Extended Double Covers and Homomorphism Bounds of Signed Graphs

Florent Foucaud[a] Reza Naserasr[b] Rongxing Xu[c,d]

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Abstract

A *signed graph* (G, σ) is a graph G together with an assignment $\sigma : E(G) \to \{+, -\}$. A homomorphism of a signed graph (G, σ) to a signed graph (H, π) is a mapping of vertices and edges of G to those of H such that signs of closed walks (products of signs of edges) are preserved. Given a class $\mathcal{C}$ of signed graphs, a connected signed graph (B, π) with no positive odd closed walk is said to be $\mathcal{C}$-*complete* if every connected signed graph in $\mathcal{C}$ with no positive odd walk whose negative girth is at least the negative girth of (B, π) admits a homomorphism to (B, π).

As a potential extension of the 4-Color Theorem we conjecture that: If a connected signed graph (B, π) has no positive odd walk and is planar-complete, then so is its extended double cover, that is a signed graph on $2|V(B)|$ vertices built from (B, π). Then, in support of this conjecture, we show that it holds on the smaller class of signed K_4-minor-free graphs. Finally, for each negative girth k, we build a signed graph (B, π) of order nearly $\frac{k^2}{2}$ which has no positive odd walk, its negative girth is k and it is complete for the class of signed K_4-minor-free graphs. This is nearly the optimal order for such graphs.

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1 Introduction

Some of the most prominent theories and conjectures in graph theory are about proving that certain graphs have rich enough structures. For example, the 4-Color Theorem

[a]Université Clermont Auvergne, CNRS, Clermont Auvergne INP, Mines Saint-Étienne, LIMOS, 63000 Clermont-Ferrand, France (`florent.foucaud@uca.fr`).
[b]Université de Paris, CNRS, IRIF, F-75006, Paris, France (`reza@irif.fr`).
[c]School of Mathematical Sciences, Zhejiang Normal University, Jinhua, Zhejiang, 32100, China (`xurongxing@zjnu.edu.cn`).
[d]School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026, China.

(4CT) is to say that K_4 is rich enough to admit a homomorphism from any simple planar graph. Attempts to build other graphs of similarly rich enough structures has led to the introduction of the notion of homomorphisms of signed graphs (see [11]). The main aim of this work is to propose and study two different ways of building such highly structured graphs.

Graphs in this work are simple and connected, unless stated otherwise. For graph terminology, we follow [13]. We work on the larger realm of signed graphs; we mention the key terminologies and refer to [11] for further details.

A *signed graph* (G, σ) is a graph G together with an assignment $\sigma : E(G) \to \{+, -\}$. If E^- is the set of negative edges of (G, σ), then we may, equivalently, use the notation (G, E^-) to denote (G, σ). The graph G is referred to as the *underlying graph* of (G, σ) and σ is called its *signature*. A *switching* at a vertex v of a signed graph (G, σ) is to multiply the signs of all edges in the cut $(v, V(G) \setminus v)$ by $-$. To switch at a set X of vertices is to switch at all the vertices of X in any sequence; this is equivalent to multiplying the signs of all edges in the cut $(X, V(G) \setminus X)$ by $-$. Two signatures σ_1 and σ_2 on G are said to be *switching equivalent*, or equivalent for short, if one can be obtained from the other one by a switching.

Given a closed walk W of (G, σ), the sign of W is the product of the signs of all the edges of W, considering multiplicity. It is clearly invariant under the switching operation. The smallest length of a negative cycle in (G, σ) is referred to the *negative girth* of (G, σ) and is denoted by $g_-(G, \sigma)$. As observed in [11], if (G, σ) is connected and has no closed walk which is positive and of odd length, then all its negative cycles are of the same parity.

One of the first theorems on the theory of signed graphs is the following.

Theorem 1 ([14])**.** *A signed graph (G, σ') is obtained from a switching at a subset X of vertices of (G, σ) if and only if for any cycle C we have $\sigma(C) = \sigma'(C)$.*

A *homomorphism* of a signed graph (G, σ) to a signed graph (H, π) is a mapping f which maps the vertices and edges of G to the vertices and edges of H, respectively, with the property that incidences, adjacencies and signs of closed walks are preserved.

For $i, j \in \mathbb{Z}_2^2$ let $g_{ij}(G, \sigma)$ be the length of a shortest closed walk in (G, σ) whose parity of length is j and whose parity of the number of negative edges is i, setting this length to be ∞ when there is no such closed walk.

Lemma 2 (No-homomorphism lemma)**.** *Given signed graphs (G, σ) and (H, π), if (G, σ) admits a homomorphism to (H, π), then for each $ij \in \mathbb{Z}_2^2$, we have $g_{ij}(G, \sigma) \geqslant g_{ij}(H, \pi)$.*

As a particular case of this lemma, we have that: If (H, π) is connected and has no positive odd walk, then (G, σ) (which maps to (H, π)), cannot have a positive closed walk either, and all its negative cycles are of the same parity as $g_-(H, \pi)$ and at least of this size.

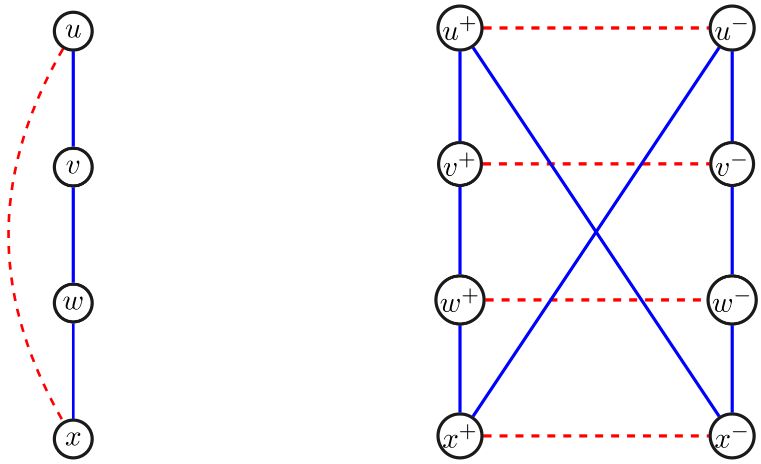
When there are no parallel edges, the vertex-mapping and the condition of preserving adjacencies uniquely determine the edge-mapping, and thus we may define a homomorphism solely by a vertex-mapping, as was originally done in [10].

In this work we are interested in graphs that admit a homomorphism from any member of a given class of graphs.

Definition 3. Given a class $\mathcal{C}$ of signed graphs and a signed graph (B, π), we say that (B, π) is $\mathcal{C}$*-complete* if every member (G, σ) of $\mathcal{C}$ satisfying $g_{ij}(G, \sigma) \geqslant g_{ij}(B, \pi)$ for every $ij \in \mathbb{Z}_2^2$, admits a homomorphism to (B, π).

The family $\mathcal{P}$ of signed planar graphs is among the most motivating families for this study. For example, the 4CT is to say that $(K_4, -)$ is $\mathcal{P}$-complete. In this work we propose that the following construction can be used to build more $\mathcal{P}$-complete signed graphs.

Definition 4 ([11])**.** Given a signed graph (G, σ), the *Extended Double Cover* of (G, σ), denoted $\mathrm{EDC}(G, \sigma)$, is defined to be the signed graph on vertex set $V^+ \cup V^-$, where $V^+ := \{v^+ : v \in V(G)\}$ and $V^- := \{v^- : v \in V(G)\}$. For each vertex x, the two vertices x^+ and x^- are connected by a negative edge; all other edges, to be described next, are positive. If vertices u and v are adjacent in (G, σ) by a positive edge, then v^+u^+ and v^-u^- are two positive edges of $\mathrm{EDC}(G, \sigma)$. If vertices u and v are adjacent in (G, σ) by a negative edge, then v^+u^- and v^-u^+ are two positive edges of $\mathrm{EDC}(G, \sigma)$.



(a) (C_{-4}) (b) $\mathrm{EDC}(C_{-4})$

Figure 1: Signed graphs C_{-4} and $\mathrm{EDC}(C_{-4})$. Dashed (red) edges are negative.

Conjecture 5. Given a planar-complete signed graph (B, π) with no positive odd walk, $EDC(B, \pi)$ is also planar-complete.

Observe that the signed graph on two vertices, connected with one positive and one negative edge, known as *digon*, admits a homomorphism from any signed bipartite graph and thus, in particular, is planar-complete. The extended double cover of the digon is switching equivalent to $(K_4, -)$, that is, a K_4 with all edges negative. Thus the 4CT is a special case of this general conjecture. The sequence of graphs built from the digon by repeatedly applying the extended double cover operation are known as the projective cubes. The restriction of the conjecture on this specific family already is quite difficult and is related to some of the most challenging questions in graph theory and combinatorics. We refer to [5] for some more details.

In this work, in support of this conjecture, we prove the following. Let $\mathcal{SP}$ denote the class of K_4-minor-free graphs, also known as Series-Parallel graphs.

Theorem 6. *If a connected signed graph (B, π) with no positive odd walk is $\mathcal{SP}$-complete, then* $\mathrm{EDC}(B, \pi)$ *is also $\mathcal{SP}$-complete.*

To prove this, we extend the notions and terminologies of signed graphs to weighted signed graphs in the next section. In Section 3 we present a proof of this theorem. Then, in Section 4, towards an optimization, we present another method of building $\mathcal{C}$-complete graphs. More precisely, for each given k we build an $\mathcal{SP}$-complete graph graph (B, π) having no closed odd walk whose shortest negative cycles is of length k. Our constructions are of order $\lfloor \frac{k^2}{2} \rfloor$, this is nearly optimal. These constructions can be viewed as a 2-dimensional analogue of the projective cubes in that they are built from grids with all positive edges by adding a negative edge between each pair of vertices at maximum distance.

2 Weighted signed graphs

A *weighted signed graph* (G, ω) is a graph G together with an assignment ω of weights in $\{\pm 1, \pm 2, \ldots, \pm k\}$ (for some integer k) to the edges of G. To emphasize on the maximum absolute value, we may refer to a weighted signed graph as a k-weighted signed graph. Observe that 1-weighted signed graphs are simply signed graphs.

The *length* of a walk, cycle or path in a weighted signed graph is the sum of absolute values of the weights of its edges. The sign of such a structure is the product of all the signs of the edges of said structure (considering multiplicity). Switching at a cut $(X, V \setminus X)$ is to change the sign of the weight of each edge in this cut while the absolute value remains the same. The walk-girth of type ij (for $ij \in \mathbb{Z}_2^2$) of a weighted signed graph (G, ω), denoted $g_{ij}(G, \omega)$, is defined similarly as before and again remains invariant under switching. As before, if G is connected and there is no positive odd walk in (G, ω), then all the negative walks are of the same parity.

A homomorphism of a weighted signed graph (G, ω) to a weighted signed graph (H, θ) is defined analogously: that is, a mapping of the vertices and edges of (G, ω) to, respectively, the edges and vertices of (H, θ) which preserves the following fundamental properties, (i). adjacencies, (ii). incidences, (iii). absolute values of weights and (iv). signs of closed walks. It can be proved similarly that this is the same as a switching (G, ω') of (G, ω) after which one must preserve the sign of each edge rather than signs of closed walks. The basic no-homomorphism lemma works here as well.

Lemma 7. *Given weighted signed graphs (G, ω) and (H, θ), if $(G, \omega) \to (H, \theta)$, then for each $ij \in \mathbb{Z}_2^2$, we have $g_{ij}(G, \omega) \geqslant g_{ij}(H, \theta)$.*

A key parameter in connected signed weighted graphs with no positive odd walk is the smallest weight of a negative closed walk.

Definition 8. For a positive integer g, a weighted signed graph (G, ω) is said to be *g-wide* if for each choice of $ij \in \mathbb{Z}_2^2$, we have $g_{ij}(G, \omega) \geqslant g_{ij}(C_{-g})$.

A special family of weighted signed graphs are the ones where the weight of each edge represents the distance in a signed graph on the same set of vertices (normally the signed graph induced by edges of weight 1 and -1). To make this definition formal enough, we use an extended notion of distance as defined in [2], where in a signed graph, we allow two vertices to have a negative distance.

Definition 9. For a signed graph (G, σ), the *algebraic distance* between two vertices u and v is defined as follows:

$$ad_{(G,\sigma)}(u,v) = \begin{cases} d_G(u,v), & \text{if there is a positive } (u,v)\text{-path of length } d_G(u,v), \\ -d_G(u,v), & \text{otherwise.} \end{cases}$$

Observe that the algebraic distance of x and y will change sign if a switching is done at one of x or y, but not both. All other switchings preserve the algebraic distance.

Definition 10. Given a signed graph (G, σ) and a weighted signed graph (G', ω), where $V(G') = V(G)$, we say that (G', ω) is a *partial* (G, σ)*-distance graph* if for every edge uv of G', $\omega(uv) = ad_{(G,\sigma)}(u,v)$. If for every edge xy of G', $|\omega(x,y)| \leqslant k$ we say that (G', ω) is a k*-partial* (G, σ)*-distance graph.*

Given a signed graph (G, σ), the partial (G, σ)-distance graph and the Extended Double Cover of such graph, which will be introduced in Section 2.1, play important roles in this paper.

2.1 Extended Double Cover of (weighted) signed graphs

We extend the notion of Extended Double Covers of signed graphs to Extended Double Covers of weighted signed graphs as follows:

Definition 11. Given a weighted signed graph (G, ω), the *Extended Double Cover* of (G, ω), denoted $\text{EDC}(G, \omega)$, is defined to be the weighted signed graph on vertex set $V^+ \cup V^-$, where $V^+ := \{v^+ : v \in V(G)\}$ and $V^- := \{v^- : v \in V(G)\}$. Vertices x^+ and x^- are adjacent by an edge of weight -1; for each pair xy of adjacent vertices of G, there will be four more edges in $\text{EDC}(G, \omega)$, namely $x^+y^+, x^+y^-, x^-y^+, x^-y^-$ whose weights are determined as follows. If xy is an edge of weight $p > 0$, then x^+y^+ and x^-y^- are both of weight p and x^+y^-, x^-y^+ are both of weight $-(p+1)$. If xy is an edge of weight $-p < 0$, then x^+y^- and x^-y^+ are both of weight p and x^+y^+, x^-y^- are both of weight $-(p+1)$.

As in the case of Extended Double Covers of signed graphs, the Extended Double Cover of a signed weighted graph adds a geometric view to the notion of switching: to switch at a vertex v of (G, ω) is equivalent to switch the role of v^+ and v^- in $\text{EDC}(G, \omega)$.

The following is a key property of this extended notion of Extended Double Cover.

Lemma 12. *Given a weighted signed graph* (G, ω) *we have:*

- $g_{01}(\text{EDC}(G, \omega)) = g_{01}(G, \omega)$.
- $g_{10}(\text{EDC}(G, \omega)) = g_{11}(G, \omega) + 1$.

- $g_{11}(\mathrm{EDC}(G,\omega)) = g_{10}(G,\omega) + 1$.

Proof. All three claims are consequences of the following observation. Given a signed closed walk W of (G,ω) there is a natural association with two closed walks, denoted $\mathrm{EDC}^+(W)$ and $\mathrm{EDC}^-(W)$ in $\mathrm{EDC}(G,\omega)$. If the starting vertex of W is x, then the starting point of $\mathrm{EDC}^+(W)$ is x^+ and that of $\mathrm{EDC}^-(W)$ is x^-. The descriptions of $\mathrm{EDC}^+(W)$, and that of $\mathrm{EDC}^-(W)$ except for the starting point, are the same. If the i^{th} vertex of W is v, then the i^{th} vertex of $\mathrm{EDC}^+(W)$ is one of v^+ or v^-, the choice of which is implied from the following procedure.

Assume that at step i of W we are at vertex v and that $v' \in \{v^+, v^-\}$ is determined as the i^{th} vertex of $\mathrm{EDC}^+(W)$. Let u be the next vertex on W. Then choose the vertex $u' \in \{u^+, u^-\}$ as follows: if the edge vu in W is positive, then u' has the same sign as v', otherwise it has the opposite sign. If W is a positive closed walk, this process ends with x^+ and we have $\mathrm{EDC}^+(W)$. But if W is a negative closed walk, this process ends with x^- in which case we must add the negative edge x^+x^- in order to have a closed walk $\mathrm{EDC}^-(W)$. In this case, $\mathrm{EDC}^-(W)$ is a negative closed walk of length 1 more than that of W, thus of different parity.

It is easily observed that each closed walk of $\mathrm{EDC}(G,\omega)$ that uses at most one negative edge is either of the form $\mathrm{EDC}^+(W)$ or of the form $\mathrm{EDC}^-(W)$ for a closed walk W of (G,ω). Furthermore, if two edges of the form v^+v^- are used, then we can create a closed walk of shorter length which is of the same sign and the same parity. Thus the minimum length closed walks of a given type can use at most one edge of type v^+v^-. The three claims then follow. □

Following the proof of Lemma 12, we have the following lemma (note that each vertex can be viewed as the starting vertex).

Lemma 13. *Let (G,σ) be a signed graph with x, y two vertices of (G,σ) in a cycle C of length g in (G,σ). If C is positive, then each of the pairs (x^+, y^+) and (x^-, y^-) is also in a positive cycle of length g in $\mathrm{EDC}(G,\sigma)$. If C is negative, then each of the pairs (x^+, y^+), (x^-, y^-), (x^+, y^-) and (x^-, y^+) is in a negative cycle of length $g+1$ in $\mathrm{EDC}(G,\sigma)$.*

Focusing on the Extended Double Covers of signed graphs we have the followings.

Observation 14. *Let (G,σ) be a signed graph with x, y two vertices of (G,σ) and an (x,y)-walk W of length p in (G,σ). If W is positive, then, in $\mathrm{EDC}(G,\sigma)$, there exist an (x^+, y^+)-walk and an (x^-, y^-)-walk, both of which are positive and of length p, and an (x^+, y^-)-walk and an (x^-, y^+)-walk, both of which are negative and of length $p+1$. If W is negative, then, in $\mathrm{EDC}(G,\sigma)$, there exist an (x^+, y^-)-walk and an (x^-, y^+)-walk, both of which are positive and of length p, and an (x^+, y^+)-walk and an (x^-, y^-)-walk, both of which are negative and of length $p+1$.*

Proposition 15. *Let (G,σ) be a g-wide signed graph with x, y two of its vertices that are in a common negative cycle of length g, and $ad_{(G,\sigma)}(x,y) = p$. Then, the following statements hold.*

- *If $p > 0$, then $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^+) = ad_{\mathrm{EDC}(G,\sigma)}(x^-, y^-) = p$. Moreover:*
  - *if $d_G(x, y) = \lfloor \frac{g}{2} \rfloor$, then $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^-) = ad_{\mathrm{EDC}(G,\sigma)}(x^-, y^+) = g - p = \lceil \frac{g}{2} \rceil$;*
  - *otherwise, $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^-) = ad_{\mathrm{EDC}(G,\sigma)}(x^-, y^+) = -p - 1$.*
- *If $p < 0$, then $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^-) = ad_{\mathrm{EDC}(G,\sigma)}(x^-, y^+) = -p$. Moreover:*
  - *if $d_G(x, y) = \lfloor \frac{g}{2} \rfloor$, then $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^+) = ad_{\mathrm{EDC}(G,\sigma)}(x^-, y^-) = g + p = \lceil \frac{g}{2} \rceil$;*
  - *otherwise, $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^+) = ad_{\mathrm{EDC}(G,\sigma)}(x^-, y^-) = p - 1$.*

Proof. Without loss of generality, by the symmetries of $\mathrm{EDC}(G, \sigma)$, we only focus on the pairs (x^+, y^+) and (x^+, y^-). As $ad_{(G,\sigma)}(x, y) = p$, and x, y are in a negative cycle of length g, by the definition of algebraic distance, we know that $|p| \leqslant \lfloor \frac{g}{2} \rfloor$. Since (G, σ) is g-wide, by Lemma 12, $\mathrm{EDC}(G, \sigma)$ is $(g + 1)$-wide.

First assume that $p > 0$. Then, there exist a positive (x, y)-path of length $d_G(x, y) = p$ and a negative (x, y)-path of length $g - p$ in (G, σ). By Observation 14, in $\mathrm{EDC}(G, \sigma)$, there exist a positive (x^+, y^+)-path Q_1 of length p, a negative (x^+, y^-)-path Q_2 of length $p+1$ and a positive (x^+, y^-)-path Q_3 of length $g-p$. So the distances between x^+ and y^+, x^+ and y^- in the underlying graph of $\mathrm{EDC}(G, \sigma)$ are p and $\min\{p+1, g-p\}$, respectively. By Definition 9, $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^+) = p$. If $p = \lfloor \frac{g}{2} \rfloor$, then $p + 1 = \lfloor \frac{g}{2} \rfloor + 1 \geqslant \lceil \frac{g}{2} \rceil = g - p$, hence $\min\{p+1, g-p\} = g-p$ and $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^-) = g-p = \lceil \frac{g}{2} \rceil$. Otherwise, we know that $p \leqslant \lfloor \frac{g}{2} \rfloor - 1$, which implies that the distance between x^+ and y^- in the underlying graph of $\mathrm{EDC}(G, \sigma)$ is $\min\{p + 1, g - p\} = p + 1$. Note that there is no positive (x^+, y^-)-path of length $p + 1$, since otherwise such path, together with Q_2, consists of a cycle of length $2(p + 1) = 2\lfloor \frac{g}{2} \rfloor \leqslant g$, a contradiction to the fact that $\mathrm{EDC}(G, \sigma)$ is $(g + 1)$-wide. Thus, by Definition 9, $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^-) = -(p + 1)$.

Now assume that $p < 0$. By similar arguments as above, there exist a negative (x, y)-path of length $d_G(x, y) = -p$ and a positive (x, y)-path of length $g + p$ in (G, σ). By Observation 14, there exist a positive (x^+, y^-)-path Q_4 of length $-p$, a negative (x^+, y^+)-path Q_5 of length $-p + 1$ and a positive (x^+, y^+)-path Q_6 of length $g + p$. Recall that $\mathrm{EDC}(G, \sigma)$ is $(g + 1)$-wide and $p \geqslant -\lfloor \frac{g}{2} \rfloor$, so the distances between x^+ and y^+, x^+ and y^- in the underlying graph of $\mathrm{EDC}(G, \sigma)$ are $\min\{-p + 1, g + p\}$ and $-p$, respectively. Thus, $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^-) = -p$. If $p = -d_G(x, y) = -\lfloor \frac{g}{2} \rfloor$, then $-p + 1 = \lfloor \frac{g}{2} \rfloor + 1 \geqslant \lceil \frac{g}{2} \rceil = g + p$, so $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^+) = g + p = \lceil \frac{g}{2} \rceil$. Otherwise, $p \geqslant -\lfloor \frac{g}{2} \rfloor + 1$, hence $\min\{-p + 1, g + p\} = -p + 1$. Also observe that there is no positive (x^+, y^+)-path of length $-p + 1$, since otherwise such path together with Q_5 consist of a negative cycle of length $2(-p + 1) \leqslant 2\lfloor \frac{g}{2} \rfloor \leqslant g$, a contradiction to the fact that $\mathrm{EDC}(G, \sigma)$ is $(g + 1)$-wide. Thus $ad_{\mathrm{EDC}(G,\sigma)}(x^+, y^+) = -(-p + 1) = p - 1$. □

3 K_4-minor-free graphs

A 2-*tree* is a graph that can be built from the complete graph K_2 in a sequence $G_0 = K_2, G_1, \ldots, G_t$ where G_i is obtained from G_{i-1} by adding a new vertex joined to two adjacent vertices of G_{i-1}, thus forming a new triangle. A partial 2-tree is a subgraph of a 2-tree. It is well-known that a graph is K_4-minor-free if and only if it is a *partial 2-tree* (see for example [6]). The class of K_4-minor-free graphs is also known as the class of *series-parallel graphs*, see for example [3]. Thus we will use the abbreviation $\mathcal{SP}$ to denote this class of graphs.

As observed from the definition of 2-trees, the triangle is the building block of edge-maximal K_4-minor-free graphs. When a girth condition is imposed on a signed K_4-minor-free graph (G, σ), then G will no longer be edge-maximal, but rather a partial 2-tree. To take advantage of the structure of such signed graphs then, in [1] and [2], weighted 2-trees are employed. Next we present these techniques in a uniform language of signed graphs (with no positive walk of odd length). We rather use the terminology developed in [4] while extending it to signed graphs.

3.1 Weighted triangles and g-wideness

Given a positive integer g with $g \geqslant 3$, and three integers p, q and r satisfying $1 \leqslant |p|, |q|, |r| \leqslant g-1$, the signed graph $T_g(p, q, r)$ is built as follows. Let $C_{g,p}$ be a negative cycle of length g with a selected pair x_1 and y_1 of vertices such that one of the two (x_1, y_1)-paths in $C_{g,p}$ is of length $|p|$ and has the same sign as p, and the other, which is of length $g - |p|$, has the opposite sign as p. Define $C_{g,q}$ similarly where selected vertices y_2 and z_1 are connected by a $|q|$-path and $C_{g,r}$ with selected vertices z_2 and x_2 which are connected by an $|r|$-path. We define the signed graph $T_g(p, q, r)$ to be the signed graph obtained from $C_{g,p}$, $C_{g,q}$ and $C_{g,r}$ by identifying x_1 and x_2 (to form the new vertex x), y_1 and y_2 (to form the new vertex y), z_1 and z_2 (to form the new vertex z). See the left picture in Figure 2 for an example.

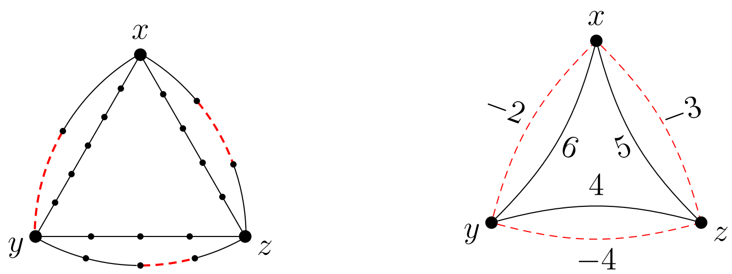


Figure 2: $T_g(p, q, r)$ with $g = 8$, $p = 6$, $q = -4$, and $r = -3$. Dashed (red) edges are negative.

We have a few immediate observations: (i). If p' is the integer satisfying $|p'| = g - |p|$ and $pp' < 0$, then $C_{g,p}$ and $C_{g,p'}$ are identical and thus $T_g(p', q, r)$ is isomorphic to $T_g(p, q, r)$. (ii). A switching at an internal vertex of an (x, y)-path would result in a different presentation of $T_g(p, q, r)$, but up to a switching they are isomorphic. (iii). A switching at x, y or z results, respectively, in $T_g(-p, q, -r)$, $T_g(-p, -q, r)$ and $T_g(p, -q, -r)$ which are isomorphic to $T_g(p, q, r)$ up to a switching.

Note that $T_g(p, q, r)$ could be simply presented as a weighted triangle (multiple edges are allowed here), see the right picture in Figure 2. But considering (i), between p and

p', we would normally choose the one whose value is positive. Therefore, we let $\triangle(p,q,r)$ denote a weighted triangle whose edges are of weight p, q and r, and extend the definition of g-wideness as follows.

Definition 16. Given a positive integer g, $g \geqslant 3$, and three integers p, q and r satisfying $1 \leqslant |p|, |q|, |r| \leqslant g-1$, we say a weighted triangle $\triangle(p,q,r)$, or the triple (p,q,r), is *g-wide* if $T_g(p,q,r)$ is g-wide.

In other words, $\triangle(p,q,r)$ is g-wide if $T_g(p,q,r)$ satisfies the following two conditions:

- There are no positive odd cycles in $T_g(p,q,r)$.
- Each of the negative cycles of $T_g(p,q,r)$ is of the same parity as g and is, furthermore, of length at least g.

In our work we will need to consider properties of triangles and edges. For a uniform writing, in the definition of a g-wide triple (p,q,r) we may allow p, q or r to be 0 as well. If $p=0$, then in the construction of $T_g(p,q,r)$ the vertices x and y are identified and a negative cycle of length g is added on the identified vertex which we may ignore. Then for $\triangle(0,q,r)$ to be g-wide the positive paths corresponding to q and r must be of the same length. Thus we may assume $q=r$. Therefore, in the rest of this work the triple $(0,r,r)$, $r \leqslant g-1$, is g-wide. Triples of the form $(0,r,r)$ will, in essence of it, represent the essential edges of the weighted graphs we will work with where r would be the weight of the corresponding edge.

That $\triangle(p,q,r)$ is g-wide depends only on the values of p, q, r and g. We have already seen that when $p=0$, triples of the form $(0,r,r)$ are the only g-wide triples. For a triple satisfying $1 \leqslant |p|, |q|, |r| \leqslant g-1$ there are a number of ways to check if it is g-wide. In this work we will use the test provided in the next proposition.

Proposition 17. *Given integers g, p, q and r satisfying $1 \leqslant |p|, |q|, |r| \leqslant g-1$, the following statements hold.*

(1) If $pqr > 0$, then the weighted triangle $\triangle(p,q,r)$ is g-wide if and only if $|p|+|q|+|r| \equiv 0 \pmod 2$ and $\max\{2|p|, 2|q|, 2|r|\} \leqslant |p|+|q|+|r| \leqslant 2g$.

(2) If $pqr < 0$, then the weighted triangle $\triangle(p,q,r)$ is g-wide if and only if $|p|+|q|+|r| \equiv g \pmod 2$ and $g \leqslant |p|+|q|+|r| \leqslant g+\min\{2|p|, 2|q|, 2|r|\}$.

Proof.
(1) $pqr > 0$. There are exactly four positive cycles in $T_g(p,q,r)$, and their lengths, which are of the same parity, are: $|p|+|q|+|r|$, $g-|p|+g-|q|+|r|$, $g-|p|+|q|+g-|r|$, and $|p|+g-|q|+g-|r|$.

Suppose first that $\triangle(p,q,r)$ is g-wide. By the definition, we have $g_{ij}(T_g(p,q,r)) \geqslant g_{ij}(C_{-g})$ and since $g_{01}(C_{-g}) = \infty$, there is no positive odd cycle, in other words $|p|+|q|+|r|$ is even. Except the four cycles we mentioned above, all the other cycles in $T_g(p,q,r)$ are negative. Moreover, the three cycles containing exactly two of $\{x,y,z\}$ are all of length g. The four negative cycles containing all three vertices x, y, z are of length $g-|p|+g-|q|+g-|r|$, $g-|p|+|q|+|r|$, $|p|+g-|q|+r$, $|p|+|q|+g-|r|$. Since the negative girth of $T_g(p,q,r)$ is g, we have: $g-|p|+g-|q|+g-|r| \geqslant g$ which is to say $|p|+|q|+|r| \leqslant 2g$. Assuming, without loss of generality, that $\max\{|p|,|q|,|r|\} = |p|$, the condition $g-|p|+|q|+|r| \geqslant g$ implies that $\max\{2|p|,2|q|,2|r|\} = 2|p| \leqslant |p|+|q|+|r|$.

Conversely, assume that $|p|+|q|+|r|$ is even and $\max\{2|p|,2|q|,2|r|\} \leqslant |p|+|q|+|r| \leqslant 2g$. We shall show that $g_{ij}(T_g(p,q,r)) \geqslant g_{ij}(C_{-g})$, for any $ij \in \{01, 10, 11\}$. The case $ij = 01$ follows from the fact that $|p|+|q|+|r|$ is even and the argument discussed in the first paragraph. The three cycles containing exactly two of x, y, z are of length g, and all of them are negative. The four negative cycles containing x, y, z are of length $g-|p|+g-|q|+g-|r|$, $g-|p|+|q|+|r|$, $g-|q|+|p|+|r|$, $g-|r|+|p|+|q|$. By the assumptions, all of these four values are of the same parity as g, and also are at least g, which implies that $g_{1j}(T_g(p,q,r)) \geqslant g_{1j}(C_{-g})$, for each $j \in \{0,1\}$.

(2) $pqr < 0$. Let $p' = -\frac{p}{|p|}(g-|p|)$, $q' = -\frac{q}{|q|}(g-|q|)$, $r' = -\frac{r}{|r|}(g-|r|)$, so $pqr < 0$ if and only if $p'q'r' > 0$. Since $T_g(p,q,r)$ is isomorphic to $T_g(p',q',r')$, $\triangle(p,q,r)$ is g-wide if and only if $T(p',q',r')$ is g-wide. Therefore, by (1), $\triangle(p,q,r)$ is g-wide if and only if $|p'|+|q'|+|r'| = 0 \pmod 2$ and $\max\{2|p'|,2|q'|,2|r'|\} \leqslant |p'|+|q'|+|r'| \leqslant 2g$. Equivalently, $g-|p|+g-|q|+g-|r| = 0 \pmod 2$ and $\max\{2g-2|p|,2g-2|q|,2g-2|r|\} \leqslant g-|p|+g-|q|+g-|r| \leqslant 2g$, after simplification, we have $|p|+|q|+|r| = g \pmod 2$ and $g \leqslant |p|+|q|+|r| \leqslant g+\min\{2|p|,2|q|,2|r|\}$. □

3.2 A test for $\mathcal{SP}$-completeness

We denote by $\mathcal{L}_g$ the set of ordered triples (p,q,r), satisfying $|p|,|q|,|r| \leqslant g-1$ and such that $\triangle(p,q,r)$ is g-wide. Observe that because of the condition $|p|,|q|,|r| \leqslant g-1$ we have less than $8g^3$ non-isomorphic weighted triangles (or edges) $\triangle(p,q,r)$, thus $[\mathcal{L}_g| \leqslant 8g^3$.

Recall that for each p with $1 \leqslant |p| \leqslant g-1$, if p' is the integer satisfying $|p'| = g-|p|$ and $pp' < 0$, then $T_g(p',q,r)$ is the same as $T_g(p,q,r)$. Thus a triple (p,q,r) can be represented in $\mathcal{L}_g$ in 8 possible ways among which there is a unique presentation where $p,q,r \geqslant 1$.

Similarly, recall that in the definition of (G,σ)-distance graph, for each weighted edge, the weight represents the algebraic distance between the two endpoints in (G,σ), which could be either positive or negative. Another special weighted signed graph (G,ω) is obtained from (G,σ) by using only positive weights: here if u and v are on a shortest negative cycle C of (G,σ), then they are connected in (G,ω) where $\omega(uv)$ is the length of the positive path in C connecting u to v.

Definition 18. Given a g-wide signed graph (G,σ) and a weighted signed graph (G',ω), where $V(G') = V(G)$ and G' is such that for each edge xy of G', the pair x, y is in a negative cycle of length g in (G,σ), we say that (G',ω) is a *girth-transformed* (G,σ)-*distance graph* if for every edge uv of G', $\omega(uv) = f_g(ad_{(G,\sigma)}(u,v))$, where f_g is defined on

$-\left\lceil \frac{g}{2} \right\rceil + 1 \leqslant x \leqslant \left\lfloor \frac{g}{2} \right\rfloor$, $x \neq 0$, as following:

$$f_g(x) = \begin{cases} x, & \text{if } x > 0, \\ g + x, & \text{otherwise.} \end{cases}$$

If for every edge xy of G', $\omega(uv) \leqslant k$, we say that (G', ω) is a *k-partial girth-transformed (G, σ)-distance graph.*

Now, with this transformation, we can address weighted signed graphs with only positive weights. Also, some known theorems can be also restated in this language. The following definition is a restatement of the "all g-good property" in [1] and [2].

Definition 19. Given a g-wide weighted signed graph (G, ω) satisfying $1 \leqslant \omega(e) \leqslant g - 1$ for every edge e, a set $\mathcal{T}$ of triangles of G is said to be *g-closed* if the following condition is satisfied:

Denoting by $\mathcal{E}$ the set of weighted edges of the triangles in $\mathcal{T}$, for each edge $xy \in \mathcal{E}$ (assuming $\omega(xy) = p$) and for each triple $(p, q, r) \in \mathcal{L}_g$, there is a triangle $xyz \in \mathcal{T}$ such that $\omega(zx) = q$ and $\omega(zy) = r$, or $\omega(zx) = g - q$ and $\omega(zy) = g - r$.

Remark 20. The only difference between all g-good property and g-closed is that in the former one, (i). $1 \leqslant |w(e)| \leqslant \lfloor \frac{g}{2} \rfloor$; (ii). the last condition is $w(zx) = q$, $w(zy) = r$ or $w(zx) = -q$, $w(zy) = -r$.

Note that in the condition above, if $q \neq r$, then the order of p, q, r matters. To be precise, in such a case, say for (p, r, q), while the definition implying existence of a vertex z satisfying $\omega(zx) = q$ and $\omega(zy) = r$, or $\omega(zx) = g - q$ and $\omega(zy) = g - r$ by consider the triple (p, r, q) instead of (p, q, r) there must also be a vertex z' satisfying $\omega(xz') = r$ and $\omega(yz') = q$, or $\omega(xz') = g - r$ and $\omega(yz') = g - q$.

The following then is a uniform restatement of results of [1] and [2]. (see also [4])

Theorem 21. *A g-wide signed graph (B, π) is $\mathcal{SP}$-complete if and only if there exists a $\lfloor \frac{g}{2} \rfloor$-partial (B, π)-distance graph with a nonempty set $\mathcal{T}$ having all g-good property.*

Then we have the following observation.

Observation 22. *Let (B, π) be a g-wide signed graph, (B', ω) be a $\lfloor \frac{g}{2} \rfloor$-partial (B, π)-distance graph with a nonempty set $\mathcal{T}$ having all g-good property. Then for every edge xy in $T \in \mathcal{T}$, x and y are in a negative cycle of length g in (B, π).*

Proof. Let $\omega(x, y) = p$ with $|p| \leqslant \lfloor \frac{g}{2} \rfloor$. By Proposition 17(2), if $p > 0$, then $(p, -\lfloor \frac{g-p}{2} \rfloor, \lceil \frac{g-p}{2} \rceil) \in \mathcal{L}_g$, and if $p < 0$, then $(p, \lfloor \frac{g+p}{2} \rfloor, \lceil \frac{g+p}{2} \rceil) \in \mathcal{L}_g$. Thus in each case, there exists a vertex z in B such that

$$|\omega(xy)| + |\omega(zx)| + |\omega(zy)| = |p| + \left\lfloor \frac{g - |p|}{2} \right\rfloor + \left\lceil \frac{g - |p|}{2} \right\rceil = g,$$

and $\omega(xy) \cdot \omega(zx) \cdot \omega(zy) < 0$, which implies that there is a negative cycle of length g in (B, π) containing both x and y. $\square$

Then Theorem 21 and Observation 22 can be restated with only positive weights as follows, respectively.

Theorem 23. *A g-wide signed graph (B,π) is $\mathcal{SP}$-complete if and only if there exists a $(g-1)$-partial girth-transformed (B,π)-distance graph (B',ω) which has a nonempty g-closed set $\mathcal{T}$ of triangles.*

Observation 24. *Let (B,π) be a g-wide signed graph, (B',ω) be a $(g-1)$-partial girth-transformed (B,π)-distance graph which has a nonempty g-closed set $\mathcal{T}$ of triangles. Then for every edge xy in $\mathcal{T}$, x and y are in a negative cycle of length g in (B,π).*

Now, we are ready to state and prove our main theorem.

Theorem 25. *For any positive integer g, if a g-wide signed graph (B,π) is $\mathcal{SP}$-complete, then so is* $\mathrm{EDC}(B,\pi)$.

Proof. Let (B,π) be a g-wide signed graph which is $\mathcal{SP}$-complete. By Theorem 23, there exists a g-partial (B,π)-distance graph (B',ω) which has a nonempty and g-closed set $\mathcal{T}$ of triangles whose vertex set is V and (weighted) edges form the set $\mathcal{E}$. We first define the weighted signed graph $(\widehat{B},\widehat{\omega})$ on vertex set $V^+\cup V^-$, where $V^+ := \{v^+ : v\in V\}$ and $V^- := \{v^- : v\in V\}$ as follows: for each $v\in V$, vertices v^+ and v^- are joined by an edge with $\widehat{\omega}(v^+v^-)=g$. If $uv\in\mathcal{E}$ with weight $\omega(uv)$, then we add four edges $\{u^+v^+,u^+v^-,u^-v^-,u^-v^+\}$ with $\widehat{\omega}(u^+v^+)=\widehat{\omega}(u^-v^-)=\omega(uv)$, and $\widehat{\omega}(u^+v^-)=\widehat{\omega}(u^-v^+)=g-\omega(uv)$.

Claim 26. *$(\widehat{B},\widehat{w})$ is a g-partial girth-transformed* $\mathrm{EDC}(B,\pi)$*-distance graph.*

Proof. Let x,y be two vertices of B' forming an edge in $\mathcal{E}$, by Observation 24, x,y are contained in a negative cycle of length g in (B,π). By Lemma 13, x^α and y^β are in a negative cycle of length $g+1$ in $\mathrm{EDC}(B,\pi)$ for any $\alpha,\beta\in\{+,-\}$. Thus, it suffices to show that $\widehat{\omega}(x^\alpha y^\beta)=f_{g+1}(ad_{\mathrm{EDC}(B,\pi)}(x^\alpha,y^\beta))$.

For each vertex x in B, $ad_{\mathrm{EDC}(B,\pi)}(x^+,x^-)=-1<0$, so $f_{g+1}(ad_{\mathrm{EDC}(B,\pi)}(x^+,x^-))=g+1-1=g=\widehat{\omega}(x^+x^-)$.

Assume x and y are two vertices in B with $ad_{(B,\pi)}(x,y)=p>0$. Thus $\omega(xy)=p$. By Proposition 15, $ad_{\mathrm{EDC}(B,\pi)}(x^+,y^+)=ad_{\mathrm{EDC}(B,\pi)}(x^-,y^-)=p$. Then we have $f_{g+1}(ad_{\mathrm{EDC}(B,\pi)}(x^+,y^+))=f_{g+1}(ad_{\mathrm{EDC}(B,\pi)}(x^-,y^-))=p=\omega(xy)=\widehat{\omega}(x^+y^+)=\widehat{\omega}(x^-y^-)$. If $d_B(x,y)=\lfloor\frac{g}{2}\rfloor$, then $\omega(xy)=p=\lfloor\frac{g}{2}\rfloor$. By Proposition 15, $ad_{\mathrm{EDC}(B,\pi)}(x^+,y^-)=ad_{\mathrm{EDC}(B,\pi)}(x^-,y^+)=\lceil\frac{g}{2}\rceil$, so we have

$$\begin{aligned} f_{g+1}(ad_{\mathrm{EDC}(B,\pi)}(x^+,y^-)) = f_{g+1}(ad_{\mathrm{EDC}(B,\pi)}(x^-,y^+)) &= \lceil\frac{g}{2}\rceil \\ &= g-\omega(xy)=\widehat{\omega}(x^+y^-)=\widehat{\omega}(x^-y^+). \end{aligned}$$

Otherwise, by Proposition 15, $ad_{\mathrm{EDC}(B,\pi)}(x^+,y^-)=ad_{\mathrm{EDC}(B,\pi)}(x^-,y^+)=-p-1<0$, so we have

$$\begin{aligned} f_{g+1}(ad_{\mathrm{EDC}(B,\pi)}(x^+,y^-)) &= f_{g+1}(ad_{\mathrm{EDC}(B,\pi)}(x^-,y^+)) \\ &= g+1+(-p-1)=g-p \\ &= g-\omega(xy)=\widehat{\omega}(x^+y^-)=\widehat{\omega}(x^-y^+). \end{aligned}$$

The case $ad_{(B,\pi)}(x, y) = p > 0$ could be verified similarly, we do not repeat again. $\square$

Let $\mathcal{T}'$ be the family of all the triangles in $\widehat{B}$ and $\mathcal{E}'$ consisting of the edges of the triangles in $\mathcal{T}'$, we shall show that $\mathcal{T}'$ is $(g+1)$-closed and nonempty.

Let p, q, r be positive integers such that $(p, q, r) \in \mathcal{L}_{g+1}$ and $e = x^{\alpha}y^{\beta} \in \mathcal{E}'$ with $\widehat{\omega}(e) = p$, where $x, y \in V(B)$, $\alpha, \beta \in \{+, -\}$ (it is possible that $x = y$, in which case, $\alpha \neq \beta$). Following Theorem 23 and Definition 19, we shall prove that there is a triangle in $\mathcal{T}'$ on e. That is equivalent to finding a vertex z^{γ}, where $z \in V(B)$, $\gamma \in \{+, -\}$, such that either

$$\widehat{\omega}(z^{\gamma}x^{\alpha}) = q, \quad \widehat{\omega}(z^{\gamma}y^{\beta}) = r, \tag{C1-1}$$

or

$$\widehat{\omega}(z^{\gamma}x^{\alpha}) = g + 1 - q, \quad \widehat{\omega}(z^{\gamma}y^{\beta}) = g + 1 - r. \tag{C1-2}$$

Note that the statement clearly holds for the case $g = 1$, indeed, in this case, (B, π) is a negative loop. Then $\mathrm{EDC}(B, \pi)$ is a digon, which is obviously $\mathcal{SP}$-complete. For the case that $g = 2$, (B, π) is a digon. In the complete $\mathrm{EDC}(B, \pi)$-distance graph (containing weighted edge uv for any $u, v \in V(B)$), there are four triangles, each of which is a weighted triangle isomorphic to $\triangle(1, 1, 2)$, and it is trivial to check that $\mathcal{T}'$ consisting of these four triangles is g-closed.

Now, consider that $g \geqslant 3$ and let e be an edge with $\widehat{\omega}(e) = p$. We first assume that $e = x^+x^-$: it follows that $p = g = \max\{|p|, |q|, |r|\}$. By Proposition 17(1), $2g \leqslant g+q+r \leqslant 2g+2$ and $g+q+r$ is even, we have $q+r \in \{g, g+2\}$.

If $q + r = g + 2$, then we have $\min\{q, r\} \geqslant 2$, so $g + 1 - q, g + 1 - r \leqslant g - 1$. Moreover, $p + g + 1 - q + g + 1 - r = p + g = 2g$ is even, thus by Proposition 17(1), $(p, g + 1 - q, g + 1 - r) \in \mathcal{L}_g$. By Theorem 23, there is a triangle $xyz \in \mathcal{T}$ such that either $\omega(zx) = q - 1$ or $\omega(zx) = g - (q - 1) = g + 1 - q$. Therefore, if $\omega(zx) = q - 1$, then $\widehat{\omega}(z^-x^+) = g + 1 - q$, and $\widehat{\omega}(z^-x^-) = q - 1 = g + 1 - r$, so (C1-2) holds with $z^{\gamma} = z^-$. If $\omega(zx) = g + 1 - q$, then $\widehat{\omega}(z^+x^+) = g + 1 - q$, and $\widehat{\omega}(z^+x^-) = q - 1 = g + 1 - r$, so (C1-2) holds with $z^{\gamma} = z^+$.

If $q + r = g$, then $(p, q, r) = (g, q, g - q) \in \mathcal{L}_g$ (as $g + q + g - q = 2g$ and by Proposition 17(1)). So there is a triangle $xyz \in \mathcal{T}$ such that either $\omega(zx) = q$ or $\omega(zx) = g - q$. If $\omega(zx) = q$, then $\widehat{\omega}(z^+x^+) = q$, and $\widehat{\omega}(z^+x^-) = g - q = r$, thus (C1-1) holds with $z^{\gamma} = z^+$. If $\omega(zx) = g - q$, then $\widehat{\omega}(z^-x^+) = q$, and $\widehat{\omega}(z^-x^-) = g - q = r$, thus (C1-1) holds with $z^{\gamma} = z^-$.

Now, without loss of generality, we assume that $e = x^+y^{\beta}$ and $x \neq y$. It follows that $\omega(xy) = p < g$ if $\beta = +$ and $\omega(xy) = g - p$ if $\beta = -$. We consider the following two cases.

Case 1. $\max\{q, r\} = g$. In this case, without loss of generality, assume that $q = g$. As by Proposition 17(1) $2g \leqslant p+g+r \leqslant 2g+2$ and $p+g+r$ is even, we have $p+r \in \{g, g+2\}$.

If $p + r = g$, then $(p, q, r) = (p, g, g - p)$. Note that $\widehat{\omega}(x^-x^+) = q = g$, and $\widehat{\omega}(x^-y^{\beta}) = g - p = r$, by setting $z^{\gamma} = x^-$, (C1-1) holds.

Next, assume that $p + r = g + 2$, i.e., $r = g + 2 - p$. Then, $\min\{p, r\} \geqslant 2$, and by Proposition 17(1) we have that $(p, 1, p - 1), (g - p, g - 1, p - 1) \in \mathcal{L}_g$.

- If $\beta = +$, then $\omega(xy) = p$. As $(p, 1, p-1) \in \mathcal{L}_g$, there is a triangle $xyz \in \mathcal{T}$ such that either $\omega(zx) = 1$ and $\omega(zy) = p-1$, or $\omega(zx) = g-1$ and $\omega(zy) = g-p+1$. For the former case, $\widehat{\omega}(z^+x^+) = 1$, and $\widehat{\omega}(z^+y^+) = p-1$; for the latter case, $\widehat{\omega}(z^-x^+) = 1$, $\widehat{\omega}(z^-y^+) = p-1$.

- If $\beta = -$, then $\omega(xy) = g-p$. As $(g-p, g-1, p-1) \in \mathcal{L}_g$, there exists a triangle $xyz \in \mathcal{T}$ such that either $\omega(zx) = g-1$ and $\omega(zy) = p-1$, or $\omega(zx) = 1$ and $\omega(zy) = g+1-p$. For the former case, $\widehat{\omega}(z^-x^+) = 1$, and $\widehat{\omega}(z^-y^-) = p-1$. For the latter case, $\widehat{\omega}(z^+x^+) = 1$, $\widehat{\omega}(z^+y^-) = p-1$.

For each case above, since $g+1-q = 1$ and $g+1-r = p-1$, (C1-2) holds.

Case 2. $\max\{q, r\} < g$. First, assume that $q + r \leqslant g+1$. Recall that $p < g$ (because the only edges of weight g in $(\widehat{B^*}, \widehat{\omega})$ are those of the form x^+x^-), so we have $p+q+r \leqslant 2g$. Thus we have $(p, q, r), (-p, -q, r) \in \mathcal{L}_g$. As $T_g(p, q, r)$ is equivalent to $T_g(-p, -q, r)$, $T_g(-p, -q, r)$ is equivalent to $T_g(g-p, g-q, r)$, so $(g-p, g-q, r) \in \mathcal{L}_g$.

- If $\beta = +$, then $\omega(xy) = p$. As $(p, q, r) \in \mathcal{L}_g$, there exists a triangle $xyz \in \mathcal{T}$ such that either $\omega(zx) = q$ and $\omega(zy) = r$, or $\omega(zx) = g-q$ and $\omega(zy) = g-r$. For the former case, $\widehat{\omega}(z^+x^+) = q$, $\widehat{\omega}(z^+y^+) = r$; for the latter case, $\widehat{\omega}(z^-x^+) = q$, $\widehat{\omega}(z^-y^+) = r$.

- If $\beta = -$, then $\omega(xy) = g-p$. As $(g-p, g-q, r) \in \mathcal{L}_g$, there exists a triangle $xyz \in \mathcal{T}$ such that either $\omega(zx) = g-q$ and $\omega(zy) = r$, or $\omega(zx) = q$ and $\omega(zy) = g-r$. For the former case, we have that $\widehat{\omega}(z^-x^+) = q$, $\widehat{\omega}(z^-y^-) = r$. For the latter case, we have that $\widehat{\omega}(z^+x^+) = q$, $\widehat{\omega}(z^+y^-) = r$.

For each case above, (C1-2) holds.

Now, assume that $q + r \geqslant g+2$, and let $q' = g+1-q$ and $r' = g+1-r$. Thus, we have that $q' + r' \leqslant g < g+1$ and $\max\{q', r'\} < g$. Then, by the previous part, that is, when $q + r \leqslant g+1$, we know that there exists a vertex such that (C1-2) holds. More precisely, there exists a vertex z^γ such that $\widehat{\omega}(z^\gamma x^\alpha) = g+1-q' = q$, and $\widehat{\omega}(z^\gamma y^\beta) = g+1-r' = r$, that is, (C1-1) holds.

This completes the proof. □

4 A class of smaller g-wide $\mathcal{SP}$-complete signed graphs

In this section, we describe a new family of $\mathcal{SP}$-complete signed graphs from $\mathcal{G}_{10} \cup \mathcal{G}_{11}$. For each $g \geqslant 2$, we will construct a g-wide $\mathcal{SP}$-complete signed graph of order $\lfloor g^2/2 \rfloor$. These graphs are smaller than the previously known examples: for odd values of g, bounds of order $(g-1)^2$ were constructed [1]; for even values of g, the only previously known examples were the signed projective cubes, of order 2^{g-1} [2].

For any pair of integers (a, b), let $\mathrm{C}(2a, b)$ denote the Cartesian product $C_{2a} \Box P_b$, viewed as a cylinder. The graph $\mathrm{C}(2a, b)$ is of diameter $a+b-1$, and for any vertex v with degree 3 in $\mathrm{C}(2a, b)$, there is a unique vertex at distance $a+b-1$ of v which is therefore called *antipodal* of v. For any pair of integers (a, b), the *Augmented Cylindrical grid* of dimensions $2a$ and b, denoted by $\mathrm{AC}(2a, b)$, is obtained from $\mathrm{C}(2a, b)$ by adding an edge between each pairs of antipodal vertices, we denote by J the set of edges between antipodal pairs (see Figure 3 for an example). More specifically, let $\mathrm{AC}(2a, b)$ be the graph defined on vertex set $\{0, 1, \ldots, 2a-1\} \times \{0, 1, \ldots, b-1\}$ such that a pair $\{(i_1, j_1), (i_2, j_2)\}$ is an edge if

- $i_1 = i_2$ and $|j_1 - j_2| = 1$ (vertical edges), or
- $j_1 = j_2$ and $|i_1 - i_2| \in \{1, 2a-1\}$ (horizontal edges), or
- $|i_2 - i_1| + |j_2 - j_1| = a + b - 1$. ($J$)

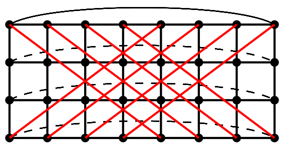


Figure 3: The augmented Cylindrical grid $AC(8, 4)$. Sloped (red) edges are the edges in J.

We note that the signed graph $(\mathrm{AC}(2a, b), J)$ could be $\mathcal{SP}_g$-complete for various choices of a and b (for example, $P_5 \square P_6$ is $\mathcal{SP}_8$-complete). Here, we will prove this for $a = \lfloor \frac{g}{2} \rfloor$, $b = \lceil \frac{g}{2} \rceil$, which results in a family of very symmetric signed graphs.

For convenience and readability, we let $\mathrm{T}(g) = \mathrm{C}(2\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil)$. Also we let $\mathrm{TT}(g) = \mathrm{AC}(2\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil)$, in short of *twisted tube of dimension g*, in the sense that $\mathrm{AC}(2\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil)$ looks like a twisted toroidal grid. Indeed, $\mathrm{AC}(2\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil)$ can be obtained from $C(2\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil + 1)$ by identifying each pair of vertices $(i, 0)$ and $(i + \lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil + 1)$, where $0 \leqslant i \leqslant \lfloor \frac{g}{2} \rfloor$, in the fashion of a *Dehn twist* studied in algebraic topology.

4.1 Properties of the signed graph $(\mathrm{TT}(g), J)$

Lemma 27. *For any integer $g \geqslant 2$, the following statements are true.*

(i) $(\mathrm{TT}(g), J)$ *is a subgraph of* $\mathrm{SPC}(g-1)$.

(ii) $(\mathrm{TT}(g), J)$ *is vertex-transitive.*

(iii) Any two vertices in $(\mathrm{TT}(g), J)$ *belong to a common negative cycle of length g.*

(iv) $(\mathrm{TT}(g), J)$ *is g-wide.*

Proof. (i) We first label the edges of $(\mathrm{TT}(g), J)$ with canonical vectors $\{e_1, e_2, \ldots, e_{g-1}\}$ of $\{0, 1\}^{g-1}$ and e_J (all coordinates are 1) as follows (indices are now to be understood modulo $2\lfloor \frac{g}{2} \rfloor$):

- $\{(i-1, j), (i, j)\}$ with label e_i if $i \leqslant \lfloor \frac{g}{2} \rfloor$ and with $e_{i - \lfloor \frac{g}{2} \rfloor}$ otherwise.

- $\{(0,j),(2\lfloor\frac{g}{2}\rfloor-1,j)\}$ with label $e_{\lfloor\frac{g}{2}\rfloor}$.
- $\{(i,j-1),(i,j)\}$ with label $e_{\lfloor\frac{g}{2}\rfloor+j}$.
- $\{(i,j),(i+\lfloor\frac{g}{2}\rfloor,j+\lfloor\frac{g}{2}\rfloor)\}$ with label e_J.

Note that the binary sum of the labels of the edges along any cycle of $\mathrm{TT}(g)$ is the all-zero vector. Conversely, if the sum of the labels along a walk is the all-zero vector, then this walk is closed. Then for any path from vertex $(0,0)$ to some vertex v of $\mathrm{TT}(g)$, the binary sum of the labels is the same. Thus we may define the mapping ϕ from the vertices of $(\mathrm{TT}(g),J)$ to the vertices of $\mathrm{SPC}(g-1)$ such that for any vertex v of $(\mathrm{TT}(g),J)$, $\phi(v)$ is the binary sum of the labels along any path from $(0,0)$ to v. Observe that the number of different coordinates between $\phi(u)$ and $\phi(v)$ is exactly the same as the number of different coordinates between u and v. On the other hand, if uv is a positive edge in $(\mathrm{TT}(G),J)$, then $uv\notin J$, and u and v differ in exactly one coordinate; if uv is a negative edge, then $uv\in J$, and u and v differ in all coordinates. Based on a definition of the signed projective cubes as Cayley signed graphs, see [5, 9], it is not difficult to show that the mapping ϕ is an injective homomorphism from $(\mathrm{TT}(g),J)$ to $\mathrm{SPC}(g-1)$, which implies that $(\mathrm{TT}(g),J)$ is a subgraph of $\mathrm{SPC}(g-1)$.

(ii). Let $v_1=(i_1,j_1)$ and $v_2=(i_2,j_2)$ be two vertices of $\mathrm{TT}(g)$. Let ϕ be the mapping

$$\phi(i,j)=\begin{cases}(i+i_2-i_1,j+j_2-j_1), & \text{if } 0\leqslant j\leqslant\left\lceil\frac{g}{2}\right\rceil-1-(j_2-j_1),\\ (i+\left\lfloor\frac{g}{2}\right\rfloor+i_2-i_1,j+j_2-j_1), & \text{if } j\geqslant\left\lceil\frac{g}{2}\right\rceil-(j_2-j_1).\end{cases}$$

where the additions and subtractions in the first coordinate are done mod$2\left\lfloor\frac{g}{2}\right\rfloor$ and in the second coordinate mod $\left\lceil\frac{g}{2}\right\rceil$.

First observe that ϕ is an automorphism of $\mathrm{TT}(g)$ mapping v_1 to v_2, as ϕ^{-1} exists, one can check it, we omit the details. On the other hand, in the signed graph $\phi(\mathrm{TT}(g),J)$, the negative edges are $\{\{(i,j_2-j_1),(i,j_2-j_1-1)\}|0\leqslant i\leqslant 2\left\lfloor\frac{g}{2}\right\rfloor-1\}$. So switching at $V=\{(i,j)|j\geqslant j_2-j_1\}$ in $\phi(TT(g),J)$ gives us $(\mathrm{TT}(g),J)$.

(iii). Since $\mathrm{TT}(g)$ is vertex-transitive, we may assume that one of these two vertices is the origin $(0,0)$. Let i be an integer between 0 and $2\left\lfloor\frac{g}{2}\right\rfloor-1$ and j be an integer between 0 and $\left\lceil\frac{g}{2}\right\rceil-1$, where $i+j\neq 0$. We need to prove that $(0,0)$ and (i,j) are in a common negative cycle of length g. By the symmetries of $\mathrm{TT}(g)$, we may assume that $i\leqslant\left\lfloor\frac{g}{2}\right\rfloor$. If we forget about the antipodal edges, there is a shortest path from $(0,0)$ to $(\left\lfloor\frac{g}{2}\right\rfloor,\left\lceil\frac{g}{2}\right\rceil-1)$ going through (i,j). Together with the antipodal edge $\{(0,0),(\left\lfloor\frac{g}{2}\right\rfloor,\left\lceil\frac{g}{2}\right\rceil-1)\}$, we get a negative cycle of length g in $(\mathrm{TT}(g),J)$.

(iv). By inductive definition of signed projective cubes $\mathrm{SPC}(k)=EDC(\mathrm{SPC}(k-1)$ (see [5]). Thus by Lemma 12, $\mathrm{SPC}(g-1)$ is g-wide. By (i) and the fact that taking subgraphs will not decrease the girths g_{ij}, the statement follows. □

In the sequel, for any two vertices $u_1=(i_1,j_1)$ and $u_2=(i_2,j_2)$ in $\mathrm{T}(g)$, we define

$$d^+_{\mathrm{T}(g)}(u_1,u_2) = |i_1-i_2| + |j_1-j_2|, \quad d^-_{\mathrm{T}(g)}(u_1,u_2) = 2\left\lfloor \frac{g}{2} \right\rfloor - |i_1-i_2| + |j_1-j_2|.$$

Intuitively speaking, $d^+_{\mathrm{T}(g)}(u_1,u_2)$ is the shortest length of a (u_1,u_2)-path that does not use any edges of type $\{(0,j),(2\lfloor \frac{g}{2} \rfloor - 1, j)\}$, while $d^-_{\mathrm{T}(g)}(u_1,u_2)$ does. As

$$d^+_{\mathrm{T}(g)}(u_1,u_2) + d^-_{\mathrm{T}(g)}(u_1,u_2) = 2\left\lfloor \frac{g}{2} \right\rfloor + 2|j_1-j_2| \leqslant 2\left\lfloor \frac{g}{2} \right\rfloor + 2(\left\lceil \frac{g}{2} \right\rceil - 1) = 2g-2,$$

we have the following observation.

Observation 28. *For any two vertices $u=(i_1,j_1)$ and $v=(i_2,j_2)$ in $\mathrm{T}(g)$,*

$$d_{\mathrm{T}(g)}(u,v) = \min\{d^+_{\mathrm{T}(g)}(u,v), d^-_{\mathrm{T}(g)}(u,v)\} \leqslant g-1.$$

Consequently, if $|i_1-i_2| \leqslant \lfloor \frac{g}{2} \rfloor$, then $d^+_{\mathrm{T}(g)}(u,v) \leqslant d^-_{\mathrm{T}(g)}(u,v)$, so $d_{\mathrm{T}(g)}(u,v) = d^+_{\mathrm{T}(g)}(u,v)$. Otherwise, $d^+_{\mathrm{T}(g)}(u,v) > d^-_{\mathrm{T}(g)}(u,v)$, so $d_{\mathrm{T}(g)}(u,v) = d^-_{\mathrm{T}(g)}(u,v)$.

The following observation is easy but useful, and also mentioned in [1, 2].

Observation 29. *Let (G,σ) be a g-wide signed graph and let C be a negative cycle of length g in (G,σ). Then, for any pair (u,v) of vertices of C, the distance in G between u and v is determined by their distance in C.*

By Observation 28, the following special case holds.

Observation 30. *Suppose $u=(i,j)$ is a vertex in $\mathrm{T}(g)$ with $0 \leqslant i < 2\lfloor \frac{g}{2} \rfloor$ and $0 \leqslant j < \lceil \frac{g}{2} \rceil$, and t is an integer satisfying $1 \leqslant t \leqslant g-1$. Then, $d_{\mathrm{T}(g)}((0,0),(i,j)) = t$ if and only if either*

- *$i \leqslant \lfloor \frac{g}{2} \rfloor$ and $i+j=t$, or*
- *$i > \lfloor \frac{g}{2} \rfloor$ and $i-j = 2\lfloor \frac{g}{2} \rfloor - t$.*

Lemma 31. *For any two vertices u and v in $(\mathrm{TT}(g),J)$, $f_g(ad_{(\mathrm{TT}(g),J)}(u,v)) = d_{\mathrm{T}(g)}(u,v)$.*

Proof. By Lemma 27(iv), $\mathrm{TT}(g)$ is g-wide, and by Lemma 27(iii), for any two vertices u,v in $\mathrm{TT}(g)$, there is a negative cycle of length g in $(\mathrm{TT}(g),J)$ containing u,v.

If $d_{\mathrm{T}(g)}(u,v) \leqslant \lfloor \frac{g}{2} \rfloor$, then by Observation 29, $d_{\mathrm{TT}(g)}(u,v) = d_{\mathrm{T}(g)}(u,v) \leqslant \lfloor \frac{g}{2} \rfloor$, which means that there exists a positive (u,v)-path of length $d_{\mathrm{T}(g)}(u,v)$ in $(\mathrm{TT}(g),J)$. Thus, by Definition 9, $ad_{(\mathrm{TT}(g),J)}(u,v) = d_{\mathrm{T}(g)}(u,v) > 0$, hence $f_g(ad_{(\mathrm{TT}(g),J)}(u,v)) = d_{\mathrm{T}(g)}(u,v)$;

If $d_{\mathrm{T}(g)}(u,v) > \lfloor \frac{g}{2} \rfloor$, then by Observation 29, $d_{\mathrm{TT}(g)}(u,v) = g - d_{\mathrm{T}(g)}(u,v) < d_{\mathrm{T}(g)}(u,v)$, which implies that there does not exist a positive (u,v)-path of length $d_{\mathrm{TT}(g)}(u,v)$ in $(\mathrm{TT}(g),J)$. Thus, by Definition 9, $ad_{(\mathrm{TT}(g),J)}(u,v) = d_{\mathrm{T}(g)}(u,v) - g < 0$, hence we conclude that $f_g(ad_{(\mathrm{TT}(g),J)}(u,v)) = g + ad_{(\mathrm{TT}(g),J)}(u,v) = d_{\mathrm{T}(g)}(u,v)$. ☐

4.2 $(\mathrm{TT}(g), J)$ is $\mathcal{SP}$-complete.

In this section, we provide a new family of smaller bounds that are $\mathcal{SP}$-complete.

Theorem 32. *For every integer $g \geqslant 2$, the signed graph $(\mathrm{TT}(g), J)$, of order $\lfloor g^2/2 \rfloor$, is $\mathcal{SP}$-complete.*

Proof. Let (B, ω) be a weighted signed graph where B is a complete graph on vertex set $V(\mathrm{TT}(g))$, and for each edge uv, $\omega(uv) = f_g(ad_{(\mathrm{TT}(g),J)}(u, v))$. By Lemma 27 (iii) and Definition 18, (B, ω) is a $(g-1)$-partial girth-transformed $(TT(g), J)$-distance graph. It is clear that the edge set of (B, ω) is non-empty. Let $\mathcal{T}$ be the collection of all triangles in (B, ω), we shall show that $\mathcal{T}$ is g-closed.

Let p, q, r be three integers satisfying $1 \leqslant p, q, r \leqslant g-1$, such that (p, q, r) is g-wide, $\mathcal{E}$ be the set of edges appeared in $\mathcal{T}$. Assume $e = xy \in \mathcal{E}$ with $\omega(e) = p$. We shall find a triangle $xyz \in \mathcal{T}$ such that either $\omega(zx) = q$ and $\omega(zy) = r$, or $\omega(zx) = g - q$ and $\omega(zy) = g - r$. By Lemma 27(ii), $\mathrm{TT}(g)$ is vertex-transitive, so we may assume that $x = (0, 0)$. By the horizontal symmetries of $\mathrm{TT}(g)$ (recall that there is an edge $\{(2\lfloor \frac{g}{2} \rfloor - 1, j), (0, j)\}$ for each j), we may assume that $y = (a, b)$, where $0 \leqslant a \leqslant \lfloor \frac{g}{2} \rfloor$, so by Observation 30, $b = p - a$. Therefore, by Lemma 31, it suffices to show that there exists a vertex $z = (c, d)$ in $\mathrm{TT}(g)$ where c and d are two integers satisfying $0 \leqslant c \leqslant 2\lfloor \frac{g}{2} \rfloor - 1$ and $0 \leqslant d \leqslant \lceil \frac{g}{2} \rceil - 1$, such that either

$$d_{\mathrm{T}(g)}(z, x) = q, \quad d_{\mathrm{T}(g)}(z, y) = r, \tag{C2-1}$$

or

$$d_{\mathrm{T}(g)}(z, x) = g - q, \quad d_{\mathrm{T}(g)}(z, y) = g - r. \tag{C2-2}$$

We may also assume that $q \leqslant \lfloor \frac{g}{2} \rfloor$, for otherwise, we can replace q with $g - q$ and replace r with $g - r$ such that (C2-1) or (C2-2) holds.

By Proposition 17(1), p, q, r satisfy the triangle-inequality. So $|a + b - q| = |p - q| \leqslant r \leqslant \min\{p + q, 2g - p - q\} = \min\{a + b + q, 2g - a - b - q\}$, also r and $p + q = a + b + q$ have the same parity. We also observe that all of $a + b + q$, $a + |b - q|$, $2\lfloor \frac{g}{2} \rfloor + b - a - q$ and $2g - a - b - q$ have the same parity, so depending on the value of r, we determine $z = (c, d)$ as follows.

(i) $|a + b - q| \leqslant r \leqslant a + |b - q|$.

- If $b \geqslant q$, then $r = a + b - q$, and we let $c = 0$ and $d = q$.
- If $b < q$, then $|a + b - q| \leqslant r \leqslant a + q - b$, let $c = \frac{a-b+q-r}{2}$ and $d = \frac{q+r-a+b}{2}$.

(ii) If $a + |b - q| + 2 \leqslant r \leqslant \min\{2\lfloor \frac{g}{2} \rfloor + b - a - q, a + b + q, 2g - a - b - q\}$, then let $c = 2\lfloor \frac{g}{2} \rfloor + \frac{a+b-q-r}{2}$ and $d = \frac{a+b+q-r}{2}$.

(iii) If $2\lfloor \frac{g}{2} \rfloor + b - a - q + 2 \leqslant r \leqslant \min\{a + b + q, 2g - a - b - q\}$, then let $c = \frac{a-b-q+r}{2}$ and $d = g - \frac{a-b+q+r}{2}$.

As (p,q,r) is g-wide, $p+q+r=a+b+q+r$ is even, and consequently, $a+b-q-r$, $a+b+q-r$, $a-b-q+r$ and $a-b+q+r$ are all even, so in each case, c and d are integers. We now proceed to prove the validity of our choices.

It is trivial to verify that (C2-1) holds for Case (i) with $b \geqslant q$.

For Case (i) with $b<q$, note that $c=a-\frac{a+b-q+r}{2}$, as $|a+b-q| \leqslant r \leqslant a+q-b$, we have

$$0=\frac{a-b+q-(a+q-b)}{2} \leqslant c \leqslant a-\frac{a+b-q+|a+b-q|}{2} \leqslant a \leqslant \left\lfloor \frac{g}{2} \right\rfloor . \tag{4.1}$$

Similarly, recall that $d=b+\frac{r-(a+b-q)}{2}$, so we have

$$b \leqslant b+\frac{|a+b-q|-(a+b-q)}{2} \leqslant d \leqslant \frac{(a+q-b)+(q+b-a)}{2}=q \leqslant \left\lfloor \frac{g}{2} \right\rfloor . \tag{4.2}$$

Observe that $c+d=q$, so by Observation 30, we have

$$d_{\mathrm{T}(g)}(z,x)=q. \tag{4.3}$$

By Inequality (4.1), we have $|a-c| \leqslant \left\lfloor \frac{g}{2} \right\rfloor$. Thus by Observation 28, Inequalities (4.1), (4.2),

$$d_{\mathrm{T}(g)}(z,y)=d^{+}_{\mathrm{T}(g)}(z,y)=(a-c)+(d-b)=r. \tag{4.4}$$

Therefore, by Inequalities (4.3), (4.4), (C2-1) holds and the case is done.

For Case (ii), note that $c=a+\left\lfloor \frac{g}{2} \right\rfloor+\frac{2\left\lfloor \frac{g}{2} \right\rfloor+b-a-q-r}{2}$ and $r \leqslant 2\left\lfloor \frac{g}{2} \right\rfloor+b-a-q$, we have

$$c \geqslant a+\left\lfloor \frac{g}{2} \right\rfloor+\frac{2\left\lfloor \frac{g}{2} \right\rfloor+b-a-q-(2\left\lfloor \frac{g}{2} \right\rfloor+b-a-q)}{2}=a+\left\lfloor \frac{g}{2} \right\rfloor \geqslant \left\lfloor \frac{g}{2} \right\rfloor .$$

On the other hand, as $c=2\left\lfloor \frac{g}{2} \right\rfloor+\frac{a+b-q-r}{2}$ and $r \geqslant a+|b-q|+2$, we have

$$c \leqslant 2\left\lfloor \frac{g}{2} \right\rfloor+\frac{a+b-q-(a+|b-q|+2)}{2}=2\left\lfloor \frac{g}{2} \right\rfloor-1+\frac{b-q-|b-q|}{2} \leqslant 2\left\lfloor \frac{g}{2} \right\rfloor-1.$$

Thus we have

$$\left\lfloor \frac{g}{2} \right\rfloor \leqslant \left\lfloor \frac{g}{2} \right\rfloor+a \leqslant c \leqslant 2\left\lfloor \frac{g}{2} \right\rfloor-1, \tag{4.5}$$

Similarly, note that $d=q+\frac{a+b-q-r}{2}$ and $r \geqslant a+|b-q|+2$, we have

$$d \leqslant q+\frac{a+b-q-(a+|b-q|+2)}{2}=q-1+\frac{b-q-|b-q|}{2} \leqslant q-1.$$

On the other hand, as $r \leqslant a+b+q$, $d \geqslant \frac{a+b+q-(a+b+q)}{2}=0$. Therefore,

$$0 \leqslant d \leqslant q-1+\frac{b-q-|b-q|}{2} \leqslant q-1. \tag{4.6}$$

Note that $c - d = 2\left\lfloor \frac{g}{2} \right\rfloor - q$, by Inequality (4.5) and Observation 30, we have

$$d_{\mathrm{T}(g)}(z, x) = q. \tag{4.7}$$

By Inequality (4.6), if $b \geqslant q$, then $d \leqslant q - 1 < b$, otherwise $d \leqslant q - 1 + (b - q) = b - 1$. So it always holds that $d < b$. Recall that Inequality (4.5) ensures that $c - a \geqslant \left\lfloor \frac{g}{2} \right\rfloor$, so by Observation 28,

$$d_{\mathrm{T}(g)}(z, y) = d^{-}_{\mathrm{T}(g)}(z, y) = 2\left\lfloor \frac{g}{2} \right\rfloor - (c - a) + (b - d) = r. \tag{4.8}$$

Thus, by Inequality (4.7) and Inequality (4.8), Inequality (C2-1) holds, and this case is done.

For Case (iii), recall that $c = \frac{a-b-q+r}{2}$ and $2\left\lfloor \frac{g}{2} \right\rfloor + b - a - q + 2 \leqslant r \leqslant a + b + q$, it follows that

$$0 < \left\lfloor \frac{g}{2} \right\rfloor - q + 1 = \frac{a - b - q + (2\left\lfloor \frac{g}{2} \right\rfloor + b - a - q + 2)}{2} \leqslant c \leqslant \frac{a - b - q + (a + b + q)}{2} = a \leqslant \left\lfloor \frac{g}{2} \right\rfloor. \tag{4.9}$$

Similarly, recall that $d = g - \frac{a-b+q+r}{2} = b + g - \frac{a+b+q+r}{2}$, and $r \leqslant 2g - a - b - q$, we have

$$d \geqslant b + g - \frac{a + b + q + (2g - a - b - q)}{2} = b \geqslant 0.$$

On the other hand, as $r \geqslant 2\left\lfloor \frac{g}{2} \right\rfloor + b - a - q + 2$, we have

$$d \leqslant g - \frac{a - b + q + (2\left\lfloor \frac{g}{2} \right\rfloor + b - a - q + 2)}{2} = \left\lceil \frac{g}{2} \right\rceil - 1.$$

Therefore, we have

$$0 \leqslant b \leqslant d \leqslant \left\lceil \frac{g}{2} \right\rceil - 1. \tag{4.10}$$

Note that $c + d = g - q$, so by Observation 30, we have

$$d_{\mathrm{T}(g)}(z, x) = g - q. \tag{4.11}$$

By Inequality (4.9), $0 \leqslant a - c \leqslant \left\lfloor \frac{g}{2} \right\rfloor$. Thus, by Observation 28, Inequality (4.9) and Inequality (4.10),

$$d_{\mathrm{T}(g)}(z, y) = d^{+}_{\mathrm{T}(g)}(z, y) = (a - c) + (d - b) = g - r. \tag{4.12}$$

Thus, by Inequality (4.11) and Inequality (4.12), we know that (C2-2) holds, and this case is done.

This completes the proof of this theorem. □

5 Concluding remarks

In this work, observing a strong connection between the notion of Extended Double Cover and some conjectures in extension of the 4CT, we proposed Conjecture 5.

This conjecture captures the following conjecture which, if verified, would be a direct strengthening of the 4CT and is in connection to some other conjectures. We refer to [9] and [5] for more details.

Conjecture 33. The signed projective cube of dimension k, denoted SPC(k) is $\mathcal{P}$-complete for each value of k.

One way of defining signed projective cubes is as follows: SPC(1) is the signed multigraph on two vertices which are connected by one positive and one negative edge. For $k \geqslant 2$, SPC(k) is the extended double cover of SPC($k-1$).

In support of Conjecture 5, we showed that the claim holds if we work with the subclass of signed K_4-minor-free graphs.

Let $\mathcal{SP}_k$ be the class of signed K_4-minor-free graphs (G,σ) satisfying $g_{ij}(G,\sigma) \geqslant g_{ij}(C_{-k})$. For even values of k, this is the class of signed bipartite K_4-minor-free graphs of negative girth at least k, and for odd values of k, the class of signed antibalanced K_4-minor-free graphs of odd-girth k.

For each given integer k, we have build a graph (B,π) of order $2\lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil$ for $\mathcal{SP}_k$ satisfying $g_{ij}(B,\pi) \geqslant g_{ij}(C_{-k})$ with the property that any K_4-minor-free signed graph (G,σ) satisfying $g_{ij}(G,\sigma) \geqslant g_{ij}(C_{-k})$ admits a homomorphism to (B,π).

For $k = 2,3,4,5,6,7$ the best possible such bounds are of order 2, 3, 6, 8, 12, 15 respectively [1, 2]. This suggest the following formula for the best order of such a bound: $\lfloor \frac{k}{2} \rfloor (\lceil \frac{k}{2} \rceil + 1)$. A quadratic lower bound for the general value k is given by W. He, R. Naserasr, and Q. Sun.

We note that when k is an odd number, the study of the homomorphism properties of $\mathcal{SP}_k$ is the same as the study of homomorphism properties of the class of series-parallel graphs of odd girth at least k. Precise bounds on the circular chromatic number in this class are given in [12], and on the fractional chromatic number, in [1, 7, 8]. The optimal bounds of order 3, 8, 15 for the cases $k = 3,5,7$ from [1] each have both circular and fractional chromatic numbers that are the same as the best bound for that of series-parallel graphs of odd girth at least k for $k = 3,5,7$, hence strengthening results on both the circular chromatic number and the fractional chromatic number. We expect that this will be the case for general odd values of k. This is an alternative motivation for finding the optimal choice for (B,π).

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