A combinatorial proof of Buryak-Feigin-Nakajima

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Abstract

Buryak, Feigin and Nakajima computed a generating function for a family of partition statistics by using the geometry of the $\mathbb{Z}/c\mathbb{Z}$ fixed point sets in the Hilbert scheme of points on $\mathbb{C}^2$. Loehr and Warrington had already shown how a similar observation by Haiman using the geometry of the Hilbert scheme of points on $\mathbb{C}^2$ could be made purely combinatorial. We extend Loehr and Warrington's techniques to also account for cores and quotients. In particular, we construct a multigraph $M_{r,s,c}$ that is a direct refinement of Loehr and Warrington's multigraphs $M_{r,s}$, retains the relevant partition data, and is preserved by an involution $I_{r,s,c}$ which we use to prove the equidistribution of a family of partition statistics. As a consequence, we obtain a purely combinatorial proof of Buryak, Feigin, and Nakajima's result.

More precisely, we define a family of partition statistics $\left\{h^+_{x,c}, x \in (0,\infty]\right\}$ and give a combinatorial proof that for all x and all positive integers c,

$$\sum q^{|\lambda|}t^{h^+_{x,c}(\lambda)} = q^{|\mu|}\prod_{i\geqslant 1}\frac{1}{(1-q^{ic})^{c-1}}\prod_{j\geqslant 1}\frac{1}{1-q^{jc}t},$$

where the sum ranges over all partitions λ with c-core μ.

Mathematics Subject Classifications: 05A17, 05E14.

1 Introduction

A *partition* λ of a positive integer n is a non-increasing sequence of positive integers $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_l$ such that $\lambda_1 + \cdots + \lambda_l = n$. We write $|\lambda| = n$. We represent partitions as *Young diagrams*, informally by drawing λ_i unit squares in a row, left to right, starting with a square with bottom left corner $(0, i-1)$.

For a square $\square$ in a Young diagram, $a(\square)$ is the number of squares to the right of $\square$ in the same row, and $l(\square)$ is the number of squares above $\square$ in the same column. For example, the square with bottom left corner $(1, 0)$ in Figure 1 has $a(\square) = 2$, $l(\square) = 1$.

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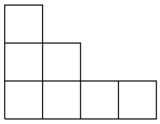


Figure 1: The Young diagram for the partition $(4, 2, 1)$ of 7.

We also define $h(\square) = a(\square) + l(\square) + 1$ and let $h_{r,s}(\lambda)$ count the number of squares in the Young diagram of λ such that $(r + s) \mid h(\square)$ and $rl(\square) = s(a(\square) + 1)$.

Buryak, Feigin, and Nakajima gave a geometric proof of the following [3, Corollary 1.3]

$$\sum_{\lambda \in \mathrm{Par}} q^{h_{r,s}(\lambda)} t^{|\lambda|} = \prod_{\substack{i \geqslant 1 \\ r+s \nmid i}} \frac{1}{1 - q^i} \prod_{i \geqslant 1} \frac{1}{1 - q^{i(r+s)} t} \tag{1}$$

where Par denotes the set of all partitions. One result of this paper is a purely combinatorial proof of the same result.

We now explain the geometric significance of generating function (1). The *Hilbert Scheme of n points on* $\mathbb{C}^2$, $\mathrm{Hilb}_n(\mathbb{C}^2)$, parametrises the ideals $I \subset \mathbb{C}[x, y]$ such that $\dim_{\mathbb{C}} (\mathbb{C}[x, y]/I) = n$. $\mathrm{Hilb}_n(\mathbb{C}^2)$ admits a torus action by lifting the $(\mathbb{C}^*)^2$ action on $\mathbb{C}^2$ given by

$$(t_1, t_2) \cdot (x, y) = (t_1 x, t_2 y) \tag{2}$$

to the action on ideals $I \subset \mathbb{C}[x, y]$ given by

$$(t_1, t_2) \cdot I = \{p(t_1^{-1} x, t_2^{-1} y) : p(x, y) \in I\}. \tag{3}$$

Let

$$\Gamma_m = \left\langle \left(e^{\frac{2\pi i}{m}}, e^{\frac{-2\pi i}{m}} \right) \right\rangle \tag{4}$$

be a finite subgroup of $\mathbb{C}^2$ of order m and let $T_{r,s}$ be the one-parameter subtorus of $\mathbb{C}^2$ given by

$$T_{r,s} = \{(t^r, t^s) : t \in \mathbb{C}^*\}. \tag{5}$$

Let $H_*^{\mathrm{BM}}(X; \mathbb{Q})$ denote the Borel-Moore homology of X with rational coefficients and let

$$P_q^{\mathrm{BM}}(X) = \sum_{i \geqslant 0} \dim H_i^{\mathrm{BM}}(X; \mathbb{Q}) q^{\frac{i}{2}}. \tag{6}$$

Buryak, Feigin and Nakajima [3, Theorem 1.2] proved that, if r, s are non-negative integers with $r + s \geqslant 1$,

$$\sum_{n \geqslant 0} P_q^{\mathrm{BM}} \left(\mathrm{Hilb}_n(\mathbb{C}^2)^{\Gamma_{r+s} \times T_{r,s}} \right) t^n = \prod_{\substack{i \geqslant 1 \\ r+s \nmid i}} \frac{1}{1 - q^i} \prod_{i \geqslant 1} \frac{1}{1 - q^{i(r+s)} t}, \tag{7}$$

where $\mathrm{Hilb}_n(\mathbb{C}^2)^{T_{r,s} \times \Gamma_{r+s}}$ is the fixed point locus of $\mathrm{Hilb}_n(\mathbb{C}^2)$ under the action of $T_{r,s} \times \Gamma_{r+s}$. The proof is split into two results. One [3, Lemma 3.1] shows that the left hand side of (7) is dependent only on $r + s$. The other [3, Lemma 3.2] computes the left hand side of (7) in the case $s = 0$. Broadly speaking, Buryak, Feigin, and Nakajima compute the dimension of the Białynicki-Birula cells when the "slope" of the acting one parameter torus is very steep, and prove that the slope itself does not affect the eigenspace.

Finally, using the methods of [2], a cell decomposition of $\mathrm{Hilb}_n(\mathbb{C}^2)^{T_{r,s}\times\Gamma_{r+s}}$ shows that the left hand side of (7) in the Grothendieck ring of varieties is given by

$$\sum_{\lambda\in\mathrm{Par}} q^{h_{r,s}(\lambda)} t^{|\lambda|}. \tag{8}$$

In [7], Loehr and Warrington gave a bijective proof that a partition statistic h_x^+ is independent of the parameter x. In a similar vein to the above, Haiman observed that h_x^+ accounts for the distribution of the dimension of the Białynicki-Birula cells associated to the action of $(\mathbb{C}^*)^2$ on $\mathrm{Hilb}_n(\mathbb{C}^2)$, i.e. the case when Γ_m is the trivial group.

We are interested in

Question 1. Is there a bijection proving (1)?

To answer this question, we also ask the following.

Question 2. Can we use Loehr and Warrington's methods to produce a related bijection that preserves the core of a partition?

We provide an affirmative answer to Question 2, and use the bijection we produce to provide a partial answer to Question 1. In particular, we define a partition statistic $h_{x,c}^+$ where $x \in [0, \infty)$ and c is a positive integer, and $h_{x,c}^+(\lambda)$ counts the number of squares $\square \in \lambda$ such that both

- the hook length $h(\square)$ is divisible by c, and
- if $a(\square)$ and $l(\square)$ denote the size of the arm and leg of $\square$ respectively,

$$\frac{a(\square)}{l(\square)+1} \leqslant x < \frac{a(\square)+1}{l(\square)}. \tag{9}$$

In the case $c = 1$, we recover Loehr and Warrington's statistic h_x^+. We then exhibit a bijection proving a refinement (Theorem 46) of [3, Lemma 3.1]. The key ingredient is a bijection at rational slope showing that $h_{x,c}^+$ is equidistributed over partitions with a fixed c-core with the statistic $h_{x,c}^-$, counting boxes $\square$ in the Young diagram such that both

- the hook length $h(\square)$ is divisible by c, and
- if $a(\square)$ and $l(\square)$ denote the size of the arm and leg of $\square$ respectively,

$$\frac{a(\square)}{l(\square)+1} < x \leqslant \frac{a(\square)+1}{l(\square)}. \tag{10}$$

Theorem 47. For all positive rational numbers x and all integers $n \geqslant 0$,

$$\sum t^{h^+_{x,c}(\lambda)} = \sum t^{h^-_{x,c}(\lambda)}$$

where both sums range over partitions λ of n with a fixed c-core μ.

To do so, we adapt Loehr and Warrington's construction of a bijection $I_{r,s}$ [7] to give a new bijection $I_{r,s,c}$ which preserves the c-core of a partition and "picks out" whether or not c divides the hook length of a cell contributing to a partition statistic. In the case $c = 1$, $I_{r,s,c}$ specialises to $I_{r,s}$. To construct $I_{r,s,c}$, we refine Loehr and Warrington's multigraph $M_{r,s}$ to a multigraph $M_{r,s,c}$ which also sees the c-core of a partition. In order to do so, we recast the c-abacus construction first introduced in [4] in terms of complete circuits of multigraphs and define an appropriate notion of homomorphism, taking $M_{r,s,c}$ to be the product of the c-abacus and $M_{r,s}$ with respect to these homomorphisms.

We then give a combinatorial proof of a result (Theorem 44), computing the distribution of $h^+_{0,c}$. This result in particular implies [3, Lemma 3.2]. Whilst our proof is combinatorial, it is not bijective, as we use a multi-counting argument. The map we define was previously defined by Walsh and Waarnar [9, §6].

Theorem 44. For all x in $[0, \infty)$,

$$\sum q^{|\lambda|} t^{h^+_{0,c}(\lambda)} = q^{|\mu|} \prod_{i \geqslant 1} \frac{1}{(1 - q^{ic})^{c-1}} \prod_{j \geqslant 1} \frac{1}{1 - q^{jc} t}$$

where the sum ranges over all partitions λ with c-core μ, henceforth denoted Par^c_μ.

Finally, our main theorem (Theorem 46) uses both Theorem 44 and the bijection $I_{r,s,c}$ to compute the following distribution, and we explain how (1) follows.

Theorem 46. For all x in $[0, \infty)$,

$$\sum q^{|\lambda|} t^{h^+_{x,c}(\lambda)} = q^{|\mu|} \prod_{i \geqslant 1} \frac{1}{(1 - q^{ic})^{c-1}} \prod_{j \geqslant 1} \frac{1}{1 - q^{jc} t}$$

where the sum is taken over all partitions $\lambda \in \mathrm{Par}^c_\mu$.

1.1 Organisation of the paper

Section 2 recalls some definitions from partition combinatorics. In particular, we recall the abacus construction (the standard reference for this is [5, §2.7]) and recall some basic generating functions. The section builds up to proving Theorem 44, which uses a bijection introduced in [9] to compute the distribution of $h^+_{0,c}$ over Par^c_μ, the set of partitions with c-core μ.

Section 3 defines the main partition statistics of interest, $\mathrm{mid}_{x,c}$, $\mathrm{crit}^-_{x,c}$, $\mathrm{crit}^+_{x,c}$, $h^+_{x,c}$ and $h^-_{x,c}$ where $h^\pm_{x,c} = \mathrm{mid}_{x,c} + \mathrm{crit}^\pm_{x,c}$. Then, we introduce our main theorem, Theorem 46. In view of Theorem 44, it remains to prove that the left hand side is independent of x. An argument analogous to that in [7] is then used to show that the independence of the left hand side from x is implied by a symmetry property when x is rational,

$$\sum_{\lambda\in\mathrm{Par}^c_\mu} q^{|\lambda|} w^{h^+_{x,c}(\lambda)} y^{h^-_{x,c}(\lambda)} = \sum_{\lambda\in\mathrm{Par}^c_\mu} q^{|\lambda|} w^{h^-_{x,c}(\lambda)} y^{h^+_{x,c}(\lambda)}. \tag{11}$$

We use this to give a set of criteria that constitute a sufficient condition for a bijection to prove Theorem 46 in Proposition 50. Finally, the section concludes with a proof that the main result of [3] is a consequence of Theorem 46.

Section 4 defines the multigraph $M_{r,s,c}(\lambda)$ corresponding to a rational $x = \frac{r}{s}$ and positive integer c, defines an ordering $<_{r,s,c}$ on partitions and multigraphs, and a special set of partitions $\lambda_{r,s,k}$. It then goes on to outline the structure of our proofs that $M_{r,s,c}$ remembers partition data. Our proof is structured somewhat differently to Loehr and Warrington's proofs that $M_{r,s}$ remembers partition data in [7]. In particular, we do not prove formulae in terms of $M_{r,s,c}$ for any partition statistics except for $\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c}$. Instead, the section works towards providing an inductive framework to prove that $M_{r,s,c}$ remembers partition data by studying how taking successor at the level of partitions and multigraphs are related, culminating in Proposition 75. One result of this section (Proposition 65) is that the map $\lambda \mapsto M_{r,s,c}(\lambda)$ is injective at the $\lambda_{r,s,k}$, so the map does not lose any data at all at these points, allowing the $\lambda_{r,s,k}$ form a family of base cases. Having outlined the key principles behind the proofs, we then defer the technical checks to Section 6.

Section 5 defines involutions $I_{r,s,c} : \mathrm{Par}^c_\mu \to \mathrm{Par}^c_\mu$ that preserve multigraphs $M_{r,s,c}(\lambda)$.

Section 6 studies how each statistic of interest in Proposition 50 changes when taking successor with respect to the ordering $<_{r,s,c}$, in particular using Proposition 75 to prove that the map $\lambda \mapsto M_{r,s,c}(\lambda)$ remembers the statistics $\mathrm{mid}_{x,c}(\lambda)$ and $\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c}$. It also proves that $I_{r,s,c}$ exchanges the statistics $\mathrm{crit}^+_{x,c}$ and $\mathrm{crit}^-_{x,c}$. Together with the results of Section 4, this completes a combinatorial proof of Theorem 46.

2 Background: partitions, cores, quotients

In this section, we recall first definitions in partition combinatorics, including the abacus construction, cores and quotients. The standard reference for the abacus construction is [5, §2.7], the abacus was first introduced in [4], cores in [8] and quotients in [6]. We take a nonstandard view of the c-core, and describe it as an equivalence class of complete circuits of a directed multigraph M_c. The language we use to describe the abacus is also nonstandard, but the construction is equivalent. We take this approach so that we have descriptions of Loehr and Warrington's construction in [7] and the c-core in terms of directed multigraphs, which allows us to formulate a simultaneous refinement of the two in Section 4. Once we have recalled this theory, we will recall a few standard generating functions and define the map G_c previously defined in [9] and use these to give a combinatorial proof of Theorem 44, which forms our base case.

Definition 3 (Partition,Young diagram)**.** A *partition* of an integer $n \geqslant 0$ is a sequence of non-increasing positive integers $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_t$ with sum n. The *size* of λ, denoted

$|\lambda|$, is n and the *length* of λ is the number of summands, written $l(\lambda) = t$. The *Young diagram* of λ consists of t rows of 1×1 boxes $\square$ in $\mathbb{R}^2$, with λ_i boxes in the ith row for each $1 \leqslant i \leqslant t$. The bottom left corner of the diagram sits at $(0, 0)$.

Example 4. The partition $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$ of 56 has the diagram given in Figure 2.

Informally, the boundary of a partition λ is the bi-infinite path traversing the y-axis from $+\infty$ until it hits a box of the partition, then follows the edge of the Young diagram until it hits the x-axis, before traversing the x-axis to $+\infty$. We split the boundary up into unit steps between lattice points, and view it as a directed multigraph where edges are additionally assigned a label indicating if they are south or east.

Definition 5 (SE directed multigraph)**.** A *SE directed multigraph* $M = (V, E, s, t, d)$ consists of a vertex set V, an edge set E, and three maps $s : E \to V$, $t : E \to V$ and $d : E \to \{\text{South}, \text{East}\}$, called source, target, and direction respectively. We say the edge e *departs from* the vertex v if $s(e) = v$ and we say that e *arrives at* the vertex w if $t(e) = w$. We call e a *south edge* if $d(e) = \text{South}$ and an *east edge* if $d(e) = \text{East}$. We sometimes abbreviate South to S and East to E in contexts where there is no danger of confusion with the edge set.

Definition 6 (Boundary graph)**.** The *boundary graph* $b(\lambda)$ of a partition λ is an SE directed multigraph. The edge set is defined as follows. For natural numbers x, y there is a south edge e with $s(e) = (x, y + 1)$, $t(e) = (x, y)$ if either

- $x = 0$ and $y \geqslant l(\lambda)$, or
- $x > 0$ and $\lambda_{y+1} = x$.

There is an east edge e with $s(e) = (x, y)$ and $t(e) = (x + 1, y)$ if either

- $y = 0$ and $x \geqslant \lambda_1$, or
- $y > 0$ and $\lambda_{y+1} \leqslant x < \lambda_y$.

The vertex set $V(b(\lambda))$ is the union of sources and targets of the edges.

Example 7. Let $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$. The boundary graph of μ is given in Figure 2, the south edges being the downward arrows and the east edges being the rightward arrows.

Note that for any edge e in the boundary graph, the value of $x - y$ at the target of e is one greater than at the source, because taking a unit step south or east increases the value of $x - y$ by 1.

So, the value of $x - y$ at the target of an edge indexes an Eulerian tour, or complete circuit, of $b(\lambda)$. For clarity, we recall the definition of a complete circuit.

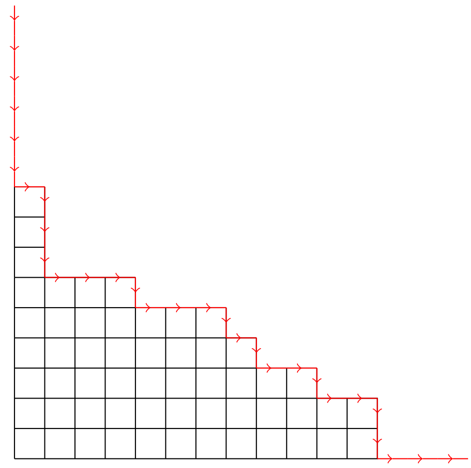


Figure 2: The Young diagram and boundary graph of $(12, 12, 10, 8, 7, 4, 1, 1, 1)$.

Definition 8 (Complete circuit)**.** Given a directed multigraph M, a *complete circuit* of M is an ordering of $E(M)$ such that if e_i and e_{i+1} are consecutive with respect to the ordering, then there is a vertex $v \in V(M)$ such that $t(e_i) = s(e_{i+1})$.

Definition 9 (Boundary tour, boundary sequence, index)**.** If an edge $e \in E(b(\lambda))$ has target (x, y), we say the *index* of e is $i(e) = x - y$. The *boundary tour* is the complete circuit of $b(\lambda)$ where the edges are ordered by index. We write the edges in this ordering as $(\ldots, e_{-2}, e_{-1}, e_0, e_1, e_2, \ldots)$. We say an edge e_j *occurs before* the edge e_k if $j < k$. The *boundary sequence* is the bi-infinite sequence $(d_i)_{i\in\mathbb{Z}}$ where $d_i = d(e_i)$. We write S and E in place of South and East respectively in the boundary sequence.

Example 10. The partition $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$ has boundary sequence

$$\ldots SSSSSSESSSE_0EESEEEESESEESEESSEEE\ldots$$

where d_0 is indicated with a 0 suffix.

2.1 Anatomy of a Young Diagram

Next, we recall some standard partition statistics and how they relate to the boundary sequence, define rimhooks, and connect to cores. We also introduce the notion of an SE directed multigraph homomorphism.

Definition 11 (Hand, foot, arm, leg, inversion, hook length)**.** A box $\square \in \lambda$ can be specified by giving the row and column of the Young diagram that the box sits in. In particular, each box in the Young diagram corresponds to a pair of edges: one south, at extreme right of the row $\square$ lies in, called the *hand of* $\square$, and another east, at the top of the column $\square$ lies in, called the *foot of* $\square$, where the foot necessarily occurs before the hand. Conversely, given an east edge s_1 departing from (x_1, y_1) and arriving at (x_1+1, y_1) and a south edge s_2 departing from (x_2, y_2) and arriving at $(x_2, y_2 - 1)$ such that $x_1 - y_1 < x_2 - y_2$, there is a unique box $\square$ in the Young diagram with bottom left corner $(x_1, y_2 - 1)$ such that s_1 and s_2 are respectively the foot and hand of $\square$. We call such a pair of south and east edges an *inversion*. Hence, we may identify a box in the Young diagram with its hand and foot in the boundary sequence.

The *arm* of $\square$ consists of the boxes that lie strictly to the right of $\square$ in the same row, and the *leg* of $\square$ consists of the boxes that lie strictly above $\square$ in the same column. We denote the number of boxes in the arm of $\square$ by $a(\square)$ and the number of boxes in the leg of $\square$ by $l(\square)$. The *hook length* of $\square$ is defined to be $h(\square) = a(\square) + l(\square) + 1$.

Example 12. The boxes in the arm and leg of the shaded box $\square$ in Figure 3 are labelled with the corresponding body part. The hand of $\square$ is the red arrow, and the foot is the blue arrow, and $a(\square) = 5$ and $l(\square) = 1$, so $h(\square) = 7$.

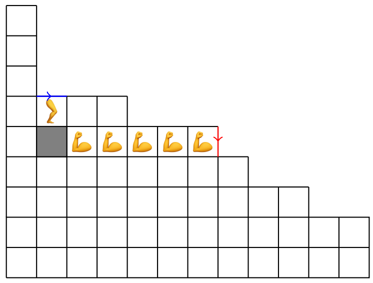


Figure 3: the arm and leg of $\square$.

Proposition 13. *Let λ be a partition. A box in the Young diagram of λ with hook length c corresponds to an inversion (d_i, d_j) in the boundary sequence of λ where $j = i + c$.*

Proof. Let h and f be the hand and foot of $\square$ in the boundary respectively. Consider the map from the arm of $\square$ to the boundary sending each box to its foot. The foot of any box in the arm of $\square$ is an east edge that occurs after f and occurs before h. Conversely, each east edge that occurs after f and occurs before h is the foot of a box in the arm of $\square$. So, $a(\square)$ counts east edges that occur after f and before h.

Analogously, $l(\square)$ counts south edges that occur after f and before h. Thus, $a(\square) + l(\square)$ counts the total number of edges that occur after f and before h. There are $h(\square) - 1$ such edges. $\square$

We now turn our attention to cores and rimhooks, first introduced by Nakayama [8].

Definition 14 (Rimhook)**.** A *rimhook* R of length c is a connected set of c boxes in λ such that removing R gives the Young diagram of a partition, and R does not contain a 2×2 box.

Corollary 15. *Rimhooks of length c are in bijection with boxes of hook length c.*

Proof. Let R be a rimhook of length c in the diagram of a partition λ. Then, by the definition of a rimhook, for every box $\square \in R$ there is an edge in the boundary graph of λ arriving at the top right corner of $\square$. Let $e_i, e_{i+1}, \ldots, e_{i+c-1}$ be the the set of all such edges (since R is connected these edges are consecutive in the boundary tour), and let e_{i+c} be the next edge in the boundary tour.

Since R is removable, $d(e_i) = E$.

We now check that $d(e_{i+c}) = S$. Since e_{i+c-1} arrives at the top right corner of the south-eastern-most square $\square$ in R, e_{i+c} departs from the top right corner of $\square$. If e_{i+c} were an east edge, there would be another box to the right of $\square$ in the same row, contradicting that R is removable. Therefore, by Proposition 13, e_i and e_{i+c} are the foot and hand respectively of a box of hook length c.

Conversely, if $\square$ is a box of hook length c, with foot e_i and hand e_{i+c} then taking the boxes with top right corners the targets of $e_i, e_{i+1}, \ldots, e_{i+c-1}$ gives a rimhook of length c. $\square$

Definition 16. A c-core of a partition λ is a partition obtained by iteratively removing rimhooks of length c from λ until a partition with no rimhooks of length c is obtained. A partition μ is called a c-core if μ has no rimhooks of length c.

Applying Corollary 15 to c-cores gives the following.

Corollary 17. *A partition λ is a c-core if and only if λ has no boxes of hook length c.*

Our aim for now will be to redefine the c-core in the language we wish to use later, and then use it to see that the result of iteratively removing rimhooks of length c is independent of the order in which rimhooks are removed. In order to do so, we need the notion of an SE directed multigraph homomorphism. Informally, these consist of two maps, one between edges, and another between vertices. We require that these maps preserve the direction (S or E) of the edges, and that they be compatible with the source and target maps.

Definition 18 (SE directed multigraph homomorphism)**.** Let $M_1 = (V_1, E_1, s_1, t_1, d_1)$, $M_2 = (V_2, E_2, s_2, t_2, d_2)$ be SE directed multigraphs. A homomorphism of SE directed multigraphs $\varphi : M_1 \to M_2$ is a pair of maps $\varphi_V : V_1 \to V_2$ and $\varphi_E : E_1 \to E_2$ such that for all edges $e \in E_1$,

$$s_2(\varphi_E(e)) = \varphi_V(s_1(e)) \tag{12}$$

$$t_2(\varphi_E(e)) = \varphi_V(t_1(e)) \tag{13}$$

$$d_2(\varphi_E(e)) = d_1(e). \tag{14}$$

In other words, φ is a quiver homomorphism that preserves direction (S or E).

Example 19. Let M_1 be the boundary graph of $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$ and let φ_V be the map taking each vertex (x, y) to $[x - y]$, the class of $x - y$ modulo 2. This map induces the homomorphism q_2 illustrated in Figure 4, with east edges coloured red and south edges coloured blue.

For ease of reading, we draw edges in the image of q_2 from left to right in order of index as $\ldots, q_2(e_{-2}), q_2(e_{-1}), q_2(e_0), q_2(e_1), q_2(e_2), \ldots$.

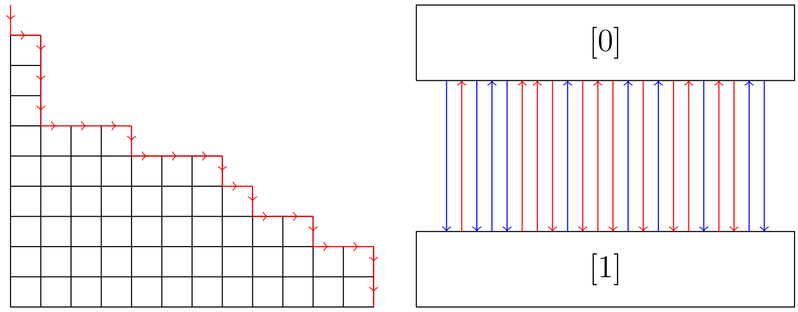


Figure 4: A portion of M_1 and the corresponding edges in $q_2(M_1)$.

We will always work with SE directed multigraph homomorphisms where the edge map φ_E is bijective, so from now on we assume φ_E is bijective for any homomorphism φ. In particular, this assumption allows us to push complete circuits through homomorphisms.

Proposition 20. *Let $\varphi : M_1 \to M_2$ be an SE directed multigraph homomorphism. Let $(e_i)_{i\in I}$ be a complete circuit of M_1. Then $(\varphi_E(e_i))_{i\in I}$ is a complete circuit of M_2.*

Proof. Since φ_E is bijective, we need only check that $s_2(\varphi_E(e_{i+1}) = t_2(\varphi_E(e_i)$ for each $i \in I$. By definition,

$$s_2(\varphi_E(e_{i+1}) = \varphi_V(s_1(e_{i+1})) \tag{15}$$
$$= \varphi_V(t_1(e_i)) \tag{16}$$
$$= t_2(\varphi_E(e_i). \tag{17}$$

□

We have seen already that rimhooks of length c correspond to boxes of hook length c which in turn correspond to inversions in the boundary sequence where, if the first term has index i, the second has index $i+c$. Intuitively enough, then, the useful homomorphism that captures all of this information is the following.

Definition 21 (***c***-abacus tour)**.** Let $(z, w) \sim_c (x, y)$ if $z - w \equiv x - y \pmod{c}$. Then, $q_c : b(\lambda) \to M_c$ is the SE directed multigraph homomorphism induced by imposing the relation $\sim_c$ on the vertices of $b(\lambda)$. The complete circuit $(q_c(e_i))_{i\in\mathbb{Z}}$ of M_c is called the c-abacus tour associated to λ.

Proposition 13 tells us that the number of boxes with hook length divisible by c can be read off from the c-abacus tour by looking at edges that correspond to a hand and foot arriving at the same vertex $(v, [i])$. So, it is sometimes useful to group the edges in a complete circuit by target. This leads us to arrival words.

Definition 22 (Arrival words, departure words)**.** Let $M = (V, E, s, t, d)$ be a directed SE multigraph and let $(e_i)_{i\in I}$ be a complete circuit of M. For $v \in V$ $I_v \subset I$ be the subset of indices such that $t(e_i) = v$. The *arrival word at* v, written v_a, is the sequence of directions $d(e_i)_{i\in I_v}$. The departure word at v is defined analogously, replacing the target map with the source map.

Notation 23. Given a sequence $(d_i)_{i\in I}$ of Ss and Es, we write $\operatorname{inv}(d_i)$ for the number of inversions.

Proposition 24. *Let λ be a partition with boundary tour $(e_i)_{i\in\mathbb{Z}}$ and let M_c have vertex set. Then, taking arrival words with respect to the complete circuit $(q_c(e_i))_{i\in\mathbb{Z}}$*

$$|\{\Box \in \lambda : c \mid h(\Box)\}| = \sum_{i=0}^{c-1} \operatorname{inv}([i]_a). \tag{18}$$

Proof. Apply Proposition 13. □

2.2 Alignment and charge

So far, we have associated to every partition a boundary sequence, a bi-infinite sequence of Ss and Es such that if we travel far enough to the left in the sequence every entry is an S, and if we travel far enough to the right, every entry is an E. We will now study these sequences in general, and identify which of them arise as boundary sequences of a partition. Then, we will define an equivalence relation on partitions, which we shall show is equivalent to having the same c-core. We will use this to show that partitions have a unique c-core, to define the c-quotients originally studied by [6], and to give a bijection between partitions of fixed c-core and c-tuples of partitions.

Definition 25 (Charge)**.** Let $D = (d_i)_{i\in\mathbb{Z}}$ be a bi-infinite sequence with $d_i \in \{S, E\}$, for each i such that for some $M \in \mathbb{N}$, $\forall m \geqslant M$, $d_{-m} = S$ and $d_m = E$. Fix an integer k. Let e_k be the number of Es in $(d_i)_{i\in\mathbb{Z}}$ with index at most k,

$$e_k = |\{d_j : d_j = E \text{ and } j \leqslant k\}|. \tag{19}$$

Similarly, let s_k be the number of Ss with index greater than k,

$$s_k = |\{d_j : d_j = S \text{ and } j > k\}|. \tag{20}$$

Then, the k-charge of D, written $\operatorname{ch}_k(D)$ is $e_k - s_k - k$.

Proposition 26. *If k and l are integers, and D is as in Definition 25, then* $\operatorname{ch}_k(D) = \operatorname{ch}_l(D)$.

Proof. We check that $\operatorname{ch}_{k+1}(D) = \operatorname{ch}_k(D)$. The proposition then follows by repeated application of the equality. Suppose $d_{k+1} = E$. Then, $e_{k+1} = e_k + 1$ and $s_{k+1} = s_k$. So,

$$\begin{aligned}
\operatorname{ch}_{k+1}(D) &= e_{k+1} - s_{k+1} - (k+1) && (21)\\
&= e_k + 1 - s_k - (k+1) && (22)\\
&= e_k - s_k - k && (23)\\
&= \operatorname{ch}_k(D). && (24)
\end{aligned}$$

Similarly, if $d_{k+1} = S$, then $e_{k+1} = e_k$ and $s_{k+1} = s_k - 1$, so $\mathrm{ch}_{k+1}(D) = \mathrm{ch}_k(D)$. Therefore, $\mathrm{ch}_k(D)$ is independent of k. □

So, in place of $\mathrm{ch}_k(D)$, we may simply write $\mathrm{ch}(D)$.

Proposition 27. *A sequence D as in Definition 25 is the boundary sequence of a partition if and only if* $\mathrm{ch}(D) = 0$.

Proof. Suppose D is the boundary sequence of a partition. Let (x_1, y_1) be the point on the line $x - y = k$ on the boundary of a partition λ. Since x_1 counts the number of south edges with index greater than k, and y_1 counts the number of east edges with index at most k, $\mathrm{ch}(D) = x_1 - y_1 - k = 0$.

If $\mathrm{ch}(D) = 0$, then we may reconstruct λ from D by placing a point at (e_k, s_k), and drawing the partition boundary in two halves: one as an infinite path departing from (e_k, s_k) taking unit steps with orientations given by $(d_i)_{i>k}$ and the other as an infinite path arriving at (e_k, s_k) taking unit steps with orientations given by $(d_i)_{i\leqslant k}$. □

Definition 28 (The relation $\boldsymbol{\sim_c}$)**.** Let λ and μ be partitions and let the arrival words taken from the c-abacus tours of λ and μ be $[0]^\lambda_a, \ldots, [c-1]^\lambda_a$, and $[0]^\mu_a, \ldots, [c-1]^\mu_a$, respectively. Define the relation $\lambda \sim_c \mu$ if, for all i with $0 \leqslant i \leqslant c-1$,

$$\mathrm{ch}([i]^\lambda_a) = \mathrm{ch}([i]^\mu_a). \tag{25}$$

Example 29. Let $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$ and refer to Figure 4. When $c = 2$, $[0]^\mu_a$ is given by

$$\cdots\ S\ S\ S\ E\ S\ E\ E\ E\ |\ E\ E\ E\ S\ E\ S\ E\ E \cdots.$$

where the bar separates terms corresponding to edges of negative or zero index from those of positive index.

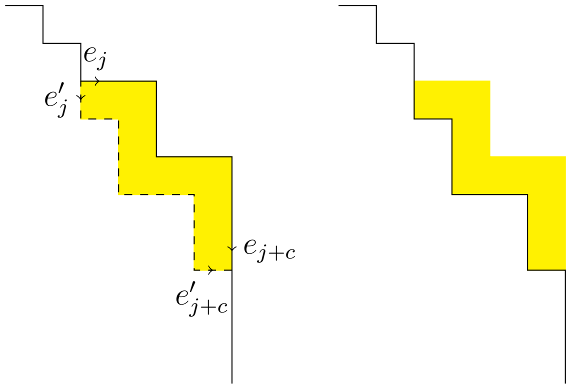
So, $\mathrm{ch}([0]^\mu_a) = 4 - 2 = 2$. Analogously, $[1]^\mu_a$ is

$$\cdots\ S\ S\ S\ S\ S\ S\ E\ S\ |\ E\ S\ S\ E\ E\ S\ E\ E\ \cdots.$$

So, $\mathrm{ch}([1]^\mu_a) = 1 - 3 = -2$.

Proposition 30. *If λ is a partition containing a rimhook R of length c and λ' is the partition obtained from λ by removing R, then $\lambda \sim_c \lambda'$.*

Proof. Let the boundary tours of λ and λ' be $(e_i)_{i\in\mathbb{Z}}$ and $(e'_i)_{i\in\mathbb{Z}}$. First, we analyse how the boundary sequences $(d(e_i))$ and $(d(e'_i))$ differ. Let R have south-eastern most box $\square_2$ and north-western most box $\square_1$. Let e_j be the east edge traversing the top edge of $\square_1$, so that e_{j+c} is the south edge traversing the right of $\square_2$.



Since we remove $\square_1$ and $\square_2$, $d(e_j) = E$, $d(e'_j) = S$, and $d(e_{j+c}) = S$ and $d(e'_{j+c}) = E$. Let $j = qc + r$ for $0 \leqslant r \leqslant c - 1$. Since the rimhook does not contain a 2×2 box and is connected, the portion of the boundary of λ' between the lines $x - y = j + 1$ and $x - y = c + j - 1$ is a translate of the original partition boundary by $(-1, -1)$, so $d(e_i) = d(e'_i)$ for all $i \notin \{j, j + c\}$. So, for all $0 \leqslant s \leqslant c - 1$ with $s \neq r$, $[s]_a^\lambda = [s]_a^{\lambda'}$, and the arrival word

$$([r]_a^{\lambda'})_i = \begin{cases} ([r]_a^\lambda)_q & i = q + 1 \\ ([r]_a^\lambda)_{q+1} & i = q \\ ([r]_a^\lambda)_i & \text{otherwise.} \end{cases} \tag{26}$$

So,

$$\begin{aligned} \mathrm{ch}([r]_a^{\lambda'}) &= \mathrm{ch}_{q-1}([r]_a^{\lambda'}) && (27) \\ &= \mathrm{ch}_{q-1}([r]_a^{\lambda}) && (28) \\ &= \mathrm{ch}([r]_a^{\lambda}). && (29) \end{aligned}$$

□

Corollary 31. *The c-core of λ is unique, and $\lambda \sim_c \mu$ if and only if λ and μ have the same c-core.*

Proof. If λ has c-core ν, then ν is obtained from λ by iteratively removing rimhooks of length c from R, so by Proposition 30, $\lambda \sim_c \nu$. Every partition has at least one c-core, so it remains to check that if μ and ν are both c-cores with $\mu \sim_c \nu$ then $\mu = \nu$. By Propositions 13 and 24, if μ and ν are both c-cores then for each i, the arrival words $[i]_a^\mu$ and $[i]_a^\nu$ do not contain any inversions. So, both consist of a string of Ss up to some index, and a string of Es thereafter. Since $\mu \sim_c \nu$, the charge of both $[i]_a^\mu$ and $[i]_a^\nu$ must be the same, and therefore $[i]_a^\mu = [i]_a^\nu$. □

The important consequence for us will be the following.

Corollary 32. *Let λ and μ be partitions. Then λ and μ have the same c-core if there is a value of m with $c \mid m$ such that for each $[i]$, both of the following hold.*

- *the arrival words in the c-abacus tour of λ and μ agree after the entry with index m;*

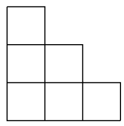
- *the portion of* $[i]_a^\lambda$ *with index at most* m *is a permutation of the portion of* $[i]_a^\mu$ *with index at most* m*.*

Example 33. We will calculate the 2-core λ of $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$. By Corollary 32 and the calculation in Example 29, λ is the unique 2-core with $\mathrm{ch}([0]_a^\lambda) = 2$ and $\mathrm{ch}([1]_a^\lambda) = -2$.

So, placing a bar in the bi-infinite string with no inversions to separate edges with positive index from those with negative or 0 index, the arrival words in $M_2(\lambda)$ at 0 and 1 respectively, are

$$\begin{array}{l} \cdots\ S\ S\ S\ S\ S\ S\ S\ S\ \mid S\ S\ E\ E\ E\ E\ E\ E\ \cdots \\ \cdots\ S\ S\ S\ S\ S\ S\ E\ E\ \mid E\ E\ E\ E\ E\ E\ E\ E \cdots . \end{array}$$

So, the 2-core is (3,2,1).



Proposition 34. *There is an bijective map* f *from* c*-core partitions to a* $\mathbb{Z}$*-module of length* $c-1$.

Proof. Consider the c-abacus of a c-core partition. The charges $(\mathrm{ch}([0]), \ldots, \mathrm{ch}([c-1]))$ specify the c-core. A c-tuple of integers $(a_0, \ldots, a_{c-1})$ represents the charges of a partition if and only if $\sum_{i=0}^{c-1} a_i = 0$. So, sending a c-core to the c-tuple of charges gives a bijective map with the $\mathbb{Z}$-module $M = \langle e_1, \ldots, e_c : \sum_{i=0}^{c-1} e_i = 0 \rangle$. □

Fix a positive integer c, a c-core μ, and a non-negative integer n. Let $\mathrm{Par}_\mu^c(n)$ denote the set of partitions of E with c-core μ. Let Par_μ^c denote the set of all partitions with c-core μ, and let Par denote the set of all partitions.

Definition 35 (Quotient)**.** The c-quotient of λ is the c-tuple of partitions given by $(q_1(\lambda), q_2(\lambda), \ldots, q_c(\lambda))$, where $q_i(\lambda)$ is the partition with boundary sequence $[i]_a$, with the index shifted so that the charge is 0.

Definition 36 (Quotient map)**.** The *quotient map* $\phi : \mathrm{Par}_\mu^c \to (\mathrm{Par})^c$ has

$$\phi(\lambda) = (q_1(\lambda), \ldots, q_c(\lambda).$$

Proposition 37. *For* $\lambda \in \mathrm{Par}_\mu^c$,

$$|\lambda| = |\mu| + c \sum_{i=1}^{c} |q_i(\lambda)|. \tag{30}$$

Proof. By Proposition 24, the number of boxes with hook length divisible by c are given by $\sum_{i=1}^{c} \mathrm{inv}[i]_a$. Starting from the c-abacus tour of μ, we can obtain the c-abacus tour of λ by adding these inversions one at a time. Adding each inversion corresponds to adding a rimhook of length c to the diagram, so contributes c to $|\lambda|$. □

2.3 The map G_c

Now we set about proving Theorem 44. We first recall three standard generating functions.

Proposition 38.

$$\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} t^{l(\lambda)} = \prod_{m \geqslant 1} \frac{1}{1 - q^m t} \tag{31}$$

$$\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} = \prod_{m \geqslant 1} \frac{1}{1 - q^m} \tag{32}$$

$$\sum_{\lambda \in \mathrm{Par}_\mu^c} q^{|\lambda|} = q^{|\mu|} \prod_{m \geqslant 1} \frac{1}{(1 - q^{mc})^c} \tag{33}$$

Proof. We may rewrite the right hand side of (31) as

$$\prod_{m \geqslant 1} 1 + q^m t + q^{2m} t^2 + q^{3m} t^3 + \dots,$$

so that picking a term $q^{km} t^k$ for each m corresponds to declaring that λ contains k parts of size m, contributing $|km|$ to λ and k to $l(\lambda)$, giving the left hand side. Setting $t = 1$ in (31) gives (32).

For (33), Proposition 37 tells us that the map ϕ gives a bijection between $\lambda \in \mathrm{Par}_\mu^c$ and c-tuples of partitions $(q_1, \dots, q_c)$ where $|\lambda| = |\mu| + c \sum_{i=1}^c |q_i(\lambda)|$. The right hand side of (33) corresponds to all choices of c-tuples $q_1, \dots, q_c \in \mathrm{Par}$, and the weighting by c corresponds to each box in q_i corresponding to c boxes in λ. □

Next, we define a partition statistic $\lambda_\square^{c*}$ that arises as a special case of one of the statistics that we study.

For a positive integer d, let $m_d(\lambda)$ denote the number of parts of λ of size d, and for fixed c let $\lambda_\square^{c*}$ denote the weighted sum

$$\lambda_\square^{c*} = \sum_{d=1}^{\infty} \left\lfloor \frac{m_d(\lambda)}{c} \right\rfloor. \tag{34}$$

In words, $\lambda_\square^{c*}$ counts the number of rectangles, of any width, of positive height divisible by c in the diagram of λ such that the whole right edge of the rectangle, and at least the rightmost step of the top edge, lies on the boundary of λ.

Example 39. Let $c = 3$. The partition $\lambda = (7, 7, 4, 4, 4, 4, 4, 4, 4, 3, 2, 2, 2, 1)$ has $m_7(\lambda) = 2$, $m_4(\lambda) = 7$, $m_3(\lambda) = 1$, $m_2(\lambda) = 3$ and $m_1(\lambda) = 1$. So, the only nonzero contributions to $\lambda_\square^{3*}$ are when $d = 2$ and $d = 4$, and

$$\lambda_\square^{3*} = \left\lfloor \frac{m_2(\lambda)}{3} \right\rfloor + \left\lfloor \frac{m_4(\lambda)}{3} \right\rfloor = 1 + 2 = 3.$$

We now define the map G_c, previously defined in [9].

[ht]

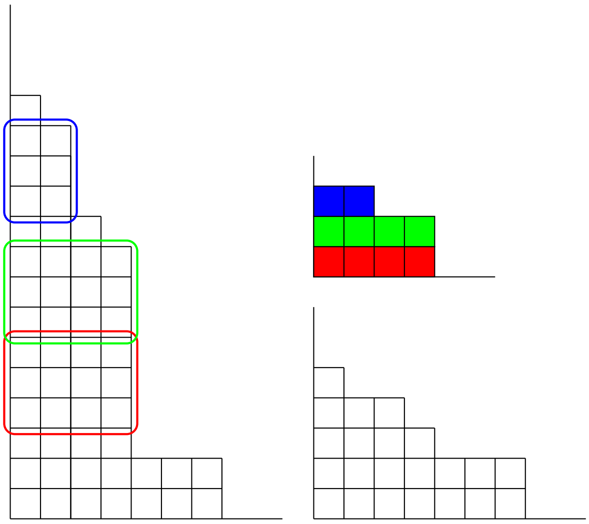


Figure 5: The partition λ has $G_3(\lambda) = ((4,4,2),(7,7,4,3,1))$.

Definition 40 (The map $\boldsymbol{G_c}$)**.** The map $G_c : \mathrm{Par} \to \mathrm{Par} \times K_c$, where $K_c = \{\lambda \in \mathrm{Par} : \lambda_{\square}^{c*} = 0\}$ is the set of partitions with no parts repeated c or more times, maps a partition λ to (ξ, ν) where for each $d \in \mathbb{N}$,

$$m_d(\xi) = \left\lfloor \frac{m_d(\lambda)}{c} \right\rfloor, \tag{35}$$

and

$$m_d(\nu) = m_d(\lambda) - c \left\lfloor \frac{m_d(\lambda)}{c} \right\rfloor. \tag{36}$$

We write $(G_c)^{-1}$ for the inverse map $(G_c)^{-1} : K_c \times \mathrm{Par} \to \mathrm{Par}$ where, for each $d \in \mathbb{N}$

$$m_d\left((G_c)^{-1}(\xi,\nu)\right) = m_d(\xi)c + m_d(\nu). \tag{37}$$

Example 41. As shown in Figure 41, the partition $\lambda = (7,7,4,4,4,4,4,4,4,3,2,2,2,1)$ has $G_3(\lambda) = ((4,4,2),(7,7,3,4,1))$.

The next proposition establishes that the c-core of a partition λ is also the c-core of the second argument of $G_c(\lambda)$, so we may restrict G_c to Par_μ^c in a way that interacts sensibly with cores.

Proposition 42. *If $\lambda \in \mathrm{Par}_\mu^c$ and $G_c(\lambda) = (\xi, \nu)$, then $\nu \in \mathrm{Par}_\mu^c$.*

Proof. Suppose the proposition is false for some λ of minimal possible size. Then, we must have $\lambda \neq \nu$, so λ must have some part of some size d repeated at least c times. The rightmost column of the rectangle of width d and height c which has all right edges and the rightmost top edge in the boundary of λ is a rimhook of size c. Let λ' be the partition formed by deleting this rimhook. Then, λ' has c-core μ and $G_c(\lambda') = (\xi', \nu)$ for some ξ'. So, since λ' is smaller than λ, $\nu \in \mathrm{Par}^c_\mu$. □

Therefore, G_c restricts to a bijection $G_c|_{\mathrm{Par}^c_\mu} : \mathrm{Par}^c_\mu \to \mathrm{Par} \times (K_c \cap \mathrm{Par}^c_\mu)$. This allows us to use G_c to prove the following.

Proposition 43. *For a positive integer c and a c-core μ, the following product formula holds.*

$$\sum_{\lambda \in K_c \cap \mathrm{Par}^c_\mu} q^{|\lambda|} = q^{|\mu|} \prod_{m \geqslant 1} \frac{1}{(1-q^{mc})^{c-1}}. \tag{38}$$

Proof. Let $\lambda \in \mathrm{Par}^\mu_c$. Then G^c_μ bijectively maps λ to a pair of partitions (ξ, ν) with $|\lambda| = |\xi| + c|\nu|$, because each part of ν corresponds to c parts of λ of the same size. So,

$$\sum_{\lambda \in \mathrm{Par}^c_\mu} q^{|\lambda|} = \sum_{\xi \in K_c \cap \mathrm{Par}^c_\mu} q^{|\xi|} \times \sum_{\nu \in \mathrm{Par}} q^{c|\nu|}. \tag{39}$$

Substituting (32) and applying Proposition 37 to (39) gives

$$q^{|\mu|} \prod_{m \geqslant 1} \frac{1}{(1-q^{mc})^{c}} = \sum_{\xi \in K_c \cap \mathrm{Par}^c_\mu} q^{|\xi|} \times \prod_{m \geqslant 1} \frac{1}{(1-q^{mc})}, \tag{40}$$

which rearranges to give (38). □

We are now in a position to prove the following identity, which forms the base case for Proposition 48.

Theorem 44. *For a fixed positive integer c,*

$$\sum_{\lambda \in \mathrm{Par}^c_\mu} q^{|\lambda|} t^{\lambda^{c*}_\square} = q^{|\mu|} \prod_{i \geqslant 1} \frac{1}{(1-q^{ic})^{c-1}} \prod_{j \geqslant 1} \frac{1}{1-q^{jc}t}. \tag{41}$$

Proof. Let $\lambda \in \mathrm{Par}^\mu_c$. Then G^c_μ bijectively maps λ to a pair of partitions (ξ, ν) with $|\lambda| = |\xi| + c|\nu|$, where each part of ν of size d corresponds to a $d \times c$ rectangle in λ contributing to $\lambda^{c*}_\square$. So,

$$\sum_{\lambda \in \mathrm{Par}^c_\mu} q^{|\lambda|} t^{\lambda^{c*}_\square} = \sum_{\xi \in K_c \cap \mathrm{Par}^c_\mu} q^{|\xi|} \times \sum_{\nu \in \mathrm{Par}} q^{c|\nu|} t^{l(\nu)}. \tag{42}$$

Substituting (38) and (31) into (42) gives (41). □

3 Further partition statistics

In this section we define the main partition statistics of interest, $h^+_{x,c}$ and $h^-_{x,c}$, where x is a real parameter and c is a positive integer. The main aim of this paper is to compute the distribution of the statistics $h^+_{x,c}$ and $h^-_{x,c}$ over Par^c_μ, given in Theorem 46. The previous section computed the distribution of $\lambda^{c*}_\square$ over Par^c_μ, giving the right hand side in Theorem 46. In this section, we connect to $\lambda^{c*}_\square$ by observing that $\lambda^{c*}_\square = h^+_{0,c}$, and then sketch a framework for piecing together a family of involutions $I_{r,s,c}$ defined on Par^c_μ to prove that the distribution $h^\pm_{x,c}$ over Par^c_μ is independent of both x and the sign. The rest of the paper will then construct the component bijections $I_{r,s,c}$.

In order to reduce the proof of Theorem 46 to the construction of appropriate bijections $I_{r,s,c}$, we first prove that Theorem 46 is implied by Theorem 47, which states that the $h^+_{x,c}$ and $h^-_{x,c}$ have the same distribution over Par^c_μ. Then, we introduce three other statistics $\mathrm{mid}_{x,c}$, $\mathrm{crit}^-_{x,c}$ and $\mathrm{crit}^+_{x,c}$ and decompose $h^+_{x,c}$ and $h^-_{x,c}$ in terms of these other statistics. Finally, we outline sufficient conditions for the bijections $I_{r,s,c}$ to prove Theorem 47 in terms of these three statistics.

We conclude the section by explaining how the main result of [3] follows from Theorem 46.

Definition 45. For a partition λ, $x \in [0,\infty]$ and a fixed $c \in \mathbb{N}$,

$$h^+_{x,c}(\lambda) = \left|\left\{\square \in \lambda \,:\, c \mid h(\square) \text{ and } \frac{a(\square)}{l(\square)+1} \leqslant x < \frac{a(\square)+1}{l(\square)}\right\}\right|, \tag{43}$$

and

$$h^-_{x,c}(\lambda) = \left|\left\{\square \in \lambda \,:\, c \mid h(\square) \text{ and } \frac{a(\square)}{l(\square)+1} < x \leqslant \frac{a(\square)+1}{l(\square)}\right\}\right|. \tag{44}$$

We interpret a fraction with denominator 0 as $+\infty$.

Note that a box $\square$ contributes to $h^+_{0,c}$ if and only if $a(\square) = 0$ and $c \mid (l(\square)+1)$. That is, $\square$ is the rightmost box in its row, and there is some m such that the row containing $\square$ and exactly $mc-1$ rows to the above all have the same height. The number of such boxes is exactly $\lambda^{c*}_\square$.

Similarly, $h^-_{\infty,c}(\lambda) = \bar{\lambda}^{c*}_\square$, where $\bar{\lambda}$ is the partition conjugate to λ.

We are now in a position to state our main result.

Theorem 46. *For all $x \in [0,\infty)$ we have*

$$\sum_{\lambda \in \mathrm{Par}^c_\mu} q^{|\lambda|} t^{h^+_{x,c}(\lambda)} = q^{|\mu|} \prod_{i \geqslant 1} \frac{1}{(1-q^{ic})^{c-1}} \prod_{j \geqslant 1} \frac{1}{1-q^{jc}t}, \tag{45}$$

and for all $x \in (0,\infty]$,

$$\sum_{\lambda \in \mathrm{Par}^c_\mu} q^{|\lambda|} t^{h^-_{x,c}(\lambda)} = q^{|\mu|} \prod_{i \geqslant 1} \frac{1}{(1-q^{ic})^{c-1}} \prod_{j \geqslant 1} \frac{1}{1-q^{jc}t}. \tag{46}$$

Proposition 48 shows that Theorem 46 is a consequence of the following result.

Theorem 47. *For all positive rational numbers x and all integers $n \geqslant 0$ we have*

$$\sum_{\lambda \in \mathrm{Par}^c_\mu(n)} t^{h^+_{x,c}(\lambda)} = \sum_{\lambda \in \mathrm{Par}^c_\mu(n)} t^{h^-_{x,c}(\lambda)}. \tag{47}$$

3.1 Reducing to Theorem 47

Proposition 48. *Theorem 47 implies Theorem 46.*

Proof. For $x \in [0, \infty)$, $c \in \mathbb{N}$ and $\delta \in \{+, -\}$ define

$$H^\delta_{x,c}(n) = \sum_{\lambda \in \mathrm{Par}^c_\mu(n)} t^{h^\delta_{x,c}(\lambda)}.$$

Suppose $H^\delta_{x,c}(n)$ is independent of both x and δ. Then

$$H^\delta_{x,c}(n) = H^+_0(n) = \sum t^{h^+_{0,c}(\lambda)} = \sum t^{\lambda^{c*}_\square}.$$

Theorem 46 then follows immediately by multiplying by q^n, adding over all $n \geqslant 0$, and applying Theorem 44. So, it suffices to prove that Theorem 47 implies that $H^\delta_{x,c}(n)$ is independent of x and δ.

For an integer n, we call a positive rational number r a *critical rational for n* if there is a partition $\mu \in \mathrm{Par}(n)$ and a box $\square \in d(\mu)$ such that $h(\square)$ is divisible by c, and $\frac{a(\square)}{l(\square)+1} = r$ or $\frac{a(\square)+1}{l(\square)} = r$. By convention, 0 and $+\infty$ are regarded as critical rationals for all n.

We denote the set of all critical rationals for n by $C(n)$. Since there are finitely many partitions of n each containing finitely many boxes in their diagrams, $C(n)$ is finite for all n. For a fixed n, write $C(n) = \{0 = r_0 < r_1 < \cdots < r_{k-1} < r_k = +\infty\}$. Define open intervals $I_j = (r_{j-1}, r_j)$ for each $1 \leqslant j \leqslant k$. Then $[0, \infty]$ decomposes into a disjoint union

$$[0, \infty] = I_1 \cup I_2 \cup \cdots \cup I_k \cup C(n).$$

Let x, x' be two elements of the same interval I_j and let $\delta, \delta' \in \{+, -\}$. Suppose λ is any partition of n. Since there are no critical rationals between x and x', $\square \in d(\lambda)$ contributes to $h^\delta_{x,c}(\lambda)$ if and only if it contributes to $h^{\delta'}_{x',c}(\lambda)$. So, $t^{h^\delta_{x,c}(\lambda)} = t^{h^{\delta'}_{x',c}(\lambda)}$. Adding over all λ, we see that if $x, x' \in I_j$,

$$H^\delta_{x,c}(n) = H^{\delta'}_{x',c}(n). \tag{48}$$

Similarly, for all $x \in I_j$,

$$H^+_{r_{j-1},c}(n) = H^\delta_{x,c}(n) = H^-_{r_j,c}(n). \tag{49}$$

On the other hand, Theorem 47 implies that

$$H^+_{r_j,c}(n) = H^-_{r_j,c}(n). \tag{50}$$

Therefore, for $\delta, \delta' \in \{+,-\}$ and $y \geqslant y'$ by applying a chain of these equalities starting with $H^{\delta}_{y,c}(n)$, one can reduce y to a critical rational and change δ to a $+$ using (49), or using (50) if y is already a critical rational. Then one may iteratively apply (50) and (49) to change δ to a $-$, and then reduce y to the next lowest critical rational and change δ back to a $+$, until an equality $H^{\delta}_{y,c}(n) = H^{-}_{r_j,c}(n)$ is obtained for $r_{j-1} \leqslant y' \leqslant r_j$. Then, applying (49) again with $x = y'$ (and (50) to flip the sign of δ if $y = r_{j-1}$ and $\delta' = -$), one obtains $H^{\delta}_{y,c}(n) = H^{\delta'}_{y',c}(n)$. □

3.2 Reducing to a symmetry property

In the case x is rational, where $h^{+}_{x,c}$ and $h^{-}_{x,c}$ may differ, it is useful to separate the boxes that contribute to both statistics from those that contribute to just one. In order to do this, we define the following statistics.

Definition 49. For $x = \frac{r}{s}$ a rational number, we have

$$\mathrm{crit}^{+}_{x,c}(\lambda) = \left|\left\{\square \in \lambda : c \mid h(\square) \text{ and } \frac{a(\square)}{l(\square+1)} = x\right\}\right|, \tag{51}$$

$$\mathrm{crit}^{-}_{x,c}(\lambda) = \left|\left\{\square \in \lambda : c \mid h(\square) \text{ and } \frac{a(\square)+1}{l(\square)} = x\right\}\right|, \tag{52}$$

$$\mathrm{mid}_{x,c}(\lambda) = \left|\{\square \in \lambda : c \mid h(\square) \text{ and } -s < sa(\square) - rl(\square) < r\}\right|. \tag{53}$$

The next proposition shows that a bijection satisfying some constraints on its behaviour with respect to these statistics will give a bijective proof of Theorem 47.

Proposition 50. *Let r, s, c be positive integers with $(r,s) = 1$ and let $x = \frac{r}{s}$. Suppose there exists a bijection $I_{r,s,c} : \mathrm{Par}^{c}_{\mu} \to \mathrm{Par}^{c}_{\mu}$ such that*

1. $|\lambda| = |I_{r,s,c}(\lambda)|$,
2. $\mathrm{mid}_{x,c}(\lambda) = \mathrm{mid}_{x,c}(I_{r,s,c}(\lambda))$,
3. $\mathrm{crit}^{+}_{x,c}(\lambda) + \mathrm{crit}^{-}_{x,c}(\lambda) = \mathrm{crit}^{+}_{x,c}(I_{r,s,c}(\lambda)) + \mathrm{crit}^{-}_{x,c}(I_{r,s,c}(\lambda))$,
4. $\mathrm{crit}^{+}_{x,c}(\lambda) = \mathrm{crit}^{-}_{x,c}(I_{r,s,c}(\lambda))$.

Then, Theorem 47 is true.

Proof. Assume that $I_{r,s,c}$ exists. Then, property 3 and 4 together imply that

$$\mathrm{crit}^{-}_{x,c}(\lambda) = \mathrm{crit}^{+}_{x,c}(I_{r,s,c}(\lambda)) \tag{54}$$

so $I_{r,s,c}$ exchanges $\mathrm{crit}^{+}_{x,c}$ and $\mathrm{crit}^{-}_{x,c}$ whilst preserving $|\lambda|$ and $\mathrm{mid}_{x,c}$.

Note that a box $\square$ contributes to $\mathrm{mid}_{x,c}$ if and only if $-s < sa(\square) - rl(\square) < r$ and the $c \mid h(\square)$. Adding $s + rl(\square)$, and dividing by $sl(\square)$, the left inequality is equivalent to

$$\frac{a(\square)+1}{l(\square)} > x. \tag{55}$$

Similar manipulation of the right inequality together with (55) shows that $\square$ contributes to $\mathrm{mid}_{x,c}$ if and only if

$$\frac{a(\square)}{l(\square)+1} < x < \frac{a(\square)+1}{l(\square)}. \tag{56}$$

So, comparing the definitions of $\mathrm{crit}^-_{x,c}$, $\mathrm{crit}^+_{x,c}$, $h^+_{x,c}$, $h^-_{x,c}$ and (56),

$$h^+_{x,c}(\lambda) = \mathrm{mid}_{x,c}(\lambda) + \mathrm{crit}^+_{x,c}(\lambda) \tag{57}$$

and

$$h^-_{x,c}(\lambda) = \mathrm{mid}_{x,c}(\lambda) + \mathrm{crit}^-_{x,c}(\lambda). \tag{58}$$

So, $I_{r,s,c}$ exchanges $h^+_{x,c}(\lambda)$ and $h^-_{x,c}(\lambda)$ whilst preserving $|\lambda|$, and hence proves Theorem 47. $\square$

3.3 Connecting to Buryak-Feigin-Nakajima

When c is divisible by $r+s$, Theorem 46 implies the following product formula. In the case $r+s=c$, this is the main combinatorial result of [3].

Corollary 51. *Let r and s be coprime integers, let $x=\frac{r}{s}$ and let $r+s \mid c$. Then*

$$\sum_{\lambda\in\mathrm{Par}} q^{|\lambda|} t^{\mathrm{crit}^+_{x,c}(\lambda)} = \prod_{\substack{i\geqslant 1\\ c\nmid i}} \frac{1}{1-q^i} \prod_{i\geqslant 1} \frac{1}{1-q^{ic}t}. \tag{59}$$

Proof. First we show that under the assumption that $r+s \mid c$, then for any partition λ, $\mathrm{mid}_{x,c}(\lambda)=0$. Suppose $\square$ were to contribute to $\mathrm{mid}_{x,c}(\lambda)$, then $\square$ would have to satisfy

$$-s < sa(\square) - rl(\square) < r. \tag{60}$$

Adding $rl+sl+s$,

$$(r+s)l(\square) < s\left(a(\square)+l(\square)+1\right) < (r+s)\left(l(\square)+1\right) \tag{61}$$

However, the upper and lower bound are consecutive multiples of $r+s$, and therefore $s(a(\square)+l(\square)+1)$ cannot be a multiple of $r+s$, so by assumption cannot be a multiple of c. So, $c\nmid h(\square)$ so $\square$ cannot contribute to $\mathrm{mid}_{x,c}(\lambda)$.

So in this case $h^+_{x,c}(\lambda) = \mathrm{crit}^+_{x,c}(\lambda)$ and Theorem 46 becomes

$$\sum_{\lambda\in\mathrm{Par}^c_\mu} q^{|\lambda|} t^{\mathrm{crit}^+_{x,c}(\lambda)} = q^{|\mu|} \prod_{i\geqslant 1} \frac{1}{(1-q^{ic})^{c-1}} \frac{1}{1-q^{ic}t}. \tag{62}$$

Summing both sides over all c-cores μ and applying Proposition 37,

$$\sum_{\lambda\in\mathrm{Par}} q^{|\lambda|}t^{\mathrm{crit}^{+}_{x,c}(\lambda)} = \prod_{i\geqslant 1}\frac{(1-q^{ic})^{c}}{1-q^{i}}\frac{1}{(1-q^{ic})^{c-1}}\frac{1}{1-q^{ic}t} \tag{63}$$

$$= \prod_{i\geqslant 1}\frac{(1-q^{ic})}{1-q^{i}}\frac{1}{1-q^{ic}t} \tag{64}$$

$$= \prod_{\substack{i\geqslant 1\\ c\nmid i}}\frac{1}{1-q^{i}}\prod_{i\geqslant 1}\frac{1}{1-q^{ic}t}. \tag{65}$$

□

4 The multigraph $M_{r,s,c}$

For the rest of the paper, $x = \frac{r}{s}$ is a rational number with r coprime to s. In this section we take our first key step in the construction of the involution $I_{r,s,c}$. First, Proposition 52 relates the statistics $\mathrm{mid}_{x,c}$, $\mathrm{crit}^{-}_{x,c}$ and $\mathrm{crit}^{+}_{x,c}$ to the boundary graph. We use this relationship to define a map from the boundary graph to a multigraph $M_{r,s,c}$ that picks out the information relevant to $\mathrm{mid}_{x,c}$ and $\mathrm{crit}^{+}_{x,c}$, much as the c-abacus tour does for the c-core. The rest of the section then outlines the method for proving that $M_{r,s,c}$ retains partition data, including the proofs that $M_{r,s,c}$ retains the c-core and the area. The proof it retains the area is a particularly easy example using the same methodology as used in the more technical proofs in Section 6, which check that $M_{r,s,c}$ retains $\mathrm{mid}_{x,c}(\lambda)$ and $\mathrm{crit}^{+}_{x,c}(\lambda) + \mathrm{crit}^{-}_{x,c}(\lambda)$.

When we define $I_{r,s,c}$ as a bijection on partitions, we build into the definition that $I_{r,s,c}$ preserves $M_{r,s,c}(\lambda)$ for any partition λ. So, together with these results it is immediate that $I_{r,s,c}$ does map Par^{c}_{μ} to Par^{c}_{μ} and satisfies hypothesis 1 in Proposition 50.

Proposition 52. *Let λ be a partition and let $\square \in \lambda$. Let e_i be the foot of $\square$, departing from $(x_1 - 1, y_1)$ and arriving at (x_1, y_1) and let e_j be the hand of λ, departing from $(x_2, y_2 + 1)$ and arriving at (x_2, y_2). Let $t = r(y_1 - y_2) + s(x_1 - x_2)$. Then*

1. *$\square$ contributes to $\mathrm{crit}^{+}_{x,c}$ if and only if $t = 0$ and $x_1 - y_1 \equiv x_2 - y_2 \pmod{c}$;*
2. *$\square$ contributes to $\mathrm{mid}_{x,c}$ if and only if $0 < t < r + s$ and $x_1 - y_1 \equiv x_2 - y_2 \pmod{c}$;*
3. *$\square$ contributes to $\mathrm{crit}^{-}_{x,c}$ if and only if $t = r + s$ and $x_1 - y_1 \equiv x_2 - y_2 \pmod{c}$.*

Proof. By the definition of index, $x_1 - y_1 = i$ and $x_2 - y_2 = j$. Let $\square \in \lambda$ have bottom left corner $(x_\square, y_\square)$. By Proposition 13, $\square$ has hook length divisible by c if and only if $j \equiv i \pmod{c}$, i.e. $x_1 - y_1 \equiv x_2 - y_2 \pmod{c}$. So, assume that $\square$ does have hook length divisible by c.

Let $k_1 = ry_1 + sx_1$ and $k_2 = ry_2 + sx_2$. Then,

$$t = k_1 - k_2, \tag{66}$$

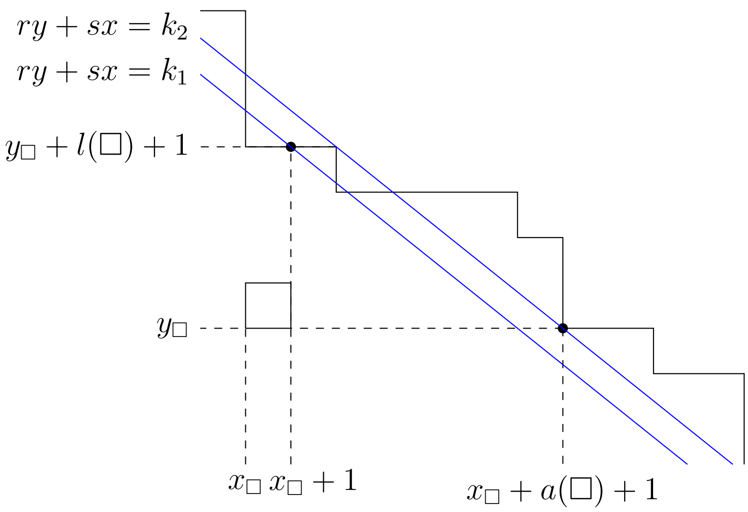


Figure 6: The box $\square$ and the lines $ry + sx = k_1$ and $ry + sx = k_2$.

$$s(x_\square + 1) + r(y_\square + l(\square) + 1) = k_1, \tag{67}$$

and

$$s(x_\square + a(\square) + 1) + ry_\square = k_2. \tag{68}$$

Subtracting (67) from (68), and substituting in (66)

$$sa(\square) - rl(\square) = r - t. \tag{69}$$

By definition, $\square$ contributes to $\mathrm{crit}^+_{x,c}$ if and only if $sa(\square) - rl(\square) = r$, that is, when $t = 0$, proving the first claim. Similarly, $\square$ contributes to $\mathrm{mid}_{x,c}$ if and only if $-s < sa(\square) - rl(\square) < r$, or equivalently $0 < t < r + s$, proving the second claim.

Finally, note $sa(\square) - rl(\square) = -s$ if and only if $t = r + s$. $\square$

We define the multigraph $M_{r,s,c}(\lambda)$ accordingly.

Definition 53 (**$M_{r,s,c}$, (r,s,c)-tour**)**.** For a partition λ the SE directed multigraph $M_{r,s,c}(\lambda)$ is obtained from $b(\lambda)$ by imposing the relation $\sim_{r,s,c}$ on the vertices, where $(x_1, y_1) \sim_{r,s,c} (x_2, y_2)$ if $ry_1 + sx_1 = ry_2 + sx_2$ and $x_2 - y_2 \equiv x_1 - y_1 \pmod{c}$. Denote the equivalence class with $ry + sx = v$ and $x - y \equiv i \pmod{c}$ by $(v, [i])$. Let $q_{r,s,c} : b(\lambda) \to M_{r,s,c}(\lambda)$ be the induced homomorphism. The (r,s,c)-tour of $M_{r,s,c}(\lambda)$ associated to λ is $(q_{r,s,c}(e_i))_{i\in\mathbb{Z}}$. At each vertex $(v,[i])$, we count the number of east edges arriving at $(v,[i])$ in the (r,s,c)-tour and denote this quantity by $\mathrm{E}_{\mathrm{in}}(v,[i])$. Similarly, we count the number of east edges departing from $(v,[i])$ in the (r,s,c)-tour and denote this quantity by $\mathrm{E}_{\mathrm{out}}(v,[i])$. We define $\mathrm{S}_{\mathrm{in}}(v,[i])$ and $\mathrm{S}_{\mathrm{out}}(v,[i])$ analogously.

Now, we explain give a useful way of drawing $M_{r,s,c}(\lambda)$ in the plane. First, we show Proposition 54, which says that when the plane is cut into strips of width $\mathrm{lcm}(c, r+s)$ by lines with $sx + ry$ constant, then there is a unique representative of each possible vertex of $M_{r,s,c}(\lambda)$ contained in the strip.

Proposition 54. *If there is a lattice point* (x, y) *satisfying both* $sx+ry = v$ *and* $x-y \equiv i \pmod{c}$*, then for any real number* m *there is exactly one such lattice point satisfying the inequality* $m \leqslant x - y < m + \mathrm{lcm}(c, r + s)$*.*

Proof. First, note that translating a lattice point (x, y) by $(r, -s)$ does not change the value of $sx+ry$. Moreover, there is no lattice point on the line $sx+ry = v$ between (x, y) and $(x + r, y - s)$, since if $(x + l_1, y - l_2)$ were such a point, we would have $sl_1 - rl_2 = 0$, so since r and s are coprime, $s \mid l_2$ and $r \mid l_1$.

Secondly, note that translating by $(r, -s)$ changes the value of $x - y$ by $r + s$. So, the translations that preserve both the value of $sx + ry$ and the residue class of $[y - x]$ modulo c are the translations by $(ar, -as)$ where $a(r+s)$ is divisible by c, that is, $a(r+s)$ is divisible by $\mathrm{lcm}(c, r+s)$. Exactly one of these translates lies in the region $m \leqslant x-y < m + \mathrm{lcm}(c, r + s)$. □

So, for a fixed integer n, we can draw the multigraph by taking the vertices to be lattice points in the portion of $\mathbb{R}^2$ in between the lines $x - y = n$ and $x - y = \mathrm{lcm}(c, r + s) + n$, with an identification along the boundary lines given by

$$(x, y) \sim \left(x + \frac{r\,\mathrm{lcm}(c, r + s)}{r + s}, y - \frac{s\,\mathrm{lcm}(c, r + s)}{r + s}\right).$$

We identify a lattice point (x, y) with the vertex $(sx + ry, [x - y])$. Then, south edges in the multigraph from $(v, [i])$ to $(v - r, [i + 1])$ are south edges between lattice points in the region described. Similarly, east edges from $(v, [i])$ to $(v + s, [i + 1])$ are east edges between lattice points. Moreover, each vertex $(v, [i])$ corresponds to a unique lattice point in the region. We can view the (r, s, c)-tour as the *cylindrical* lattice path tour obtained by collapsing the boundary of the partition onto this cylinder.

Example 55. When $c = 2$, $r = 3$ and $s = 2$ we may draw the (r, s, c)-multigraph of $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$ as in Figure 7.

Remark 56. As with the boundary graph, but unlike the c-abacus, the direction of an edge in $M_{r,s,c}$ can be read off from its source and target. If an edge e has $s(e) = (v, [i])$ and $t(e) = (w, [i + 1])$ then either $d(e) = E$ and $w = v + s$ or $d(e) = S$ and $w = v - r$.

Remark 57. If we act on $\mathbb{C}[x, y]$ by $T \times \mathbb{Z}/c\mathbb{Z}$ where $T = \{(t^s, t^r) : t \in \mathbb{C}^*\}$ and lift to ideals as in (2) and (3), and colour boxes according to the weight of the corresponding monomial with respect to this representation, the colouring carries the same information as the multigraph.

The first property that we check is that $M_{r,s,c}(\lambda)$ determines the c-core of λ.

Proposition 58. *If* λ *and* μ *are partitions with* $M_{r,s,c}(\lambda) = M_{r,s,c}(\mu)$ *then* λ *and* μ *have the same* c*-core.*

Proof. Let v be large enough so that $\left(\left\lceil \frac{v}{s} \right\rceil, 0\right)$ is on the boundary of both λ and μ. Fix $m > \left\lceil \frac{v}{s} \right\rceil$ such that $c \mid m$. Then, in both the boundary tour of μ and the boundary tour

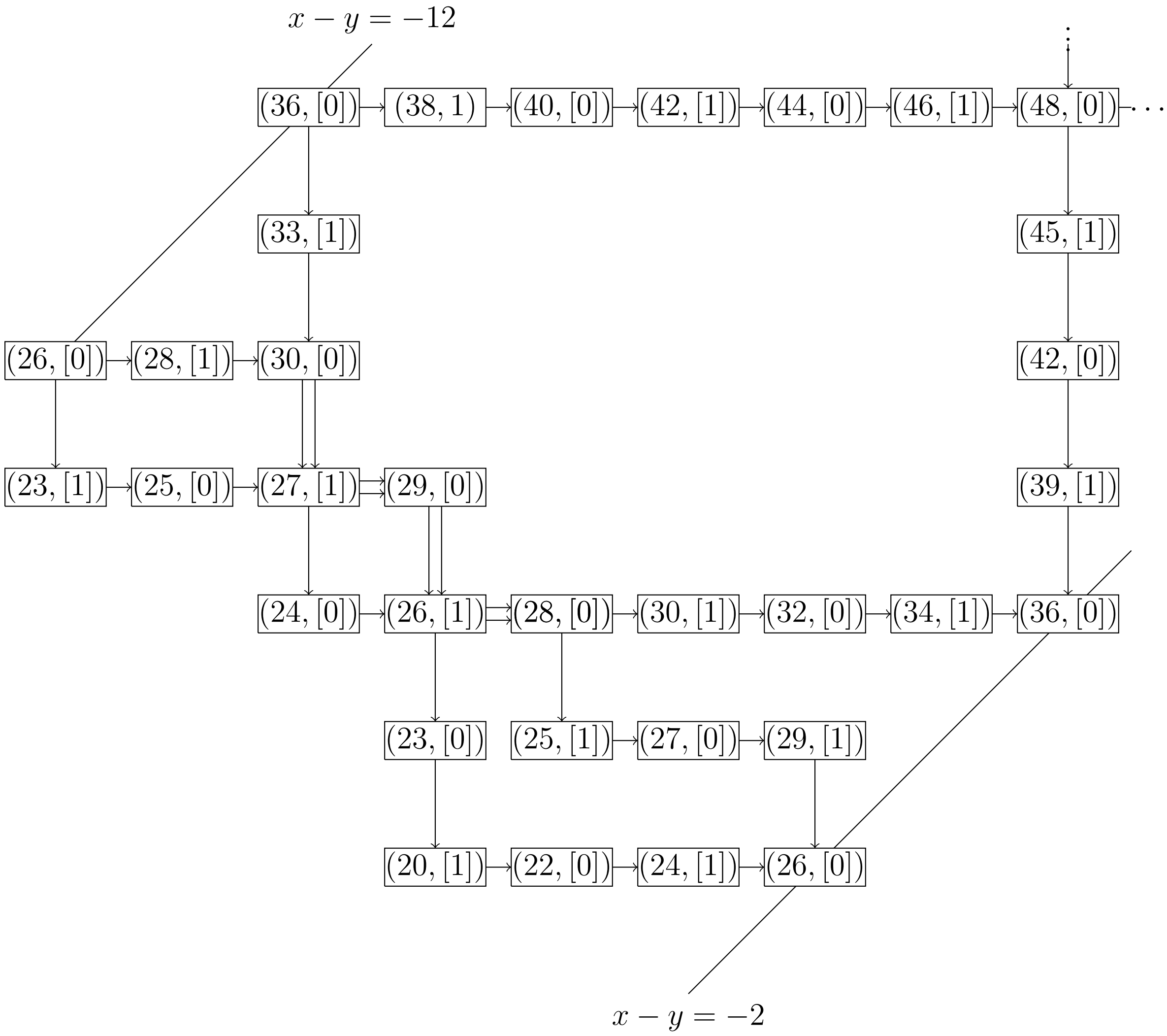


Figure 7: $M_{3,2,2}(12, 12, 10, 8, 7, 4, 1, 1, 1)$.

of λ, every edge with index at least m is an east edge. These edges account for every E in an arrival word at a vertex $(w, [j])$ with $w \geqslant sm$.

For each $[i]$, the number of east edges with index less than m, for both λ and μ, is given by

$$\sum_{w<sm} \mathrm{E}_{\mathrm{in}}(w, [i]).$$

Therefore λ, μ and m satisfy the hypotheses of Corollary 32, and so λ and μ have the same c-core. □

Next, we show how to read $\mathrm{crit}^{+}_{x,c}(\lambda)$ and $\mathrm{crit}^{-}_{x,c}(\lambda)$ off the (r, s, c)-tour of $M_{r,s,c}(\lambda)$. Rephrasing the first part of Proposition 52 in terms of the (r, s, c)-tour gives

Corollary 59. *Let λ be a partition. Then*

$$\mathrm{crit}^{+}_{x,c}(\lambda) = \sum_{(v,[i])\in M_{r,s,c}(\lambda)} \mathrm{inv}(v, [i])_a. \tag{70}$$

A similar formula with the departure words holds for $\mathrm{crit}^{-}_{x,c}(\lambda)$.

Corollary 60.

$$\mathrm{crit}^{-}_{x,c}(\lambda) = \sum_{(v,[i])\in M_{r,s,c}(\lambda)} \mathrm{inv}(v, [i])_d. \tag{71}$$

Proof. By the third part of Proposition 52, □ contributes to $\mathrm{crit}^{-}_{x,c}(\lambda)$ if and only if the foot and hand arrive at points (x_1, y_1) and (x_2, y_2) respectively with

$$r + s = r(y_1 - y_2) + s(x_1 - x_2) \tag{72}$$

and

$$x_1 - y_1 \equiv x_2 - y_2 \pmod{c}. \tag{73}$$

The foot and hand arrive at (x_1, y_1) and (x_2, y_2) respectively if and only if they depart from points $(x_1 - 1, y_1)$ and $(x_2, y_2 + 1)$ respectively. The condition (73) is equivalent to

$$(x_1 - 1) - y_1 \equiv x_2 - (y_2 + 1) \pmod{c}. \tag{74}$$

The condition (72) is equivalent to

$$r + s = r(y_1 - (y_2 + 1)) + s((x_1 - 1) - x_2) + r + s, \tag{75}$$

so subtracting $r + s$ from both sides,

$$r(y_1 - (y_2 + 1)) + s((x_1 - 1) - x_2) = 0. \tag{76}$$

□

We now outline a framework for inductive proofs that the statistics in hypotheses 1-3 of Proposition 50 are determined by the multigraph $M_{r,s,c}$, using an ordering $<_{r,s,c}$ on partitions and multigraphs. The key result in this direction is Proposition 75.

4.1 The order $<_{r,s,c}$

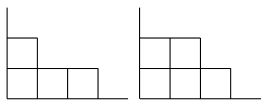
The structure of the proofs that $M_{r,s,c}(\lambda)$ determines each property of λ will be proven by induction on $|\lambda|$, adding a box at each step. Since the structure of $M_{r,s,c}(\lambda)$ is somewhat delicate, we have to be somewhat careful when choosing a box to add. The following ordering on partitions gives us a framework for adding boxes.

If (x_1, y_1) and (x_2, y_2) are two points in $\mathbb{N}^2$, say $(x_1, y_1) <_{r,s,c} (x_2, y_2)$ if either of the following hold.

- $sx_1 + ry_1 < sx_2 + ry_2$;
- $sx_1 + ry_1 = sx_2 + ry_2$, and $x_1 - y_1 \equiv x_2 - y_2 \pmod{c}$, and $x_1 - y_1 < x_2 - y_2$.

The partial order $>_{r,s,c}$ on points in the plane induces a partial order $>_{r,s,c}$ on partitions as follows. Say that $\lambda' >'_{r,s,c} \lambda$ if λ' can be obtained from λ by adding a box with bottom left corner (x, y) minimal with respect to $>_{r,s,c}$ over all possible bottom left corners of boxes that can be added to λ to obtain a partition. Then for partitions μ, λ say that $\mu >_{r,s,c} \lambda$ if there is a sequence of partitions $\lambda = \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_m = \mu$ such that for each i, $\lambda_i <'_{r,s,c} \lambda_{i+1}$. If $\mu >'_{r,s,c} \lambda$, say that μ is a *successor* for λ with respect to $>_{r,s,c}$. Every partition has a successor with respect to $>_{r,s,c}$, but successors are not necessarily unique.

Example 61. Let $r = 3$, $s = 2$, and $c = 2$. There are three boxes that could be added to the Young diagram of $(3, 1)$ to give another partition. They have bottom left corners at $(3, 0)$, $(1, 1)$, and $(0, 2)$, with values of $2x + 3y$ of $6, 5$ and 6 respectively. So $(3, 1)$ has a unique successor with respect to $<_{3,2,2}$, which is $(3, 2)$.



For $(3, 2)$, the boxes that could be added to the diagram have bottom left corners $(3, 0)$, $(2, 1)$ and $(0, 2)$, with values of $2x + 3y$ of $6, 7$ and 6 respectively. The values of $x - y$ for $(0, 3)$ and $(2, 0)$ have different parity so $(2, 0) \not<_{3,2,2} (0, 3)$, and both $(4, 2)$ and $(3, 2, 1)$ are successors of $(3, 2)$. Note that $(4, 1) \not>_{3,2,2} (3, 1)$.

Note that if $\mu >_{r,s,c} \lambda$, then all boxes of the Young diagram of λ are also boxes of the Young diagram of μ, but as Example 61 shows the converse is not true in general.

Definition 62 (Accumulation point)**.** For a partition μ with the property that whenever μ strictly contains λ, we also have $\mu >_{r,s,c} \lambda$, we call μ an *accumulation point for* $>_{r,s,c}$.

The next section describes a family of accumulation points and proves some key properties.

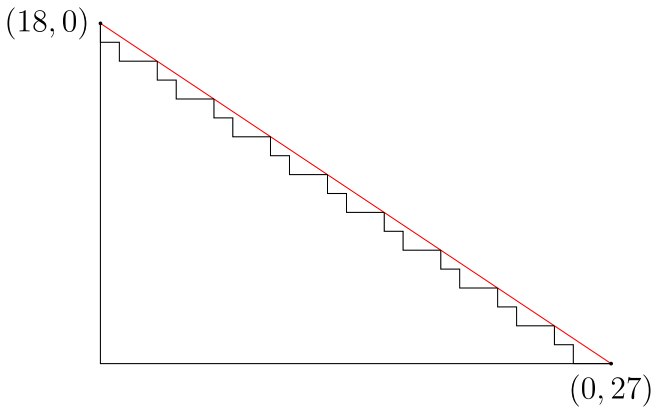


Figure 8: the Young diagram of $\lambda_{3,2,54}$.

4.2 The accumulation points $\boldsymbol{\lambda_{r,s,k}}$

Definition 63 (The partition $\boldsymbol{\lambda_{r,s,k}}$)**.** For a given natural number k, the partition $\lambda_{r,s,k}$ is the partition with Young diagram consisting of all boxes with top right corners on or below the line $sx + ry = k$.

Example 64. The Young diagram for $\lambda_{3,2,54}$ is given in Figure 8.

Proposition 65. *Let r, s, k be positive integers. Let μ be a partition with diagram strictly contained in the diagram of $\lambda_{r,s,k}$. Then, any successor μ^+ of μ with respect to $>_{r,s,c}$ has diagram contained in the diagram of $\lambda_{r,s,k}$. In particular, $\lambda_{r,s,k}$ is an accumulation point for $\lambda_{r,s,c}$.*

Proof. If (x, y) is the top right corner of a box in μ, then since the diagram for μ is contained in the diagram of $\lambda_{r,s,k}$, $sx+ry \leqslant k$. So, the bottom left corner of the same box is at $(x-1, y-1)$ with $s(x-1)+r(y-1) \leqslant k-r-s$. Since the containment of μ in λ is strict, there is at least one box in the diagram of λ, not contained in the diagram of μ, with bottom left corner $(x-1, y-1)$ satisfying $s(x-1)+r(y-1) \leqslant k-r-s$. Moreover, since translating a box with top right corner (x, y) left or down decreases $s(x-1) + r(y-1)$, there is a box $\square_1$ with bottom left corner $(x-1, y-1)$ that can be added to μ to give a valid partition diagram that satisfies $s(x-1) + r(y-1) \leqslant k-r-s$. Now, if μ^+ is not contained in λ, then μ^+ contains some box $\square_2$ with top right corner (z, w) such that $sz+rw > k$, so the bottom left corner $(z-1, w-1)$ satisfies $s(z-1)+r(w-1) > k-r-s$. This is a contradiction, as $(z-1, w-1) >_{r,s,c} (x-1, y-1)$, and $\square_1$ can be added to μ. $\square$

The accumulation points $\lambda_{r,s,k}$ will be extremely useful for two reasons. Firstly, as we check in Proposition 69, $M_{r,s,c}(\lambda_{r,s,k})$ admits a unique (r, s, c)-tour whenever $rsc \mid k$, so that $M_{r,s,c}$ must determine any partition statistic in these cases, as it determines the partition itself. Secondly, as we check in Proposition 67, if we take successor with respect to $<_{r,s,c}$ iteratively on a given partition, we will eventually hit an accumulation point. This allows us to use the $\lambda_{r,s,k}$ as a base case for iterative proofs that statistics are independent of the choice of (r,s,c)-tour, and reduces the problem of understanding how a statistic interacts with $M_{r,s,c}$ to understanding how it behaves when we take successor.

The $\lambda_{r,s,k}$ are not necessarily the only accumulation points. However, they suffice for our purposes.

Example 66. The partition $(1,1)$ is an accumulation point when $r=s=1$ and $c=2$, but is not a $\lambda_{1,1,k}$. Indeed, the only successor of the empty partition is (1), and the only successor of (1) is $(1,1)$ since the bottom left corners of the boxes addable to (1) are $(1,0)$ and $(0,1)$ with $1-0 \equiv 0-1 \pmod 2$.

Proposition 67. *If the diagram of a partition μ is contained in the diagram of $\lambda_{r,s,k}$ for some k, then for any sequence*

$$\mu = \mu_0 <'_{r,s,c} \mu_1 <'_{r,s,c} \cdots <'_{r,s,c} \mu_m$$

where $m = |\lambda_{r,s,k}| - |\mu|$, we must have $\mu_m = \lambda_{r,s,k}$.

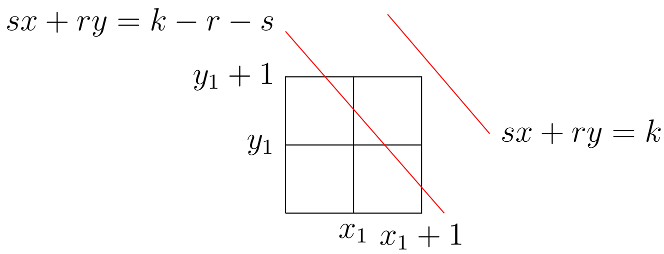
Proof. Applying Proposition 65 to $\mu_0, \mu_1, \ldots, \mu_m$, the diagram of μ_m must be contained in the diagram of $\lambda_{r,s,k}$, and $|\mu_m| = |\mu_0| + m = |\lambda_{r,s,k}|$, so $\mu_m = \lambda_{r,s,k}$. □

We now work towards proving that, in the case $rsc \mid k$, if $M_{r,s,c}(\lambda) = M_{r,s,c}(\lambda_{r,s,k})$, then $\lambda = \lambda_{r,s,k}$. First, we collect some restrictions on the arrival words that arise in the (r,s,c)-tour corresponding to $\lambda_{r,s,k}$. The condition that $rsc \mid k$ does not damage the capacity of the $\lambda_{r,s,k}$ to act as base cases, as to contain the diagram of a partition we just need k to be large enough.

Proposition 68. *Let $rsc \mid k$ and let $k_1 = \frac{k}{rs}$. The vertices $(v,[i])$ in the multigraph of $\lambda_{r,s,k}$ all satisfy $v > k-r-s$. Moreover, we have the following constraints on the arrival words at a vertex $(v,[i])$.*

- *If $k-r-s < v \leqslant k-r$, then all letters in the arrival word are Ss.*
- *If $k-r < v < k$, all letters in the arrival word are Es.*
- *If $v = k$ there the arrival word at $(k,[0])$ has first letter S and all other letters E. For $[i] \neq [0]$, all letters in the arrival word at $(k,[i])$ are Es.*

Proof. If a box □ has top right corner (x_1, y_1) with $ry_1 + sx_1 \leqslant k-r-s$, then the 2×2 box with centre (x_1,y_1) contains □, along with three other boxes with top right corners (x_1+1, y_1), (x_1, y_1+1) and (x_1+1, y_1+1).



These points satisfy $ry_1 + s(x_1+1) \leqslant k - r < k$, $r(y_1+1) + sx_1 \leqslant k - s < k$, and $r(y_1+1)+s(x_1+1) \leqslant k$. Since $\lambda_{r,s,k}$ contains *all* boxes with top right corners on or below the line $sx + ry = k$, the entire 2×2 box with centre (x, y) is contained in $\lambda_{r,s,k}$ so the boundary never visits (x, y).

Suppose $k-r-s < v \leqslant k-r$. Any east letter in the arrival word at a vertex $(v, [i])$ is also an east letter in the departure word of some vertex $(v-s, [i-1])$, but $v-s \leqslant k-r-s$, so there is no such vertex.

Suppose now that $k - r < v \leqslant k$. Any south letter in the arrival word at a vertex $(v, [i])$ arriving at a point (x_1, y_1) with $ry_1 + sx_1 = v$ is also a south letter in the departure word of some vertex $(v + r, [i-1])$. We have that $v + r > k$, so the south edge cannot be the right edge of a box in the Young diagram of $\lambda_{r,s,k}$ and must be along the y axis. Therefore, $x_1 = 0$ and $v = ry_1$ is divisible by r. However, by assumption k is divisible by r and therefore $k - r$ and k are consecutive multiples of r. So, this is only possible if $v = k$. Since the value of ry decreases as the boundary progresses south down the y-axis, there is only one such edge, namely, the edge departing from $\left(0, \frac{k}{r}+1\right)$ and arriving at $\left(0, \frac{k}{r}\right)$. □

We are now in a position to check our base case. We will show that, if $rsc \mid k$ and $M_{r,s,c}(\mu) = M_{r,s,c}(\lambda_{r,s,k})$ then $\mu = \lambda_{r,s,k}$. So, the accumulation point $\lambda_{r,s,k}$ act as a base case for a claim that any statistic is independent of the choice of (r, s, c)-tour.

Proposition 69. *For fixed integers r, s, c, k with $k = rsk_1$ and $c \mid k_1$, there is a unique (r, s, c)-tour of $M_{r,s,c}(\lambda_{r,s,k})$.*

Proof. Suppose we pick a different (r, s, c)-tour of $M_{r,s,c}(\lambda_{r,s,k})$ corresponding to a partition μ. First, we will show that the partition boundary of μ must leave the y-axis earlier than the boundary of $\lambda_{r,s,k}$. Let $(v, [i])$ be the vertex with v maximal such that the arrival word at $(v, [i])$ changes. Such a vertex certainly exists because any partition boundary differs in finitely many edges from the boundary of the empty partition. Let $(v, [i])^\lambda_a$ and $(v, [i])^\mu_a$ be the arrival words at $(v, [i])$ in the tour corresponding to $\lambda_{r,s,k}$ and μ respectively. Then, $(v, [i])^\mu_a$ must be a permutation of $(v, [i])^\lambda_a$, so since $(v, [i])^\lambda_a \neq (v, [i])^\mu_a$, $(v, [i])^\lambda_a$ must contain both Es and Ss. Proposition 68 then tells us that either

- $v > k$, in which case any letter in the arrival word at $(v, [i])$ must correspond to an edge on a co-ordinate axes. Since the value of v decreases as the boundary steps south along the y axis, and increases as it steps east along the x-axis, we must have $(v, [i])^{\lambda_{r,s,k}}_a = SE$.
- $(v, [i]) = (k, [0])$, in which case Proposition 68 implies $(v, [i])^{\lambda_{r,s,k}}_a$ is an S followed by a string of Es, where the S corresponds to an edge on the y-axis.

In either case, $(v, [i])^\mu_a$ must begin with an E. So, the boundary of μ must step east off the y-axis before it hits the lattice point on the y-axis corresponding to $(v, [i])$ - otherwise $(v, [i])^\mu_a$ would have first letter S. So, the boundary of μ does step east off the axis earlier than the boundary of $\lambda_{r,s,k}$. In particular, the boundary of μ never visits the point $(0, k_1 s)$.

Now consider the arrival words $(k,[0])_a^{\lambda_{r,s,k}}$ and $(k,[0])_a^{\mu}$. Let Z be the set of points (x,y) in the plane in the equivalence class $(k,[0])$ with respect to $\sim_{r,s,c}$,

$$Z = \{(x,y) : x, y \in \mathbb{Z}_{\geqslant 0}, sx + ry = k \text{ and } x - y \equiv 0 \pmod{c}\}. \tag{77}$$

The length of the arrival words $(k,[0])_a^{\lambda_{r,s,k}}$ and $(k,[0])_a^{\mu}$ count the number of times the boundaries of $\lambda_{r,s,k}$ and μ respectively visit points in Z. Both arrival words have the same length (they are permutations of each other) so the boundaries of $\lambda_{r,s,k}$ and μ must visit the same number of lattice points in Z. By the definition of $\lambda_{r,s,k}$, the boundary of $\lambda_{r,s,k}$ visits *all* of the points in Z, so the boundary of μ must also visit all $|Z|$ of these points. But the boundary of μ does not visit the point $(0, k_1 s)$, a contradiction. □

Next, we check that there is a sensible pull back of the ordering $>_{r,s,c}$ to (r,s,c)-multigraphs, so that taking successor can be understood to mean something at both the level of the partition and at the level of the multigraph. We abuse notation and write $>_{r,s,c}$ for the ordering on multigraphs and partitions.

Proposition 70. *Given an (r,s,c)-multigraph M, let V_S be the set of vertices $(w,[i])$ with at least one south edge arriving at $(w,[i])$. Let $(v,[i]) \in V_S$ such that v is minimal. Then there is an edge from $(v,[i])$ to $(v+s,[i+1])$.*

Proof. At least one edge arrives at $(v,[i])$ so at least one edge departs from $(v,[i])$. Any south edge departing from $(v,[i])$ would arrive at $(v-r,[i+1])$, so $(v-r,[i+1])$ would be in V_S, contradicting the minimality of v. Therefore at least one east edge departs from $(v,[i])$, and arrives at $(v+s,[i+1])$. □

Definition 71 (Multigraph successors)**.** Given an (r,s,c)-multigraph M, let $(v,[i]) \in V_S$ as in the previous proposition. Then we say M^+ is a *successor* of M if M^+ can be obtained from M by deleting one south edge from $(v+r,[i-1])$ to $(v,[i])$ and one east edge from $(v,[i])$ to $(v+s,[i+1])$, and adding one east edge from $(v+r,[i-1])$ to $(v+r+s,[i])$ and one south edge from $(v+r+s,[i])$ to $(v+s,[i+1])$. Sometimes we emphasize the vertex $(v,[i])$ and say M^+ is a successor of M that *changes from* $(v,[i])$.

At the level of multigraphs, we will only need the notion of successors, but for completeness we also explicitly define $<_{r,s,c}$ at the level of multigraphs.

Definition 72 (Ordering on multigraphs)**.** Given (r,s,c)-multigraphs $M = M_{r,s,c}(\lambda)$ and $M' = M_{r,s,c}(\lambda')$ we say $M <_{r,s,c} M'$ if there is a sequence of (r,s,c)-multigraphs $M = M_1, \ldots, M_n = M'$ such that M_{i+1} is a successor of M_i^+ for each i.

Corollary 73. *If λ is a partition with $M_{r,s,c}(\lambda) = M$ and M^+ is a successor of M changing from $(v,[i])$, then in M, $\mathrm{E_{in}}(v,[i]) = \mathrm{S_{out}}(v,[i]) = 0$.*

Proof. Identical to the proof of Proposition 70. □

The next proposition shows that this definition of successors at the level of multigraphs aligns with our definition at the level of partitions.

Proposition 74. *Let λ be a partition with $M_{r,s,c}(\lambda) = M$. If M^+ is a successor of M that changes at $(v, [i])$, then there is a unique partition λ^+ such that $\lambda^+ >'_{r,s,c} \lambda$ and $M^+ = M_{r,s,c}(\lambda^+)$.*

Proof. Let λ' be any successor of λ. Then, the Young diagram of λ' consists of all boxes in the Young diagram of λ and one additional box $\square$. Let the bottom left corner of $\square$ have co-ordinate (x_1, y_1), where $x_1 - y_1 \equiv i \pmod{c}$ and $ry_1 + sx_1 = l$. Then by definition of a successor, if we take minima over the points (x, y) in $b(\lambda)$,

$$l = \min(sx + ry) \tag{78}$$

and

$$x_1 - y_1 = \min_{\substack{sx+ry=l \\ [x-y]=[i]}} (x - y). \tag{79}$$

In particular, l and $[i]$ are sufficent to determine $x_1 - y_1$. Let s_1 and s_2 be the edges in $b(\lambda)$ arriving at and departing from (x_1, y_1) respectively, and let s'_1 and s'_2 be the edges in $b(\lambda')$ arriving at and departing from $(x_1 + 1, y_1 + 1)$ respectively, as shown in Figure 9. Then, the multigraph of λ' differs from the multigraph of λ only in that one edge from

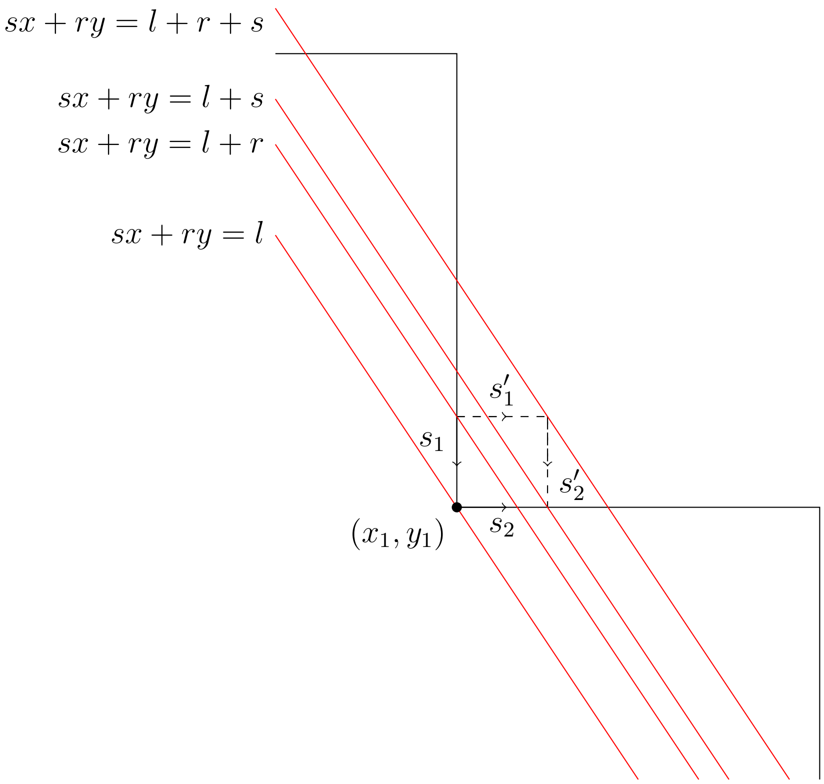


Figure 9: a partition and its successor differ by replacing s_1 and s_2 with s'_1 and s'_2.

$(l + r, [i - 1])$ to $(l, [i])$, and one edge from $(l, [i])$ to $(l + s, [i + 1])$ corresponding to s_1

and s_2 respectively, are deleted, and one edge from $(l+r,[i-1])$ to $(l+r+s,[i])$, and one edge from $(l+r+s,[i])$ to $(l+s,[i+1])$, corresponding to s_1' and s_2' respectively are added. That is, $M_{r,s,c}(\lambda')$ is the successor of M changing from $(l,[i])$.

For uniqueness, given that M^+ changes from M at $(l,[i])$, any successor of λ with multigraph M^+ must be λ' by (79) because the value of $x-y$ increases by 1 at every consecutive point visited in the boundary.

For existence, if M^+ changes from M at $(v,[j])$ then v is minimal such that there is a south edge into $(v,[j])$ and an east edge out of $(v,[j])$. So, $v=\min_{(x,y)\in b(\lambda)}(sx+ry)$ and there is at least one point (x,y) on the boundary such that $[x-y]=[j]$. So, letting (x_2,y_2) minimise $x-y$ over all such points, and adding a box with bottom left corner (x_2,y_2) gives a successor λ^+ of λ with multigraph M^+. □

We are now in a position to prove our key structural proposition.

Proposition 75. *Let $f:\mathrm{Par}\to\mathbb{R}$. Suppose there is a function $g:\{M_{r,s,c}(\lambda)\mid\lambda\in\mathrm{Par}\}^2\to\mathbb{R}$ such that, if λ is a partition, and λ^+ is a successor of λ, where λ and λ^+ have (r,s,c)-multigraphs M and M^+ respectively,*

$$f(\lambda^+)-f(\lambda)=g\left(M^+,M\right). \tag{80}$$

Then, for any partitions μ_1 and μ_2 with $M_{r,s,c}(\mu_1)=M_{r,s,c}(\mu_2)$, $f(\mu_1)=f(\mu_2)$.

Proof. Let $M=M_{r,s,c}(\mu_1)=M_{r,s,c}(\mu_2)$. There is a sequence of multigraphs $M=M_0,M_1,\dots$ where M_j is a successor of M_{j-1} for each j. Set $\lambda_0=\mu_1$ and $\nu_0=\mu_2$. Then, by Proposition 74 there are sequences of partitions $\lambda_0,\lambda_1,\dots$ and $\nu_0,\nu_1,\dots$ such that $M_j=M_{r,s,c}(\lambda_j)=M_{r,s,c}(\nu_j)$,

$$\lambda_0<'_{r,s,c}\lambda_1<'_{r,s,c}\dots, \tag{81}$$

and

$$\nu_0<'_{r,s,c}\nu_1<'_{r,s,c}\dots. \tag{82}$$

Let k be divisible by rsc and large enough so that all boxes in the Young diagrams of μ_1 or μ_2 lie below the line $sx+ry=k$. By Proposition 67, there is some m such that $M_m=M_{r,s,c}(\lambda_{r,s,k})$. By Proposition 69, $\lambda_m=\nu_m=\lambda_{r,s,k}$. Then,

$$f(\mu_1)=f(\lambda_{r,s,k})-\sum_{i=1}^{m}\left(f(\lambda_i)-f(\lambda_{i-1})\right) \tag{83}$$

$$=f(\lambda_{r,s,k})-\sum_{i=1}^{m}g(M_i,M_{i-1}) \tag{84}$$

$$=f(\lambda_{r,s,k})-\sum_{i=1}^{m}\left(f(\nu_i)-f(\nu_{i-1})\right) \tag{85}$$

$$=f(\mu_2). \tag{86}$$

□

Armed with Proposition 75, checking that $M_{r,s,c}$ determines the area of a partition is particularly straightforward.

Corollary 76. *If λ and μ are partitions with $M_{r,s,c}(\lambda) = M_{r,s,c}(\mu)$ then $|\lambda| = |\mu|$.*

Proof. Apply Proposition 75 with $g(M^+, M) = 1$. □

Having outlined the structure of the proofs that $M_{r,s,c}$ determines partition statistics, we defer the checks that $M_{r,s,c}$ determines $\text{mid}_{x,c}$ and $\text{crit}^+_{x,c} + \text{crit}^-_{x,c}$ to Section 6. We now turn our attention to defining $I_{r,s,c}$.

5 The involution $I_{r,s,c}$

In this section we construct the bijection $I_{r,s,c}$ and check that it is well defined. In order to do so, we first need to understand how to recover a partition from a family of arrival words.

5.1 Recovering a partition from the arrival words

Thus far we have constructed $M_{r,s,c}(\lambda)$ and an (r, s, c)-tour from $b(\lambda)$. We will define $I_{r,s,c}$ as an involution that preserves $M_{r,s,c}$ but changes the (r, s, c)-tour, in fact by changing the order in which some of the letters appear in the arrival words. In order to check the result is well defined, we need to understand how to recover a boundary sequence from a family of arrival words, and indeed have a criterion for when it is possible to do so if the family of arrival words does not a priori arise from a partition.

If v is minimal such that all boxes in the partition have top right corner on or below the line $sx + ry = v$, then we have that for all $w > v$, any arrival at a vertex $(w, [i])$ must be on a co-ordinate axis. So,

$$(w, [i])_a = \begin{cases} SE & \text{if } r \mid w, s \mid w, \frac{w}{s} \equiv \frac{-w}{r} \equiv i \pmod c \\ E & \text{if } s \mid w, c \mid \left(\frac{w}{s} - i\right) \text{ and either } r \nmid w \text{ or } c \nmid (\frac{-w}{r} - i) \\ S & \text{if } r \mid w, c \mid \left(\frac{-w}{r} - i\right) \text{ and either } s \nmid w \text{ or } c \nmid (\frac{w}{s} - i) \\ \emptyset & \text{otherwise.} \end{cases} \tag{87}$$

Moreover, v is uniquely specified as the largest vertex where the arrival word at $(v, [i])$ does *not* satisfy (87) for some $i \in \{1, 2, \ldots, c\}$.

So, we can identify v and fill in the co-ordinate axes above or to the right of the line $sx + ry = v$ as part of the partition boundary. We may then fill in the remainder working backwards from the arrival words - we outline the method below by example.

Example 77. Suppose we have $r = 3$, $s = 2$, $c = 2$, and the set of arrival words specified below

(20,[1])	S	(22,[0])	E	(23,[0])	S	(23,[1])	S
(24,[0])	S	(24,[1])	E	(25,[0])	E	(25,[1])	S
(26,[0])	SE	(26,[1])	SSE	(27,[0])	E	(27,[1])	SSE
(28,[0])	EE	(28,[1])	E	(29,[0])	SS	(29,[1])	E
(30,[0])	SE	(30,[1])	E				

Empty for all other vertices $(w,[j])$ with $w < 30$. Then for $w > 30$,

$$(w,[j])_a = \begin{cases} SE & \text{if } 6 \mid w, \frac{w}{2} \equiv \frac{-w}{3} \equiv j \pmod 2 \\ E & \text{if } 2 \mid w, 2 \mid \left(\frac{w}{2} - j\right) \text{ and either } 3 \nmid w \text{ or } 2 \nmid \left(\frac{-w}{3} - j\right) \\ S & \text{if } 3 \mid w, 2 \mid \left(\frac{-w}{3} - j\right) \text{ and either } 3 \nmid w \text{ or } 2 \nmid \left(\frac{w}{2} - j\right) \\ \emptyset & \text{otherwise.} \end{cases}$$

Looking at the vertex $(30,[0])$, with $w = 30$ and $j = 0$ we have $\frac{w}{2} \not\equiv j \pmod 2$, so 30 is maximal such that there is vertex $(30,[i])$ that does not satisfy (87) for some i. So, $v = 30$ and we draw a ray along the positive x-axis beginning at $(15,0)$, and a ray along the positive y-axis beginning at $(0,10)$. It then remains to fill in the boundary between the points $\left(\left\lceil \frac{v}{s} \right\rceil, 0\right)$, and $\left(0, \left\lceil \frac{v}{r} \right\rceil\right)$. To do this, we look first at the arrival word at $\left(s\left\lceil \frac{v}{s} \right\rceil, \left[\left\lceil \frac{v}{s} \right\rceil\right]\right)$, $(30,[1])$ in our example, corresponding to the point on the x-axis at which the ray begins. The last letter of this word tells us what kind of edge we should add to the boundary to arrive at $\left(\left\lceil \frac{v}{s} \right\rceil, 0\right)$, in this case an E, so we add an edge from $(14,0)$ to $(15,0)$. and delete the last E from $(30,[1])_a$. The same logic allows the rest of the boundary to be filled out edge by edge, as in Figure 10.

5.2 The first arrival tree

Next we lay out a criterion for a family of arrival words to arise from a partition. We already know that any family of arrival words arising from a partition must satisfy (87) for $w > v$ large enough. We now give a criterion on the arrival words at the remaining vertices with $w \leqslant v$ to arise from a partition.

Definition 78 (First arrival graph)**.** Let λ be a partition and let $M_{r,s,c}(\lambda) = M$. Let V and E be the vertex set and edge set of M respectively. Let the (r,s,c)-tour of M corresponding to λ have arrival word $(v,[i])_a$ at each vertex $(v,[i]) \in V$. Suppose there is another family of arrival words

$$S = \{(v,[i])'_a : (v,[i]) \in V\}, \tag{88}$$

such that for each $(v,[i])$, $(v,[i])'_a$ is a permutation of $(v,[i])_a$. Denote the first letter of the arrival word $(v,[i])'_a$ by $(v,[i])'_1$, and let the first arrival edge $e_1(v,[i])$ with respect to S be any edge e with $t(e) = (v,[i])$ and $d(e) = (v,[i])'_1$. Let T_S be the subgraph of M with vertex set V and directed edge set

$$E(T_S) = \{e_1(v,[i])'_a : (v,[i]) \in M\}. \tag{89}$$

In this case we call T_S the *first arrival graph with respect to S*.

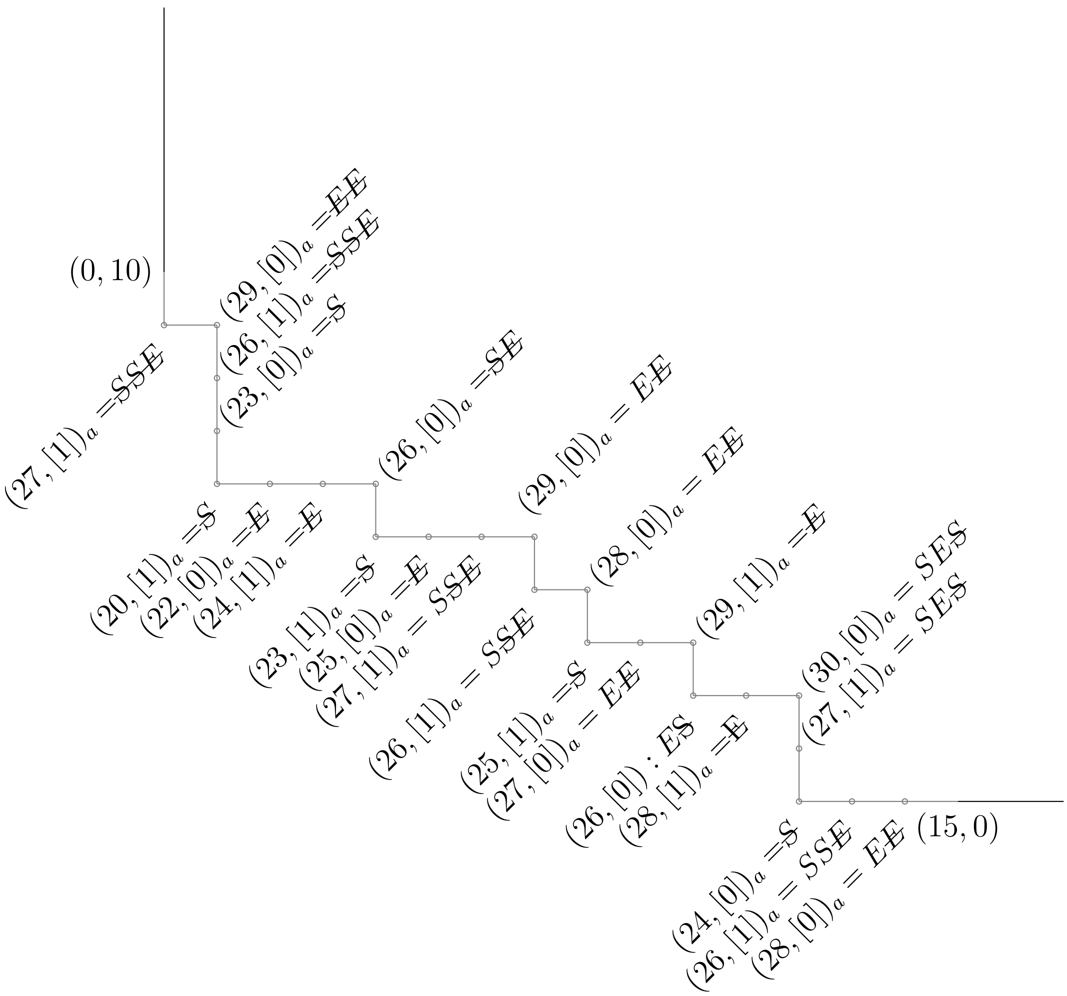


Figure 10: the arrival words in Example 77 give the partition $(12, 12, 10, 8, 7, 4, 1, 1, 1)$.

Definition 79 (The graphs $\boldsymbol{M^{\leqslant k}}$ and $\boldsymbol{T^{\leqslant k}}$)**.** For an integer k, and an (r, s, c)-multigraph M, let $M^{\leqslant k}$ be the induced subgraph of M with vertex set

$$V(M^{\leqslant k}) = \{(v, [i]) : (v, [i]) \in V(M) \text{ and } v \leqslant k\}.$$

For a family of arrival words S, let $T_S^{\leqslant k}$ be the induced subgraph of T_S with vertex set $V(M^{\leqslant k})$.

We require some preparation before proving Proposition 81, as we make use of [1, Thm 5]. The proof is not hard, but could possibly be disruptive to the flow of this paper, so the interested reader is referred to [1] for a full proof. The notion we will need is that of a T-graph.

Definition 80 ($\boldsymbol{T}$-graph)**.** A T-graph is a finite directed multigraph such that at each vertex, the number of edges arriving is the same as the number of edges departing.

Theorem 5a of [1] says that, given a complete circuit of a T-graph, starting and ending at a vertex v, the set of edges given by the last departures from any given vertex give a spanning tree of the T-graph rooted at v. Reversing the direction of all edges, equivalantly, the first arrival graph arising from a complete circuit of a T-graph is a spanning tree. Conversely, [1, Thm 5b] says that any spanning tree rooted at v gives rise to a complete circuit with last departures (or equivalently, first arrivals) agreeing with the edges of the spanning tree.

In order to apply these theorems to our situation, we need to separate M into a T-graph and a well understood complement, which is how we prove Proposition 81.

Proposition 81. *Let λ be a partition and let $M_{r,s,c}(\lambda) = M$. Suppose there is a family of arrival words $S = \{(v,[i])'_a : (v,[i]) \in V\}$ assigned to M. Let T_S be the first arrival graph with respect to S. Then there is a partition μ with an (r,s,c)-tour having arrival words S if and only if both of the following hold.*

1. *There exists some v such that for all $w > v$, and all j, $(w,[j])'_a$ satisfies* (87)*.*

2. *T_S is a spanning tree of M.*

Proof. The first condition has already been shown to be necessary, so we prove that assuming the first condition holds, the second condition is equivalent to the existence of μ. Fix v such that for all $w \geqslant v$, (87) holds for both $(w,[j])_a$ and $(w,[j])'_a$. Let $k \geqslant v$ be such that $k = rsk_1$ for some integer k_1 with $c \mid k_1$. Let $M^{\leqslant k}$ and $M^{>k}$ be the induced subgraphs of M with vertex sets given by

$$V(M^{\leqslant k}) = \{(v,[i]) \in V(M) : v \leqslant k\}, \tag{90}$$

$$V(M^{>k}) = \{(k,[0])\} \cup \{(v,[i]) \in V(M) : v > k\}, \tag{91}$$

and let $T_S^{\leqslant k}$ and $T_S^{>k}$ be the induced subgraphs of T_S with vertex sets $V(M^{\leqslant k})$ and $V(M^{>k})$. Then $T_S^{>k}$ is a spanning tree of $M^{>k}$.

Let $x = (k_1 r, 0)$ and $y = (0, k_1 s)$ on the boundary. Then, the edges in $M^{>k}$ correspond to the rays along the axes starting at x and y. The (r,s,c)-tour corresponding to λ restricted to $M^{\leqslant k}$ is a complete circuit starting and finishing at $(k,[0])$, and each edge corresponds to an edge in the boundary of λ that occurs after the south edge arriving at y and occurs before the east edge departing from x. So, $M^{\leqslant k}$ contains $|x|$ south edges and $|y|$ east edges. Therefore, there is a partition μ with arrival words S if and only if there is a complete circuit of $M^{\leqslant k}$ such that the arrival words agree with S.

Assume that a complete circuit of $M^{\leqslant k}$ with arrival words as given in S exists. The (r,s,c)-tour of M corresponding to λ consists of a circuit of $M^{>k}$ and $M^{\leqslant k}$, so the in-degree of any vertex v of $M^{\leqslant k}$ is equal to the out-degree of v in $M^{\leqslant k}$ and $M^{\leqslant k}$ is connected. In particular, $M^{\leqslant k}$ is a T-graph. So, [1, Thm 5a] implies that $T_S^{\leqslant k}$ is a tree rooted at $(k,[0])$. Therefore, T_S is a spanning tree of M.

Now assume that T_S is a spanning tree of M. Then, $T_S^{\leqslant k}$ is a tree rooted at $(k,[0])$ and [1, Thm 5b] implies that there is a complete circuit of $M^{\leqslant k}$ with arrival words agreeing with S. □

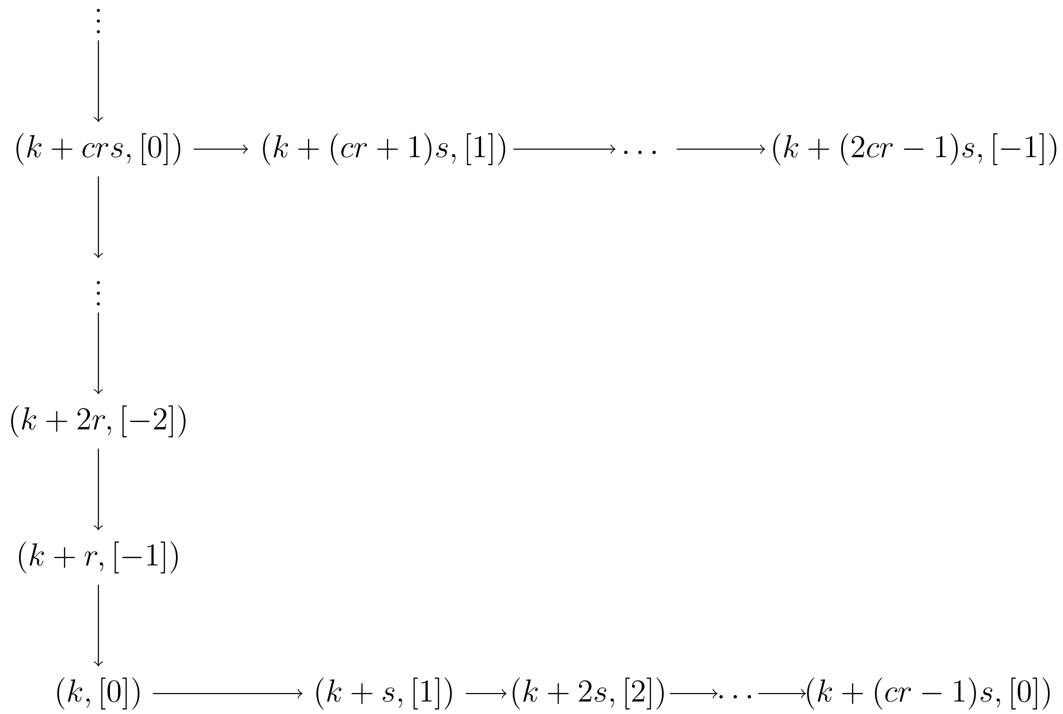


Figure 11: $T_S^{>k}$ in the case $(c, r+s) = 1$.

From now on, let $\lambda <_{r,s,c} \lambda_{r,s,k}$ where $k = rsk_1$ and $c \mid k_1$. Let $M_{r,s,c}(\lambda) = M$, let S be the family of arrival words corresponding to λ. We will now refer to T_S as the first arrival tree.

When we have a drawing of $M_{r,s,c}(\lambda)$ on the cylinder defined in Proposition 54, and a directed path p from $(v, [i])$ to $(w, [j])$, we define the *winding number of* p to be the number of times strictly after leaving $(v, [i])$ and before or on arriving at $(w, [j])$ that p arrives at a vertex on the upper boundary strip. We will be particularly interested in the case where $(v, [i]) = (k, [-k_1 s])$ and p is the unique path in the first arrival tree from $(k, [-k_1 s])$ to $(w, [j])$.

Definition 82 (Switch, eastern vertex, southern vertex)**.** Given a partition λ with $\lambda <_{r,s,c} \lambda_{r,s,k}$, and (r, s, c)-multigraph M, let T denote the first arrival tree of M corresponding to λ. Let $(v, [i]) \in V(M)$ have $v \leqslant k$ and $(v, [i]) \neq (k, [0])$. Then $(v, [i])$ is a *switch* if $(v + r, [i-1])$ and $(v - s, [i-1])$[1] are both vertices of M, and the distances in T from the vertex $(k, [0])$ to $(v + r, [i-1])$, $(v - s, [i-1])$ are equal. Now drop the condition that $v \leqslant k$ and $(v, [i]) \neq (k, [0])$. If $(v, [i])$ is not a switch and the first letter in the arrival word is E, say $(v, [i])$ is *eastern*, and let Ea be the set of all eastern vertices $(v, [i])$ with $v \leqslant k$. If $(v, [i])$ is not a switch and the first letter in the arrival word is S, say $(v, [i])$ is

[1]these are the two equivalence classes that, if they are vertices of M, could form a tail of an edge to $(v, [i])$

southern, and let So be the set of all southern vertices $(v, [i])$ with $v \leqslant k$.

Example 83. For $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$, the first arrival tree of $M_{3,2,2}(\mu)$ is as in Figure 12.

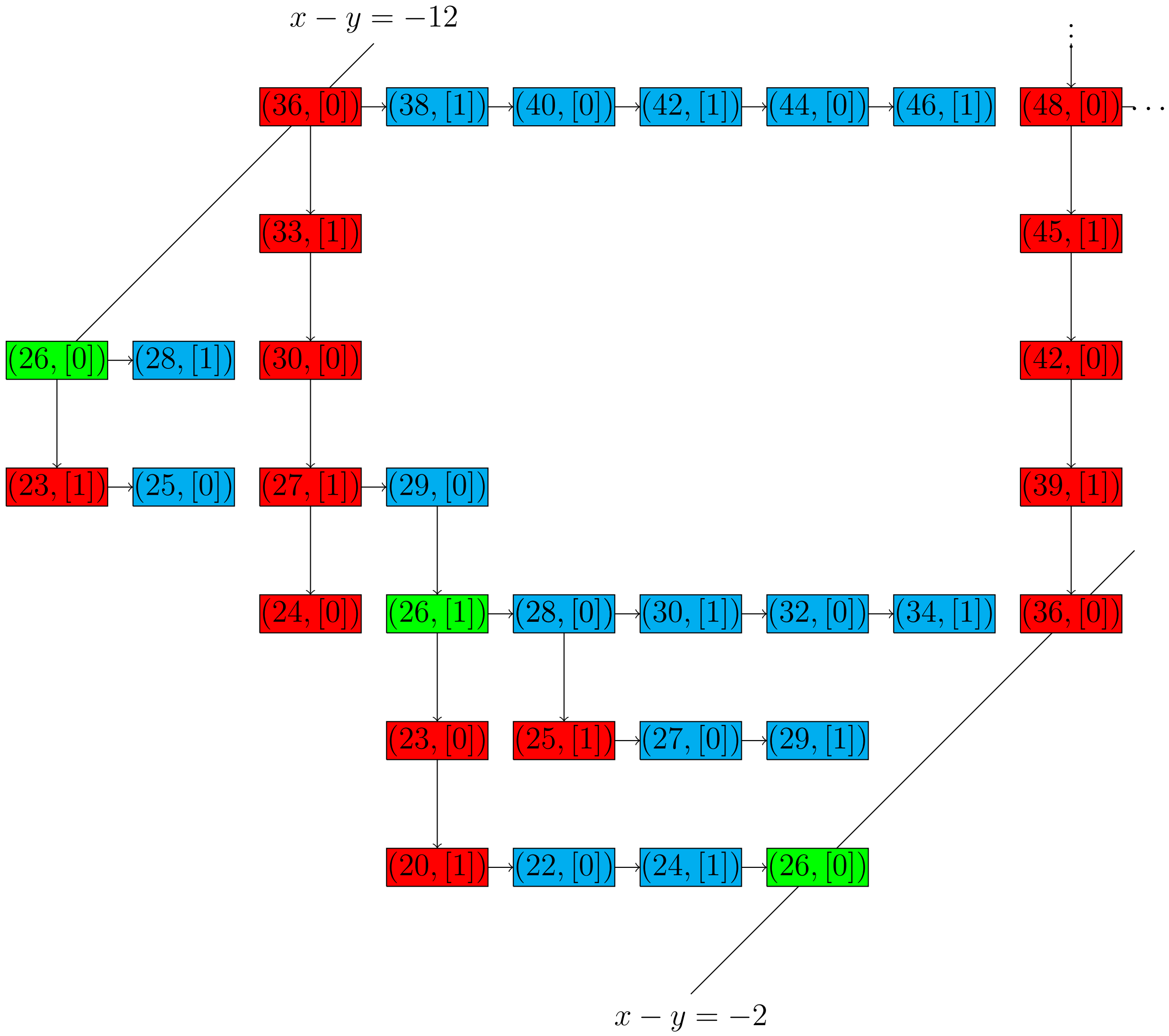


Figure 12: The first arrival tree in $M_{3,2,2}(12, 12, 10, 8, 7, 4, 1, 1, 1)$ with the eastern vertices coloured blue, the southern vertices coloured red, and the switches coloured green.

The paths in the first arrival tree from $(36, [0])$ to $(26, [0])$, $(23, [1])$, $(28, [1])$ and $(25, [0])$ have winding number 1, whilst the other vertices $(v, [i])$ for which there is a path in the first arrival tree from $(36, [0])$ to $(v, [i])$ have winding number 0. The switches are coloured green, the southern vertices red, and the eastern vertices blue (compare with Figure 7 to verify the colouring).

5.3 Definition of $I_{r,s,c}$

Given a partition λ with $\lambda \leqslant_{r,s,c} \lambda_{r,s,k}$, with multigraph M and first arrival tree T, we define the partition $I_{r,s,c}(\lambda)$ as follows. The multigraph of $I_{r,s,c}(\lambda)$ is also given by M.

Now, we obtain the (r, s, c)-tour of $I_{r,s,c}(\lambda)$ by, at each switch, reversing the arrival word, and at each vertex that is not a switch, fixing the first letter of the arrival word and reversing the rest of the arrival word.

To see that $I_{r,s,c}(\lambda)$ is well defined, we need to check that taking the first arrival at each vertex of M gives a spanning tree. We do this by checking that in T, the move of deleting an east (respectively south) edge arriving at a switch $(v, [i])$ and adding a new south (respectively east) edge arriving at $(v, [i])$ gives another spanning tree T'. There are still edges arriving at every vertex we had edges arriving at before, but now the edge arriving at $(v, [i])$ might be departing from a different vertex. So, it suffices to check that $(v, [i])$ is still connected to each of $(v - s, [i - 1])$ and $(v + r, [i - 1])$, and that we have not introduced a cycle by adding the new edge. For the former, it suffices to check $(v - s, [i - 1])$ and $(v + r, [i - 1])$ are still connected to each other. In T, $(v + r, [i - 1])$ and $(v - s, [i - 1])$ are both connected to $(k, [0])$ by paths. Moreover, the distance in T to $(k, [0])$ strictly decreases with each step along the path we take, so $(v, [i])$ is not a vertex on either of these paths. So, both of these paths exist T', and $(v + r, [i - 1])$ and $(v - s, [i - 1])$ are connected to one another. To see that the new edge does not introduce a cycle, observe that if we had introduced a cycle, we would now have two distinct paths from $(k, [0])$ to $(v, [i])$. Since the only edge into $(v, [i])$ is from its new neighbour, we must have had two distinct paths from $(k, [0])$ to the new neighbour in T originally. But then we had a cycle in T originally, a contradiction.

Hence, we may permute the letters in any arrival word at any vertex and the result will still correspond to a partition as long as we do not change the first letter in the arrival word at a vertex that is not a switch. Since we defined $I_{r,s,c}(\lambda)$ to fix the first letter in the arrival word at any vertex that is not a switch, $I_{r,s,c}(\lambda)$ is well defined. Moreover, we can recover λ from $I_{r,s,c}(\lambda)$ by doing the same operation again, as each operation is self-inverse and preserves switches.

Since $I_{r,s,c}$ does not change $M_{r,s,c}$, we can apply Proposition 58 and Corollary 76 respectively to obtain

$$|I_{r,s,c}(\lambda)| = |\lambda|, \tag{92}$$

and

$$\mathrm{core}_c(I_{r,s,c}(\lambda)) = \mathrm{core}_c(\lambda). \tag{93}$$

Moreover, the map sending λ to $I_{r,s,c}(\lambda)$ is an involution - it is immediate from the definition that a vertex is a switch after this reassignment if and only if it were a switch before the reassignment.

6 Further statistics determined by $M_{r,s,c}$

This section checks that $I_{r,s,c}$ satisfies hypotheses 2–4 in Proposition 50.

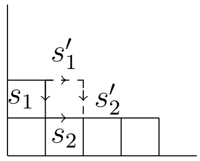


Figure 13: the partition λ with $(l,[i]) = (4,[0])$ and s_1, s_2, s'_1 and s'_2 labelled

First, we use the method introduced in Section 4 to prove that $M_{r,s,c}$ determines $\text{mid}_{x,c}$ and $\text{crit}^+_{x,c} + \text{crit}^-_{x,c}$. Then we check that $I_{r,s,c}$ exchanges $\text{crit}^+_{x,c}$ and $\text{crit}^-_{x,c}$, concluding the proof of Theorem 46. In the language of Proposition 75, Propositions 86 and 88 calculate $g(M^+, M)$ for $f(\lambda) = \text{mid}_{x,c}(\lambda)$ and $f(\lambda) = \text{crit}^+_{x,c}(\lambda) + \text{crit}^-_{x,c}(\lambda)$ respectively.

6.1 $M_{r,s,c}$ determines $\text{mid}_{x,c}$

Notation 84. Let λ, M, M^+, (x,y), $(l,[i])$, s_1, s_2 s'_1 and s'_2 be as in the proof of Proposition 74. For any edge e in the boundary of λ, write $\text{E}_{\text{in}}{}^{\to e}(w,[j])$ for the number of Es in the arrival word at a vertex $(w,[j])$ that correspond to east edges in the boundary of λ that occur before e. Define $\text{S}_{\text{in}}{}^{\to e}(w,[j])$ analogously for the number of Ss. Write $\text{E}_{\text{in}}{}^{e\to}(w,[j])$ for the number of Es in the arrival word at a vertex $(w,[j])$ that correspond to east edges in the boundary of λ that occur after e. Define $\text{S}_{\text{in}}{}^{e\to}(w,[j])$ analogously for the number of Ss. Analogously define $\text{S}_{\text{out}}{}^{\to e}(w,[j])$, $\text{E}_{\text{out}}{}^{\to e}(w,[j])$, $\text{S}_{\text{out}}{}^{\to e}(w,[j])$, and $\text{E}_{\text{out}}{}^{\to e}(w,[j])$ for the departure words. We will use this notation with $e = s_1$ or $e = s_2$. Finally, write $\text{E}^+_{\text{in}}(w,[j])$ for the number of Es in the arrival word at $(w,[j])$ in M^+ and define analogously $\text{S}^+_{\text{in}}, \text{E}^+_{\text{out}}$ and S^+_{out}.

We work in the ring R of functions $V(M) \to \mathbb{Z}$. Practically, the only consequence of this is that we write $fg(v,[i])$ for the pointwise product $f(v,[i])g(v,[i])$ and $(f+g)(v,[i])$ for $f(v,[i]) + g(v,[i])$. There should be no confusion between composition and product of functions as functions from $V(M)$ to $\mathbb{Z}$ are not composable.

Example 85. Let $\lambda = (4,1)$, $r = 3$, $s = 1$ and $c = 2$. Then $\min_{(x,y)\in b(\lambda)}(3y+x) = 4$ achieved at $(4,0)$ and $(1,1)$. Since $4 - 0 \equiv 1 - 1 \pmod 2$ and $1 - 1 < 4 - 0$, we have $(1,1) <_{3,1,2} (4,0)$ so $(4,2)$ is the only $(3,1,2)$-successor of λ. So, $(l,[i]) = (4,[0])$. As an example of the use of Notation 84, $(\text{E}_{\text{in}}{}^{\to s_1}\text{S}_{\text{out}}{}^{s_1\to} + \text{E}^+_{\text{out}})(7,[1]) = 1 \times 1 + 1 = 2$.

Proposition 86. *Let λ be a partition with $M_{r,s,c}(\lambda) = M$. Let M^+ be a successor of M that changes from $(l,[i])$. If $\lambda^+ >'_{r,s,c} \lambda$ and $M^+ = M_{r,s,c}(\lambda^+)$,*

$$\text{mid}_{x,c}(\lambda^+) - \text{mid}_{x,c}(\lambda) = \sum_{w=l+1}^{l+s+r-1} (\text{E}_{\text{out}} - \text{E}_{\text{in}})(w,[i]), \tag{94}$$

where $x = \frac{r}{s}$.

Proof. By Proposition 74, the Young diagram of λ^+ is obtained from that of λ by adding a box with bottom corner (x_1, y_1) where $ry_1 + sx_1 = l$ and $[x_1 - y_1] = [i]$.

Proposition 52 gives a formula for $\text{mid}_{x,c}(\lambda)$: it is the number of pairs of edges e_1, e_2 in the boundary sequence such that e_1 is an east edge arriving at a point (x, y) satisfying $sx + ry = v$ and $[x - y] = [j]$ for some j, and e_2 is a south edge occurring after e_1 arriving at a point (x', y') satisfying $ry' + sx' = w$ and $[x' - y'] = [j]$, where w and v satisfy $-s - r < w - v < 0$.

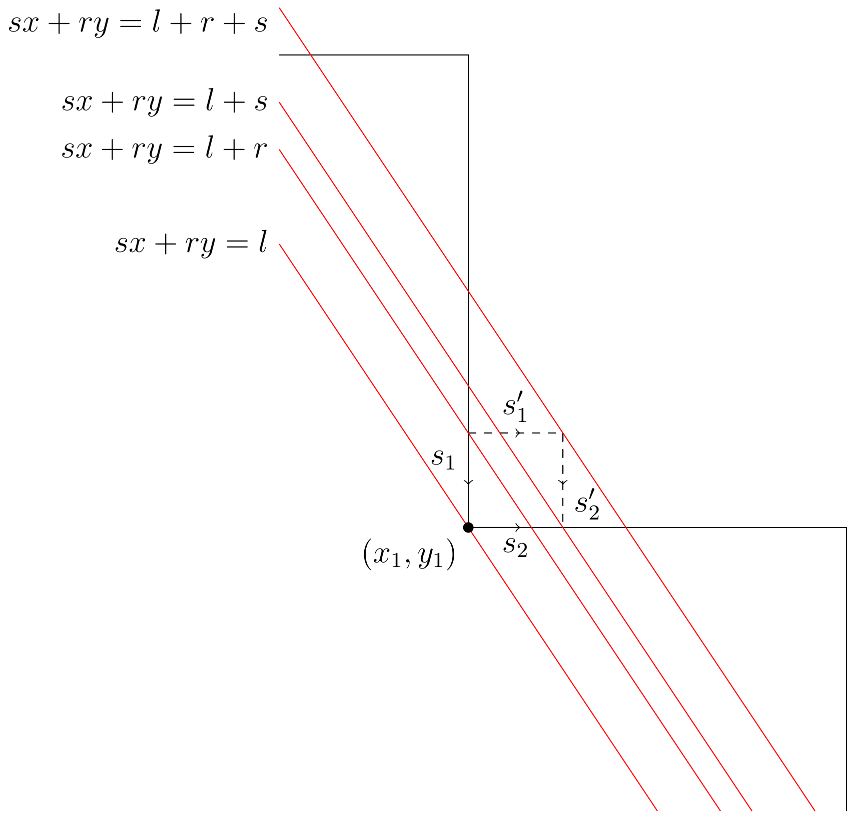


Figure 14: the diagrams of λ and λ^+.

We account for the change in the number of such pairs when changing s_1 to s_1' and s_2 to s_2' below: the only changes to $\text{mid}_{x,c}$ will be when $e_1 \in \{s_2, s_1'\}$ or $e_2 \in \{s_1, s_2'\}$.

By adding s_1' we gain the number of south edges after s_1, arriving at points (x, y) on lines $sx + ry = w$, such that $-s - r < w - (l + r + s) < 0$ and $[x - y] = [i]$. By deleting s_1 we lose the number of east edges occurring before s_1 arriving at points (x, y) on lines $sx + ry = v$ such that $-s - r < l - v < 0$ and $[x - y] = [i]$ So, the contribution to $\text{mid}_{x,c}(\lambda^+) - \text{mid}_{x,c}(\lambda)$ from switching s_1 to s_1' is S_1 where

$$S_1 = \sum_{w=l+1}^{l+r+s-1} \left(\text{S}_{\text{in}}{}^{s_1 \to} - \text{E}_{\text{in}}{}^{\to s_1}\right)(w, [i]). \tag{95}$$

By adding s_2' we gain the number of east edges before s_2, arriving at points (x, y) on lines $sx + ry = v$, such that $-s - r < l + s - v < 0$, and $x - y \equiv i + 1 \pmod{c}$. By deleting s_2 we lose the number of south edges occurring after s_2 arriving at points (x, y) on lines $sx + ry = w$ such that $-s - r < w - (l + s) < 0$ and $[x - y] = [i + 1]$. So, the contribution to $\mathrm{mid}_{x,c}(\lambda^+) - \mathrm{mid}_{x,c}(\lambda)$ from switching s_2 to s_2' is S_2 where

$$S_2 = \sum_{v=l+s+1}^{l+r+2s-1} \mathrm{E_{in}}^{\to s_2}(v, [i+1]) - \sum_{v=l-r+1}^{l+s-1} \mathrm{S_{in}}^{s_2 \to}(v, [i+1]). \tag{96}$$

So,

$$\mathrm{mid}_{x,c}(\lambda^+) - \mathrm{mid}_{x,c}(\lambda) = S_1 + S_2. \tag{97}$$

Now, note that an east edge into $(v, [i + 1])$ is also an east edge out of $(v - s, [i])$, and a south edge into $(w, [i+1])$ is also a south edge out of $(w+r, [i])$. Applying this reasoning to (96),

$$S_2 = \sum_{w=l+1}^{l+r+s-1} \left(\mathrm{E_{out}}^{\to s_2} - \mathrm{S_{out}}^{s_2 \to}\right)(w, [i]). \tag{98}$$

Substituting (95) and (98) into (97),

$$\mathrm{mid}_{x,c}(\lambda^+) - \mathrm{mid}_{x,c}(\lambda) = \sum_{w=l+1}^{l+s+r-1} \left(\mathrm{S_{in}}^{s_1 \to} - \mathrm{E_{in}}^{\to s_1} + \mathrm{E_{out}}^{\to s_2} - \mathrm{S_{out}}^{s_2 \to}\right)(w, [i]). \tag{99}$$

Since s_2 is an east edge occuring immediately after s_1, $\mathrm{S_{out}}^{s_2 \to} = \mathrm{S_{out}}^{s_1 \to}$, and since s_1 is a south edge immediately preceding s_2, $\mathrm{E_{out}}^{\to s_2} = \mathrm{E_{out}}^{\to s_1}$. So,

$$\mathrm{mid}_{x,c}(\lambda^+) - \mathrm{mid}_{x,c}(\lambda) = \sum_{w=l+1}^{l+s+r-1} \left(\mathrm{S_{in}}^{s_1 \to} - \mathrm{E_{in}}^{\to s_1} + \mathrm{E_{out}}^{\to s_1} - \mathrm{S_{out}}^{s_1 \to}\right)(w, [i]). \tag{100}$$

Now, note that at any vertex $(v, [j])$ except $(l, [i])$, we have that

$$\left(\mathrm{S_{in}}^{s_1 \to} + \mathrm{E_{in}}^{s_1 \to}\right)(v, [j]) = \left(\mathrm{S_{out}}^{s_1 \to} + \mathrm{E_{out}}^{s_1 \to}\right)(v, [j]), \tag{101}$$

because after s_1 we depart every vertex after we arrive at it, the left hand side counting arrivals at the vertex after s_1 and the right side counting departures. Rearranging gives

$$\left(\mathrm{S_{in}}^{s_1 \to} - \mathrm{S_{out}}^{s_1 \to}\right)(v, [j]) = \left(\mathrm{E_{out}}^{s_1 \to} - \mathrm{E_{in}}^{s_1 \to}\right)(v, [j]). \tag{102}$$

Substituting (102) into (100),

$$\mathrm{mid}_{x,c}(\lambda^+) - \mathrm{mid}_{x,c}(\lambda) = \sum_{w=l+1}^{l+s+r-1} \left(\mathrm{E_{out}}^{s_1 \to} - \mathrm{E_{in}}^{s_1 \to} - \mathrm{E_{in}}^{\to s_1} + \mathrm{E_{out}}^{\to s_1}\right)(w, [i]). \tag{103}$$

Since s_1 is a south edge, $\mathrm{E_{in}}^{s_1 \to} + \mathrm{E_{in}}^{\to s_1} = \mathrm{E_{in}}$ and $\mathrm{E_{out}}^{s_1 \to} + \mathrm{E_{out}}^{\to s_1} = \mathrm{E_{out}}$, so

$$\mathrm{mid}_{x,c}(\lambda^+) - \mathrm{mid}_{x,c}(\lambda) = \sum_{w=l+1}^{l+s+r-1} \left(\mathrm{E_{out}} - \mathrm{E_{in}}\right)(w, [i]). \tag{104}$$

□

Corollary 87. *If* λ *and* μ *are partitions with* $M_{r,s,c}(\lambda) = M_{r,s,c}(\mu)$*, then* $\mathrm{mid}_{x,c}(\lambda) = \mathrm{mid}_{x,c}(\mu)$*.*

Proof. Apply Proposition 75 with

$$g(M^+, M) = \sum_{w=l+1}^{l+s+r-1} (\mathrm{E_{out}} - \mathrm{E_{in}})\,(w, [i]), \tag{105}$$

where M^+ is the successor of M that changes from $(l, [i])$. □

6.2 $M_{r,s,c}$ determines $\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c}$

Proposition 88. *Let* λ *be a partition with* $M_{r,s,c}(\lambda) = M$ *and let* M^+ *be a successor of* M *that changes from* $(l, [i])$*. Then, if* λ^+ *is the successor of* λ *with multigraph* M^+*,*

$$\mathrm{crit}^+_{x,c}(\lambda^+) + \mathrm{crit}^-_{x,c}(\lambda^+) - \mathrm{crit}^+_{x,c}(\lambda) - \mathrm{crit}^-_{x,c}(\lambda) \tag{106}$$

is equal to

$$\mathrm{S_{in}}(l, [i]) - 1 + (\mathrm{S_{in}} - \mathrm{S_{out}})(l + r + s, [i]), \tag{107}$$

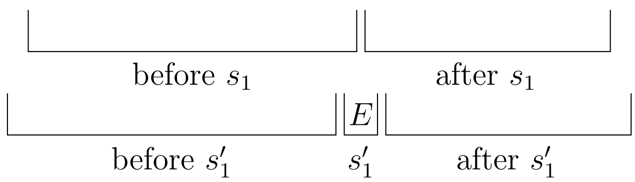
where $x = \frac{r}{s}$*.*

Proof. First, we compute $\mathrm{crit}^+_{x,c}(\lambda^+) - \mathrm{crit}^+_{x,c}(\lambda)$. Corollary 59 implies that

$$\mathrm{crit}^+_{x,c}(\lambda^+) - \mathrm{crit}^+_{x,c}(\lambda) = \sum_{v \in M_{r,s,c}(\lambda^+)} \mathrm{inv}(v_a) - \sum_{v \in M_{r,s,c}(\lambda)} \mathrm{inv}(v_a) \tag{108}$$

We keep the notation of the previous proposition and reference Figure 14 throughout. The only nonzero terms in the difference (108) come from $v \in \{(l, [i]), (l + r + s, [i]), (l + s, [i + 1])\}$. We work case-by-case through these vertices.

- We delete the first arrival at $(l, [i])$, corresponding to deleting s_1. All arrivals at $(l, [i])$ are Ss by Corollary 73, so this does not affect $\mathrm{inv}(l, [i])_a$.
- We add an E to the arrival word at $(l + r + s, [i])$, corresponding to adding s_1'.



  This E is the first letter in an inversion with second letter any S occuring after s_1', so $(l + r + s, [i])$ contributes $\mathrm{S_{in}}^{s_1 \to}(l + r + s, [i])$ to (108).

- We replace the first E in the arrival word at $(l + s, [i + 1])$ (corresponding to s_2) with an S (corresponding to s_2'). Therefore, we lose all inversions with the replaced E edge as their first letter passing from λ to λ^+. There are $\mathrm{S_{in}}^{s_2 \to}(l + s, [i + 1])$ such inversions.

$SSSSSSSS\ldots SSSS$	E	
before s_2	s_2	after s_2
$SSSSSSSS\ldots SSSS$	S	
before s'_2	s'_2	after s'_2

We gain no inversions from the new S edge, because s_2 was the first east departure from $(l,[i])$ in the tour corresponding to λ. So, $(l+s,[i+1])$ contributes $-{\mathrm{S_{in}}}^{s_2\to}(l+s,[i+1])$ to (108).

So,

$$\mathrm{crit}^+_{x,c}(\lambda^+)-\mathrm{crit}^+_{x,c}(\lambda)={\mathrm{S_{in}}}^{s_1\to}(l+r+s,[i])-{\mathrm{S_{in}}}^{s_2\to}(l+s,[i+1]). \tag{109}$$

A south arrival before (respectively after) s_2 at $(l+s,[i+1])$ is a south departure before (respectively after) s_2 from $(l+r+s,[i])$. Combining this logic with (109),

$$\mathrm{crit}^+_{x,c}(\lambda^+)-\mathrm{crit}^+_{x,c}(\lambda)=\left({\mathrm{S_{in}}}^{s_1\to}-{\mathrm{S_{out}}}^{s_2\to}\right)(l+r+s,[i]). \tag{110}$$

We now analyse

$$\mathrm{crit}^-_{x,c}(\lambda^+)-\mathrm{crit}^-_{x,c}(\lambda)=\sum_{v\in M_{r,s,c}(\lambda^+)}\mathrm{inv}(v_d)-\sum_{v\in M_{r,s,c}(\lambda)}\mathrm{inv}(v_d). \tag{111}$$

The departure words at every vertex except for $(l,[i])$, $(l+r+s,[i])$, and $(l+r,[i-1])$ are unchanged so the only nonzero terms in (111) come from $v\in\{(l,[i]),(l+r+s,[i]),(l+r,[i-1])\}$. An analogous argument to the above shows that the contribution of $(l+r,[i-1])$ to (111) is $\left({\mathrm{S_{out}}}^{s_1\to}-{\mathrm{E_{out}}}^{\to s_1}\right)(l+r,[i-1])$, the contribution of $(l+r+s,[i])$ is ${\mathrm{E_{out}}}^{\to s_2}(l+r+s,[i])$, and $(l,[i])$ does not contribute. So,

$$\mathrm{crit}^-_{x,c}(\lambda^+)-\mathrm{crit}^-_{x,c}(\lambda)=\left({\mathrm{S_{out}}}^{s_1\to}-{\mathrm{E_{out}}}^{\to s_1}\right)(l+r,[i-1])+{\mathrm{E_{out}}}^{\to s_2}(l+r+s,[i]). \tag{112}$$

An east departure from $(l+r,[i-1])$ is an east arrival at $(l+r+s,[i])$, so

$$\mathrm{crit}^-_{x,c}(\lambda^+)-\mathrm{crit}^-_{x,c}(\lambda)={\mathrm{S_{in}}}^{s_1\to}(l,[i])-\left({\mathrm{E_{in}}}^{\to s_1}-{\mathrm{E_{out}}}^{\to s_2}\right)(l+r+s,[i]). \tag{113}$$

Now, since s_1 is the first edge to arrive at $(l,[i])$,

$${\mathrm{S_{in}}}^{s_1\to}(l,[i])=\mathrm{S_{in}}(l,[i])-1. \tag{114}$$

Since s_1 does not arrive at $(l+r+s,[i])$, we leave $(l+r+s,[i])$ before s_1 the same number of times as we arrive before s_1. So,

$$({\mathrm{E_{in}}}^{\to s_1}+{\mathrm{S_{in}}}^{\to s_1})(l+r+s,[i])=({\mathrm{E_{out}}}^{\to s_2}+{\mathrm{S_{out}}}^{\to s_2})(l+r+s,[i])). \tag{115}$$

Rearranging,

$${\mathrm{E_{in}}}^{\to s_1}(l+r+s,[i])=({\mathrm{E_{out}}}^{\to s_2}+{\mathrm{S_{out}}}^{\to s_2}-{\mathrm{S_{in}}}^{\to s_1})(l+r+s,[i]). \tag{116}$$

Substituting (116) and (114) into (113),

$$\mathrm{crit}^-_{x,c}(\lambda^+) - \mathrm{crit}^-_{x,c}(\lambda) = \mathrm{S}_{\mathrm{in}}(l,[i]) - 1 + (\mathrm{S}_{\mathrm{in}}{}^{\to s_1} - \mathrm{S}_{\mathrm{out}}{}^{\to s_2})(l+r+s,[i]). \tag{117}$$

Since $(l+r+s,[i])$ is not an endpoint of s_1 or s_2,

$$(\mathrm{S}_{\mathrm{in}}{}^{\to s_1} + \mathrm{S}_{\mathrm{in}}{}^{s_1\to})(l+r+s,[i]) = \mathrm{S}_{\mathrm{in}}(l+r+s,[i]) \tag{118}$$

and

$$(\mathrm{S}_{\mathrm{out}}{}^{\to s_2} + \mathrm{S}_{\mathrm{out}}{}^{s_2\to})(l+r+s,[i]) = \mathrm{S}_{\mathrm{out}}(l+r+s,[i]). \tag{119}$$

Adding (117) and (110), and then applying (119) and (118) completes the proof. □

Corollary 89. *If λ and μ are partitions such that $M_{r,s,c}(\mu) = M_{r,s,c}(\lambda)$ then*

$$\mathrm{crit}^+_{x,c}(\lambda) + \mathrm{crit}^-_{x,c}(\lambda) = \mathrm{crit}^+_{x,c}(\mu) + \mathrm{crit}^-_{x,c}(\mu). \tag{120}$$

Proof. Apply Proposition 75 with

$$g(M^+, M) = \mathrm{S}_{\mathrm{in}}(l,[i]) - 1 + (\mathrm{S}_{\mathrm{in}} - \mathrm{S}_{\mathrm{out}})(l+r+s,[i]). \tag{121}$$

where the calculations S_{in} and $\mathrm{S}_{\mathrm{out}}$ are done with respect to the multigraph M, and M^+ is the successor of M changing from $(l,[i])$. □

So, we know that $M_{r,s,c}(\lambda)$ determines the c-core of λ, $|\lambda|$, $\mathrm{mid}_{x,c}(\lambda)$ and $\mathrm{crit}^+_{x,c}(\lambda) + \mathrm{crit}^-_{x,c}(\lambda)$, and that any bijection preserving $M_{r,s,c}$ therefore satisfies hypotheses 1-3 of Proposition 50. It will be useful in our final remaining check, that $I_{r,s,c}$ satisfies the fourth criterion in Proposition 50, to have a formula for $\mathrm{crit}^+_{x,c}(\lambda) + \mathrm{crit}^-_{x,c}(\lambda)$ in terms of $M_{r,s,c}(\lambda)$. This is what Proposition 90 computes.

Proposition 90. *Let λ be a partition. If $k = rsk_1$ where $c \mid k_1$ and $\lambda <_{r,s,c} \lambda_{r,s,k}$, then*

$$(\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c})(\lambda) = \sum_{\substack{(v,[j])\\ v\leqslant k}} \mathrm{E}_{\mathrm{in}}\mathrm{S}_{\mathrm{in}}(v,[j]) - \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c, s+r)} \right\rfloor. \tag{122}$$

Proof. First, we prove that (122) holds when $\lambda = \lambda_{r,s,k}$.

We will show that for all boxes $\square \in \lambda_{r,s,k}$, $-s < sa(\square) - rl(\square) < r$, and hence that the left hand side of (122) is zero at $\lambda_{r,s,k}$. We will then check that the right hand side of (122) is zero at $\lambda_{r,s,k}$.

The ith part of $\lambda_{r,s,k}$ corresponds to a row with top right corner (x_i, i) where x_i is maximal such that $sx_i + ri \leqslant k$. So,

$$x_i = \left\lfloor \frac{k - ri}{s} \right\rfloor = \left\lfloor \frac{k_1 rs - ri}{s} \right\rfloor = k_1 r - \left\lceil \frac{ri}{s} \right\rceil. \tag{123}$$

Similarly, the number of parts of $\lambda_{r,s,k}$ of size at least j corresponds to a column with top right corner (j, y_j) where y_j is maximal such that $sj + ry_j \leqslant k$, so

$$y_j = \left\lfloor \frac{k - sj}{r} \right\rfloor = \left\lfloor \frac{k_1 rs - sj}{r} \right\rfloor = k_1 s - \left\lceil \frac{sj}{r} \right\rceil. \tag{124}$$

Now, let $\square \in \lambda$ be a box with top right corner (i, j). Then, the arm of $\square$ is given by $x_i - j$ and the leg of $\square$ is given by $y_j - i$. So,

$$
\begin{aligned}
sa(\square) - rl(\square) &= s(x_i - j) - r(y_j - i) && (125)\\
&= k_1 rs - s\left\lceil \frac{ri}{s} \right\rceil - sj - k_1 rs + r\left\lceil \frac{sj}{r} \right\rceil + ri && (126)\\
&= \left(r\left\lceil \frac{sj}{r} \right\rceil - sj \right) - \left(s\left\lceil \frac{ri}{s} \right\rceil + ri \right). && (127)
\end{aligned}
$$

Now, consider the two bracketed quantities separately, setting $x = \left(r\left\lceil \frac{sj}{r} \right\rceil - sj \right)$ and $y = -\left(s\left\lceil \frac{ri}{s} \right\rceil + ri \right)$. For the first bracket we have that

$$
r\left(\frac{sj}{r} \right) \leqslant r\left\lceil \frac{sj}{r} \right\rceil < r\left(\frac{sj}{r} + 1 \right), \tag{128}
$$

so

$$
0 \leqslant r\left\lceil \frac{sj}{r} \right\rceil - sj < r. \tag{129}
$$

Similarly for the second bracket,

$$
-s < ri - s\left\lceil \frac{ri}{s} \right\rceil \leqslant 0. \tag{130}
$$

So, $sa(\square) - rl(\square)$ can be written as $x + y$ for $x \in [0, r)$ and $y \in (-s, 0]$ and therefore $-s < sa(\square) - rl(\square) < r$.

Therefore,

$$
\mathrm{crit}^{+}_{x,c}(\lambda_{r,s,k}) + \mathrm{crit}^{-}_{x,c}(\lambda_{r,s,k}) = 0. \tag{131}
$$

Next we evaluate the right hand side of (122) at $\lambda_{r,s,k}$. Proposition 68 tells us that for all vertices $(v, [i])$ such that $0 \leqslant v < k$, the arrival word at $(v, [i])$ in $M_{r,s,c}(\lambda_{r,s,k})$ does not contain both an E and a S. So, for all such $(v, [i])$ we have $\mathrm{E_{in}S_{in}}(v, [i]) = 0$. So, the right hand side of (122) simplifies to

$$
\sum_{i=0}^{c-1} \mathrm{E_{in}S_{in}}(k, [i]) - \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c, s+r)} \right\rfloor \tag{132}
$$

Proposition 68 also tells us that $\mathrm{S_{in}}(k, [i]) = 0$ unless $[i] = [0]$, and that $\mathrm{S_{in}}(k, [0]) = 1$, so we can rewrite (132) as

$$
\mathrm{E_{in}}(k, [0]) - \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c, s+r)} \right\rfloor. \tag{133}
$$

So, it suffices to show that $\mathrm{E_{in}}(k, [0]) = \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c, s+r)} \right\rfloor$. The east edges in the boundary of $\lambda_{r,s,k}$ arriving at vertices $(k, [i])$ for some i correspond to points (x, y) with $x > 0$ and $y \geqslant 0$ such that $sx + ry = k$. These points have coordinates $\{(r, s(k_1 - 1)), (2r, s(k_1 -$

$2)), \ldots, ((k_1-1)r, s), (k_1 r, 0)\}$. Now, $\mathrm{E_{in}}(k,[0])$ counts the number of these points that also lie on a line $x-y=i$ for $[i]=[0]$. The set of values of $x-y$ for this set of points is $\{r+s-k_1 s, 2(r+s)-k_1 s, \ldots, k_1(r+s)-k_1 s\}$. Letting $l(r+s) = \mathrm{lcm}(c, r+s)$, the values of $x-y$ that give us the same congruence class as 0 when taken modulo c are of the form $ml(r+s)-k_1 s$ for some integer m. The number of values of this form in the given set is indeed $\left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c,s+r)} \right\rfloor$.

Now suppose $\lambda <_{r,s,c} \lambda_{r,s,k}$ is maximal with respect to $>_{r,s,c}$ such that the proposition is false. In particular, the proposition holds for any successor $\lambda^+ >'_{r,s,c} \lambda$. Let M^+ be a successor of M that changes from $(l,[i])$, and let λ^+ be the successor of λ with multigraph M^+. Then, $(\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c})(\lambda^+) - (\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c})(\lambda)$ can be written as Δ_1, where

$$\Delta_1 = \mathrm{S_{in}}(l,[i]) - 1 + (\mathrm{S_{in}} - \mathrm{S_{out}})(l+r+s,[i]). \tag{134}$$

By assumption,

$$(\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c})(\lambda^+) = \sum_{\substack{(v,[j])\\ v \leqslant k}} \mathrm{E^+_{in}}\,\mathrm{S^+_{in}}(v,[j]) - \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c,s+r)} \right\rfloor \tag{135}$$

So, combining (134) and (135),

$$(\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c})(\lambda) = \sum_{\substack{(v,[j])\\ v \leqslant k}} \mathrm{E^+_{in}}\,\mathrm{S^+_{in}}(v,[j]) - \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c,s+r)} \right\rfloor - \Delta_1. \tag{136}$$

First, we note that a vertex $(v,[j])$ contributes the same to the sums

$$\sum_{\substack{(v,[j])\\ v \leqslant k}} \mathrm{E_{in}}\mathrm{S_{in}}(v,[j])$$

taken over the multigraphs M or M^+ unless $(v,[j]) \in \{(l,[i]), (l+r+s,[i], (l+s,[i+1])\}$. In fact, since there are no east edges into $(l,[i])$ in M or M^+, we only need consider terms with $(v,[j]) \in \{(l+r+s,[i], (l+s,[i+1])\}$. So,

$$(\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c})(\lambda) = \sum_{\substack{(v,[j])\\ v \leqslant k}} \mathrm{E_{in}}\mathrm{S_{in}}(v,[j]) - \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c,s+r)} \right\rfloor - \Delta_1 + \Delta_2 \tag{137}$$

where

$$\Delta_2 = (\mathrm{E^+_{in}}\,\mathrm{S^+_{in}} - \mathrm{E_{in}}\mathrm{S_{in}})(l+r+s,[i]) + (\mathrm{E^+_{in}}\,\mathrm{S^+_{in}} - \mathrm{E_{in}}\mathrm{S_{in}})(l+s,[i+1]). \tag{138}$$

Because M^+ changes from M at $(l,[i])$, $\mathrm{E^+_{in}}(l+s,[i+1]) = \mathrm{E_{in}}(l+s,[i+1]) - 1$, $\mathrm{S^+_{in}}(l+s,[i+1]) = \mathrm{S_{in}}(l+s,[i+1]) + 1$, $\mathrm{E^+_{in}}(l+r+s,[i]) = \mathrm{E_{in}}(l+r+s,[i]) + 1$ and $\mathrm{S^+_{in}}(l+r+s,[i]) = \mathrm{S_{in}}(l+r+s,[i])$ so (138) simplifies to

$$\Delta_2 = \mathrm{E_{in}}(l+s,[i+1]) - \mathrm{S_{in}}(l+s,[i+1]) + \mathrm{S_{in}}(l+r+s,[i]) - 1. \tag{139}$$

A south arrival at $(l+s,[i+1])$ is the same as a south departure from $(l+r+s,[i])$, and an east arrival at $(l+s,[i+1])$ is the same as an east departure from $(l,[i])$, so

$$\Delta_2 = \mathrm{E}_{\mathrm{out}}(l,[i]) - (\mathrm{S}_{\mathrm{out}} - \mathrm{S}_{\mathrm{in}})\,(l+r+s,[i]) - 1. \tag{140}$$

By Corollary 73, all edges leaving $(l,[i])$ are east edges and all edges arriving are south edges. The same number of edges arrive and leave, so $\mathrm{E}_{\mathrm{out}}(l,[i]) = \mathrm{S}_{\mathrm{in}}(l,[i])$. So,

$$\Delta_2 = (\mathrm{S}_{\mathrm{in}} - \mathrm{S}_{\mathrm{out}})(l+r+s,[i]) + \mathrm{S}_{\mathrm{in}}(l,[i]) - 1 = \Delta_1. \tag{141}$$

Substituting (141) into (137) completes the proof. □

It remains to check that $\mathrm{crit}^+_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^-_{x,c}(\lambda)$ and $\mathrm{crit}^-_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^+_{x,c}(\lambda)$.

First, we make some make some straightforward but important observations about $M_{r,s,c}(\lambda)$ and winding numbers in Proposition 91. Then, we apply these to the first arrival tree to prove some formulae about distances between consecutive vertices in the (r,s,c)-tour with respect to the first arrival tree, depending on whether the vertex is eastern, southern, or a switch in Proposition 92. Finally, we apply these to proving $\mathrm{crit}^+_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^-_{x,c}(\lambda)$ and $\mathrm{crit}^-_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^+_{x,c}(\lambda)$ in Proposition 93.

Proposition 91. *Let $(v,[i])$ and $(w,[j])$ be two vertices of $M_{r,s,c}(\lambda)$, and let p_1 and p_2 be directed paths between $(v,[i])$ and $(w,[j])$. Suppose p_1 is given by the sequence of vertices $(v,[i]) = (v_0,[i_0]),\dots,(v_{|p_1|,[i_0+|p_1|]}) = (w,[j])$. Then,*

1. *$|p_1| - |p_2|$ is divisible by $\mathrm{lcm}(c, r+s)$.*
2. *Let $(v,[i])$ be m lattice steps below the upper boundary of the cylinder, and let $|p_1| = q\,\mathrm{lcm}(c,r+s) + u$ where $-m < u \leqslant \mathrm{lcm}(c,r+s) - m$. The winding number of p_1 is q.*

Proof. The first point follows from Proposition 54: p_1 and p_2 are lattice paths from points (x_1,y_1) and (x_1+ar, y_1-as) respectively to points (x_2,y_2) and (x_2+br, y_2-bs) respectively, where $\mathrm{lcm}(c,r+s)$ divides $a(r+s)$ and $b(r+s)$. We have that $|p_1| = x_2 - x_1 + y_1 - y_2$ and $|p_2| = x_2 + br - x_1 - ar + y_1 - as - y_2 + bs$, so $|p_1| - |p_2| = (r+s)(a-b)$, which is divisible by $\mathrm{lcm}(c,r+s)$.

The second point follows because as we trace out a directed path, the value of $x-y$ moves cyclically through the residue classes modulo $\mathrm{lcm}(c,r+s)$, incrementing by 1 with each step. □

Proposition 92. *Let $(k,[0]) = v_0, v_1 \dots, v_{(r+s)k_1} = (k,[0])$ be the vertices visited, in order, by the (r,s,c)-tour, corresponding to the section of the boundary of λ between $(0,k_1 s)$ and $(k_1 r, 0)$. Let d_i denote the distance in the first arrival tree T from $(k,[0])$ to v_i.*

1. *If v_i is a switch, or if there is a copy of the edge (v_{i-1},v_i) in T, then $d_i - d_{i-1} = 1$.*
2. *If v_i is an eastern vertex and there is no copy of (v_{i-1},v_i) in $E(T)$, then $d_i - d_{i-1} = 1 + \mathrm{lcm}(c,r+s)$.*

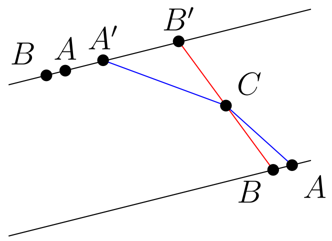
3. *If v_i is a southern vertex and there is no copy of (v_{i-1}, v_i) in $E(T)$, then $d_i - d_{i-1} = 1 - \operatorname{lcm}(c, r+s)$.*

Proof. Write p_i for the path in T from $(k, [0])$ to v_i, so that $|p_i| = d_i$. The first point follows immediately from the definition of a switch and the definition of T.

In general, the winding number of a vertex v is the same as the winding number of the last vertex on the upper boundary strip that T before v. So, drawing T on the cylinder and then forgetting the identification of the two boundary lines, the connected components form sets of vertices of equal winding number.

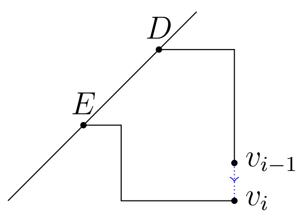
Moreover, if $\operatorname{wind}(p_i) = \operatorname{wind}(p_{i-1})$, then by the second part of Proposition 91, $||p_i| - |p_{i-1}|| < \operatorname{lcm}(c, r+s)$. Since there is a path of length 1 (not necessarily in T) connecting v_{i-1} and v_i, then by the first part of Proposition 91, $|p_i| - |p_{i-1}| \equiv 1 \pmod{\operatorname{lcm}(c, r+s)}$. Therefore, $d_i - d_{i-1} = 1$, so v_i is a switch or there is a copy of (v_{i-1}, v_i) in $E(T)$.

For 2 and 3, we first prove that as we scan southwest along the upper boundary strip, the winding numbers of the paths from $(k, [0])$ to the vertices on the strip weakly increase. We proceed by induction.



Suppose A and B are vertices on the upper boundary strip and B is southwest of A, and let p_A and p_B be the paths in T from $(k, [0])$ to A and B respectively. We will show $\operatorname{wind}(p_B) \geqslant \operatorname{wind}(p_A)$. If $A = (k, [0])$ then we are done, so suppose not. There is a copy of both A and B on the lower boundary strip, with B still southwest of A. Moreover, p_A and p_B run from points A' and B' respectively on the upper boundary strip to A and B, where we possibly have $A' = B'$. However, A' cannot be strictly southwest of B', as otherwise p_A and p_B would have to cross at a vertex C, introducing a cycle from $(k, [0])$ following p_A to C and then following p_B back to $(k, [0])$. Let p'_B and p'_A be p_B and p_A shortened to finish at B' and A' respectively. Then, by strong induction, $\operatorname{wind}(p'_B) \geqslant \operatorname{wind}(p'_A)$. Adding 1 to both sides, $\operatorname{wind}(p_B) \geqslant \operatorname{wind}(p_A)$.

Now, in the case that v_i is eastern, and there is no copy of (v_{i-1}, v_i) in $E(T)$, (v_{i-1}, v_i) must be a south edge, and v_{i-1} and v_i lie in different connected components. Since $\operatorname{wind}(p_i) \neq \operatorname{wind}(p_{i-1})$, $\operatorname{wind}(p_i) > \operatorname{wind}(p_{i-1})$. Let D and E be the last vertices on the upper boundary strip on p_i and p_{i-1} respectively.



Since all paths in T have vertices at lattice points and do not intersect with each other, there can be no path that starts at a vertex on the upper boundary strip between E and D that crosses all the way to the lower boundary strip. Hence, the copy of E on the lower boundary strip either lies in the same connected component as D or in a component northeast of D. So, $\mathrm{wind}(p_i) = \mathrm{wind}(p_{i-1}) + 1$. Let q_i be the path obtained by extending p_{i-1} by the south edge (v_{i-1}, v_i). Then $|q_i| = d_{i-1} + 1$. The second part of Proposition 91 tells us that $|p_i|$ and $|q_i|$ agree modulo $\mathrm{lcm}(c, r+s)$ and therefore $d_i = d_{i-1}+1+\mathrm{lcm}(c, r+s)$.

An analogous argument proves the third formula. □

Proposition 93. *Let λ be a partition. Then*

$$\mathrm{crit}^{+}_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^{-}_{x,c}(\lambda) \tag{142}$$

and

$$\mathrm{crit}^{-}_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^{+}_{x,c}(\lambda). \tag{143}$$

Proof. We will check that $\mathrm{crit}^{+}_{x,c}(\lambda) = (\mathrm{crit}^{+}_{x,c} + \mathrm{crit}^{-}_{x,c})(M_{r,s,c}(\lambda)) - \mathrm{crit}^{+}_{x,c}(I_{r,s,c}(\lambda))$.

Recall that $\mathrm{crit}^{+}_{x,c}$ counts the total number of inversions in the arrival word at vertices in $M_{r,s,c}(\lambda)$. Suppose the arrival word at vertex $(v, [i])$ has a south edges and b east edges. If v is a switch, then I reverses the arrival word at $(v, [i])$, so the pairs of S, E edges that contribute to $\mathrm{crit}^{+}_{x,c}(I(\lambda))$ are exactly those that do not contribute to $\mathrm{crit}^{+}_{x,c}(\lambda)$, so the contributions over $I(\lambda)$ and λ at $(v, [i])$ sum to ab.

Note that if $(v, [i]) \in \mathrm{Ea}$ then $I(\lambda)$ has inversions in the arrival word at $(v, [i])$ using the first E and any S in the arrival word, and then any other pair of south and east edges contribute to $I(\lambda)$ if and only if they do not contribute to λ, so the two contributions sum to $ab + a$. Similarly, if $(v, [i]) \in \mathrm{So}$ then the contributions sum to $ab - b$. Hence, we have that the total $\mathrm{crit}^{+}_{x,c}(\lambda) + \mathrm{crit}^{+}_{x,c}(I_{r,s,c}(\lambda))$ can be written as $S_1 + S_2 + S_3$ where

$$S_1 = \sum_{(v,[i])\text{is a switch}} \mathrm{E_{in}}(v, [i])\mathrm{S_{in}}(v, [i])$$

$$S_2 = \sum_{(v,[i])\in \mathrm{Ea}} \mathrm{E_{in}}(v, [i])\mathrm{S_{in}}(v, [i]) + \mathrm{S_{in}}(v, [i])$$

$$S_3 = \sum_{(v,[i])\in \mathrm{So}} \mathrm{E_{in}}(v, [i])\mathrm{S_{in}}(v, [i]) - \mathrm{E_{in}}(v, [i]).$$

Now, note first that no vertex $(v, [i])$ with $v > k$ contributes to any of these sums. Indeed, no such vertex is a switch, and the arrival word at any such $(v, [i])$ has length 0, 1 or 2, containing at most one S and at most one E. If the arrival word is empty there is nothing to prove. If the arrival word is E then the vertex is eastern, and $\mathrm{E_{in}}(v, [i])\mathrm{S_{in}}(v, [i]) + \mathrm{S_{in}}(v, [i]) = 0$. If the arrival word is S then the vertex is southern and $\mathrm{E_{in}}(v, [i])\mathrm{S_{in}}(v, [i]) - \mathrm{E_{in}}(v, [i]) = 0$. The only other possible arrival word is SE, in which case the vertex is southern and $\mathrm{E_{in}}(v, [i])\mathrm{S_{in}}(v, [i]) - \mathrm{E_{in}}(v, [i]) = 1 - 1 = 0$. So, we may restrict our sum to vertices $(v, [i])$ with $v \leqslant k$.

Proposition 88 proves (122),

$$(\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c})(M_{r,s,c}(\lambda)) = \sum_{v=0}^{k}\sum_{i=0}^{c-1} \mathrm{E}_{\mathrm{in}}(v,[i])\mathrm{S}_{\mathrm{in}}(v,[i]) - \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c,r+s)} \right\rfloor .$$

We wish to show that (122) is equal to $S_1 + S_2 + S_3$, and therefore it suffices to check that

$$\sum_{(v,[i])\in \mathrm{So}} \mathrm{E}_{\mathrm{in}}(v,[i]) - \sum_{(v,[i])\in \mathrm{Ea}} \mathrm{S}_{\mathrm{in}}(v,[i]) = \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c,s+r)} \right\rfloor . \tag{144}$$

Note that the east edges entering southern vertices and the south edges entering eastern vertices are exactly the edges in $M_{r,s,c}(\lambda)$ arriving at non-switch vertices that are *not* a copy of an edge in the first arrival tree T. Hence, if we let n_0 denote the number of edges e entering vertices $(v,[i])$ with $v \leqslant k$ such that either

- $(v,[i])$ is a switch, or
- $(v,[i])$ is not a switch and there is a copy of e in the first arrival tree T,

then

$$n_0 + \sum_{(v,[i])\in \mathrm{So}} \mathrm{E}_{\mathrm{in}}(v,[i]) + \sum_{(v,[i])\in \mathrm{Ea}} \mathrm{S}_{\mathrm{in}}(v,[i]) = k_1(r+s). \tag{145}$$

Now, let $(k,[-k_1 s]) = v_0, v_1 \ldots, v_{(r+s)k_1} = (k,[k_1 r])$ be the vertices visited, in order, possibly with repetition, by the (r,s,c)-tour. Let d_i denote the distance in the first arrival tree from $(k,[-k_1 s])$ to v_i. Now $c \mid k_1$ by assumption, and thus $c \mid (r+s)k_1$, so we have that

$$0 = d_{(r+s)k_1} = \sum_{i=1}^{(r+s)k_1} d_i - d_{i-1}. \tag{146}$$

Substituting the formulae for $d_i - d_{i-1}$ proven in Proposition 92 into (146) and writing l for $\mathrm{lcm}(c, r+s)$,

$$n_0 + (1+l)\sum_{(v,[i])\in \mathrm{Ea}} \mathrm{S}_{\mathrm{in}}(v,[i]) + (1-l)\sum_{(v,[i])\in \mathrm{So}} \mathrm{E}_{\mathrm{in}}(v,[i]) = 0. \tag{147}$$

Subtracting (147) from (145) gives

$$k_1(r+s) = \mathrm{lcm}(c,r+s)\left(\sum_{(v,[i])\in \mathrm{So}} \mathrm{E}_{\mathrm{in}}(v,[i]) - \sum_{(v,[i])\in \mathrm{Ea}} \mathrm{S}_{\mathrm{in}}(v,[i])\right). \tag{148}$$

Now, since k is divisible by rsc, $k_1 = \frac{k}{rs}$ is divisible by c, so $\mathrm{lcm}(c, r+s)$ divides $k_1(r+s)$. Therefore,

$$\left\lfloor \frac{k_1(r+s)}{\mathrm{lcm}(c,r+s)} \right\rfloor = \frac{k_1(r+s)}{\mathrm{lcm}(c,r+s)} = \sum_{(v,[i])\in \mathrm{So}} \mathrm{E}_{\mathrm{in}}(v,[i]) - \sum_{(v,[i])\in \mathrm{Ea}} \mathrm{S}_{\mathrm{in}}(v,[i])),$$

which is (144), which completes the proof. □

6.3 Extended Example

Let $c = 2$ and $n = 7$, and $\mu = (2, 1)$. Then

$$\mathrm{Par}^2_\mu(7) = \{(6,1), (4,3), (4,1,1,1), (2,2,2,1), (2,1,1,1,1,1)\}. \tag{149}$$

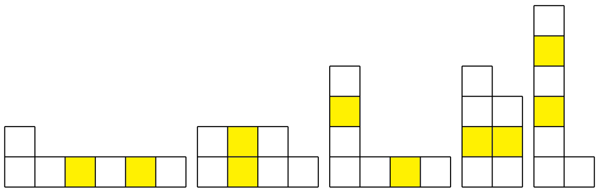


Figure 15: the partitions in $\mathrm{Par}^2_\mu(7)$ with boxes of even hook length coloured yellow.

For the shaded cells, the set of values of $\frac{a(\square)}{l(\square)+1}$ is $\left\{3, 1, 0, \frac{1}{3}\right\}$, and the set of values of $\frac{a(\square)+1}{l(\square)}$ is $\left\{\infty, 3, 1, \frac{1}{3}\right\}$. So, the critical rationals are $\left\{0, \frac{1}{3}, 1, 3, \infty\right\}$.

In this example, we will verify that

$$\sum_{\lambda \in \mathrm{Par}^2_\mu(7)} t^{h^+_{4,2}(\lambda)} = \sum_{\mathrm{Par}^2_\mu(7)} t^{\lambda^{2*}_\square}.$$

Recall

$$h^+_{4,2}(\lambda) = \left|\left\{\square \in \lambda : 2 \mid h(\square) \text{ and } \frac{a(\square)}{l(\square)+1} \leqslant 4 < \frac{a(\square)+1}{l(\square)}\right\}\right|. \tag{150}$$

From our computation of the critical rationals, given that $2 \mid h(\square)$ for some box in a partition $\lambda \in \mathrm{Par}^2_\mu(7)$, $4 < \frac{a(\square)+1}{l(\square)}$ if and only if $3 < \frac{a(\square)+1}{l(\square)}$, and $\frac{a(\square)}{l(\square)+1} \leqslant 4$ if and only if $\frac{a(\square)}{l(\square)+1} \leqslant 3$. So, $h^+_{4,2}(\lambda) = h^+_{3,2}(\lambda)$. Now we use $I_{3,1,2} : \mathrm{Par}^2_\mu(7) \to \mathrm{Par}^2_\mu(7)$. Because $\mathrm{mid}_{3,2}(\lambda) = \mathrm{mid}_{3,2}(I_{3,1,2}(\lambda))$ and $\mathrm{crit}^\pm_{3,2}(\lambda) = \mathrm{crit}^\mp_{3,2}(\lambda)$, $I_{3,1,2}$ is a bijection exchanging $h^+_{3,2}$ and $h^-_{3,2}$, so

$$\sum_{\lambda \in \mathrm{Par}^2_\mu(7)} t^{h^+_{3,2}(\lambda)} = \sum_{\lambda \in \mathrm{Par}^2_\mu(7)} t^{h^-_{3,2}(\lambda)}.$$

We now explicitly compute $I_{3,1,2}(\lambda)$ for $\lambda = (6, 1)$.

The diagram of $(6, 1)$ lies below the line $3y + x = 9$. So, we choose the smallest value $k \geqslant 9$ such that $3 \times 2 \times 1 \mid k$, $k = 12$. Then, $k_1 = \frac{12}{3} = 4$.

The (r, s, c)-tour of $M_{3,1,2}((6, 1))$ is defined by the following family of arrival words.

(4,[0])	S	(5,[1])	E	(6,[0])	SSE
(7,[1])	EEE	(8,[0])	E	(9,[1])	SEE

and for $w > 9$,

$$(w,[i])_a = \begin{cases} SE & 3 \mid w, w \equiv \frac{-w}{3} \equiv i \pmod 2 \\ E & 2 \mid (w-i) \text{ and either } 3 \nmid w \text{ or } 2 \nmid (\frac{-w}{3} - i) \\ S & 3 \mid w, 2 \mid \left(\frac{-w}{3} - i\right), 2 \nmid (w-i) \\ \text{empty} & \text{otherwise} \end{cases}. \tag{151}$$

The multigraph is given in Figure 16 with the edges in the first arrival tree in bold.

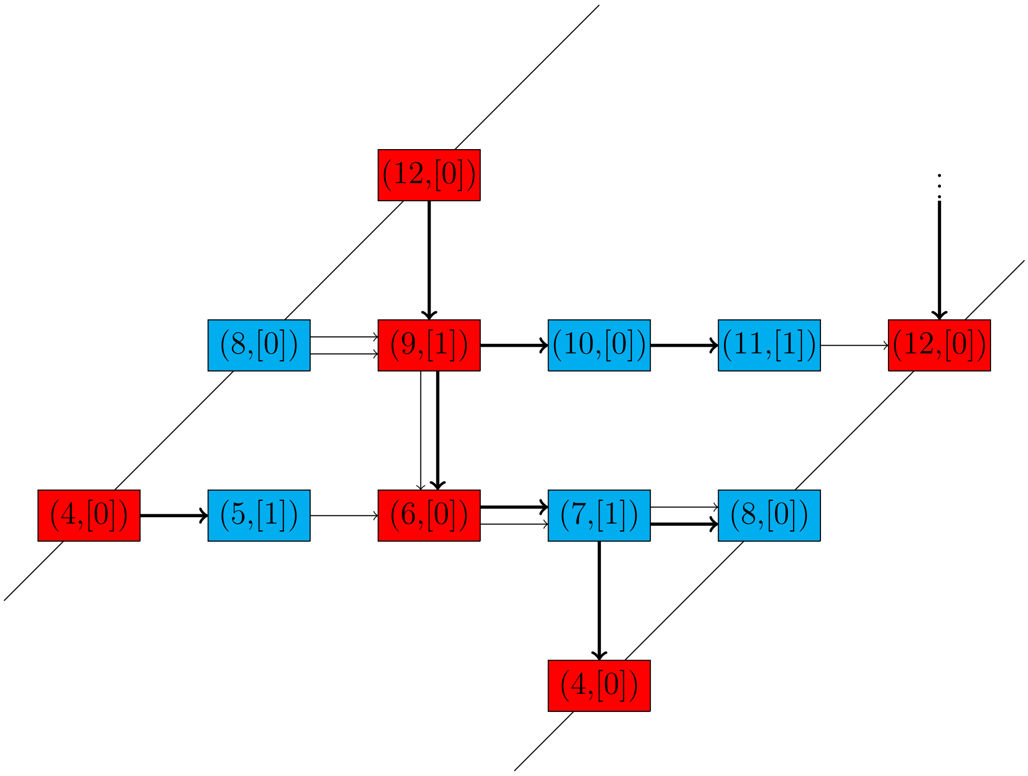


Figure 16: $M_{3,1,2}((6,1))$ with the edges of the first arrival tree in bold.

After applying $I_{3,1,2}$ the arrival words are

(4,[0])	S	(5,[1])	E	(6,[0])	SSE
(7,[1])	EEE	(8,[0])	EE	(9,[1])	SEE

with all arrival words at $(w,[i])$ with $w > 9$ unchanged. These arrival words correspond to the partition $(4,3)$. So, $h^+_{3,2}((6,1)) = h^-_{3,2}((4,3))$.

From our computation of the critical rationals, given that $2 \mid h(\square)$ for some box in a partition $\lambda \in \mathrm{Par}^2_\mu(7)$, $3 \leqslant \frac{a(\square)+1}{l(\square)}$ if and only if $1 < \frac{a(\square)+1}{l(\square)}$, and $\frac{a(\square)}{l(\square)+1} < 3$ if and only if $\frac{a(\square)}{l(\square)+1} \leqslant 1$. So, $h^-_{3,2}(\lambda) = h^+_{1,2}(\lambda)$ for all $\lambda \in \mathrm{Par}^2_\mu(7)$. Now, $I_{1,1,2}$ exchanges $h^+_{1,2}$ and $h^-_{1,2}$, and $I_{1,1,2}((4,3)) = (2,2,2,1)$, so $h^+_{4,2}((6,1)) = h^+_{1,2}((2,2,2,1))$. Using the same logic again

$h^+_{1,2}(\lambda) = h^-_{\frac{1}{3},2}(\lambda)$ for each $\lambda \in \mathrm{Par}^2_\mu(7)$. Using $I_{1,3,2}$, $I_{1,3,2}((2,2,2,1)) = (2,1,1,1,1,1)$, so $h^+_{4,2}((6,1)) = h^-_{\frac{1}{3},2}((2,1,1,1,1,1))$. Finally, for any partition $\lambda \in \mathrm{Par}^2_\mu(7)$, $\frac{1}{3} \leqslant \frac{a(\square)+1}{l(\square)}$ if and only if $0 < \frac{a(\square)+1}{l(\square)}$, and $\frac{a(\square)}{l(\square)+1} < \frac{1}{3}$ if and only if $a(\square) = 0$, if and only if $\frac{a(\square)}{l(\square)+1} \leqslant 0$, so $h^-_{\frac{1}{3},2}(\lambda) = h^+_{0,2}(\lambda)$.

Therefore, since

$$I_{1,3,2} \circ I_{1,1,2} \circ I_{3,1,2}((6,1)) = (2,1,1,1,1,1),$$

we have that $h^+_{4,2}(6,1) = h^+_{0,2}(2,1,1,1,1,1)$. For the other partitions in $\mathrm{Par}^2_\mu(7)$,

$$I_{1,3,2} \circ I_{1,1,2} \circ I_{3,1,2}(4,3) = I_{1,3,2} \circ I_{1,1,2}((6,1)) = I_{1,3,2}(2,1,1,1,1,1) = (2,2,2,1),$$

$$I_{1,3,2} \circ I_{1,1,2} \circ I_{3,1,2}(4,1,1,1) = I_{1,3,2} \circ I_{1,1,2}((4,1,1,1)) = I_{1,3,2}(4,1,1,1) = (4,1,1,1).$$

$$I_{1,3,2} \circ I_{1,1,2} \circ I_{3,1,2}(2,2,2,1) = I_{1,3,2} \circ I_{1,1,2}((2,2,2,1)) = I_{1,3,2}(4,3) = (4,3).$$

$$I_{1,3,2} \circ I_{1,1,2} \circ I_{3,1,2}(2,1,1,1,1,1) = I_{1,3,2} \circ I_{1,1,2}((2,1,1,1,1,1)) = I_{1,3,2}(6,1) = (6,1).$$

Hence we can verify the equidistribution of $h^+_{x,2}$ with $h^-_{x,2}$ over $\mathrm{Par}^2_\mu(7)$ for each $x \in \mathbb{R}_{>0}$, thus verifying Theorem 3.3 in this case.

7 Further Work

We note here that Problem 8.9 in [9] may be amenable to similar techniques.

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