An A_α-Spectral Erdős-Sós Theorem

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Abstract

Let G be a graph and let α be a real number in $[0,1]$. In 2017, Nikiforov proposed the A_α-matrix for G as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and the degree diagonal matrix of G, respectively. The largest eigenvalue of $A_\alpha(G)$ is called the A_α-index of G. The famous Erdős-Sós conjecture states that every n-vertex graph with more than $\frac{1}{2}(k-1)n$ edges must contain every tree on $k+1$ vertices. In this paper, we consider an A_α-spectral version of this conjecture. For $n > k$, let $S_{n,k}$ be the join of a clique on k vertices with an independent set of $n-k$ vertices and denote by $S_{n,k}^+$ the graph obtained from $S_{n,k}$ by adding one edge. We show that for fixed $k \geqslant 2$, $0 < \alpha < 1$ and $n \geqslant \frac{88k^2(k+1)^2}{\alpha^4(1-\alpha)}$, if a graph on n vertices has A_α-index at least as large as $S_{n,k}$ (resp. $S_{n,k}^+$), then it contains all trees on $2k+2$ (resp. $2k+3$) vertices, or it is isomorphic to $S_{n,k}$ (resp. $S_{n,k}^+$). These extend the results of Cioabă, Desai and Tait (2022), in which they confirmed the adjacency spectral version of the Erdős-Sós conjecture.

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

In this paper, we consider only simple and finite graphs. Unless otherwise stated, we follow the traditional notation and terminology (see, for instance, Bollobás [2], Godsil and Royle [17]).

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Let F be a fixed graph. We say that G is *F-free* if it does not contain F as a subgraph. As usual, let P_n, C_n and K_n be the path, the cycle and the complete graph on n vertices, respectively. And let $K_{a,b}$ be the complete bipartite graph with the sizes of partite sets being a and b, respectively.

In 2017, Nikiforov [29] proposed the *A_α-matrix* of G, which is a convex combination of $D(G)$ and $A(G)$, i.e.,

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), \ \ \alpha \in [0,\, 1],$$

where $A(G)$ and $D(G)$ are, respectively, the adjacency matrix and the degree diagonal matrix of G (see below). It is obvious that $A_0(G) = A(G)$, $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$ and $A_1(G) = D(G)$, where $Q(G) = D(G) + A(G)$ is the *signless Laplacian matrix* of G. The largest eigenvalue of $A_\alpha(G)$ is called the *A_α-index* of G, denoted by $\lambda_\alpha(G)$ as usual. The A_0-index (resp. twice of $A_{\frac{1}{2}}$-index) of G is usually referred to as the *index* (resp. *Q-index*) of G, denoted by $\rho(G)$ (resp. $q(G)$).

In 2013, Füredi and Simonovits [15] posed the following problem:

Problem 1 (Füredi-Simonovits type problem)**.** Assume $\mathbb{U}$ is a family of graphs and G is in $\mathbb{U}$. For a specific pair of parameters (τ, υ) on G, our aim is to maximize the second parameter υ under the condition that G is F-free and its first parameter τ is given.

For a simple graph $G = (V(G), E(G))$, we use $n := |V(G)|$ and $m := |E(G)|$ to denote the *order* and the *size* of G, respectively. With no confusion, we also use the *size* to denote the cardinality of a set. If the pair of parameters above are the order and size of a graph, i.e., $(\tau, \upsilon) = (n, m)$, then the Füredi-Simonovits type problem is just the classical Turán type problem: determine the maximum number of edges, $\mathrm{ex}(n, F)$, of an n-vertex F-free graph. The value $\mathrm{ex}(n, F)$ is called the *Turán number* of F. The research for the Turán number attracts much attention, and it has become to be one of the most attractive fundamental problems in extremal graph theory (see [15, 28] for surveys).

For each non-bipartite graph F, the celebrated Erdős-Stone-Simonovits theorem [11, 12] determines an asymptotic formula for $\mathrm{ex}(n, F)$. It is natural and interesting for us to study the Turán number of a bipartite graph. For more advances in this topic, readers may refer to the survey [15]. In particular, let T be a tree on $k+1$ vertices, considering the disjoint copies of K_k shows that $\mathrm{ex}(n, T) \geqslant \lfloor \frac{n}{k} \rfloor \binom{k}{2}$.

The following Erdős-Sós conjecture predicts that $\mathrm{ex}(n, T) \leqslant \frac{1}{2}(k-1)n$ for all trees T of order $k+1$.

Conjecture 2 (Erdős-Sós [10])**.** Let G be an n-vertex graph of size m. If $m > \frac{1}{2}(k-1)n$, then G contains all trees of order $k+1$.

The Erdős-Sós conjecture is still open, however it has been confirmed in some special cases. For example, the conjecture is true when G is C_4-free [31], P_{k+5}-free [9]; when k is large compared to n [42, 32, 37, 35, 38, 19]; and for some special classes of trees [13, 14, 16, 23].

In Problem 1, if one lets $(\tau, \upsilon) = (n, \rho(G))$, i.e., the pair of parameters are the order and the index on $\mathbb{U}$, then it becomes to be the spectral Turán type problem (also known as Brualdi-Solheid-Turán type problem, see [3]): what is the maximal index of an F-free graph of order n? Over the past decade, much attention has been paid to the Brualdi-Solheid-Turán type problem. For more details, one may consult the references, such as for $F = K_r$ [24, 36], $F = K_{s,t}$ [1, 24, 26], $F = P_k$ [27], $F = C_{2k}$ [25, 41, 39, 8], whereas when F is a K_r-minor, or $K_{s,t}$-minor, we may consult [33, 40].

For two graphs G and H, we define $G \cup H$ to be their disjoint union. The *join* $G \vee H$ is the graph obtained from $G \cup H$ by joining every vertex of G with every vertex of H. Then define $S_{n,k} = K_k \vee (n-k)K_1$ and $S^+_{n,k} = K_k \vee (K_2 \cup (n-k-2)K_1)$.

In 2010, Nikiforov [27] posed a conjecture, which is an adjacency spectral version of Erdős-Sós conjecture.

Conjecture 3 (Nikiforov [27])**.** Let $k \geqslant 2$ and G be a graph of sufficiently large order n.

(a) If $\rho(G) \geqslant \rho(S_{n,k})$, then G contains all trees of order $2k+2$ unless $G = S_{n,k}$;

(b) If $\rho(G) \geqslant \rho(S^+_{n,k})$, then G contains all trees of order $2k+3$ unless $G = S^+_{n,k}$.

There are many results involving Conjecture 3, see [21, 22, 27]. Very recently, Conjecture 3 was completely resolved by Cioabă, Desai and Tait [7].

In this paper, we consider an A_α-spectral version of Erdős-Sós conjecture for $\alpha \in (0,1)$, which extends the main results of Cioabă, Desai and Tait [7]. Our main results can be stated as:

Theorem 4. *Let $0 < \alpha < 1$, $k \geqslant 2$, and G be a graph of order $n \geqslant \frac{88k^2(k+1)^2}{\alpha^4(1-\alpha)}$.*

(a) *If $\lambda_\alpha(G) \geqslant \lambda_\alpha(S_{n,k})$, then G contains all trees of order $2k+2$ unless $G = S_{n,k}$;*

(b) *If $\lambda_\alpha(G) \geqslant \lambda_\alpha(S^+_{n,k})$, then G contains all trees of order $2k+3$ unless $G = S^+_{n,k}$.*

Over the past decade, the signless Laplacian spectral extremal problems were studied extensively. For more details, one may refer to [4, 5, 6, 20, 30] and the references therein. Taking $\alpha = \frac{1}{2}$ in Theorem 4 resolves a signless Laplacian spectral version of Erdős-Sós conjecture.

Theorem 5. *Let $k \geqslant 2$ and G be a graph of order $n \geqslant 2816k^2(k+1)^2$.*

(a) *If $q(G) \geqslant q(S_{n,k})$, then G contains all trees of order $2k+2$ unless $G = S_{n,k}$;*

(b) *If $q(G) \geqslant q(S^+_{n,k})$, then G contains all trees of order $2k+3$ unless $G = S^+_{n,k}$.*

Our paper is organized as follows. In the remainder of this section, we introduce some necessary notation and terminology. In Section 2, we give some necessary preliminaries. In Section 3, we progressively refine the structure of our extremal graphs $G_{n,k,\alpha}$, $G'_{n,k,\alpha}$ (see below) and complete the proof of Theorem 4, finally. Some concluding remarks are given in the last section.

For two disjoint vertex subsets V_1 and V_2 of $V(G)$, denote by $G[V_1]$ a subgraph of G induced on V_1 and $G[V_1, V_2]$ a subgraph of G with one end vertex in V_1 and the other in

V_2. Then the number of edges of $G[V_1]$ and $G[V_1, V_2]$ can be abbreviated to $e(V_1)$ and $e(V_1, V_2)$, respectively. The *neighborhood* of a vertex v (in a graph G) is denoted by $N(v)$. The *degree* $d(v)$ of a vertex v (in a graph G) is the number of edges incident with it. Then the *maximum degree* of G is denoted by $\Delta(G)$.

We say that two vertices u and v in G are *adjacent* (or *neighbours*) if they are joined by an edge and we write it as $u \sim v$. Then the *adjacency matrix* of G is defined as an $n \times n$ $(0,1)$-matrix $A(G) = (a_{ij})$ with $a_{ij} = 1$ if and only if $v_i \sim v_j$. The *degree diagonal matrix* of G is defined as an $n \times n$ diagonal matrix $D(G) = \mathrm{diag}(d(v_1), \dots, d(v_n))$.

Noting that $A_\alpha(G)$ is real symmetric, its *eigenvalues* are real. If G is connected and $\alpha \neq 1$, then $A_\alpha(G)$ is a non-negative and irreducible matrix. From Perron-Frobenius theory, there exists a unique positive eigenvector of $A_\alpha(G)$ corresponding to $\lambda_\alpha(G)$ for $\alpha \in [0, 1)$, and we call this vector the *Perron vector* of $A_\alpha(G)$.

For $k \geqslant 2$, let $\mathcal{T}_k$ (resp. $\mathcal{T}'_k$) denote the set of all trees of order $2k+2$ (resp. $2k+3$), and let $\mathcal{G}_{n,k}$ (resp. $\mathcal{G}'_{n,k}$) denote the set of graphs of order n not containing at least one tree in $\mathcal{T}_k$ (resp. $\mathcal{T}'_k$). For $\alpha \in (0, 1)$, let $G_{n,k,\alpha}$ (resp. $G'_{n,k,\alpha}$) be a graph with maximum A_α-index among all graphs in $\mathcal{G}_{n,k}$ (resp. $\mathcal{G}'_{n,k}$).

2 Preliminaries

In this section we give some preliminary results, which will be used to prove our main results.

The following upper bound for the Turán number of trees is well-known.

Lemma 6 ([7]). *For all trees T of order t, one has*

$$\mathrm{ex}(n, T) \leqslant (t-2)n.$$

Lemma 7 ([7]). *For $k \geqslant 2$, the graphs $K_{k+1,2k+1}$ and $K^{+}_{k,2k+1} := kK_1 \vee ((2k-1)K_1 \cup K_2)$ contain all trees in $\mathcal{T}_k$; the graphs $K_{k+1,2k+2}$, $K'_{k,2k+2} := kK_1 \vee ((2k-1)K_1 \cup P_3)$ and $K''_{k,2k+2} := kK_1 \vee ((2k-2)K_1 \cup 2K_2)$ contain all trees in $\mathcal{T}'_k$.*

Lemma 8 ([29]). *For $0 \leqslant \alpha \leqslant 1$, if G is a graph of order n, then $\lambda_\alpha(G) \geqslant \alpha\Delta(G)$.*

Let M be an $n \times n$ real symmetric matrix and let $\pi : V = V_1 \cup V_2 \cup \cdots \cup V_s$ be a partition of $V = \{1, 2, \dots, n\}$. Then corresponding to the partition π, M can be partitioned into the following block matrix:

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1s} \\ M_{21} & M_{22} & \cdots & M_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ M_{s1} & M_{s2} & \cdots & M_{ss} \end{pmatrix}.$$

The *quotient matrix* of M with respect to π is the matrix $M_\pi = (b_{ij})_{s \times s}$, where b_{ij} is the average row sum of the block M_{ij}. The partition π is said to be *equitable* if each block M_{ij} has constant row sums for $i, j \in \{1, 2, \dots, s\}$.

Lemma 9 ([18]). *Let M be a real symmetric matrix and let M_π be an equitable quotient matrix of M. Then the eigenvalues of M_π are also the eigenvalues of M. Furthermore, if M is nonnegative and irreducible, then $\lambda(M) = \lambda(M_\pi)$, where $\lambda(M)$ and $\lambda(M_\pi)$ are the largest eigenvalues of M and M_π, respectively.*

Recall that $G_{n,k,\alpha}$ and $G'_{n,k,\alpha}$ are the extremal graphs with the maximum A_α-index among all graphs in $\mathcal{G}_{n,k}$ and $\mathcal{G}'_{n,k}$, respectively, where $\alpha \in (0,1)$. The obvious fact $\mathcal{G}_{n,k} \subseteq \mathcal{G}'_{n,k}$ implies

$$\lambda_\alpha(G'_{n,k,\alpha}) \geqslant \lambda_\alpha(G_{n,k,\alpha}). \tag{1}$$

Lemma 10. *Let $0 < \alpha < 1$, $k \geqslant 2$, and $n \geqslant \frac{9k^2}{\alpha^4}$. Then*

$$\lambda_\alpha(G_{n,k,\alpha}) \geqslant \max\left\{\alpha n+\frac{k}{\alpha}-k-1-\frac{2k(k+1)}{\alpha^3 n-\alpha^2(k+1+\alpha)+\alpha k}, \alpha n+\frac{k}{\alpha}-k-1-\alpha,\ \alpha(n-1)\right\}.$$

Proof. As $S_{n,k}$ does not contain P_{2k+2} (in $\mathcal{T}_k$), one has $S_{n,k} \in \mathcal{G}_{n,k}$. Then the definition of $G_{n,k,\alpha}$ implies

$$\lambda_\alpha(G_{n,k,\alpha}) \geqslant \lambda_\alpha(S_{n,k}). \tag{2}$$

On the other hand, by Lemma 8, one has

$$\lambda_\alpha(S_{n,k}) \geqslant \alpha\Delta(S_{n,k}) = \alpha(n-1). \tag{3}$$

Furthermore, let $V_1 = \{v \in V(S_{n,k}) | d_{S_{n,k}}(v) = n-1\}$ and $V_2 = V(S_{n,k})\backslash V_1$. Considering the partition $V(S_{n,k}) = V_1 \cup V_2$, we have the following equitable quotient matrix of $A_\alpha(S_{n,k})$

$$B = \begin{pmatrix} k-1+\alpha(n-k) & (1-\alpha)(n-k) \\ (1-\alpha)k & \alpha k \end{pmatrix}.$$

According to Lemma 9, one has $\lambda_\alpha(S_{n,k}) = \lambda(B)$, the largest eigenvalue of B, i.e.,

$$\lambda_\alpha(S_{n,k}) = \frac{\alpha n+k-1+\sqrt{\alpha^2 n^2+(4k-6\alpha k-2\alpha)n+(4\alpha-3)k^2+(4\alpha-2)k+1}}{2}.$$

Together with $n \geqslant \frac{9k^2}{\alpha^4}$, one has

$$\lambda_\alpha(S_{n,k}) \geqslant \alpha n+\frac{k}{\alpha}-k-1-\frac{2k(k+1)}{\alpha^3 n-\alpha^2(k+1+\alpha)+\alpha k} \geqslant \alpha n+\frac{k}{\alpha}-k-1-\alpha. \tag{4}$$

Then (2)-(4) give us

$$\lambda_\alpha(G_{n,k,\alpha}) \geqslant \max\left\{\alpha n+\frac{k}{\alpha}-k-1-\frac{2k(k+1)}{\alpha^3 n-\alpha^2(k+1+\alpha)+\alpha k}, \alpha n+\frac{k}{\alpha}-k-1-\alpha,\ \alpha(n-1)\right\}.$$

This completes the proof. □

3 The proof of Theorem 4

In this section, in order to prove Theorem 4, we need the following preliminary.

Lemma 11. *Let G be a connected graph and let $\mathbf{y}$ be a Perron vector of $A_\alpha(G)$. Then, for each $v \in V(G)$, one has*

$$\lambda_\alpha^2(G)\mathbf{y}_v = \alpha d(v)\lambda_\alpha(G)\mathbf{y}_v + \alpha(1-\alpha)\sum_{u\sim v} d(u)\mathbf{y}_u + (1-\alpha)^2\sum_{w\sim v}\sum_{u\sim w}\mathbf{y}_u. \tag{5}$$

Proof. By the definition of $A_\alpha(G)$ one has

$$\begin{aligned} A_\alpha^2(G) =&[\alpha D(G) + (1-\alpha)A(G)]^2 \\ =&\alpha^2 D^2(G) + \alpha(1-\alpha)D(G)A(G) + \alpha(1-\alpha)A(G)D(G) + (1-\alpha)^2 A^2(G). \end{aligned}$$

Based on $A_\alpha(G)\mathbf{y} = \lambda_\alpha(G)\mathbf{y}$, for each $v \in V(G)$, one has

$$\lambda_\alpha(G)\mathbf{y}_v = (A_\alpha(G)\mathbf{y})_v = \alpha d(v)\mathbf{y}_v + (1-\alpha)\sum_{u\sim v}\mathbf{y}_u \tag{6}$$

and

$$\begin{aligned} \lambda_\alpha^2(G)\mathbf{y}_v =&(A_\alpha^2(G)\mathbf{y})_v \\ =&(\alpha^2 D^2(G)\mathbf{y})_v + (\alpha(1-\alpha)D(G)A(G)\mathbf{y})_v + (\alpha(1-\alpha)A(G)D(G)\mathbf{y})_v \\ &+ ((1-\alpha)^2 A^2(G)\mathbf{y})_v \\ =&\alpha^2 d^2(v)\mathbf{y}_v + \alpha(1-\alpha)d(v)\sum_{u\sim v}\mathbf{y}_u + \alpha(1-\alpha)\sum_{u\sim v} d(u)\mathbf{y}_u + (1-\alpha)^2\sum_{w\sim v}\sum_{u\sim w}\mathbf{y}_u \\ =&\alpha d(v)\left[\alpha d(v)\mathbf{y}_v + (1-\alpha)\sum_{u\sim v}\mathbf{y}_u\right] + \alpha(1-\alpha)\sum_{u\sim v} d(u)\mathbf{y}_u + (1-\alpha)^2\sum_{w\sim v}\sum_{u\sim w}\mathbf{y}_u \\ =&\alpha d(v)\lambda_\alpha(G)\mathbf{y}_v + \alpha(1-\alpha)\sum_{u\sim v} d(u)\mathbf{y}_u + (1-\alpha)^2\sum_{w\sim v}\sum_{u\sim w}\mathbf{y}_u. \qquad \text{(by (6))} \end{aligned}$$

This completes the proof. □

In what follows, all the lemmas in this section are applied to both $G_{n,k,\alpha}$ and $G'_{n,k,\alpha}$. For brevity, we write all the proofs only for $G'_{n,k,\alpha}$. The same results for $G_{n,k,\alpha}$ follow by replacing $G'_{n,k,\alpha}$ with $G_{n,k,\alpha}$ in the proofs.

Fix $0 < \alpha < 1$, $k \geqslant 2$ and $n \geqslant \frac{88k^2(k+1)^2}{\alpha^4(1-\alpha)}$. Denote by $\lambda_\alpha := \lambda_\alpha(G'_{n,k,\alpha})$, and let $\mathbf{x}$ be an eigenvector corresponding to λ_α whose maximum entry is equal to 1 (here $\mathbf{x}$ is the Perron vector of $A_\alpha(G'_{n,k,\alpha})$ if $G'_{n,k,\alpha}$ is connected). Further, let z be a vertex with $\mathbf{x}_z = 1$ (if there are at least two such vertices, then we choose one arbitrarily). Take $\sigma = \frac{1}{11k^2}$ and let $V := V(G'_{n,k,\alpha})$. Then let L be the set of vertices in V having "large" eigenvector entries, and let $S = V\backslash L$ be the set of vertices in V having "small" eigenvector entries, i.e.,

$$L = \{v \in V|\mathbf{x}_v \geqslant \sigma\}, \quad S = \{v \in V|\mathbf{x}_v < \sigma\}.$$

The following lemma shows that our extremal graph $G'_{n,k,\alpha}$ is connected, and so $\mathbf{x}$ is positive.

Lemma 12. *The graph $G'_{n,k,\alpha}$ is connected.*

Proof. Suppose to the contrary that $G'_{n,k,\alpha}$ has $t \geqslant 2$ components $G_1, \ldots, G_t$. Then let G_1 be a component of G such that $\lambda_\alpha = \lambda_\alpha(G_1)$. Let $\mathbf{y}$ be the Perron vector of $A_\alpha(G_1)$ whose maximum entry is equal to 1, and set $\mathbf{x} = (\mathbf{y}^T, \mathbf{0}^T)^T$, where $\mathbf{0}$ is a zero-vector of dimension $n - |V(G_1)|$. It is easy to see that $\mathbf{x}$ is an eigenvector corresponding to λ_α. So $z \in V(G_1)$. According to $A_\alpha(G'_{n,k,\alpha})\mathbf{x} = \lambda_\alpha \mathbf{x}$, one has

$$\lambda_\alpha = \lambda_\alpha \mathbf{x}_z = \alpha d(z)\mathbf{x}_z + (1-\alpha)\sum_{u\sim z}\mathbf{x}_u = \alpha d(z) + (1-\alpha)\sum_{u\sim z}\mathbf{x}_u \leqslant d(z).$$

Then by (1) and Lemma 10,

$$d(z) \geqslant \lambda_\alpha \geqslant \alpha(n-1).$$

Take a vertex $u \in V(G_2)$. Define a graph $\hat{G}'_{n,k,\alpha}$ which is obtained from $G'_{n,k,\alpha}$ by deleting all edges adjacent to u and adding the edge uz. Then G_1 is a proper subgraph of some component of $\hat{G}'_{n,k,\alpha}$, and so by the Perron-Frobenius theory,

$$\lambda_\alpha = \lambda_\alpha(G_1) < \lambda_\alpha(\hat{G}'_{n,k,\alpha}). \tag{7}$$

On the other hand, we have the following claim.

Claim 13. *$\hat{G}'_{n,k,\alpha}$ does not contain any trees in $\mathcal{T}'_k$ which are not contained in $G'_{n,k,\alpha}$.*

Proof of Claim 13. Suppose to the contrary that there exists a tree $T \in \mathcal{T}'_k$ such that T is contained in $\hat{G}'_{n,k,\alpha}$ but is not contained in $G'_{n,k,\alpha}$. Then uz is a pendant edge of T. As T is a tree on $2k+3$ vertices and $d_{\hat{G}'_{n,k,\alpha}}(z) \geqslant d_{G'_{n,k,\alpha}}(z) \geqslant \alpha(n-1)$, one see that $\alpha(n-1) - (2k+1) \geqslant 1$ for $n \geqslant \frac{88k^2(k+1)^2}{\alpha^4(1-\alpha)}$. Hence, there exists at least one vertex, say w, not in $V(T)$ such that $w \sim z$. Thus one finds that $T - zu + zw$ is an isomorphic copy of T in $G'_{n,k,\alpha}$, a contradiction. □

Now, by Claim 13 we see $\hat{G}'_{n,k,\alpha} \in \mathcal{G}'_{n,k}$, and so (7) gives a contradiction to the choice of $G'_{n,k,\alpha}$. Therefore, $G'_{n,k,\alpha}$ is connected. □

Lemma 14. *For all $v \in L$, one has $d(v) \geqslant \left(1 - \frac{1}{2(k+1)}\right)n$.*

Proof. Suppose to the contrary that there is a vertex $v \in L$ such that $d(v) < (1 - \frac{1}{2(k+1)})n$. Applying (5) to v gives

$$\begin{aligned}\lambda_\alpha^2 \mathbf{x}_v =& \alpha d(v)\lambda_\alpha \mathbf{x}_v + \alpha(1-\alpha)\sum_{u\sim v} d(u)\mathbf{x}_u + (1-\alpha)^2 \sum_{w\sim v}\sum_{u\sim w}\mathbf{x}_u \\ \leqslant& \alpha d(v)\lambda_\alpha \mathbf{x}_v + 2\alpha(1-\alpha)m(G'_{n,k,\alpha}) + 2(1-\alpha)^2 m(G'_{n,k,\alpha}).\end{aligned} \tag{8}$$

Note that there exists a tree T in $\mathcal{T}'_k$ such that $G'_{n,k,\alpha}$ is T-free. By Lemma 6, one has

$$m(G'_{n,k,\alpha}) \leqslant \mathrm{ex}(n, T) \leqslant (2k+1)n. \tag{9}$$

Then (8) implies

$$\lambda_\alpha(\lambda_\alpha - \alpha d(v))\mathbf{x}_v \leqslant 2(1-\alpha)(2k+1)n. \tag{10}$$

In view of (1), Lemma 10 and $d(v) < (1-\frac{1}{2(k+1)})n$, we obtain

$$\lambda_\alpha(\lambda_\alpha - \alpha d(v))\mathbf{x}_v > \alpha(n-1)\left[\alpha(n-1) - \alpha\left(1-\frac{1}{2(k+1)}\right)n\right]\mathbf{x}_v.$$

Together with (10) and $\mathbf{x}_v \geqslant \sigma$ for all $v \in L$, we obtain

$$\frac{\alpha^2\sigma n^2}{2(k+1)} - \left[2(1-\alpha)(2k+1) + \alpha^2\sigma + \frac{\alpha^2\sigma}{2(k+1)}\right]n + \alpha^2\sigma < 0,$$

a contradiction for $n \geqslant \frac{88k^2(k+1)^2}{\alpha^4(1-\alpha)} \geqslant \frac{8(k+1)^2}{\sigma\alpha^2}$. □

For convenience, let

$$f := \alpha n + \frac{k}{\alpha} - k - 1 - \alpha \text{ with } 0<\alpha<1,\ k \geqslant 2 \text{ and } n > 0. \tag{11}$$

Lemma 15. $|L| = k$.

Proof. If $|L| \geqslant k+1$, then there is a subset L' of L such that $|L'| = k+1$. Let N be the set of common neighbors of the vertices in L'. From Lemma 14, we see that $|N| \geqslant (k+1)(1-\frac{1}{2(k+1)})n - kn = \frac{n}{2} \geqslant 2k+2$ for $n \geqslant \frac{88k^2(k+1)^2}{\alpha^4(1-\alpha)} \geqslant 4k+4$. Now the graph $G'_{n,k,\alpha}[L', N]$ contains $K_{k+1,2k+2}$, that is to say, $K_{k+1,2k+2} \subseteq G'_{n,k,\alpha}$. By Lemma 7, $G'_{n,k,\alpha}$ contains all trees in $\mathcal{T}'_k$, a contradiction.

In what follows, we show that $|L| \leqslant k-1$ will never happen. Recall that z is the vertex with $\mathbf{x}_z = 1$. If $|L| \leqslant k-1$, then applying (5) to the vertex z gives

$$\begin{aligned}
\lambda_\alpha^2 = \lambda_\alpha^2\mathbf{x}_z =& \alpha d(z)\lambda_\alpha\mathbf{x}_z + \alpha(1-\alpha)\sum_{u\sim z} d(u)\mathbf{x}_u + (1-\alpha)^2\sum_{w\sim z}\sum_{u\sim w}\mathbf{x}_u\\
=& \alpha\lambda_\alpha d(z) + \alpha(1-\alpha)\left(\sum_{\substack{u\sim z\\ u\in L}} d(u)\mathbf{x}_u + \sum_{\substack{u\sim z\\ u\in S}} d(u)\mathbf{x}_u\right)\\
&+ (1-\alpha)^2\left(\sum_{\substack{w\sim z\\ w\in S}}\sum_{\substack{u\sim w\\ u\in S}}\mathbf{x}_u + \sum_{\substack{w\sim z\\ w\in S}}\sum_{\substack{u\sim w\\ u\in L}}\mathbf{x}_u + \sum_{\substack{w\sim z\\ w\in L}}\sum_{\substack{u\sim w\\ u\in S}}\mathbf{x}_u + \sum_{\substack{w\sim z\\ w\in L}}\sum_{\substack{u\sim w\\ u\in L}}\mathbf{x}_u\right)\\
\leqslant& \alpha\lambda_\alpha d(z) + \alpha(1-\alpha)[(k-2)n + 2m(G'_{n,k,\alpha})\sigma]\\
&+ (1-\alpha)^2[2m(G'_{n,k,\alpha})\sigma + (k-1)n + (k-2)n\sigma + 2e(L)].
\end{aligned}$$

Then, by (9) one has

$$\lambda_\alpha(\lambda_\alpha - \alpha d(z)) \leqslant [(1-\alpha)k - 1 + \alpha^2]n + (1-\alpha)5kn\sigma + (1-\alpha)^2(k-1)(k-2). \tag{12}$$

In view of Lemma 10, (1), (11) and $d(z) \leqslant n-1$, we obtain

$$\lambda_\alpha(\lambda_\alpha - \alpha d(z)) \geqslant f\left[f + \alpha - \frac{2k(k+1)}{\alpha^2 f} - \alpha(n-1)\right].$$

Together with (12), we obtain

$$\begin{aligned}&[\alpha^2-(k+1)\alpha+k]n + \left(\frac{k}{\alpha}-k-1-\alpha\right)\left(\frac{k}{\alpha}-k-1+\alpha\right) - \frac{2k(k+1)}{\alpha^2}\\ \leqslant&[\alpha^2-(k+1)\alpha+k-1+\alpha]n + (1-\alpha)5kn\sigma + (1-\alpha)^2(k-1)(k-2),\end{aligned}$$

which implies

$$(1-\alpha)(1-5k\sigma)n \leqslant \frac{2k(k+1)}{\alpha^2} - \left(\frac{k}{\alpha}-k-1-\alpha\right)\left(\frac{k}{\alpha}-k-1+\alpha\right) + (1-\alpha)^2(k-1)(k-2) < \frac{4k^2}{\alpha^2},$$

a contradiction to $n \geqslant \frac{88k^2(k+1)^2}{\alpha^4(1-\alpha)} \geqslant \frac{4k^2}{\alpha^2(1-\alpha)(1-5k\sigma)}$.

This completes the proof. □

By Lemmas 14 and 15, we have

$$|L| = k \quad \text{and} \quad d(v) \geqslant \left(1 - \frac{1}{2(k+1)}\right)n \geqslant \left(1-\frac{1}{2k}\right)n \text{ for all } v \in L.$$

Let N be the set of common neighbors of the vertices in L and $V' = V\backslash(N \cup L)$. Then $|N| \geqslant k(1-\frac{1}{2k})n - (k-1)n = \frac{n}{2}$, and so $|V'| \leqslant \frac{n}{2}$. In order to show that $V' = \emptyset$, (and so $G'_{n,k,\alpha}$ contains $K_{k,n-k}$ as a spanning subgraph), we need the following lemma.

Lemma 16. *It holds that $\mathbf{x}_v \geqslant 1 - \frac{1}{k}$ for all $v \in L$.*

Proof. Suppose to the contrary that there is a vertex $v \in L$ such that $\mathbf{x}_v < 1 - \frac{1}{k}$. Recall that z is the vertex with $\mathbf{x}_z = 1 > 1 - \frac{1}{k}$. Therefore, $v \neq z$. Applying (5) to the vertex z gives

$$\begin{aligned}\lambda_\alpha^2 =&\lambda_\alpha^2\mathbf{x}_z = \alpha d(z)\lambda_\alpha \mathbf{x}_z + \alpha(1-\alpha)\sum_{u\sim z} d(u)\mathbf{x}_u + (1-\alpha)^2\sum_{w\sim z}\sum_{u\sim w}\mathbf{x}_u\\ \leqslant&\alpha\lambda_\alpha d(z) + \alpha(1-\alpha)\left(\sum_{\substack{u\sim z\\ u\in L\backslash\{v\}}} d(u)\mathbf{x}_u + d(v)\mathbf{x}_v + \sum_{\substack{u\sim z\\ u\in S}} d(u)\mathbf{x}_u\right)\\ &+(1-\alpha)^2\left(\sum_{\substack{w\sim z\\ w\in S}}\sum_{\substack{u\sim w\\ u\in S}}\mathbf{x}_u + \sum_{\substack{w\sim z\\ w\in S}}\sum_{\substack{u\sim w\\ u=v}}\mathbf{x}_u + \sum_{\substack{w\sim z\\ w\in S}}\sum_{\substack{u\sim w\\ u\in L\backslash\{v\}}}\mathbf{x}_u + \sum_{\substack{w\sim z\\ w\in L}}\sum_{\substack{u\sim w\\ u\in S}}\mathbf{x}_u + \sum_{\substack{w\sim z\\ w\in L}}\sum_{\substack{u\sim w\\ u\in L}}\mathbf{x}_u\right)\\ \leqslant&\alpha\lambda_\alpha d(z) + \alpha(1-\alpha)\left[(k-2)n + d(v)\mathbf{x}_v + 2m(G'_{n,k,\alpha})\sigma\right] + (1-\alpha)^2\left[2m(G'_{n,k,\alpha})\sigma\right.\\ &\left.+d(v)\mathbf{x}_v + (k-1)n + (k-1)n\sigma + 2e(L)\right].\end{aligned}$$

In view of (9), $d(v) < n$ and $\mathbf{x}_v < 1 - \frac{1}{k}$, we obtain

$$\begin{aligned}\lambda_\alpha(\lambda_\alpha - \alpha d(z)) <& [(1-\alpha)k + \alpha^2 - 1]n + (1-\alpha)(5k+1)n\sigma \\ &+ (1-\alpha)^2 k(k-1) + (1-\alpha)n(1-\frac{1}{k}).\end{aligned} \tag{13}$$

In view of Lemma 10, (1), (11) and $d(z) \leqslant n-1$, we obtain

$$\lambda_\alpha(\lambda_\alpha - \alpha d(z)) \geqslant f\left[f + \alpha - \frac{2k(k+1)}{\alpha^2 f} - \alpha(n-1)\right].$$

Together with (13), we have

$$\begin{aligned}&[\alpha^2 - (k+1)\alpha + k]n + \left(\frac{k}{\alpha} - k - 1 - \alpha\right)\left(\frac{k}{\alpha} - k - 1 + \alpha\right) - \frac{2k(k+1)}{\alpha^2} \\ <& [(1-\alpha)k + \alpha^2 - 1]n + (1-\alpha)(5k+1)n\sigma + (1-\alpha)^2 k(k-1) + (1-\alpha)n\left(1 - \frac{1}{k}\right),\end{aligned}$$

i.e.,

$$\begin{aligned}\left(\frac{1}{k} - (5k+1)\sigma\right)(1-\alpha)n <& \frac{2k(k+1)}{\alpha^2} + (1-\alpha)^2 k(k-1) \\ &- \left(\frac{k}{\alpha} - k - 1 - \alpha\right)\left(\frac{k}{\alpha} - k - 1 + \alpha\right) < \frac{4k^2}{\alpha^2},\end{aligned}$$

a contradiction to $n \geqslant \frac{88k^2(k+1)^2}{\alpha^4(1-\alpha)} \geqslant \frac{4k^2}{\alpha^2(1-\alpha)[\frac{1}{k} - (5k+1)\sigma]}$. Therefore, for all $v \in L$, we have $\mathbf{x}_v \geqslant 1 - \frac{1}{k}$. □

Lemma 17. *The set V' is empty, and so $G'_{n,k,\alpha}$ contains $K_{k,n-k}$ as a spanning subgraph.*

Proof. Suppose to the contrary that $V' \neq \emptyset$. Then the following (a)-(d) hold.

(a) For each $v \in V'$, $e(\{v\}, L) \leqslant k-1$. Otherwise, v is a common neighbor of the vertices in L, and so $v \in N$, a contradiction.

(b) For each $v \in V'$, $e(\{v\}, N) \leqslant 2k+1$. Otherwise, $G'_{n,k,\alpha}[L \cup \{v\}, N]$ contains $K_{k+1,2k+2}$, and so by Lemma 7, $G'_{n,k,\alpha}$ contains all trees in $\mathcal{T}'_k$, a contradiction.

(c) The number of edges of $G'_{n,k,\alpha}[V']$ satisfies $e(V') \leqslant (2k+1)|V'|$. Otherwise, by Lemma 6, $G'_{n,k,\alpha}[V']$ contains all trees in $\mathcal{T}'_k$, and so $G'_{n,k,\alpha}$ contains all trees in $\mathcal{T}'_k$, a contradiction.

(d) There is a vertex $v \in V'$ satisfying $d_{G'_{n,k,\alpha}[V']}(v) \leqslant 5k$. Otherwise, $2e(V') > 5k|V'|$, and so $e(V') > \frac{5}{2}k|V'| \geqslant (2k+1)|V'|$, a contradiction to (c).

According to (d), we can choose a vertex $v \in V'$ such that $d_{G'_{n,k,\alpha}[V']}(v) \leqslant 5k$. Then construct a new graph $\hat{G}'_{n,k,\alpha}$, which is obtained from $G'_{n,k,\alpha}$ by deleting all the edges incident to v and then adding the edges uv for all $u \in L$. We see that $\hat{G}'_{n,k,\alpha}$ does not contain any trees in $\mathcal{T}'_k$ which are not contained in $G'_{n,k,\alpha}$.

Suppose to the contrary that there is a tree $T \in \mathcal{T}'_k$ such that T is contained in $\hat{G}'_{n,k,\alpha}$ but is not contained in $G'_{n,k,\alpha}$. Then T contains v. As N is the set of common neighbors of the vertices in L, and $|N| \geqslant \frac{n}{2} \geqslant 2k+3 = |V(T)|$ for $n \geqslant \frac{88k^2(k+1)^2}{\alpha^4(1-\alpha)}$. By replacing v with a vertex w in $N \backslash V(T)$, one finds an isomorphic copy of T in $G'_{n,k,\alpha}$, a contradiction. This implies $\hat{G}'_{n,k,\alpha} \in \mathcal{G}'_{n,k}$.

Furthermore, by the Courant-Fischer theorem (see [18, Section 2.6]), one has

$$\begin{aligned}\lambda_\alpha(\hat{G}'_{n,k,\alpha}) - \lambda_\alpha \geqslant & \frac{\mathbf{x}^T(A_\alpha(\hat{G}'_{n,k,\alpha}) - A_\alpha(G'_{n,k,\alpha}))\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \\ = & \frac{\sum_{u\in L}[\alpha \mathbf{x}_u^2 + 2(1-\alpha)\mathbf{x}_u\mathbf{x}_v + \alpha\mathbf{x}_v^2] - \sum_{u\sim v}[\alpha \mathbf{x}_u^2 + 2(1-\alpha)\mathbf{x}_u\mathbf{x}_v + \alpha\mathbf{x}_v^2]}{\mathbf{x}^T\mathbf{x}}.\end{aligned}$$

Then

$$\begin{aligned}[\lambda_\alpha(\hat{G}'_{n,k,\alpha}) - \lambda_\alpha]\mathbf{x}^T\mathbf{x} \geqslant & \sum_{\substack{u\nsim v\\ u\in L}}[\alpha \mathbf{x}_u^2 + 2(1-\alpha)\mathbf{x}_u\mathbf{x}_v + \alpha\mathbf{x}_v^2] \\ & - \sum_{\substack{u\sim v\\ u\in N}}[\alpha \mathbf{x}_u^2 + 2(1-\alpha)\mathbf{x}_u\mathbf{x}_v + \alpha\mathbf{x}_v^2] - \sum_{\substack{u\sim v\\ u\in V'}}[\alpha \mathbf{x}_u^2 + 2(1-\alpha)\mathbf{x}_u\mathbf{x}_v + \alpha\mathbf{x}_v^2]. \qquad (14)\end{aligned}$$

In view of Lemma 16 and (a), we have

$$\sum_{\substack{u\nsim v\\ u\in L}}[\alpha \mathbf{x}_u^2 + 2(1-\alpha)\mathbf{x}_u\mathbf{x}_v + \alpha\mathbf{x}_v^2] \geqslant \alpha(1-\frac{1}{k})^2 + 2(1-\alpha)(1-\frac{1}{k})\mathbf{x}_v + \alpha\mathbf{x}_v^2. \qquad (15)$$

In view of (b), $d_{G'_{n,k,\alpha}[V']}(v) \leqslant 5k$ and $\mathbf{x}_u < \sigma$ for each $u \in V' \cup N$, we obtain

$$\sum_{\substack{u\sim v\\ u\in N}}[\alpha \mathbf{x}_u^2 + 2(1-\alpha)\mathbf{x}_u\mathbf{x}_v + \alpha\mathbf{x}_v^2] + \sum_{\substack{u\sim v\\ u\in V'}}[\alpha \mathbf{x}_u^2 + 2(1-\alpha)\mathbf{x}_u\mathbf{x}_v + \alpha\mathbf{x}_v^2] < 8k[\alpha\sigma^2 + 2(1-\alpha)\sigma\mathbf{x}_v + \alpha\sigma^2].$$

Together with (14) and (15), we have

$$[\lambda_\alpha(\hat{G}'_{n,k,\alpha}) - \lambda_\alpha]\mathbf{x}^T\mathbf{x} > \alpha[(1-\frac{1}{k})^2 - 16k\sigma^2] + 2(1-\alpha)(1-\frac{1}{k} - 8k\sigma)\mathbf{x}_v.$$

In view of $k \geqslant 2$, $\sigma = \frac{1}{11k^2}$ and $\mathbf{x}_v > 0$, we obtain $\lambda_\alpha(\hat{G}'_{n,k,\alpha}) > \lambda_\alpha$. This gives a contradiction to the choice of $G'_{n,k,\alpha}$.

Therefore, $V' = \emptyset$, and so $G'_{n,k,\alpha}$ contains $K_{k,n-k}$ as a spanning subgraph. □

Proof of Theorem 4. It follows from Lemma 17 that both $G_{n,k,\alpha}$ and $G'_{n,k,\alpha}$ contain $K_{k,n-k}$ as a spanning subgraph, where the part on k vertices is the set L and the part on $n-k$ vertices is the set N. By Lemma 7, $e(N) = 0$ in $G_{n,k,\alpha}$ and $e(N) \leqslant 1$ in $G'_{n,k,\alpha}$. Therefore,

$G_{n,k,\alpha} \subseteq S_{n,k}$ and $G'_{n,k,\alpha} \subseteq S^+_{n,k}$. By the Perron-Frobenius theory, one has $\lambda_\alpha(G_{n,k,\alpha}) \leqslant \lambda_\alpha(S_{n,k})$, $\lambda_\alpha(G'_{n,k,\alpha}) \leqslant \lambda_\alpha(S^+_{n,k})$. And $\lambda_\alpha(G_{n,k,\alpha}) = \lambda_\alpha(S_{n,k})$ if and only if $G_{n,k,\alpha} \cong S_{n,k}$, $\lambda_\alpha(G'_{n,k,\alpha}) = \lambda_\alpha(S^+_{n,k})$ if and only if $G'_{n,k,\alpha} \cong S^+_{n,k}$. On the other hand, $S_{n,k} \in \mathcal{G}_{n,k}$ and $S^+_{n,k} \in \mathcal{G}'_{n,k}$ imply $G_{n,k,\alpha} \cong S_{n,k}$ and $G'_{n,k,\alpha} \cong S^+_{n,k}$. This completes the proof of Theorem 4. □

4 Concluding remarks

In this paper, we confirm the A_α spectral version of the Erdős-Sós conjecture. Consequently, the signless Laplacian spectral version of the Erdős-Sós conjecture is also confirmed (see Theorem 5). In 2017, Nikiforov [29] posed the following two problems:

Problem 18 ([29])**.** Given a graph F, what is the maximum $\lambda_\alpha(G)$ of a graph G of order n, with no subgraph isomorphic to F?

Problem 19 ([29])**.** Solve Problem 18 if F is a path or a cycle of given order.

Nikiforov [29] solved Problem 18 when F is a complete graph; Tian, Chen and Cui [34] solved Problem 18 and so Problem 19 when F is C_4 for $\frac{1}{2} \leqslant \alpha < 1$, $n \geqslant 10$ and when F is C_5 for $0 \leqslant \alpha < \frac{1}{2}$, $n > \frac{11}{1-2\alpha} + 4$.

As $S_{n,k}$ (resp. $S^+_{n,k}$) has no subgraph isomorphic to P_{2k+2} (resp. P_{2k+3}), the following corollary is a direct consequence of our main result (i.e., Theorem 4) in this paper.

Corollary 20. *Let $0 < \alpha < 1$, $k \geqslant 2$ and G be a graph of order $n \geqslant \frac{88k^2(k+1)^2}{\alpha^4(1-\alpha)}$.*

(a) *If $\lambda_\alpha(G) \geqslant \lambda_\alpha(S_{n,k})$, then G contains P_{2k+2} unless $G = S_{n,k}$;*

(b) *If $\lambda_\alpha(G) \geqslant \lambda_\alpha(S^+_{n,k})$, then G contains P_{2k+3} unless $G = S^+_{n,k}$.*

Then the value $\lambda_\alpha(S_{n,k})$ (resp. $\lambda_\alpha(S^+_{n,k})$) is the maximum $\lambda_\alpha(G)$ of a graph G of order n, with no subgraph isomorphic to P_{2k+2} (resp. P_{2k+3}). Therefore, our results solved Problem 18 and so Problem 19 when F is P_ℓ for $\ell \geqslant 6$, $0 < \alpha < 1$, and $n \geqslant \frac{88(\lfloor \frac{\ell}{2} \rfloor -1)^2 \lfloor \frac{\ell}{2} \rfloor^2}{\alpha^4(1-\alpha)}$.

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References

[1] L. Babai, B. Guiduli, Spectral extrema for graphs: the Zarankiewicz problem. *Electron. J. Combin.*, 16(1): #R123, 8 pp, 2009.

[2] B. Bollobás, Modern Graph Theory. *Springer Science and Business Media*, 1998.

[3] R. A. Brualdi, E. S. Solheid, On the spectral radius of complementary acyclic matrices of zeros and ones. *SIAM J. Algebraic Discrete Methods*, 7(2): 265–272, 1986.

[4] M.-Z. Chen, A.-M. Liu, X.-D. Zhang, The signless Laplacian spectral radius of graphs with forbidding linear forests. *Linear Algebra Appl.*, 591: 25–43, 2020.

[5] M.-Z. Chen, X.-D. Zhang, On the signless Laplacian spectral radius of $K_{s,t}$-minor free graphs. *Linear Multilinear Algebra*, 69(10): 1922–1934, 2021.

[6] W.-W. Chen, B. Wang, M.-Q. Zhai, Signless Laplacian spectral radius of graphs without short cycles or long cycles. *Linear Algebra Appl.*, 645: 123–136, 2022.

[7] S. Cioabă, D. N. Desai, M. Tait, A spectral Erdős-Sós theorem, arXiv:2206.03339, 2022.

[8] S. Cioabă, D. N. Desai, M. Tait, The spectral even cycle problem, arXiv:2205.00990, 2022.

[9] N. Eaton, G. Tiner, On the Erdős-Sós conjecture for graphs having no path with $k + 4$ vertices. *Discrete Math.*, 313(16): 1621–1629, 2013.

[10] P. Erdős, Extremal problems in graph theory. *Proc. Sympos. Smolenice*, pp. 29–36, 1963, *Publ. House Czechoslovak Acad. Sci.*, Prague, 1964.

[11] P. Erdős, M. Simonovits, A limit theorem in graph theory. *Studia Sci. Math. Hungar*, 1: 51–57, 1966.

[12] P. Erdős, A. H. Stone, On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52: 1087–1091, 1946.

[13] G. H. Fan, Z. H. Huo, The Erdős-Sós conjecture for spiders of four legs. *J. Comb.*, 7: 271–283, 2016.

[14] G. H. Fan, L. L. Sun, The Erdős-Sós conjecture for spiders. *Discrete Math.*, 307(23): 3055–3062, 2007.

[15] Z. Füredi, M. Simonovits, The history of degenerate (bipartite) extremal graph problems. *Erdős centennial, Bolyai Soc. Math. Stud.*, 25: 169–264, 2013.

[16] A. Gilbert, G. Tiner, On the Erdős-Sós conjecture for spiders in graphs having no path with $2k + 1$ vertices. *Ars Combin.*, 139: 297–313, 2018.

[17] C. Godsil, G. Royle, Algebraic Graph Theory. *vol. 207 of Graduate Texts in Mathematics, Springer-Verlag, New York*, 2001.

[18] C. D. Godsil, Algebraic Combinatorics. *Chapman and Hall, New York*, 1993.

[19] A. Görlich, A. Żak, On Erdős-Sós conjecture for trees of large size. *Electron. J. Combin.*, 23(1): #P1.52, 14 pp, 2016.

[20] B. He, Y.-L. Jin, X.-D. Zhang, Sharp bounds for the signless Laplacian spectral radius in terms of clique number. *Linear Algebra Appl.*, 438(10): 3851–3861, 2013.

[21] X. M. Hou, B. Y. Liu, S. C. Wang, J. Gao, C. H. Lv, The spectral radius of graphs without trees of diameter at most four. *Linear Multilinear Algebra*, 69(8): 1407–1414, 2021.

[22] X. X. Liu, H. J. Broersma, L. G. Wang, On a conjecture of Nikiforov involving a spectral radius condition for a graph to contain all trees. *Discrete Math.*, 345(12): 113112, 2022.

[23] A. McLennan, The Erdős-Sós conjecture for trees of diameter four. *J. Graph Theory*, 49(4): 291–301, 2005.

[24] V. Nikiforov, Bounds on graph eigenvalues II. *Linear Algebra Appl.*, 427: 183–189, 2007.

[25] V. Nikiforov, The maximum spectral radius of C_4-free graphs of given order and size. *Linear Algebra Appl.*, 430: 2898–2905, 2009.

[26] V. Nikiforov, A contribution to the Zarankiewicz problem. *Linear Algebra Appl.*, 432: 1405–1411, 2010.

[27] V. Nikiforov, The spectral radius of graphs without paths and cycles of specified length. *Linear Algebra Appl.*, 432(9): 2243–2256, 2010.

[28] V. Nikiforov, Some new results in extremal graph theory: In surveys in Combinatorics 2011. *London Math. Society Lecture Note Ser.*, 392: 141–181, 2011.

[29] V. Nikiforov, Merging the A-and Q-spectral theories. *Appl. Anal. Discrete Math.*, 11: 81–107, 2017.

[30] V. Nikiforov, X. Y. Yuan, Maxima of the Q-index: graphs without long paths, *Electron. J. Linear Algebra* 27: 504–514, 2014.

[31] J.-F. Saclé, M. Woźniak, The Erdős-Sós conjecture for graphs without C_4. *J. Combin. Theory Ser. B*, 70(2): 367–372, 1997.

[32] P. J. Slater, S.K. Teo, H. P. Yap, Packing a tree with a graph of the same size. *J. Graph Theory*, 9(2): 213-216, 1985.

[33] M. Tait, The Colin de Verdière parameter, excluded minors, and the spectral radius. *J. Combin. Theory Ser. A*, 166: 42–58, 2019.

[34] G.-X. Tian, Y.-X. Chen, S.-Y. Cui, The extremal α-index of graphs with no 4-cycle and 5-cycle. *Linear Algebra Appl.*, 619: 160–175, 2021.

[35] G. Tiner, On the Erdős-Sós conjecture for graphs on $n = k+3$ vertices. *Ars Combin.*, 95: 143–150, 2010.

[36] H. Wilf, Spectral bounds for the clique and independence numbers of graphs. *J. Combin. Theory Ser. B* 40: 113–117, 1986.

[37] M. Woźniak, On the Erdős-Sós conjecture. *J. Graph Theory*, 21(2): 229–234, 1996.

[38] L.-T. Yuan, X.-D. Zhang, On the Erdős-Sós conjecture for graphs on $n = k + 4$ vertices. *Ars Math. Contemp.*, 13(1): 49–61, 2017.

[39] M. Q. Zhai, H. Q. Lin, Spectral extrema of graphs: forbidden hexagon. *Discrete Math.*, 343(10): 112028, 2020.

[40] M. Q. Zhai, H. Q. Lin, Spectral extrema of $K_{s,t}$-minor free graph–On a conjecture of M. Tait. *J. Combin. Theory Ser. B*, 157: 184–215, 2022.

[41] M. Q. Zhai, B. Wang, Proof of a conjecture on the spectral radius of C_4-free graphs. *Linear Algebra Appl.*, 437: 1641–1647, 2012.

[42] B. Zhou, A note on the Erdős-Sós Conjecture. *Acta Math. Sci.*, 4(3): 287–289, 1984.