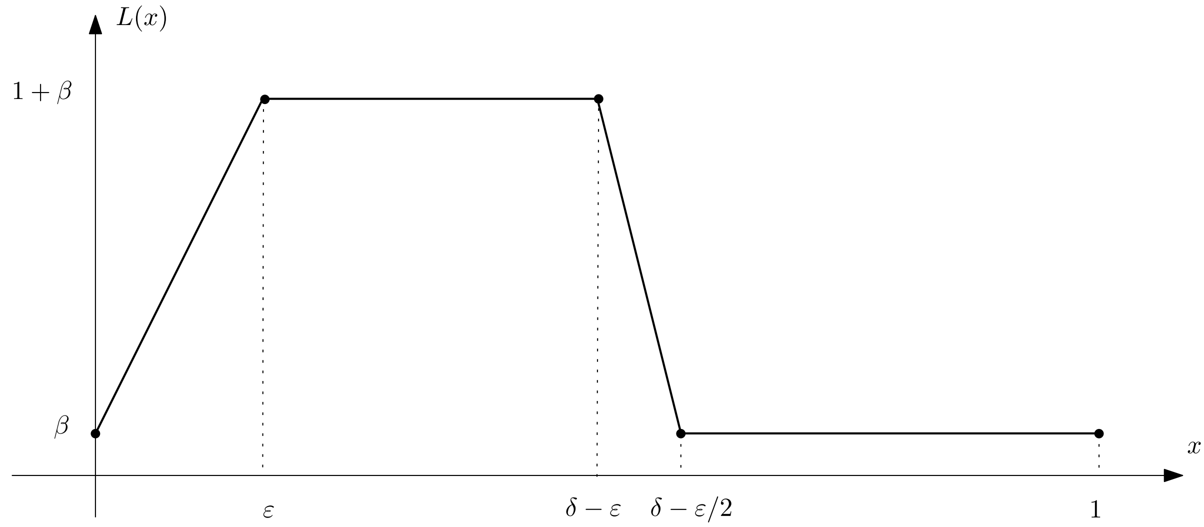

Figure 1: The graph of the function L.

Since the function L is continuous, the Stone-Veierstrass Theorem gives a polynomial $p(x) = \sum_{i=1}^{D} a_i x^i$ such that $|p(x) - L(x)| \leqslant \beta$ for every $x \in [0,1]$. This polynomial clearly has the following properties:

$$p(x) \geqslant 0, \qquad \text{for all } x \in [0,1], \tag{20}$$

$$p(x) \leqslant 2\beta, \qquad \text{for all } x \in \{0\} \cup [\delta-\varepsilon/2, 1], \tag{21}$$

$$p(x) \geqslant 1, \qquad \text{for all } x \in [\varepsilon, \delta-\varepsilon]. \tag{22}$$

For a k-hypergraphon U, define $Q_U := \int_{[0,1]^{r[\ell]}} q_U(\mathbf{x})\, \mathrm{d}\mathbf{x}$ to be the average of

$$q_U(\mathbf{x}) := p\left(t_{\mathbf{x}}(\mathrm{Edge}_{k,\ell}, U)\right)$$

over uniform $\mathbf{x} \in [0,1]^{r[\ell]}$. Note by (19) that

$$q_U(\mathbf{x}) = \sum_{i=0}^{D} a_i \left(t_{\mathbf{x}}(\mathrm{Edge}_{k,\ell}, U)\right)^i = a_0 + \sum_{i=1}^{D} a_i\, t_{\mathbf{x}}(\mathrm{Edge}_{k,\ell}^i, U), \quad \text{for all } \mathbf{x} \in [0,1]^{r[\ell]},$$

Thus, by the linearity of integral and by (17),

$$Q_U \;=\; a_0 + \sum_{i=0}^{D} a_i \int_{[0,1]^{r[\ell]}} t_{\mathbf{x}}(\mathrm{Edge}_{k,\ell}^i, U)\,\mathrm{d}\mathbf{x} = a_0 + \sum_{i=1}^{D} a_i\, t(\llbracket \mathrm{Edge}_{k,\ell}^i \rrbracket, U).$$

Let $U_n := W^{G_n}$ be the hypergraphon of G_n. As G_n converges to W, we have that, for every $i \in [D]$, the (unlabelled) k-graph density $t(\llbracket \mathrm{Edge}_{k,\ell}^i \rrbracket, U_n)$ converges to $t(\llbracket \mathrm{Edge}_{k,\ell}^i \rrbracket, W)$ and thus

$$\lim_{n\to\infty} Q_{U_n} = Q_W. \tag{23}$$

If we take $n \to \infty$ and evaluate Q_{U_n} then, with $m := |V(G_n)|$, the outer integral becomes the average value of the polynomial p evaluated at the (obviously defined) rooted density $t_{v_1,\dots,v_\ell}(\mathrm{Edge}_{k,\ell}, G_n)$ where $v_1,\dots,v_\ell$ are independent uniformly chosen vertices of G_n. For each of these evaluations of p, its argument is either 0 (if some two v_i's coincide or the ℓ-set $\{v_1,\dots,v_\ell\}$ is not covered by any edge of G_n) or at least $\delta_\ell^+(G_n)\cdot(k-\ell)!/m^{k-\ell}$ (in all other cases). Thus, if n is large enough, then by $\delta_\ell^+(G_n)\cdot(k-\ell)!/m^{k-\ell} = \delta + o(1) > \delta - \varepsilon/2$ and by (21) each computed value of p is at most 2β. Thus we have that $Q_{U_n} \leqslant 2\beta$ for all large n.

On the other hand, since $q_W(\mathbf{x}) \geqslant 0$ for every $\mathbf{x}$ by (20), we have by (22) that

$$Q_W \geqslant \int_{X_\varepsilon} q_W(\mathbf{x})\,\mathrm{d}\mathbf{x} \geqslant 1 \cdot \mu(X_\varepsilon) \geqslant \varepsilon.$$

Since $2\beta < \varepsilon$, Q_W cannot be be the limit of Q_{U_n}, a contradiction to (23) proving Lemma 6. $\square$

Proof of Lemma 7. Let $\delta := \delta_\ell^+(W)$, $n \to \infty$ and let $G \sim \mathbb{G}(n, W)$. We have to show that $\delta_\ell^+(G)$ is unlikely to be far from $\delta\binom{n-\ell}{k-\ell}$. Assume that W is non-zero as otherwise G is empty with probability 1 and the lemma trivially holds.

Consider the vertex exposure martingale $(Y_0,\dots,Y_n)$ with $Y_n = |E(G)|$, where for $i = 1,\dots,n$ we expose all x_A's with $\max A = i$ as well as all edges of G whose maximal element is i. We have that $|Y_i - Y_{i-1}| \leqslant \binom{n-1}{k-1}$ for every $i \in [n]$ because if we change our choices at Step i this will affect only edges of G containing i. (Note that, unlike in the proof of Proposition 1, we do not have to worry about the measure-0 event that some different x_i's coincide.) Also,

$$Y_0 = \mathbb{E}(Y_n) = t(\mathrm{Edge}_{k,0}, W)\binom{n}{k},$$

which is $\Omega(n^k)$ since W is non-zero. Azuma's inequality gives that the probability that G spans no edges (i.e. $Y_n = 0$) is $\mathrm{e}^{-\Omega(n)} < \varepsilon/3$.

Next, let us show that, for every fixed ℓ-set $L \subseteq [n]$, the probability that its degree $\deg_L(G)$ in $G \sim \mathbb{G}(n, W)$ is positive but less than $(\delta - \varepsilon)\binom{n-\ell}{k-\ell}$ is at most $(\varepsilon/3)\binom{n}{\ell}^{-1}$. By symmetry, assume that $L = [\ell]$. Take any $\mathbf{x} \in [0,1]^{r[\ell]}$. By ignoring a set of $\mathbf{x}$ of measure 0, we have that $\deg_W(\mathbf{x})$ is 0 or at least δ. In the former case, $\deg_L(G) = 0$ with probability 1. So suppose that $\deg_W(\mathbf{x}) \geqslant \delta$. Given $\mathbf{x}$, consider the natural vertex exposure martingale $(X_0, \ldots, X_{n-\ell})$ for $\deg_L(G)$. It holds that $|X_i - X_{i-i}| \leqslant \binom{n-\ell-1}{k-\ell-1}$ for each $i \in [n-\ell]$. Since

$$X_0 = \mathbb{E}[X_{n-\ell}] = \deg_W(\mathbf{x})\binom{n-\ell}{k-\ell} \geqslant \delta\binom{n-\ell}{k-\ell},$$

Azuma's inequality gives that the probability of $X_{n-\ell} < X_0 - \varepsilon\binom{n-\ell}{k-\ell}$ is $\mathrm{e}^{-\Omega(n)} < (\varepsilon/3)\binom{n}{\ell}^{-1}$, as claimed.

The Union Bound over all $\binom{n}{\ell}$ choices of an ℓ-set $L \subseteq [m]$ shows that the probability that G is empty or $\delta_\ell^+(G) < (\delta - \varepsilon)\binom{n-\ell}{k-\ell}$ is at most $2\varepsilon/3$.

Finally, it remains to upper bound the probability that $\delta_\ell^+(G)$ is too large. Since this part of the lemma is not used anywhere else in the paper, we will be rather brief. By the definition of $\delta = \delta_\ell^+(W)$ and since W is non-zero, the set

$$Y := \{\mathbf{x} \in [0,1]^{r[l]} : 0 < \deg_W(\mathbf{x}) \leqslant \delta + \varepsilon/2\}$$

has positive measure. By decreasing $\varepsilon > 0$ if necessary, assume that

$$Z := \{\mathbf{x} \in [0,1]^{r[l]} : \varepsilon < \deg_W(\mathbf{x}) \leqslant \delta + \varepsilon/2\}$$

has positive measure. Take a uniform $\mathbf{x} \in [0,1]^{r([n],k-1)}$. The expected number of ℓ-sets $L \subseteq [n]$ such that $\mathbf{x}_{r(L)} \in Z$ is $\Omega(n^\ell)$. By Azuma's inequality, the probability that no L satisfies this is $\mathrm{e}^{-\Omega(n)} < \varepsilon/6$. Furthermore, if we take any L with $\mathbf{x}_{r(L)} \in Z$ and condition on $\mathbf{x}_{r(L)}$ then Azuma's equality, when applied to the martingale where we expose for each vertex $i \in [n] \setminus L$ all information up to i, shows that the probability of $\deg_G(L)$ being at least $(\varepsilon/2)\binom{n-\ell}{k-\ell}$ away from its expected value $\deg_W(\mathbf{x})\binom{n-\ell}{k-\ell}$ is $\mathrm{e}^{-\Omega(n)} < \varepsilon/6$. Thus with probability at least $1 - \varepsilon/3$ there is an ℓ-set $L \subseteq [n]$ with $0 < \deg_G(L) \leqslant (\delta + \varepsilon)\binom{n-\ell}{k-\ell}$, which implies that $\delta_\ell^+(G) \leqslant (\delta + \varepsilon)\binom{n-\ell}{k-\ell}$.

By putting all together, we conclude that, with probability at least $1 - \varepsilon$, the random n-sample $G \sim \mathbb{G}(n, W)$ has positive ℓ-degree $(\delta \pm \varepsilon)\binom{n-\ell}{k-\ell}$, as desired. □

5 Concluding remarks

One can also consider extremal limits for the ℓ-degree Turán problem. Namely, let $\mathcal{W}_\ell(\mathcal{F})$ consist of all k-hypergraphons W such that there is a sequence $(G_n)_{n=1}^\infty$ of $\mathcal{F}$-free k-graphs convergent to W such that $|V(G_n)| \to \infty$ and the *mininum ℓ-degree*

$$\delta_\ell(G_n) := \min\{\deg_{G_n}(L) : L \subseteq [n],\ |L| = \ell\}$$

is $(\gamma_\ell(\mathcal{F})+o(1))\binom{|V(G_n)|-\ell}{k-\ell}$ as $n \to \infty$, where $\gamma_\ell(\mathcal{F})$ is defined in (2). Also, let the *minimum ℓ-degree* of a k-hypergraphon W be

$$\delta_\ell(W) := \operatorname*{ess\text{-}inf}_{\mathbf{x}\in[0,1]^{r[\ell]}} \deg_W(\mathbf{x}).$$

Our proof of Theorem 5 can be easily adapted to to produce the following result (in fact, the proof is simpler since we do not have to treat ℓ-sets of zero degree in a special way).

Theorem 8. *Let $k > \ell \geqslant 0$ be integers and let $\mathcal{F}$ be a k-graph family. Then $\gamma_\ell(\mathcal{F})$ is the maximum of $\delta_\ell(W)$ over all $\mathcal{F}$-free k-hypergraphons W. Moreover, $W \in \mathcal{W}_\ell(\mathcal{F})$ if and only if W is $\mathcal{F}$-free and $\delta_\ell(W) = \gamma_\ell(\mathcal{F})$.*

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References

[1] N. Alon and J. H. Spencer, *The probabilistic method*, 4th ed., Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2016.

[2] J. Balogh, F. C. Clemen, and B. Lidický, *Hypergraph Turán problems in ℓ_2-norm*, Surveys in combinatorics 2022, London Math. Soc. Lecture Note Ser., vol. 481, Cambridge Univ. Press, Cambridge, 2022, pp. 21–63.

[3] J. Balogh, N. Lemons, and C. Palmer, *Maximum size intersecting families of bounded minimum positive co-degree*, SIAM J. Discr. Math. **35** (2021), 1525–1535.

[4] G. Elek and B. Szegedy, *A measure-theoretic approach to the theory of dense hypergraphs*, Advances in Math. **231** (2012), 1731–1772.

[5] V. Falgas-Ravry, O. Pikhurko, E. R Vaughan, and J. Volec, *The codegree threshold of K_4^-*, J. Lond. Math. Soc. **107** (2023), 1660–1691.

[6] R. Glebov, D. Král', and J. Volec, *A problem of Erdős and Sós on 3-graphs*, Israel J. Math. **211** (2016), 349–366.

[7] A. Halfpap, N. Lemons, and C. Palmer, *Positive co-degree density of hypergraphs*, arXiv:2207.05639, 2022.

[8] G. O. H. Katona, *A theorem on finite sets*, Theory of Graphs. Proc. Colloq. Tihany, Akademiai Kiado. Academic Press, New York, 1966, pp. 187–207.

[9] G. O. H. Katona, T. Nemetz, and M. Simonovits, *On a graph problem of Turán (In Hungarian)*, Mat. Fiz. Lapok **15** (1964), 228–238.

[10] P. Keevash, *Hypergraph Turán problem*, Surveys in Combinatorics (R. Chapman, ed.), London Math. Soc. Lecture Notes Ser., vol. 392, Cambridge Univ. Press, 2011, pp. 83–140.

[11] J. B. Kruskal, *The number of simplices in a complex*, Mathematical Optimization Techniques (R. Bellman, ed.), Univ. California Press, Berkeley, 1963, pp. 251–278.

[12] A. Lo and K. Markström, *ℓ-degree Turán density*, SIAM J. Discr. Math. **28** (2014), 1214–1225.

[13] L. Lovász, *Combinatorial problems and exercises*, North-Holland, 1992.

[14] D. Mubayi and Y. Zhao, *Co-degree density of hypergraphs*, J. Combin. Theory (A) **114** (2007), 1118–1132.

[15] C. Reiher, V. Rödl, and M. Schacht, *On a Turán problem in weakly quasirandom 3-uniform hypergraphs*, J. Europ. Math. Soc **20** (2018), 1139–1159.

[16] A. Sidorenko, *What we know and what we do not know about Turán numbers*, Graphs Combin. **11** (1995), 179–199.

[17] E. M. Stein and R. Shakarchi, *Real analysis*, Princeton Lectures in Analysis, vol. 3, Princeton University Press, Princeton, NJ, 2005, Measure theory, integration, and Hilbert spaces.

[18] Z. Wu, *Positive co-degree Turán number for C_5 and C_5^-*, `arXiv:2212.12815`, 2022.

[19] Y. Zhao, *Hypergraph limits: a regularity approach*, Random Struct. Algorithms **47** (2015), 205–226.