Proving a Directed Analog of the Gyárfás-Sumner Conjecture for Orientations of P_4

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Abstract

An oriented graph is a digraph that does not contain a directed cycle of length two. An (oriented) graph D is *H-free* if D does not contain H as an induced sub(di)graph. The Gyárfás-Sumner conjecture is a widely-open conjecture on simple graphs, which states that for any forest F, there is some function f such that every F-free graph G with clique number $\omega(G)$ has chromatic number at most $f(\omega(G))$. Aboulker, Charbit, and Naserasr [Extension of Gyárfás-Sumner Conjecture to Digraphs, Electron. J. Comb., 2021] proposed an analog of this conjecture to the dichromatic number of oriented graphs. The *dichromatic number* of a digraph D is the minimum number of colors required to color the vertex set of D so that no directed cycle in D is monochromatic.

Aboulker, Charbit, and Naserasr's $\overrightarrow{\chi}$-boundedness conjecture states that for every oriented forest F, there is some function f such that every F-free oriented graph D has dichromatic number at most $f(\omega(D))$, where $\omega(D)$ is the size of a maximum clique in the graph underlying D. In this paper, we perform the first step towards proving Aboulker, Charbit, and Naserasr's $\overrightarrow{\chi}$-boundedness conjecture by showing that it holds when F is any orientation of a path on four vertices.

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1 Introduction

In a simple graph, the size of a maximum clique gives a lower bound on its chromatic number. But if a graph contains no large cliques, does it necessarily have small chromatic

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number? This question has been answered in the negative. In the mid-twentieth century, Mycielski [21] and Zykov [27] gave constructions for triangle-free graphs with arbitrarily large chromatic number. Hence, we may ask the following question instead: Given some fixed graph H, do graphs with a bounded clique number that do not contain H *as an induced subgraph* have bounded chromatic number? In 1959, Erdős showed that there exist graphs with arbitrarily high girth and arbitrarily high chromatic number [12]. Hence, the answer to the previous question is "no" whenever H contains a cycle, and thus we need only consider the question when H is a forest. Around the 1980s, Gyárfás and Sumner independently conjectured [14, 26] that for any forest H, all graphs with bounded clique number and no induced copy of H have bounded chromatic number. The conjecture has been proven for some specific classes of forests but remains largely open; see [24] for a survey of related results. This paper concerns an analog of the Gyárfás-Sumner conjecture to directed graphs proposed by Aboulker, Charbit, and Naserasr [5]. We will state the Gyárfás-Sumner conjecture and its analog for directed graphs more formally after introducing some necessary terminology.

Throughout the paper, for integers i, j, we let $[i, j] = \{i, \ldots, j\}$. A directed graph, or *digraph*, is a pair $D = (V, E)$ where V is the vertex set and E is a set of ordered pairs of vertices in V called the arc set. We call a digraph *oriented* if it has no *digon* (directed cycle of length two). This paper will focus on finite, simple, oriented graphs.

For a digraph $D = (V, E)$ we define the *underlying graph* of D to be the graph $D^* = (V, E^*)$ where E^* is the set obtained from E by replacing each arc $e \in E$ by an undirected edge between the same two vertices. We say two vertices in D are *adjacent* or *neighbors* if they are adjacent in D^*. If (v, w) is an arc of D we say that v is an *in-neighbor* of w and that w is an *out-neighbor* of v. We denote the set of neighbors of a vertex $v \in V(D)$ by $N(v)$ and we denote $N(v) \cup \{v\}$ by $N[v]$. For a set of vertices $S \subseteq V(D)$ we let $N(S)$ and $N[S]$ denote the sets $\cup_{v \in S} N(v) \setminus S$ and $\cup_{v \in S} N[v]$. We call $N(S)$ the *neighborhood* of S and $N[S]$ the *closed neighborhood* of S. For a subdigraph $H \subseteq D$ we let $N(H)$ denote the set $N(V(H))$.

We let P_t denote the path on t vertices. We say an oriented path is a *directed path* if its vertices are $p_1, p_2, \ldots, p_t$, and its arcs are given by $\{(p_i, p_{i+1}),\ i \in [1, t-1]\}$. We let $\overrightarrow{P_t}$ denote the directed path on t vertices. We sometimes describe orientations of the path using symbols $\leftarrow$ and $\rightarrow$, these should be seen as the arcs of the given orientation. The arcs follow the order of the vertices in the underlying path, which we sometimes omit. For example, we may refer to a directed path $\overrightarrow{P_4}$ on vertices p_1, p_2, p_3, p_4 as $p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_4$ or as $\rightarrow\rightarrow\rightarrow$. We say a digraph D is *strongly connected* if for every $v, w \in V(D)$ there is a directed path starting at v and ending at w. An induced subdigraph H of a digraph D is a *strongly connected component* of D if it is strongly connected and every induced subgraph H' of D such that $H \subsetneq H'$ is not strongly connected. We call a strongly connected component H a source (sink) component of D if every arc between $V(H)$ and $V(D \setminus H)$ begins (ends) in $V(H)$.

A *tournament* is an orientation of a complete graph. A *transitive tournament* is an acyclic tournament. Given a (di)graph G and $S \subseteq V$, we denote the sub(di)graph of G induced by S as $G[S]$. We say that a (di)graph G *contains* a (di)graph H if G contains H

as an induced sub(di)graph. If G does not contain a (di)graph H we say that G is *H-free*. If G does not contain any of the (di)graphs $H_1, H_2, \ldots, H_k$ we say G is *$(H_1, H_2, \ldots, H_k)$-free*. The *clique number* and the *chromatic number* of a digraph are the chromatic number and clique number of its underlying graph, respectively. We denote the clique number and the chromatic number of a (di)graph G by $\omega(G)$ and $\chi(G)$, respectively. We say that a graph H is *χ-bounding* if there exists a function f with the property that every H-free graph G satisfies $\chi(G) \leqslant f(\omega(G))$. In this language, [12] implies all χ-bounding graphs are forests. We are now ready to state the Gyárfás-Sumner conjecture more formally.

Conjecture 1 (The Gyárfás-Sumner conjecture [14, 26])**.** Every forest is χ-bounding.

Today, the conjecture is only known to hold for restricted classes of forests. For example, Gyárfás showed that it holds for paths [15] via a short and elegant proof. Subsequently, the conjecture was proven for other classes of forests. For example, the following classes of trees have been proven to be χ-bounding:

- Trees of radius two by Kierstead and Penrice in 1994 [18],
- Trees that can be obtained from a tree of radius two by subdividing *every* edge incident to the root exactly once by Kierstead and Zhu in 2004 [20], and
- Trees that can be obtained from a tree of radius two by subdividing *some of the* edges incident to the root exactly once by Scott and Seymour in 2020 [23].

Note that the class of trees described by the third bullet contains the classes described in both the first and second bullet. See the survey of Scott and Seymour [24] for an overview of the state of the conjecture from 2020.

How can the Gyárfás-Sumner conjecture be adapted to the directed setting? A first idea is to call an oriented graph H *χ-bounding* if there exists a function f with the property that every H-free oriented graph D satisfies $\chi(D) \leqslant f(\omega(D))$. Then, once again, by [12], all χ-bounding oriented graphs are oriented forests. Note that if an oriented graph H is χ-bounding, its underlying graph H^* is also χ-bounding. However, the converse does not hold, as, for instance, P_4 is χ-bounding, but there exist orientations of P_4 that are not χ-bounding. There are four different orientations of P_4, up to reversing the order of the vertices on the whole path:

$$\to\to\to,\ \to\leftarrow\to,\ \to\leftarrow\leftarrow,\ \leftarrow\leftarrow\to$$

Only the last two oriented graphs in the list are χ-bounding:

- Recall, we denote the oriented P_4 with orientation $\to\to\to$ by $\overrightarrow{P_4}$. In 1991, Kierstead and Trotter [19], showed that $\overrightarrow{P_4}$ is not χ-bounding. Their construction was inspired by Zykov's construction of triangle-free graphs with a high chromatic number [27], and builds $\overrightarrow{P_4}$-free oriented graphs with arbitrarily large chromatic number and no clique of size three.

- Around 1990, Gyárfás pointed out that $\leftarrow\rightarrow\leftarrow$ is not χ-bounding, as witnessed by an orientation of the shift graphs on pairs [16]. We will denote the P_4 with orientation $\leftarrow\rightarrow\leftarrow$ by $\overrightarrow{A_4}$.

- Chudnovsky, Scott and Seymour [10] showed that $\rightarrow\leftarrow\leftarrow$ and $\leftarrow\leftarrow\rightarrow$ are both χ-bounding in 2019. In the same article, the authors show that orientations of stars are also χ-bounding (stars are the class of complete bipartite graphs $K_{1,t}$ for any $t \geqslant 1$). We will denote $\rightarrow\leftarrow\leftarrow$ and $\leftarrow\leftarrow\rightarrow$ by $\overrightarrow{Q_4}$ and $\overrightarrow{Q_4}'$, respectively.

This attempt at adapting the Gyárfás-Sumner conjecture to oriented graphs does not hold for oriented paths such as $\overrightarrow{P_4}$ and $\overrightarrow{A_4}$. Hence, we focus on a different approach proposed by Aboulker, Charbit, and Naserasr [5] which uses a concept called "dichromatic number". Directed coloring, or dicoloring, is a weakening of coloring defined on digraphs and was proposed by Neumann-Lara and subsequently developed by Erdős and Neumann-Lara [13, 22]. A *dicoloring* of a digraph D is a partition of $V(D)$ into classes, or colors, such that each class induces an acyclic digraph (that is, there is no monochromatic directed cycle). The *dichromatic number* of D, denoted as $\overrightarrow{\chi}(D)$, is the minimum number of colors needed for a dicoloring of D. Throughout the paper, given a digraph D, we sometimes identify vertex set $X \subseteq V(D)$ and the subgraph $D[X]$. In particular, for $X \subseteq V(D)$, we let $\omega(X) = \omega(D[X])$, $\chi(X) = \chi(D[X])$, and $\overrightarrow{\chi}(X) = \overrightarrow{\chi}(D[X])$. Notice that every coloring of a directed graph D is also a dicoloring, thus $\overrightarrow{\chi}(D) \leqslant \chi(D)$.

Much prior research has been done to understand which induced subdigraphs must exist in digraphs of large dichromatic number. Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé gave an explicit description of all the tournaments H for which the class of H-free tournaments has bounded dichromatic number [6]. Similar questions for general digraphs remain open. A set $\mathcal{F}$ of digraphs is called *heroic* if the class of $\mathcal{F}$-free digraphs has bounded dichromatic number. It is easy to see that every heroic set must forbid some digraph with a digon and that every heroic set must forbid some oriented graph. However, we are far from a characterization such as the one in [6] despite significant work towards better understanding heroic sets (such as [1, 3, 5, 9]). It was shown in [5] that: For every oriented forest F, if the set consisting of a digon, F, and a digraph H is heroic, then F is the disjoint union of oriented stars or H is a transitive tournament. Motivated by this, Aboulker, Charbit, and Naserasr conjectured that for every oriented forest F, and any transitive tournament K, the set consisting of the digon, F and K is heroic in [5]. This conjecture, which we call the "ACN $\overrightarrow{\chi}$-boundedness conjecture", is the main subject of this paper.

In order to highlight how the ACN $\overrightarrow{\chi}$-boundedness is analogous to the Gyarfas-Sumner conjecture, we will restate the ACN $\overrightarrow{\chi}$-boundedness in terms of "$\overrightarrow{\chi}$-boundedness". We say a class of digraphs $\mathcal{D}$ is $\overrightarrow{\chi}$*-bounded* if there exists a function f such that every $D \in \mathcal{D}$ satisfies $\overrightarrow{\chi}(D) \leqslant f(\omega(D))$, and we call such an f a $\overrightarrow{\chi}$*-binding function* for $\mathcal{D}$. We say a digraph H is $\overrightarrow{\chi}$*-bounding* if the class of H-free oriented graphs is $\overrightarrow{\chi}$-bounded. Note, that every tournament of order 2^k contains a transitive tournament of order k. Thus, if a digraph H is $\overrightarrow{\chi}$-bounding, the set consisting of a digon, H, and K is heroic for any choice of a transitive tournament K, so we can restate the conjecture as follows:

Conjecture 2 (The ACN $\vec{\chi}$-boundedness conjecture [5])**.** Every oriented forest is $\vec{\chi}$-bounding.

The converse of the ACN $\vec{\chi}$-boundedness conjecture holds; all $\vec{\chi}$-bounding digraphs must be oriented forests. Indeed, Harutyunyan and Mohar proved that there exist oriented graphs of arbitrarily large undirected girth and dichromatic number [17]. Oriented graphs of sufficiently large undirected girth do not contain any fixed digraph that is not an oriented forest. Hence, no digraph containing a digon or a cycle in its underlying graph is $\vec{\chi}$-bounding. Moreover, for any finite list of digraphs $D_1, D_2, \ldots, D_k$, if the class of $(D_1, D_2, \ldots, D_k)$-free oriented graphs is $\vec{\chi}$-bounded then one of D_1, $D_2, \ldots, D_k$ must be a forest. One might ask whether the situation changes when we forbid an infinite list of oriented graphs. We list some results related to this:

- In [8], Carbonero, Hompe, Moore, and Spirkl provided a construction for oriented graphs with clique number at most three, arbitrarily high dichromatic number, and no induced directed cycles of odd length at least 5. They use this to show that there exist graphs G with arbitrarily large chromatic such that every induced triangle-free subgraph of G has chromatic number at most four, disproving a well-known conjecture.

- In [4], Aboulker, Bousquet, and de Verclos showed that the class of chordal oriented graphs, that is, oriented graphs forbidding induced directed cycles of length greater than three, is not $\vec{\chi}$-bounded, answering a question posed in [8].

- In [7], Carbonero, Hompe, Moore, and Spirkl extended the result of [8] to t-chordal graphs. A digraph is *t-chordal* if it does not contain an induced directed cycle of length other than t. In [7] the authors showed that t-chordal graphs are not $\vec{\chi}$-bounded, but t-chordal $\vec{P_t}$-free graphs are $\vec{\chi}$-bounded.

Note that Conjecture 2 only considers *oriented* graphs. This is the only sensible case. Indeed, if F contains a digon, the class of F-free oriented graphs is the class of all oriented graphs, which is not $\vec{\chi}$-bounded (because for example tournaments are oriented graphs). If F contains no digons and at least one arc, then the class of F-free *digraphs* is not $\vec{\chi}$-bounded; Any digraph obtained from a graph by replacing every edge with a digon does not contain any oriented graph with at least one edge as an induced subgraph. Hence, by [21, 27], for any choice of an oriented graph with at least one edge F, there exist F-free digraphs (with digons) that have arbitrarily high dichromatic number and do not contain a triangle in their underlying graph.

The ACN $\vec{\chi}$-boundedness conjecture is still widely open. It is not known whether the conjecture holds for any orientation of any tree T on at least five vertices that is not a star. In particular, it is not known whether the conjecture holds for oriented paths. In contrast, Gyárfás showed that every path is χ-bounding in the 1980s [14, 15]. We will introduce some terminology before discussing the status of the ACN $\vec{\chi}$-boundedness conjecture for oriented paths in more detail. For $t \leqslant 3$, P_t is $\vec{\chi}$-bounding. (This can be proven by, for example, noting that for $t \leqslant 3$, the graph P_t is a star and applying Chudnovsky, Scott, and Seymour's result [10] that every orientation of a star is χ-bounding and therefore also $\vec{\chi}$-bounding.) However, for $t \geqslant 4$, the picture gets more complicated:

- Let T be any fixed orientation of K_3. In [5], Aboulker, Charbit and Naserasr showed that class of (T, $\vec{P_4}$)-free oriented graphs have bounded dichromatic number. The authors also show that $\vec{P_4}$-free oriented graphs with clique number at most three have bounded dichromatic number.

- Let $\vec{K_t}$ denote the transitive tournament on t vertices. In [25], Steiner showed that the class of ($\vec{K_3}$, $\vec{A_4}$)-free oriented graphs has bounded dichromatic number. In the same paper Steiner asked whether the class of $(H, \vec{K_t})$-free oriented graphs has bounded dichromatic number for $t \geqslant 4$ and $H \in \{\vec{P_4}, \vec{A_4}\}$. We explain in the next subsection that our main result answers this question in the affirmative.

1.1 Our contributions

In this paper, we show that every orientation of P_4 is $\vec{\chi}$-bounding and thus the ACN $\vec{\chi}$-boundedness conjecture holds for all orientations of P_4. The ACN $\vec{\chi}$-boundedness conjecture is open for any orientation of P_t for $t \geqslant 5$. Our main novel result is that $\vec{P_4}$ and $\vec{A_4}$ are both $\vec{\chi}$-bounding. Chudnovsky, Scott and Seymour showed that both $\vec{Q_4}$ and $\vec{Q_4}'$ are χ-bounding and thus also $\vec{\chi}$-bounding in [10]. We include in a new proof that $\vec{Q_4}$ and $\vec{Q_4}'$ are both $\vec{\chi}$-bounding and improve the $\vec{\chi}$-binding function for the classes of $\vec{Q_4}$-free oriented graphs and $\vec{Q_4}'$-free oriented graphs. To summarize, our main result is the following:

Theorem 3. *Let H be an oriented P_4. Then, the class of H-free oriented graphs is $\vec{\chi}$-bounded. In particular, for any H-free oriented graph D,*

$$\vec{\chi}(D) \leqslant (\omega(D) + 7)^{(\omega(D)+8.5)}.$$

Our result also answers the question of [25] in the affirmative, that is, for $H \in \{\vec{P_4}, \vec{A_4}\}$ and any $k \geqslant 4$ the class of H-free oriented graphs not containing a transitive tournament of order k has bounded dichromatic number.

The ACN $\vec{\chi}$-boundedness conjecture and other questions raised in [5] are aimed at characterizing heroic sets of cardinality three. If we ignore the degenerate cases where heroic sets include the empty graph or the graph consisting of a single vertex, there are no heroic sets consisting of only one element and the only heroic set of cardinality two consists of an arc and a digon. Therefore, heroic sets of cardinality three are the first interesting case. Then, Theorem 3 can be restated by saying every set $\{\overleftrightarrow{K_2}, H, K\}$, where $\overleftrightarrow{K_2}$ denotes a digon, H is an orientation of P_4, and K is a transitive tournament is heroic. Further explanation of how the ACN $\vec{\chi}$-boundedness Conjecture is motivated by questions about heroic sets is given in [5].

Structure of the paper and proof overview. Let H be any orientation of P_4. We prove Theorem 3 by induction on the clique number. We fix an integer $\omega(D) \geqslant 2$. We define a function f and assume that H-free oriented graphs with clique number ω' where $1 \leqslant \omega' < \omega$ have dichromatic number at most $f(\omega')$. We then consider an oriented graph with clique number ω and show that D can be dicolored using at most $f(\omega)$ colors.

Our strategy to bound $\overrightarrow{\chi}(D)$ crucially relies on a tool we call *dipolar sets* which were introduced by the name "nice sets" in [5]. Dipolar sets have the following useful property [5]: In order to bound the dichromatic number of a class of oriented graphs closed under taking induced subgraphs, it suffices to exhibit a dipolar set of bounded dichromatic number for each of the members in the class. We give a few preliminary observations as well as an introduction to dipolar sets in Section 2.

In Section 3, we show how to construct a dipolar set for any H-free oriented graph D of clique number ω. Our goal is to obtain a bound for the dichromatic number of this set. The backbone of our construction is an object we call a closed tournament.

Definition 4 (path-minimizing closed tournament)**.** We say K and P *form a closed tournament* $C = K \cup V(P)$ if K is a tournament of order $\omega(D)$ and P is a directed path from a sink component to a source component in K.

Given a digraph D, we say $C = K \cup P$ forms a *path-minimizing closed tournament* if it is a closed tournament such that K is a maximum clique of D, and $|P|$ is minimized amongst all possible choices of K, P that form a closed tournament.

It follows from the definition of closed tournament that the graph induced by a closed tournament is strongly connected and that every strongly connected oriented graph has a path-minimizing closed tournament. We will define a set S consisting of the closed neighborhood of a path-minimizing closed tournament C and a subset of the second neighbors of C. We will show that if D is H-free, then S is a dipolar set. This proof will rely heavily on the fact that C is strongly connected.

The strong connectivity of C is a powerful property in showing that S is a dipolar set. However, ensuring C is strongly connected by adding P to K makes it harder to bound the dichromatic number of $N(C)$. We explain in Section 2 that we can easily bind the dichromatic number of the first neighborhood of any bounded cardinality set. Unfortunately, we have no control over the cardinality of P in a path-minimum closed tournament. In fact, P, and thus C, might be arbitrarily large with respect to ω. This makes the task of bounding the dichromatic number of $N(C)$ significantly harder. Fortunately, since D is H-free and we may choose C to be a *path-minimizing* closed tournament, there are a lot of restrictions on what arcs may exist between vertices of $N(C)$. Ultimately, our goal is to exploit these restrictions to bind the dichromatic number of $N(C)$.

Interestingly, we can define S and prove that it is a dipolar set in the same way for each possible choice of an oriented P_4. We describe our construction of a dipolar set S in Section 3. However, we used different (but similar) proofs to show that S has bounded dichromatic number for $H = \overrightarrow{P_4}$, $\overrightarrow{A_4}$, and $\overrightarrow{Q_4}$. The proof that S has bounded dichromatic number when H is $\overrightarrow{Q_4}$ implies the result when H is $\overrightarrow{Q_4}'$.

In Section 4, we bound the dichromatic number of C, the vertices of S in the second neighborhood of C, and $N(K)$ for H-free graphs where H is an arbitrary choice of an orientated P_4. In Section 5, we bound the dichromatic number of the vertices in S not handled in Section 4. These remaining vertices are the set $N(P) \setminus N[K]$. Here we use separate (but similar) proofs for $H = \overrightarrow{P_4}, \overrightarrow{A_4}, \overrightarrow{Q_4}$. In Section 6, we put the pieces together to obtain our main result that any orientation of P_4 is $\overrightarrow{\chi}$-bounding. We discuss some related open questions in Section 7.

2 Preliminaries

In this section, we lay the groundwork for our proof by making a few observations useful in later sections and introducing dipolar sets. In the rest of the paper, we will only consider strongly connected oriented graphs since the dichromatic number of an oriented graph is equal to the maximum dichromatic number of one of its strongly connected components. In particular, we will work with the following assumptions:

Scenario 5 (Inductive Hypothesis)**.** *Let H be an oriented P_4 and let $\omega > 1$ be an integer. We let γ be the maximum of $\overrightarrow{\chi}(D')$ over every H-free oriented graph D' satisfying $\omega(D') < \omega$, and assume γ is finite. We let D be an H-free oriented graph with clique number ω and assume D is strongly connected.*

We will aim to bound the $\overrightarrow{\chi}(D)$ in terms of γ and ω. We begin with some easy observations about the dichromatic number of the neighborhood of any sets of vertices in D. For any vertex $v \in V(D)$, by definition $\omega(N(v)) \leqslant \omega - 1$ as otherwise D would contain a tournament of size greater than ω. Hence, for any $v \in V(D)$, $\overrightarrow{\chi}(N(v)) \leqslant \gamma$. This can be directly extended to bounding the dichromatic number of the neighborhood of a set of a given size as follows:

Observation 6. *Let D be an oriented graph and let γ be the maximum value of $\overrightarrow{\chi}(N(v))$ for any $v \in V(D)$. Then every $X \subseteq V(D)$ satisfies:*

$$\overrightarrow{\chi}(N(X)) \leqslant \overrightarrow{\chi}\left(\bigcup_{x\in X} N(x)\right) \leqslant |X| \cdot \gamma.$$

We now formally define dipolar sets, one of the main tools used in this paper. Note, dipolar sets were first introduced in [4] as "nice sets".

Definition 7 (dipolar set)**.** A *dipolar set* of an oriented graph D is a nonempty subset $S \subseteq V(D)$ that can be partitioned into S^+, S^- such that no vertex in S^+ has an out-neighbor in $V(D \setminus S)$ and no vertex in S^- has an in-neighbor in $V(D \setminus S)$.

We will use the following lemma from [5] which reduces the problem of bounding the dichromatic number of D to bounding the dichromatic number of a dipolar set in every induced oriented subgraph of D.

Lemma 8 (Lemma 17 in [5])**.** *Let $\mathcal{D}$ be a family of oriented graphs closed under taking induced subgraphs. Suppose there exists a constant c such that every $D \in \mathcal{D}$ has a dipolar set S with $\overrightarrow{\chi}(S) \leqslant c$. Then every $D \in \mathcal{D}$ satisfies $\overrightarrow{\chi}(D) \leqslant 2c$.*

3 Building a dipolar set

In this section we give a construction for a dipolar set in an H-free oriented graph D where H is an oriented P_4. We will then show that the dipolar set we construct has bounded dichromatic number if D satisfies the properties given in Scenario 5.

3.1 Closed Tournaments

The simplest case for our construction is when D contains a *strongly connected* tournament J of order $\omega(D)$. Then, we can build a dipolar set consisting of the union of J and a subset of vertices at distance at most two from K.

Let K be a tournament of order $\omega(D)$ contained in D. By definition every vertex $v \in N(K)$ has a non-neighbor in K. Hence, the graph underlying $D[K \cup \{v\}]$ contains an induced P_3. Now, suppose K is strongly connected. Then we get an even more powerful property: Since K is strongly connected there is both an arc from $K \setminus N(v)$ to $N(v) \cap K$ and to $K \setminus N(v)$ from $N(v) \cap K$. This means that $D[K \cup \{v\}]$ contains an induced P_3 starting at v whose last edge is oriented as $\rightarrow$ and an induced P_3 starting at v whose last edge is oriented as $\leftarrow$. This property will give us more power to build specific induced orientations of P_3 in $N[K]$. In particular, this restricts the way vertices at distance at most two interact with the rest of the graph and allows us to exhibit a dipolar set.

To overcome the fact that D may not contain a strongly connected tournament of order $\omega(D)$, we use closed tournaments. By definition of closed tournament every strongly connected oriented graph has a path-minimizing closed tournament. We will base our construction of a dipolar set on some path-minimizing tournament in order to gain some additional structure that we can use to bound the dichromatic number of our dipolar set. In the next subsection we formally give the definition of our dipolar set.

3.2 Extending a closed tournament into a dipolar set

In order to build a dipolar set from a closed tournament, we need to make some distinctions between different types of neighbors of a set of vertices. For a set of vertices A and $v \in N(A)$ we say v is a *strong neighbor* of A if v has both an in-neighbor and an out-neighbor in A. Then, the *strong neighborhood* of A is the set of strong neighbors of A.

Given a closed tournament C, we let X denote the set of strong neighbors of C. The following lemma proves that $N[C \cup X]$ is a dipolar set.

Lemma 9. *Let H be an orientation of P_4 and D be an H-free oriented graph. Let C be a closed tournament in D and let X denote the strong neighborhood of C. Then $N[C \cup X]$ is a dipolar set.*

Proof. Let Z denote the neighbors of C that are not strong, and let $Y = N(X) \setminus N[C]$. These sets satisfy $N[C \cup X] = C \cup X \cup Z \cup Y$ and the graph on $N[C \cup X]$ is illustrated in Figure 1.

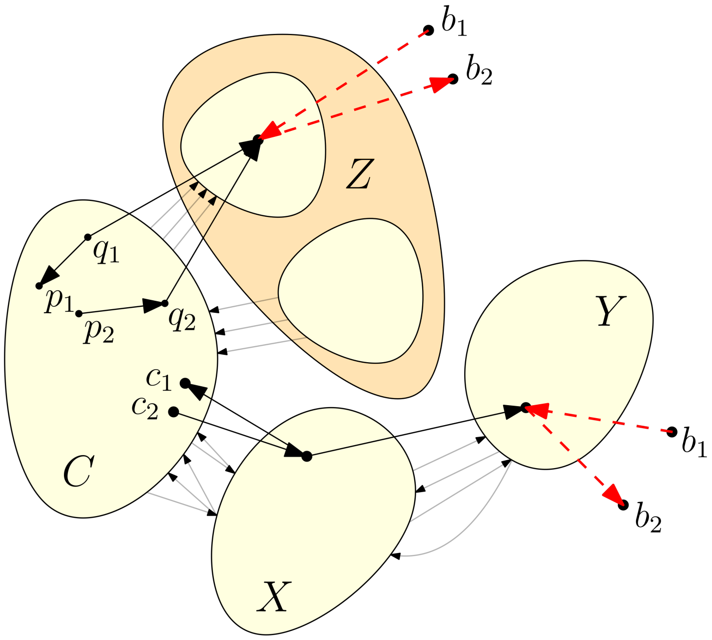

Figure 1: An illustration of the extension of a closed tournament C into the dipolar set $N[C \cup X]$. Highlighted in blue, Z consists of neighbors of C that are not strong, i.e., do not have both an in-neighbor and an out-neighbor in C. The set X consists of the strong neighborhood of C, while set Y contains all neighbors of X not in $N[C]$. Note that arcs between Z and X or Y are not represented here. In Lemma 9, we prove that if there is some vertex in $N[C \cup X]$ with both an in-neighbor and an out-neighbor in the rest of the oriented graph (drawn in dashed red), then $N[C \cup X] \cup \{b_1, b_2\}$ contains all orientations of P_4 as an induced oriented subgraph.

Then by definition, $N[C \cup X] = N[C] \cup Y$ and the only vertices of $N[C \cup X]$ with neighbors in $V(D) \setminus N[C \cup X]$ are in $Y \cup Z$. Suppose for a contradiction that some $v \in Z \cup Y$ has both an in-neighbor b_1 and an out-neighbor b_2 in $V(D) \setminus (N[C] \cup Y)$. Let us first deal with the case where $v \in Y$.

$$\textit{If } v \in Y, \textit{ then } D[C \cup X \cup Y \cup \{b_1, b_2\}] \textit{ contains } H. \tag{1}$$

Suppose $v \in Y$. Then, by definition, the following statements all hold:

- There is some $x \in X$ such that x and v are adjacent.
- There are vertices $c_1, c_2 \in C$ where c_1 is an in-neighbor of x and c_2 is an out-neighbor of x.
- b_1, b_2 are not adjacent to any of x, c_1, c_2.

Thus, for some choice of $i, j \in \{1, 2\}$ the set $\{c_i, x, v, b_j\}$ induces a copy of H. (See Figure 1.) This proves (1).

Since D is an H-free oriented graph, it follows from (1) that $v \in Z$. Then by definition of Z, the neighbors of v in C are either all in-neighbors of v or all out-neighbors of v.

There exist arcs $(q_1, p_1), (p_2, q_2) \in E(C)$ such that v is adjacent to q_1, q_2 and non-adjacent to p_1, p_2. (2)

It follows from the fact that $\omega(C) = \omega(D)$ that v has some non-neighbor in C. Since C is strongly connected, $N(v) \cap C$ must have both an incoming arc and an outgoing arc from $C \setminus N(v)$. Let p_1, p_2 be vertices of $C \setminus N(v)$ witnessing this fact and let q_1, q_2 their respective neighbors in $N(v) \cap C$. This proves (2).

It follows that for some $i, j \in \{1, 2\}$ the graph induced by $\{p_i, q_i, v, b_j\}$ is a copy of H, a contradiction. (See Figure 1). □

4 First steps towards bounding the dichromatic number of $N[C \cup X]$

For brevity, we will fix the following variables for the remainder of the paper.

Definition 10. Let $D, H, \gamma, K, P, C, X, Y$ be defined as follows:

- Let H be an oriented P_4.
- Let D be a digraph satisfying the assumptions of Scenario 5 with respect to H. Let γ be as in Scenario 5.
- We choose a tournament K of order $\omega(D)$ and a directed path P that form a path-minimizing tournament C in D.
- Let X be the strong neighborhood of C and $Y = N(X) \setminus N[C]$.

In the previous section we showed that $N[C \cup X]$ is a dipolar set. Thus, by Lemma 8 we can prove that all orientations of P_4 are $\vec{\chi}$-bounding by proving that $\vec{\chi}(N[C \cup X])$ is bounded in terms of $\omega(D)$ and γ, the maximum value of $\vec{\chi}(D')$ for any H-free D' with clique number less than $\omega(D)$.

Since $N[C \cup X] \subseteq N[K] \cup V(P) \cup (N(P) \setminus N[K]) \cup Y$, the following inequality holds by definition of the dichromatic number:

$$\vec{\chi}(N[C \cup X]) \leqslant \vec{\chi}(N[K]) + \vec{\chi}(V(P)) + \vec{\chi}(N(P) \setminus N[K]) + \vec{\chi}(Y) \tag{3}$$

We will bound $\vec{\chi}(N[C \cup X])$ by bounding each of the terms on the right-hand side of the inequality. We bound the dichromatic number of $N[K]$ and P in Subsection 4.1 and we bound the dichromatic number of Y in Subsection 4.2. We are able to use the same techniques for each choice of H when proving these bounds.

As already hinted, bounding $\overrightarrow{\chi}(N[C \cup X])$ is non-trivial because we have no control over the cardinality of P. Hence, we cannot obtain a useful bound on $\overrightarrow{\chi}(N(P) \setminus N[K])$ by simply applying Observation 6. In the next section, we will show how to bound $\overrightarrow{\chi}(N(P) \setminus N[K])$. We will require separate proofs for $H = \overrightarrow{Q_4}, \overrightarrow{P_4}, \overrightarrow{A_4}$.

4.1 Bounding the dichromatic number of $V(P)$ and $N[K]$

We bound the dichromatic number of $V(P)$ and $N[K]$ by an easy observation about "forward-induced" paths. We say a directed path $p_1 \to p_2 \to \cdots \to p_t$ is *forward-induced* if no arc of the form (p_i, p_j) exists where $j > i + 1$ and $i, j \in [1, t]$.

Observation 11. *Let P be a forward-induced directed path in some oriented graph. Then $\overrightarrow{\chi}(P) \leqslant 2$.*

Proof. Let the vertices of P be $p_1 \to p_2 \to \ldots \to p_\ell$, in order. We assign colors to the vertices of P by alternating the colors along P. Suppose there is some monochromatic directed cycle Q in the oriented graph induced by $V(P)$. Then Q contains no arc of P. Hence, Q must contain some arc (p_i, p_j) with $i, j \in [1, \ell]$ and $j > i + 1$, contradicting the defintion of forwards-induced. □

Now, we turn to bound the dichromatic number of our dipolar set, $N[C \cup X]$ by bounding $\overrightarrow{\chi}(N[K] \cup V(P))$.

Observation 12. *It holds that*

$$\overrightarrow{\chi}(N[K]) \leqslant \omega \cdot \gamma$$

Moreover,

$$\overrightarrow{\chi}(N[C]) \leqslant \overrightarrow{\chi}(N(P) \setminus N[K]) + \omega \cdot \gamma + 2.$$

Proof. Since $|K| = \omega(D) > 1$, we have $N[K] = \bigcup_{x \in K} N(x)$, and hence $\overrightarrow{\chi}(N[K]) \leqslant \omega \cdot \gamma$ by Observation 6. This proves the first statement. By definition, $N[C] = (N(P) \setminus N[K]) \cup N[K] \cup P$. Since C is path-minimizing, P is forward-induced. Thus, we obtain the second statement by Observation 11. □

Thus, by inequality (3), it only remains to bound the dichromatic number of Y and $N(P) \setminus N[K]$ in order to bound the dichromatic number of our dipolar set $N[C \cup X]$.

4.2 Bounding the dichromatic number of Y

In this subsection, we bound the dichromatic number of $Y = N(X) \setminus N[C]$. We first state a more general lemma, which gives the bound on $\overrightarrow{\chi}(Y)$ as a direct corollary.

Lemma 13. *Let H be an oriented P_4 and let D be an H-free oriented graph. Suppose there is a partition of $V(D)$ into sets Q, R, S such that there is no arc between Q and S, every $r \in R$ has both an in-neighbor and an out-neighbor in Q, and every $s \in S$ has a neighbor in R. Let γ be a positive integer such that for every $r \in R$, we have $\overrightarrow{\chi}(N(r)) \leqslant \gamma$. Then $\overrightarrow{\chi}(S) \leqslant 2\gamma$.*

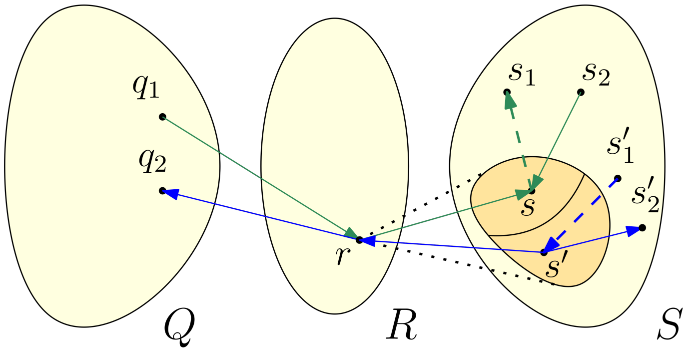

Figure 2: Vertex sets Q, R, S, such that no arc lies between Q and S, the vertices in R are all strong neighbors of Q and S is a subset of neighbors of R. We illustrate the case of graphs forbidding $\overrightarrow{P_4}$. Other orientations behave symmetrically. In green, a vertex $r \in R$ is depicted with an out-neighbor $s \in S$. Then if s has an out-neighbor $s_1 \in S \setminus N(r)$ there would be an induced $\overrightarrow{P_4}$, a contradiction. Symmetrically in blue, an in-neighbor $s' \in S$ of r cannot admit an in-neighbor $s_2' \in S \backslash N(r)$.

Proof. Note that if an oriented graph D with partition of $V(D)$ into Q, R, S satisfies the conditions of the lemma, for every $S' \subseteq S$ the oriented graph induced by $Q \cup R \cup S'$ also satisfies the conditions of the lemma with partition Q, R, S'. Hence, by Lemma 8 it is enough to show that for every oriented graph D with partition Q, R, S satisfying the conditions of the lemma and $S \neq \emptyset$, the oriented graph induced by S contains a dipolar set with dichromatic number at most γ.

Let Q, R, S be as in the statement of the lemma and suppose $S \neq \emptyset$. Then, there is some $r \in R$ with a neighbor in S. By assumption, r has an in-neighbor q_1 and an out-neighbor q_2 in Q. Suppose some $s \in N(r) \cap S$ has both an in-neighbor s_i and an out-neighbor s_j in $S \setminus N(r)$. Then there is a copy of H induced by $\{q_i, r, s, s_j\}$ for some choice of $i, j \in \{1, 2\}$, a contradiction. (See Figure 2.) Hence, $N(r) \cap S$ is a dipolar set. Moreover, $\vec{\chi}(N(r) \cap S) \leqslant \vec{\chi}(N(r)) \leqslant \gamma$ by definition of γ. □

Corollary 14. *We have that that:*

$$\vec{\chi}(Y) \leqslant 2\gamma.$$

Proof. By definition, we may apply Lemma 13 to the induced subgraph $D[C \cup X \cup Y]$ with $Q \coloneqq C$, $R \coloneqq X$ and $S \coloneqq Y$ (see Figure 1). Hence, $\vec{\chi}(Y) \leqslant 2\gamma$. □

At this point, we only need to bound $\vec{\chi}(N(P) \setminus N[K])$ in order to bound the dichromatic number of our dipolar set $N[C \cup X]$ by Corollary 14. This will be the purpose of the next section.

5 Completing the bound on the dichromatic number of our dipolar set

Here, we keep the definitions of D, C, K, P, X, Y and H, γ as fixed in Definition 10. In this section, we bound the dichromatic number of $N(P) \setminus N[K]$, thus completing the bound on the terms of the right-hand side of inequality (3). By Corollary 14, this will imply that every oriented graph which forbids some orientation of P_4 has a dipolar set of bounded dichromatic number. Thus, by Lemma 8, this will give us our main result.

By definition of path-minimizing closed tournament, P is a forward-induced directed path. In Subsection 5.1, we start by giving some structural properties on properties of the neighborhood of forwards-induced paths. Then, in Subsection 5.2, we show how to use these properties to bound the dichromatic number of the first neighborhood of C for $\overrightarrow{Q_4}$-free graphs. (Recall, the bound for $\overrightarrow{Q_4}$-free oriented graphs implies the bound for $\overrightarrow{Q_4}'$-free oriented graphs.) When H is one of the other two orientations, $\overrightarrow{P_4}$ and $\overrightarrow{A_4}$, we required a finer analysis of $N(P)$ in order to bound $\overrightarrow{\chi}(N(P))$. We handle this case in Subsections 5.3.1–5.3.3.

5.1 Forbidden arcs amongst neighbors of a forward-induced directed path

We define two partitions of the first neighborhood of a directed path and show how to forbid some of the arcs between classes of each partition in an H-free oriented graph. For the rest of this paper we will refer to the vertex set of P as $P = p_1 \to p_2 \to \cdots \to p_\ell$. Many of the results in the following sections actually hold for arbitrary forward-induced paths in D, but we state them for P as this suffices in our proof that orientations of P_4 are $\overrightarrow{\chi}$-bounding.

Definition 15. For brevity, for any $v \in N(P)$ and $i, j \in [1, \ell]$ we say p_i is the *first* neighbor of v on P if v is adjacent to p_i and non-adjacent to $p_{i'}$ for each $1 \leqslant i' < i \leqslant \ell$. Similarly, p_j is the *last* neighbor of v on P if v is adjacent to p_j and non-adjacent to each $p_{j'}$ for each $1 \leqslant j < j' \leqslant \ell$. We will define two partitions of $N(P)$ according to their first and last neighbors in $V(P)$, respectively.

- For each $i \in [1, \ell]$ we say $v \in N(P)$ is in F_i if p_i is the first neighbor of v on P. This yields partition $(F_1, F_2, \ldots, F_\ell)$, which we call *the partition of $N(P)$ by* first attachment *(on P)*.
- Symmetrically, for each $j \in [1, \ell]$ we say $v \in L_j$ if p_j is the last neighbor of v on P. This yields partition $(L_1, L_2, \ldots, L_\ell)$, which we call the *partition of $N(P)$ by last attachment (on P)*.

For each $i \in [1, \ell]$ we refine each partition by dividing each F_i, L_i into the in-neighbors and out-neighbors of v. We define F_i^+ and L_i^+ to be the sets consisting of all the in-neighbors of p_i in F_i, L_i, respectively. Similarly, we define F_i^- and L_i^- to be the sets consisting of all the out-neighbors of p_i in F_i, L_i, respectively.

Observation 16. *Let $2 \leqslant i < j \leqslant \ell - 1$. Then the following statements all hold:*

- *If D is $\overrightarrow{Q_4}$-free, there are no arcs from F_j to F_i.*
- *If D is $\overrightarrow{P_4}$-free, there are no arcs from F_i^- to F_j, and no arcs from L_i to L_j^+.*
- *If D is $\overrightarrow{A_4}$-free, there are no arcs from F_i^+ to F_j, and no arcs from L_i to L_j^-.*

Proof. Let $2 \leqslant i < j \leqslant \ell - 1$. We prove each statement individually.

$$\text{If } D \text{ is } \overrightarrow{Q_4}\text{-free, there are no arcs from } F_j \text{ to } F_i. \tag{4}$$

Suppose for some $v \in F_j$ and $w \in F_i$ that $(v, w) \in E(D)$. Then the vertices p_{i-1}, p_i, w, v, induce a P_4 in D with orientation $p_{i-1} \to p_i \to w \leftarrow v$ or orientation $p_{i-1} \to p_i \leftarrow w \leftarrow v$ depending on whether $w \in F_i^+$ or $w \in F_i^-$. In either case we obtain an induced $\overrightarrow{Q_4}$ on p_{i-1}, p_i, w, v. This proves (4).

$$\text{If } D \text{ is } \overrightarrow{P_4}\text{-free, there are no arcs from } F_i^- \text{ to } F_j\text{, and no arcs from } L_i \text{ to } L_j^+. \tag{5}$$

Suppose for some $v \in F_i^-$ and $w \in F_j$ that $(v, w) \in E(D)$. Then $p_{i-1} \to p_i \to v \to w$ is an induced $\overrightarrow{P_4}$ (see the dark blue arcs in Figure 3). Hence, D is not $\overrightarrow{P_4}$-free. This proves the first part of the statement (5). The argument that there are no arcs from L_i to L_j^+ in a $\overrightarrow{P_4}$-free graph is symmetric. This proves (5).

$$\text{If } D \text{ is } \overrightarrow{A_4}\text{-free, there are no arcs from } F_i^+ \text{ to } F_j\text{, and no arcs from } L_i \text{ to } L_j^-. \tag{6}$$

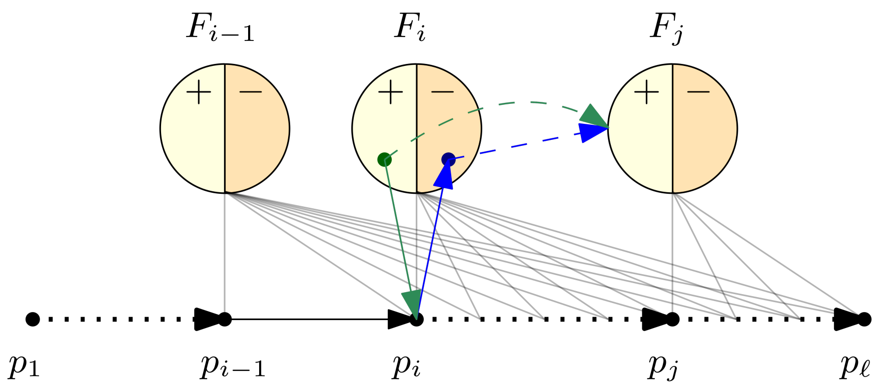

Figure 3: A depiction of $P = p_1 \to \cdots \to p_\ell$ along with the partition $(F_1, \ldots, F_\ell)$ of $N(P)$ by first attachment on P. Note that the setting is symmetric for the partition of $N(P)$ by the last attachment. Each class of the partition F_i is represented as a circle and further split into F_i^+ in yellow and F_i^- in orange, all possible arcs towards P are drawn in gray. An arc from F_i^- to F_j with $j > i$ would induce a $\overrightarrow{P_4}$ using (p_{i-1}, p_i), as highlighted in blue. An arc from F_i^+ to F_j would induce a $\overrightarrow{A_4}$, represented in green.

By symmetry, it is enough to show that if D is $\overrightarrow{A_4}$-free then there is no arc from F_i^+ to F_j. Suppose for some $v \in F_i^+$ and $w \in F_j$ that $(v, w) \in E(D)$. Then $p_{i-1} \to p_i \leftarrow v \to w$ is an induced $\overrightarrow{A_4}$ in D (see the dark green arcs in Figure 3). This proves (6). $\square$

In the Subsection 5.2 we use Observation 16 to bound the dichromatic number of $N(P) \setminus N[K]$ in the $\overrightarrow{Q_4}$-free case. In the $\overrightarrow{P_4}$-free case and the $\overrightarrow{A_4}$-free case we need to perform a more careful analysis of $N(P) \setminus N[K]$ in order to bound its dichromatic number because the conditions guaranteed by Observation 16 are weaker in these two cases. In Subsection 5.3.1, we use Observation 16 to bound the dichromatic number of the following subsets of $N(P) \setminus N[K]$

$$W^p = \left(F_2^- \cup F_3^- \cup \cdots \cup F_{\ell-1}^-\right) \cup \left(L_2^+ \cup L_3^+ \cup \cdots \cup L_{\ell-1}^+\right) \tag{7}$$

when D is $\overrightarrow{P_4}$-free and

$$W^a = \left(F_2^+ \cup F_3^+ \cup \cdots \cup F_{\ell-1}^+\right) \cup \left(L_2^- \cup L_3^- \cup \cdots \cup L_{\ell-1}^-\right) \tag{8}$$

when D is $\overrightarrow{A_4}$-free. The vertices in $N(P) \setminus (N[K] \cup W^p)$ and $N(P) \setminus (N[K] \cup W^a)$ have restrictions on how they may have neighbors in $V(P)$. We will use this to bound their dichromatic number in Subsections 5.3.2 and 5.3.3, respectively.

5.2 The $\overrightarrow{Q_4}$-free case

In this section, we bound the dichromatic number of a path-minimizing closed tournament in D when D is a $\overrightarrow{Q_4}$-free oriented graph satisfying the conditions of Scenario 5.

Lemma 17. *If D is $\overrightarrow{Q_4}$-free, $\overrightarrow{\chi}(N(P) \setminus N(\{p_1, p_\ell\})) \leqslant \gamma$.*

Proof. Assume D is a $\overrightarrow{Q_4}$-free oriented graph. By definition $N(P) \setminus N(\{p_1, p_\ell\}) \subseteq F_2 \cup F_3 \cup \cdots \cup F_{\ell-1}$. By Observation 16 every directed cycle in $D[F_2 \cup F_3 \cup \cdots \cup F_{\ell-1}]$ is completely contained in $D[F_i]$ for some $i \in [2, \ell - 1]$. By definition $F_i \subseteq N(p_i)$ so $\overrightarrow{\chi}(F_i) \leqslant \gamma$ for each $i \in [1, \ell]$. Hence, we may use the same set of γ colors for each of $F_2, F_3, \ldots, F_{\ell-1}$. Thus, $\overrightarrow{\chi}(N(P) \setminus N(\{p_1, p_\ell\}) \leqslant \gamma$. $\square$

Lemma 17 allows us to demonstrate a bound on our dipolar set $N[C \cup X]$ as follows:

Lemma 18. *Let D be $\overrightarrow{Q_4}$-free. Then D has a dipolar set with dichromatic number at most $(\omega(D) + 3) \cdot \gamma + 2$*

Proof. Recall $C = K \cup P$ is a path-minimizing closed tournament of D and p_1, p_ℓ denote the ends of P. By definition, $p_1, p_\ell \in K$. By Lemma 9, $N[C \cup X]$ is a dipolar set. Then, by inequality (3), bounding the terms on the right-hand side by Observation 6, Observation 11, Lemma 17, and Corollary 14 we obtain:

$$\overrightarrow{\chi}(N[C \cup X]) \leqslant +(\omega(D) + 3) \cdot \gamma + 2. \qquad \square$$

5.3 The $\overrightarrow{P_4}$-free case and the $\overrightarrow{A_4}$-free case

In this subsection, we bound the dichromatic number of our dipolar set in the case where D is $\overrightarrow{P_4}$-free or $\overrightarrow{A_4}$-free.

5.3.1 Bounding $\vec{\chi}(W^p)$ and $\vec{\chi}(W^a)$

In this subsection, we bound the dichromatic number of W^p and W^a using Observation 16 in $\overrightarrow{P_4}$-free and $\overrightarrow{A_4}$-free oriented graphs, respectively.

Lemma 19. *If D is $\overrightarrow{P_4}$-free, then $\vec{\chi}(W^p) \leqslant 2\gamma$. Similarly, if D is $\overrightarrow{A_4}$-free, then $\vec{\chi}(W^a) \leqslant 2\gamma$.*

Proof. We begin by proving the first statement. Suppose D is $\overrightarrow{P_4}$-free. Then by Observation 16, every directed cycle in $D[F_2^- \cup F_3^- \cup \cdots \cup F_{\ell-1}^-]$ is completely contained in $D[F_i^-]$ for some $i \in [2, \ell-1]$. By assumption, $\vec{\chi}(N(p_i)) \leqslant \gamma$ for every $p_i \in P$. Hence, we may use the same set of γ colors for each of $F_2^-, F_3^-, \ldots, F_{\ell-1}^-$. So $\vec{\chi}(F_2^- \cup F_3^- \cup \cdots \cup F_{\ell-1}^-) \leqslant \gamma$. By symmetry, $\vec{\chi}(L_2^+ \cup L_2^+ \cup \cdots \cup L_{\ell-1}^+) \leqslant \gamma$. Therefore, since W^p is the union of these two sets, we obtain $\vec{\chi}(W^p) \leqslant 2\gamma$.

The case is symmetric when D is $\overrightarrow{A_4}$-free. The third item of Observation 16 allows us to use the same set of colors for each of $F_2^+, F_3^+, \ldots, F_{\ell-1}^+$, and the same set of colors for each of $L_2^-, L_3^-, \ldots, L_{\ell-1}^-$. Hence, $\vec{\chi}(F_2^+ \cup F_3^+ \cup \cdots \cup F_{\ell-1}^+) \leqslant \gamma$ and $\vec{\chi}(L_2^- \cup L_3^- \cup \cdots \cup L_{\ell-1}^-) \leqslant \gamma$. Since W^a is the union of these two sets, $\vec{\chi}(W^a) \leqslant 2\gamma$. $\square$

5.3.2 Completing the bound on the dichromatic number of our dipolar set in the $\overrightarrow{P_4}$-free case

Recall, we keep the definitions of D, C, K, P, X, Y and H, γ as fixed in Definition 10. We will assume $H = \overrightarrow{P_4}$ for the remainder Subsubsection 5.3.2. In this section, we will consider the dichromatic number of the following set of vertices.

Definition 20. We let

$$R^p = N(P) \setminus (N(\{p_1, p_2, p_\ell\}) \cup W^p)$$

Then, $N(P) \setminus N[K] \subseteq W^p \cup R^p \cup N(p_2)$ and we can bound $\vec{\chi}(W^p)$ in terms of $\omega(D)$ and γ. Since $\omega(D) = \omega$, it follows that $\omega(N(v)) < \omega$ for each $v \in V(D)$. Hence, $\vec{\chi}(N(p_2)) \leqslant \gamma$. Thus, by Lemma 9 and Corollary 14, we only need to bound $\vec{\chi}(R^p)$ in terms of ω and γ in order to demonstrate that D is a dipolar set of bounded dichromatic number.

By definition, W^p is the set of vertices in $N(P) \setminus N(\{p_1, p_\ell\})$ whose first neighbor on P is an in-neighbor or whose last neighbor in $V(P)$ is an out-neighbor. Hence, R^p consists exactly of the vertices in $N(P) \setminus N(\{p_1, p_2, p_\ell\})$ whose first neighbor in $V(P)$ is an out-neighbor and whose last neighbor in $V(P)$ is an in-neighbor.

We will show that since $C = K \cup V(P)$ is a *path-minimizing* closed tournament there is no tournament of order ω in R^p and thus $\vec{\chi}(R^p) \leqslant \gamma$. In particular, we will show that for a contradiction, if R^p has a tournament J of order ω, then we can find a directed path

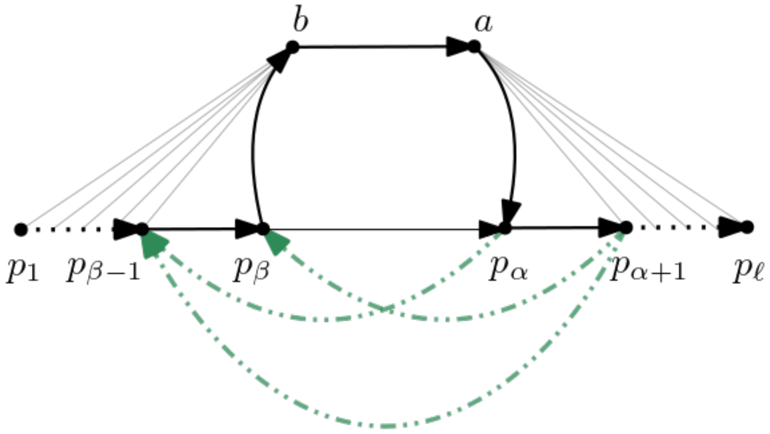

Figure 4: On the bottom, P and an arc (b, a) between neighbors of P in R^p. Illustrated here is the case where the last neighbor p_β of b on P appears just before the first neighbor p_α of a, all possible arcs are shown in gray. Then, path $p_{\beta-1}, p_\beta, p_\alpha, p_{\alpha+1}$ cannot induce a $\overrightarrow{P_4}$. Any arc possibly preventing this, shown in dash-dotted green, yields a path from a to b of length at most five.

P' that is *shorter than* P such that J and P' form a closed tournament. In order to prove this, we will need the following lemma, which will allow us to exhibit a relatively short path between two adjacent vertices in R^p. Note that the following lemma does not use any property of P other than that it is forward-induced.

Lemma 21. *Let $a, b \in R^p$, if $(b, a) \in E(D)$, there is a directed path from a to b on at most* $\max\{6, \ell - 1\}$ *vertices.*

Proof. Let $\alpha, \beta \in \{1, 2, \ldots, \ell\}$ such that p_α is the *first* neighbor of a in $V(P)$ and p_β is the *last* neighbor of b in $V(P)$. Then, since $a, b \notin \cup_{i=1}^{\ell} F_i^- \cup L_i^+$, the corresponding arcs are $(a, p_\alpha), (p_\beta, b) \in E(D)$. By definition of R^p, we obtain that $3 \leqslant \alpha, \beta \leqslant \ell - 1$. Hence, we may assume that $\beta < \alpha$, for otherwise $a \to p_\alpha \to p_{\alpha+1} \to \cdots \to p_\beta \to b$ is a directed path from a to b with at most $\ell - 1$ vertices, as desired.

Since $\beta < \alpha$ the vertices a, b have no common neighbors in $V(P)$. Now, consider the directed path $p_\beta \to b \to a \to p_\alpha$. Since D is $\overrightarrow{P_4}$-free it cannot be induced. Thus, $(p_\beta, p_\alpha) \in E(D)$ or $(p_\alpha, p_\beta) \in E(D)$.

Suppose that $(p_\alpha, p_\beta) \in E(D)$. Then, $v \to p_\alpha \to p_\beta \to b$ is a directed path from a to b of length three, as desired. Hence, we may assume that $(p_\beta, p_\alpha) \in E(D)$.

Since P is a forward-induced directed path and $\beta < \alpha$, it follows that $\alpha = \beta + 1$. Consider the directed path $p_{\beta-1} \to p_\beta \to p_\alpha \to p_{\alpha+1}$. Since D is $\overrightarrow{P_4}$-free it cannot be induced. Therefore, the vertices $p_{\beta-1}$ and p_α, the vertices p_β and $p_{\alpha+1}$, or the vertices $p_{\beta-1}$ and $p_{\alpha+1}$ are adjacent. Furthermore, since P is a shortest path, this means that at least one of $(p_{\alpha+1}, p_{\beta-1})$, $(p_{\alpha+1}, p_\beta)$,$(p_\alpha, p_{\beta-1})$ is an arc of D, see Figure 4. We consider each case separately:

- Suppose $(p_\alpha, p_{\beta-1})$ is an arc of D. Then $a \to p_\alpha \to p_{\beta-1} \to p_\beta \to b$ is a path of D.
- Suppose $(p_{\alpha+1}, p_\beta)$ is an arc of D. Then $a \to p_\alpha \to p_{\alpha+1} \to p_\beta \to b$ is a path of D.
- Suppose $(p_{\alpha+1}, p_{\beta-1})$ is an arc of D. Then $a \to p_\alpha \to p_{\alpha+1} \to p_{\beta-1} \to p_\beta \to b$ is a path of D.

In every case, the oriented graph induced by $\{a, p_\alpha, p_{\alpha+1}, p_{\beta-1}, p_\beta, b\}$ contains a directed path from a to b on at most six vertices. Since one of the cases must hold, this completes the proof. □

With the last lemma in hand, we are ready to bind the dichromatic number of R^p.

Lemma 22. *Suppose D is $\overrightarrow{P_4}$-free. Then D contains a dipolar set of dichromatic number at most $(\omega + 6) \cdot \gamma + 2$.*

Proof. Let K be a maximum tournament and P be a directed path in D such that K and P form a path-minimizing closed tournament C in D. Then by Lemma 9, $N[C \cup X]$ is a dipolar set. We will use Lemma 21 to bound $\overrightarrow{\chi}(R^p)$. Then we will combine this bound with the results from the previous sections to bound $\overrightarrow{\chi}(N[C \cup X])$.

$$\textit{If } P \textit{ contains at least 7 vertices, then } \overrightarrow{\chi}(R^p) \leqslant \gamma. \tag{9}$$

Suppose $\ell \geqslant 7$. If $\omega(R^p) < \omega(D)$ then by assumption $\overrightarrow{\chi}(R^p) \leqslant \gamma$, so we may assume that there is an $\omega(D)$-tournament $J \subseteq R$. Since C is a minimum closed tournament and P is non-empty, J is not strongly connected. Hence, there must be exactly one strongly connected component of J that is a sink and exactly one strongly connected component of J that is a source (and they are not equal). Let v be a vertex in the sink component of J and w be a vertex in the source component of J. Therefore, $(w, v) \in E(D)$. Thus, by Lemma 21 there is a path Q from v to w of length less than that of P. Hence, J, P' form a closed tournament. By definition since K, P were chosen to form a path-minimizing closed tournament P' cannot be shorter than P, a contradiction. This proves (9).

$$\overrightarrow{\chi}(N(P) \setminus N[K]) \leqslant 4\gamma \tag{10}$$

Let the vertices of P be $p_1 \to p_2 \to \ldots \to p_\ell$, in order. By definition, $p_1, p_\ell \in K$. If $\ell \leqslant 6$, then by Observation 6, $\overrightarrow{\chi}(N(P) \setminus N[K]) \leqslant 4\gamma$, as desired. Hence, we may assume this is not the case.

Let W^p, R^p be defined with respect to P. Then,

$$\overrightarrow{\chi}(N(P) \setminus N[K]) \leqslant \overrightarrow{\chi}(W^p) + \overrightarrow{\chi}(R^p) + \overrightarrow{\chi}(N(p_2)).$$

Hence, by Lemma 19, Observation 6, and (9) we obtain $\overrightarrow{\chi}(N(P) \setminus N[K]) \leqslant 4\gamma$. This proves (10).

Then, by inequality (3), bounding the terms on the right-hand side by Observation 6, Observation 11, inequality (10), and Corollary 14, we have

$$\overrightarrow{\chi}(N[C \cup X]) \leqslant \overrightarrow{\chi}(N(P) \setminus N[K]) + (\omega + 2) \cdot \gamma + 2 \leqslant (\omega + 6) \cdot \gamma + 2. \qquad \square$$

5.3.3 Completing the bound on the dichromatic number of our dipolar set in the $\overrightarrow{A_4}$-free case

In this section, we prove a bound on the remaining vertices of $N(P) \setminus N[K]$ and use the results of the previous sections to show that D contains a dipolar set of bounded dichromatic number in the $\overrightarrow{A_4}$-free case.

Definition 23. Let P be a shortest path with vertices $p_1 \to p_2 \to \cdots \to p_\ell$, in order. Then

$$R^a = N(P) \setminus (N(\{p_1, p_\ell\}) \cup W^a).$$

Recall $W^a = \cup_{i=2}^{\ell-1} F_i^+ \cup L_i^-$, that is, vertices in $N(P) \setminus N(\{p_1, p_\ell\})$ whose first neighbor on P is an out-neighbor or whose last neighbor in $V(P)$ is an in-neighbor. Hence, R^a consists exactly of the vertices in $N(P) \setminus N(\{p_1, p_\ell\})$ whose first neighbor in $V(P)$ is an in-neighbor and whose last neighbor in $V(P)$ is an out-neighbor.

We bound $\overrightarrow{\chi}(R^a)$ using a similar technique to the one we used to bound the dichromatic number of W^a. Recall that we need to bound the dichromatic number of the union of the sets $F_i \backslash W^a$. To this end, we prove that there are no arcs between these sets with indices differing by more than three. We first make the following observation, holding for any shortest directed path.

Observation 24. *Let D be an $\overrightarrow{A_4}$-free oriented graph. Let $P = p_1 \to p_2 \to \ldots \to p_\ell$ be a shortest directed path from p_1 to p_ℓ in D. Let $i, j \in [1, \ell]$ with $j > i + 3$. Suppose $v, w \in N(P)$ such that $(p_i, v), (w, p_j) \in E(D)$. Then $(v, w) \notin E(D)$.*

Proof. If $(v, w) \in E(D)$ then we may replace the path $p_i \to p_{i+1} \to p_{i+2} \to p_{i+3} \to \cdots \to p_j$ with the path $p_i \to v \to w \to p_j$ in P to obtain a shorter directed path from p_1 to p_ℓ, a contradiction. □

The previous observation allows us to prove that some arcs between vertices in R^a are forbidden.

Observation 25. *Let D be an $\overrightarrow{A_4}$-free oriented graph. Then, for any two integers in $i, j \in [2, \ell - 1]$ satisfying $i + 2 < j$ there is no arc from a vertex in $F_i \setminus W^a$ to a vertex in $F_j \setminus W^a$.*

Proof. Let i, j be integers $i, j \in [2, \ell - 1]$ satisfying $i + 2 < j$. Suppose $v \in F_i \setminus W^a$ and $w \in F_j \setminus W^a$. By definition of W^a, for every $r \in N(P) \setminus W^a$, the first neighbor of r in $V(P)$ is an in-neighbor of r and the last neighbor of $r \in N(P)$ is an out-neighbor of r. Hence, there is some $x \in [j+1, \ell-1]$ such that p_x is an out-neighbor of w. Thus, $x > i+3$ and by Observation 24, $(v, w) \notin E(D)$, see Figure 5. □

With the last observation in hand, we are ready to bound $\overrightarrow{\chi}(R^a)$, which we do through a similar argument to the case of W^a. The main difference is that here, we use 3 disjoint pallets of γ colors each and choose a color palette for $F_i \backslash W^a$ according to the index of i modulo 3 (where $i \in [2, \ell - 1]$).

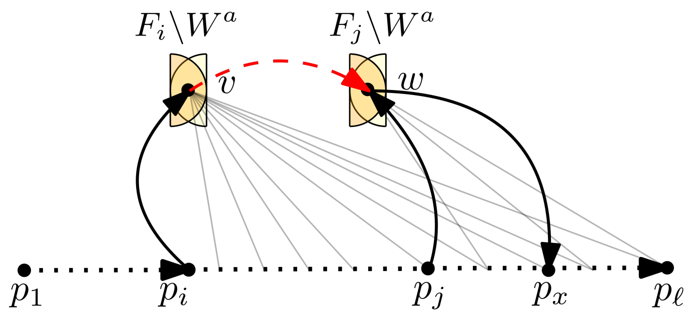

Figure 5: A depiction of P, along with $v \in F_i \setminus W^a$ and $w \in F_j \setminus W^a$ of R^a. Since vertex $w \notin W^a$, it must also belong to some L_x^+ with $x > j$, meaning its last neighbor on P is an out-neighbor. Then, since $x > j > i + 2$, an arc (v, w) would yield a shorter path from p_1 to p_ℓ.

Lemma 26. *If D is $\overrightarrow{A_4}$-free, then $\overrightarrow{\chi}(R^a)) \leqslant 3\gamma$.*

Proof. By definition $\overrightarrow{\chi}(F_i) \leqslant \gamma$ for each $i \in [1, \ell]$. We fix three disjoint sets S_0, S_1, S_2 of γ colors and dicolor each $F_j \setminus W^a$ for $j \in [2, \ell - 1]$ with set S_i where $i = j \mod 3$. By Observation 25, $D[F_2 \cup F_3 \cup \cdots \cup F_{\ell-1} \setminus W^a]$ does not contain any monochromatic directed cycle. Hence, $\overrightarrow{\chi}(F_2 \cup F_3 \cup \cdots \cup F_{\ell-1} \setminus W^p) \leqslant 3\gamma$, as desired. □

We combine the previous observation with the results of the previous sections to show that $\overrightarrow{A_4}$-free oriented graphs have a dipolar set of bounded dichromatic number.

Lemma 27. *Suppose D is an oriented graph satisfying Scenario 5 with $H = \overrightarrow{A_4}$. Then D contains a dipolar set of dichromatic number at most $(\omega + 7) \cdot \gamma + 2$.*

Proof. By Lemma 9, $N[C \cup X]$ is a dipolar set. Now, $N(P) \setminus N[K] \subseteq W^a \cup R^a$. Thus, by combining Lemmas 19 and 26, we obtain $\overrightarrow{\chi}(N(P) \setminus N[K]) \leqslant 5\gamma$. Then, by combining this bound with inequality (3), Observation 6, Observation 11, and Corollary 14 we obtain:

$$\overrightarrow{\chi}(N[C \cup X]) \leqslant (\omega(D) + 7) \cdot \gamma + 2.$$

□

6 Orientations of P_4 are $\overrightarrow{\chi}$-bounding

In this section, we consider an oriented graph D satisfying Scenario 5. The previous sections show that D has a dipolar set of bounded dichromatic number. We will use this result and Lemma 8 to show that oriented graphs not containing some orientation of P_4 are $\overrightarrow{\chi}$-bounded.

6.1 D contains a dipolar set with bounded dichromatic number

In the previous sections, it is proved that if D does not contain some $H \in \{\overrightarrow{Q_4}, \overrightarrow{P_4}, \overrightarrow{A_4}\}$, then D has a dipolar set of bounded dichromatic number. These results can be summarized in the following lemma.

Lemma 28. *Let $\omega > 1$ be an integer. Let $H \in \{\overrightarrow{Q_4}, \overrightarrow{P_4}, \overrightarrow{A_4}\}$. Let γ be the maximum value of $\overrightarrow{\chi}(D')$ for any H-free oriented graph D' with $\omega(D') < \omega$. Let D be a strongly connected H-free oriented graph with clique number ω. Then,*

- *If $H = \overrightarrow{Q_4}$, then D contains a dipolar set with dichromatic number at most $(\omega + 3) \cdot \gamma + 2$.*
- *If $H = \overrightarrow{P_4}$, then D contains a dipolar set with dichromatic number at most $(\omega + 6) \cdot \gamma + 2$.*
- *If $H = \overrightarrow{A_4}$, then D contains a dipolar set with dichromatic number at most $(\omega + 7) \cdot \gamma + 2$.*

Proof. The result for $H = \overrightarrow{Q_4}, \overrightarrow{P_4}, \overrightarrow{A_4}$ is given in Lemmas 18, 22 and 27, respectively. □

6.2 Computing the $\overrightarrow{\chi}$-binding function

We will show that an element from the following family of functions is a $\overrightarrow{\chi}$-binding function for any class of oriented graphs forbidding a particular orientation of P_4.

Definition 29. For any integer $c \geqslant 3$ we let

$$f_c(x) = 2^x(x+c)! + \sum_{i=0}^{x} \frac{2^{i+2}(x+c)!}{(x+c-i)!}$$

for any non-negative integer x.

We will need that f_c satisfies the following recursive properties in order to show that for some c the function f_c is χ-bounding for any class of oriented graph forbidding a particular orientation of P_4.

Observation 30. *Let $c \geqslant 3$. Then:*

- *$f_c(x) = 2(x+c)f_c(x-1) + 4$ for any integer $x \geqslant 2$, and*
- *$f_c(1) > 1$,*
- *$f_c(x) \leqslant (x+c)^{x+c+1.5}$ for any integer $x \geqslant 1$.*

Proof. By definition since $c \geqslant 3$ we obtain that $f_c(1) > 2c! > 1$. Hence, the second bullet holds, and we will now prove the first bullet. Let $x \geqslant 2$ be an integer. Then,

$$f_c(x) = 2(x+c)\left(2^{x-1}(x+c-1)! + \frac{1}{2(x+c)}\sum_{i=0}^{x} \frac{2^{i+2}(x+c)!}{(x+c-i)!}\right).$$

By definition,

$$\sum_{i=0}^{x} \frac{2^{i+2}(x+c)!}{(x+c-i)!} = 2(x+c)\left(\sum_{i=1}^{x} \frac{2^{i+1}(x+c-1)!}{(x+c-i)!}\right) + 4 = 2(x+c)\left(\sum_{i=0}^{x-1} \frac{2^{i+2}(x+c-1)!}{(x+c-i-1)!}\right) + 4.$$

Thus, by combining the previous two equations we obtain that $f_c(x) = 2(x+c)f_c(x-1)+4$. This proves the first bullet.

We will complete the proof by showing the third bullet holds. Let $x \geqslant 1$. By definition,

$$\sum_{i=0}^{x} \frac{2^{i+2}(x+c)!}{(x+c-i)!} = 2^2 + 2^3(x+c) + 2^4(x+c)(x+c-1) + \cdots + 2^{x+2}\frac{(x+c)!}{c!}.$$

Since $c! > 4$ every $i \in [0,x]$ satisfies $\frac{2^{i+2}(x+c)!}{(x+c-i)!} \leqslant 2^x(x+c)!$. Hence, we obtain

$$\sum_{i=0}^{x} \frac{2^{i+2}(x+c)!}{(x+c-i)!} \leqslant 2^x(x+1)(x+c)!.$$

$$f_c(x) < 2^x(x+c)! + 2^x(x+1)(x+c)! \leqslant (x+2)2^x(x+c)!. \tag{11}$$

We will use the following well-known equation called *Stirling's Formula* to complete the proof.

Every $n \geqslant 1$ *satisfies*

$$\left(n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}\right). \tag{12}$$

Since $c \geqslant 3$ we obtain the following by combining (11) and (12).

$$(x+2)2^x(x+c)! < (x+2)2^x\sqrt{2\pi(x+c)}\left(\frac{(x+c)}{e}\right)^{x+c} e^{\frac{1}{12(x+c)}} < (x+c)^{x+c+1.5}.$$

This proves the third bullet. □

6.3 $\overrightarrow{\chi}$-boundedness

We are now ready to prove the following more precise version of our main result, Theorem 3.

Theorem 31. *Let H be an orientation of P_4, then H-free graphs are $\overrightarrow{\chi}$-bounded. Specifically,*

- *If D is $\overrightarrow{Q_4}$-free or $\overleftarrow{Q_4}$-free, then $\overrightarrow{\chi}(D) \leqslant (\omega(D)+3)^{\omega(D)+4.5}$,*
- *If D is $\overrightarrow{P_4}$-free, then $\overrightarrow{\chi}(D) \leqslant (\omega(D)+6)^{\omega(D)+7.5}$, and*
- *If D is $\overrightarrow{A_4}$-free, then $\overrightarrow{\chi}(D) \leqslant (\omega(D)+7)^{\omega(D)+8.5}$.*

Proof. Let H be an orientation of P_4 Note $\overrightarrow{Q_4}$ can be obtained from $\overrightarrow{Q_4}'$ by reversing the orientation of every edge. Hence, the theorem holds for $\overrightarrow{Q_4}$ if and only if it holds for $\overrightarrow{Q_4'}$. Thus, may assume $H \in \{\overrightarrow{Q_4}, \overrightarrow{P_4}, \overrightarrow{A_4}\}$.

We let $c = 3$ if $H = \overrightarrow{Q_4}$, $c = 6$ if $H = \overrightarrow{P_4}$ and $c = 7$ if $H = \overrightarrow{A_4}$. Then by the third bullet of Observation 30, it is enough to show that the class of H-free oriented graphs is $\overrightarrow{\chi}$-bounded by f_c.

We have $f_c(1) > 1$ by the first bullet of Observation 30, so the statement holds for oriented graphs with no arcs. We complete the proof by induction on the clique number. Let $\omega > 1$ be an integer. Suppose every H-free oriented graph D' with clique number less than ω satisfies $\overrightarrow{\chi}(D') \leqslant f(\omega(D'))$. Let D be an H-free oriented graph with clique number equal to ω. We will show $\overrightarrow{\chi}(D) \leqslant f_c(\omega)$. We may assume by induction on the number of vertices that D is strongly connected.

By Lemma 28, D has a dipolar set S with $\overrightarrow{\chi}(S) \leqslant (\omega + c) \cdot f_c(\omega - 1) + 2$. Then by Lemma 8,

$$\overrightarrow{\chi}(D) \leqslant 2 \cdot \overrightarrow{\chi}(S) \leqslant 2(\omega + c) \cdot f_c(\omega - 1) + 4.$$

Since $\omega \geqslant 2$ this implies $\overrightarrow{\chi}(D) \leqslant f_c(\omega)$ by the second bullet of Observation 30. This completes the proof. $\square$

7 Conclusion

Our result is an initial step towards resolving the ACN $\overrightarrow{\chi}$-boundedness conjecture for orientation of paths in general. However, we think we are still far from this result. Our construction of a dipolar set with bounded chromatic number relies heavily on the length of P_4, and we do not expect that our techniques can be directly extended to show that any oriented P_t for $t \geqslant 5$ is $\overrightarrow{\chi}$-bounding. It would already be interesting to hear the answer to the easier question: Does there exists an integer $t \geqslant 5$ and an orientation H of P_t such that the class of oriented graphs forbidding H and all tournaments of size 3 has unbounded dichromatic number? (Note in print: Since submitting our paper, Aboulker, Aubian, Charbit, and Thomassé showed that $\overrightarrow{P_6}$-free oriented graphs with clique number at most two have bounded dichromatic number [2].)

Recall that the classes of $\overrightarrow{Q_4}$-free oriented graphs and $\overrightarrow{Q_4}'$-free oriented graphs were already shown to be χ-bounded in [10]. The χ-binding function f' for these two classes from [10] is defined using recurrence

$$f'(x) := 2(3f'(x-1))^5$$

which leads to a double-exponential bound on χ, and cannot guarantee a better bound on $\overrightarrow{\chi}$. In this paper, Theorem 3 provides an improved $\overrightarrow{\chi}$-binding function when any orientation of P_4 is forbidden. It would interest us to know of any improvements to the $\overrightarrow{\chi}$ function. In particular, we would like to know whether any orientation of P_4 is *polynomially* $\overrightarrow{\chi}$-bounding. In other words, is there some oriented P_4 so that the class of oriented graphs forbidding it has a *polynomial* $\overrightarrow{\chi}$-binding function?

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