Edge Separators for Graphs Excluding a Minor

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Abstract

We prove that every n-vertex K_t-minor-free graph G of maximum degree Δ has a set F of $O(t^2(\log t)^{1/4}\sqrt{\Delta n})$ edges such that every component of $G - F$ has at most $n/2$ vertices. This is best possible up to the dependency on t and extends earlier results of Diks, Djidjev, Sýkora, and Vrťo (1993) for planar graphs, and of Sýkora and Vrťo (1993) for bounded-genus graphs. Our result is a consequence of the following more general result: The line graph of G is isomorphic to a subgraph of the strong product $H \boxtimes K_{\lfloor p \rfloor}$ for some graph H with treewidth at most $t-2$ and $p = \sqrt{(t-3)\Delta|E(G)|} + \Delta$.

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1 Introduction

A *balanced vertex separator* of an n-vertex graph G is a set $X \subseteq V(G)$ such that every component of $G - X$ has at most $n/2$ vertices.[1] The well-known Planar Separator Theorem by Lipton and Tarjan [7] states that every n-vertex planar graph has a balanced vertex separator of size $O(\sqrt{n})$. Alon, Seymour, and Thomas [1] showed that every n-vertex K_t-minor-free graph has a balanced vertex separator of size at most $t^{3/2}\sqrt{n}$.

In this paper, we study balanced edge separators. A *balanced edge separator* of an n-vertex graph G is a set $F \subseteq E(G)$ such that every component of $G - F$ has at most $n/2$ vertices. The aforementioned classes of graphs with balanced vertex separators of size $O(\sqrt{n})$ do not admit balanced edge separators of size $o(n)$; indeed, the smallest balanced edge separator of the n-vertex star $K_{1,n-1}$ has size $\lceil n/2 \rceil$.

The star, however, has a vertex of degree $n - 1$. If we assume that the maximum degree Δ of G is sublinear in n, then in some cases we can retrieve sublinear balanced edge separators. Diks, Djidjev, Sýkora, and Vrťo [3] showed that if G is planar, then G

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[1] Note that it implies that the components of $G - X$ can be grouped into two sets each with at most $2n/3$ vertices; this is sometimes used as the definition of 'balanced vertex separator' in the literature.

has a balanced edge separator of size $O(\sqrt{\Delta n})$, and Sýkora and Vrťo [8] showed that if the Euler genus of G is g, then there exists a balanced edge separator of size $O(\sqrt{g\Delta n})$. These results are true also in the weighted setting where each vertex $x \in V(G)$ is assigned a weight $w(x)$ with $0 \leqslant w(x) \leqslant \frac{1}{2}$, the total weight of all vertices is 1, and the edge separator should split G into components of weight at most $\frac{1}{2}$.

Lasoń and Sulkowska [6] showed that in the weighted setting, if G is an n-vertex K_t-minor-free graph of maximum degree $\Delta = o(n)$ and the vertices are weighted proportionally to their degrees, then there exists a balanced edge separator of size $o(n)$. Their proof relies on spectral methods—more precisely, on an upper bound for the second smallest eigenvalue of the Laplacian matrix of K_t-minor-free graphs due to Biswal, Lee, and Rao [2]—and only works for these specific weights. They asked if one can always find a balanced edge separator of size $O_t(\sqrt{\Delta n})$ for any weights (including uniform weights), as is the case for planar graphs and graphs of bounded genus. In this paper, we give an affirmative answer to this question.

Theorem 1. *Let $t \geqslant 3$, let G be an n-vertex K_t-minor-free graph of maximum degree Δ, and let $w\colon V(G) \to [0, \frac{1}{2}]$ be a weight function such that $\sum_{x\in V(G)} w(x) = 1$. Then there exists a set $F \subseteq E(G)$ with*

$$|F| \leqslant (t-1)\left\lfloor \sqrt{(t-3)|E(G)|\Delta} + \Delta \right\rfloor \in O\left(t^2(\log t)^{1/4} \cdot \sqrt{\Delta n}\right)$$

such that $\sum_{x\in V(C)} w(x) \leqslant \frac{1}{2}$ for each component C of $G - F$.

This result is best possible up to dependency on t: Sýkora and Vrťo [8] showed that there exist n-vertex planar graphs G of maximum degree Δ such that every balanced edge separator has size $\Omega(\sqrt{\Delta n})$.

We actually prove the following stronger result.

Theorem 2. *Let $t \geqslant 3$ and let G be a K_t-minor-free graph of maximum degree Δ with m edges. Then the line graph of G is isomorphic to a subgraph of the strong product $H \boxtimes K_{\lfloor p\rfloor}$ for some graph H with $\operatorname{tw}(H) \leqslant t-2$ and $p = \sqrt{(t-3)\Delta m} + \Delta$.*

The *strong product $H \boxtimes K$* of graphs H and K is a graph on $V(H) \times V(K)$ where (x_1, y_1) and (x_2, y_2) are adjacent if $x_1 = x_2$ and $y_1y_2 \in E(K)$, or $x_1x_2 \in E(H)$ and $y_1 = y_2$, or $x_1x_2 \in E(H)$ and $y_1y_2 \in E(K)$. (When K is a complete graph on p vertices, taking the strong product of H with K amounts to 'blowing up' each vertex of H by a clique of size p.) Theorem 2 directly implies the following upper bound on the treewidth of the line graph of G.

Theorem 3. *Let $t \geqslant 3$, and let G be a K_t-minor free graph of maximum degree Δ with m edges. Then the line graph of G has treewidth at most $(t-1)\lfloor\sqrt{(t-3)\Delta m} + \Delta\rfloor - 1$.*

Theorem 1 then follows from Theorem 3 by a simple argument on the tree-decomposition provided by the latter theorem (see Section 3).

Theorem 2 can be thought of as an 'edge version' of the following recent strengthening of the balanced vertex separator result by Alon, Seymour and Thomas [1] due to

Illingworth, Scott and Wood [4]: Every n-vertex K_t-minor free graph is isomorphic to a subgraph of the strong product $H \boxtimes K_{\lfloor p \rfloor}$ where $\mathrm{tw}(H) \leqslant t-2$ and $p = 2\sqrt{(t-3)n}$. The authors of [4] established their result by modifying the proof in [1]. Our proof of Theorem 2 is likewise a modification of the proof in [1] and relies heavily on the insights from [4]; the main work consists in adapting to the edge setting.

One consequence of Theorem 1 is an upper bound for the isoperimetric number (a.k.a. edge expansion or Cheeger constant) of K_t-minor-free graphs. The *isoperimetric number* $\phi(G)$ of a graph G is defined as

$$\phi(G) = \min\left\{\frac{|\{xy \in E(G) : x \in S, y \notin S\}|}{|S|} : S \subseteq V(G), 1 \leqslant |S| \leqslant |V(G)|/2\right\}.$$

Corollary 4. *For $t \geqslant 3$, every n-vertex K_t-minor-free graph G with maximum degree Δ satisfies*

$$\phi(G) = O\left(t^2(\log t)^{1/4} \cdot \sqrt{\tfrac{\Delta}{n}}\right)$$

Proof. Let F be a balanced edge separator of G with $|F| = O(t^2(\log t)^{1/4} \cdot \sqrt{\Delta n})$ given by Theorem 1. Since each component of $G - F$ has at most $n/2$ vertices, we may choose a subset of these components so that the union S of their vertex sets satisfies $\frac{1}{3}n \leqslant |S| \leqslant \frac{1}{2}n$. It follows

$$\phi(G) \leqslant \frac{|\{xy \in E(G) : x \in S, y \notin S\}|}{|S|} \leqslant \frac{|F|}{n/3} = O\left(t^2(\log t)^{1/4} \cdot \sqrt{\frac{\Delta}{n}}\right). \qquad \square$$

The bound in Corollary 4 is best possible up to the dependence on t, and extends previous bounds for planar graphs [3] and bounded-genus graphs [8].

Our proofs are constructive, and in particular, there exists a polynomial time algorithm, which given a graph G and an integer t, outputs a set F as in Theorem 1 and a graph H as in Theorem 2 together with an isomorphism between the line graph of G and a subgraph of $H \boxtimes K_{\lfloor p \rfloor}$ where $p = \sqrt{(t-3)\Delta m} + \Delta$.

In Section 2 we introduce all the necessary definitions, notations and preliminary results, and in Section 3 we prove Theorems 1, 2 and 3.

2 Preliminaries

We consider simple finite undirected graphs G with vertex set $V(G)$ and edge set $E(G)$. For subsets $X, Y \subseteq V(G)$, we denote by $E_G(X, Y)$ the set of all edges $xy \in E(G)$ with $x \in X$ and $y \in Y$. We denote by $N_G(X)$ the open neighborhood of a set $X \subseteq V(G)$, i.e. the set of all vertices outside X that are adjacent to at least one vertex from X. We drop the subscripts from the notations $E_G(X, Y)$ and $N_G(X)$ when the graph G is clear from the context.

A set of vertices $U \subseteq V(G)$ is *connected* in a graph G if the induced subgraph $G[U]$ is connected. For $t \geqslant 1$, a *K_t-model* in G is a sequence $(U_1, \ldots, U_t)$ of pairwise disjoint connected subsets of $V(G)$ such that $E(U_i, U_j) \neq \emptyset$ for distinct $i, j \in \{1, \dots, t\}$. Note that G contains K_t as a minor if and only if G has a K_t-model.

A *tree-decomposition* of a graph G is a family $(B_u)_{u \in V(T)}$ of subsets of $V(G)$ called *bags*, indexed by nodes of a tree T, such that

- $V(G) = \bigcup_{u \in V(T)} B_u$,
- for every $xy \in E(G)$ there exists $u \in V(T)$ with $\{x, y\} \subseteq B_u$, and
- For all $u_1, u_2, u_3 \in V(T)$ such that u_2 lies on the path between u_1 and u_3 in T, we have $B_{u_1} \cap B_{u_3} \subseteq B_{u_2}$.

The *width* of $(B_u)_{u \in V(T)}$ is $\max\{|B_u| - 1 : u \in V(T)\}$. The *treewidth* of a graph G, denoted $\operatorname{tw}(G)$, is the minimum width of its tree-decompositions. The following fact summarizes some simple properties of tree-decompositions that we use in our proof.

Fact 5. *Given a graph G with a tree-decomposition $(B_u)_{u \in V(T)}$ of width at most k, the following properties hold:*

(i) For every clique X in G there exists $u \in V(T)$ with $X \subseteq B_u$.

(ii) For every connected set $U \subseteq V(G)$, the set $\{u \in V(T) : B_u \cap U \neq \emptyset\}$ is connected in T.

(iii) For every graph G' such that $\operatorname{tw}(G') \leqslant k$ and $G \cap G'$ is a (possibly empty) complete graph, we have $\operatorname{tw}(G \cup G') \leqslant k$.

(iv) Every graph G' obtained from G by adding a new vertex adjacent to a clique of G of size at most k has treewidth at most k.

The *line graph $L(G)$* of a graph G is a graph whose vertex set is $E(G)$ and in which distinct edges $e, e' \in E(G)$ are adjacent if they share a common end in G.

Given a graph G, a *graph-partition* of G is a graph H such that the vertex set of H is a partition of $V(G)$ into nonempty parts, and for all distinct $X, Y \in V(H)$ we have $XY \in E(H)$ if $E_G(X, Y) \neq \emptyset$. (Note that XY is allowed to be an edge of H even if $E_G(X, Y) = \emptyset$) For an integer k and a real p, we call a graph-partition H of G a *(k, p)-partition* if $\operatorname{tw}(H) \leqslant k$ and $|X| \leqslant p$ for each $X \in V(H)$. Observe that if a graph G has a (k, p)-partition then G is isomorphic to a subgraph of $H \boxtimes K_{\lfloor p \rfloor}$ for some graph H with $\operatorname{tw}(H) \leqslant k$. In the proofs, we will often consider a graph-partition H of a graph G together with some distinguished clique $\{X_1, \dots, X_h\}$ of H; in this case, we say that H is *rooted* at $\{X_1, \dots, X_h\}$.

3 The proofs

We need the following lemma by Alon, Seymour and Thomas [1].

Lemma 6 ([1]). *Let G be a graph, let $A_1, \ldots, A_h$ be h subsets of $V(G)$, and let r be a real number with $r \geqslant 1$. Then either:*

- *there is a tree T in G (not necessarily induced) with $|V(T)| \leqslant r$ such that $V(T) \cap A_i \neq \emptyset$ for every $i \in \{1, \ldots, h\}$, or*
- *there exists $Z \subseteq V(G)$ with $|Z| \leqslant (h-1)|V(G)|/r$ such that no component of $G - Z$ intersects all of $A_1, \ldots, A_h$.*

We deduce an edge variant of this lemma.

Lemma 7. *Let G be a graph without isolated vertices, let $A_1, \ldots, A_h$ be h subsets of $V(G)$, and let r be a real number with $r \geqslant 1$. Then either:*

- *there is a tree T in G with $|E(T)| \leqslant r$ such that $V(T) \cap A_i \neq \emptyset$ for every $i \in \{1, \ldots, h\}$, or*
- *there exists $F \subseteq E(G)$ with $|F| \leqslant (h-1)|E(G)|/r$ such that no component of $G - F$ intersects all of $A_1, \ldots, A_h$.*

Proof. Apply Lemma 6 to $L(G)$, the sets $E_G(A_1, V(G)), \ldots, E_G(A_h, V(G))$ and r. If $L(G)$ contains a tree T_0 with $|V(T_0)| \leqslant r$ such that $V(T_0) \cap E_G(A_i, V(G)) \neq \emptyset$ for every $i \in \{1, \ldots, h\}$, then G contains a tree T such that $E(T) = V(T_0)$, and thus $|E(T)| = |V(T_0)| \leqslant r$ and $V(T) \cap A_i \neq \emptyset$ for every $i \in \{1, \ldots, h\}$.

Now suppose that $L(G)$ contains a set $Z \subseteq V(L(G))$ with $|Z| \leqslant (h-1)|V(L(G))|/r = (h-1)|E(G)|/r$ such that no component of $L(G) - Z$ intersects each $E_G(A_i, V(G))$ with $i \in \{1, \ldots, h\}$. Let $F := Z$. If no component of $G - F$ intersects all of $A_1, \ldots, A_h$, then F satisfies the lemma, and we are done. Assume thus that some component C of $G - F$ intersects all of $A_1, \ldots, A_h$. By our assumption on F, there exists $i \in \{1, \ldots, h\}$ such that $E(C) \cap E_G(A_i, V(G)) = \emptyset$. Hence, C must consist of a single vertex that belongs to all of $A_1, \ldots, A_h$, and therefore the tree $T := C$ satisfies the lemma. □

The following lemma is the heart of the proofs of our results.

Lemma 8. *Let t, Δ, m, h be integers with $t \geqslant 3$ and $\Delta, m, h \geqslant 1$, and let $p := \sqrt{(t-3)\Delta m} + \Delta$. Let G be a connected K_t-minor-free graph of maximum degree at most Δ with m edges, let C be a proper induced subgraph of G with $|V(C)| \geqslant 1$, and let $E_1, \ldots, E_h$ be disjoint nonempty subsets of $E(G) \setminus E(C)$ such that $|E_i| \leqslant p$ for each $i \in \{1, \ldots, h\}$. If there exists a K_h-model $(U_1, \ldots, U_h)$ in $G - V(C)$ such that $N(V(C)) \subseteq U_1 \cup \cdots \cup U_h$ and $E(V(C), U_i) \subseteq E_i$ for each $i \in \{1, \ldots, h\}$, then $L(G)[E(C) \cup E_1 \cup \cdots \cup E_h]$ admits a $(t-2, p)$-partition H rooted at $\{E_1, \ldots, E_h\}$.*

Proof. We prove the lemma by induction on the value $2|V(C)| + h$. Since there exists a K_h model in the K_t-minor free graph, we have $h \leqslant t - 1$. Therefore, if $|V(C)| = 1$, and thus $E(C) = \emptyset$, then the lemma is satisfied by the partition H which is a complete graph on $\{E_1, \ldots, E_h\}$. In particular, the lemma holds in the base case, when $2|V(C)| + h = 3$. From now on, we assume

$$|V(C)| > 1.$$

For the induction step, suppose that $2|V(C)|+h > 3$. Suppose that C is disconnected, say C is the union of vertex disjoint graphs C_1 and C_2 with $|V(C_\alpha)| \geqslant 1$ for $\alpha \in \{1,2\}$. For each $\alpha \in \{1,2\}$, we have $2|V(C_\alpha)| + h < 2|V(C)| + h$, $N(V(C_\alpha)) \subseteq N(V(C))$ and $E(V(C_\alpha), U_i) \subseteq E(V(C), U_i)$ for each $i \in \{1,\dots,h\}$, so we may apply the induction hypothesis to C_α to obtain a $(t-2,p)$-partition H_α of $L(G)[E(C_\alpha) \cup E_1 \cup \cdots \cup E_h]$ rooted at $\{E_1,\dots,E_h\}$. By Fact 5(iii), $H = H_1 \cup H_2$ is the desired $(t-2,p)$-partition of $L(G)[E(C)\cup E_1\cup\cdots\cup E_h)]$. Therefore, we may assume that C is connected. In particular, $(V(C), U_1, \dots, U_h)$ is a K_{h+1}-model in the K_t-minor-free graph G, so

$$h \leqslant t-2.$$

For each $i \in \{1,\dots,h\}$, let $A_i = V(C) \cap N(U_i)$. Suppose that some A_i is empty, say without loss of generality $A_h = \emptyset$. Since G is connected, not all sets A_i are empty, so $h \geqslant 2$. We have $2|V(C)| + (h-1) < 2|V(C)| + h$ and $N(V(C)) \subseteq U_1 \cup \cdots \cup U_{h-1}$ because $N(V(C)) \cap U_h = \emptyset$, so we may apply the induction hypothesis to C and the sets $E_1,\dots,E_{h-1}$ to obtain a $(t-2,p)$-partition H_0 of $L(G)[E(C) \cup E_1 \cup \cdots \cup E_{h-1}]$ rooted at $\{E_1,\dots,E_{h-1}\}$. Since $A_h = \emptyset$, no edge of E_h is incident to an edge in $E(C)$, and thus the line graph $L(G)$ does not contain any edges between E_h and $E(C)$. Hence, by Fact 5(iv), the desired $(t-2,p)$-partition H of $L(G)[E(C) \cup E_1 \cup \cdots \cup E_h]$ can be obtained from H_0 by adding E_h as a new vertex adjacent to $E_1,\dots,E_{h-1}$. Therefore, we may assume that the sets $A_1,\dots,A_h$ are nonempty.

Suppose that C contains a tree T on at most $\sqrt{(h-1)m/\Delta}+1$ vertices that contains at least one vertex in A_i for each $i \in \{1,\dots,h\}$. Let $U_{h+1} = V(T)$, so that $(U_1,\dots,U_{h+1})$ is a K_{h+1}-model, and let $E_{h+1} := E(V(T), V(C))$. Note that E_{h+1} is nonempty, since C is connected and $|V(C)| > 1$. Observe that

$$|E_{h+1}| \leqslant \Delta \cdot |V(T)| \leqslant \Delta \cdot \left(\sqrt{(h-1)m/\Delta}+1\right) \leqslant \sqrt{(t-3)\Delta m} + \Delta = p.$$

If $V(T) = V(C)$ then the complete graph on $\{E_1,\dots,E_{h+1}\}$ gives the desired $(t-2,p)$-partition H of $L(G)[E(C) \cup E_1 \cup \cdots \cup E_h]$. (Recall that $h \leqslant t-2$.)

If $V(T) \neq V(C)$, then we have $2 \cdot |C - V(T)| + (h+1) \leqslant 2(|V(C)|-1) + (h+1) < 2|V(C)|+h$, so we may apply the induction hypothesis to $C - V(T)$, $(U_1,\dots,U_{h+1})$ and the sets E_1, …, E_{h+1} to obtain a $(t-2,p)$-partition H of $L(G)[E(C-V(T))\cup E_1\cup\cdots\cup E_{h+1}]$ rooted at $\{E_1,\dots,E_{h+1}\}$. Since $E(C) = E(C-V(T))\cup E_{h+1}$, the partition H is a $(t-2,p)$-partition of $L(G)[E(C) \cup E_1 \cup \cdots \cup E_h]$ rooted at $\{E_1,\dots,E_h\}$, as desired.

It remains to consider the case when no tree in C with at most $\sqrt{(h-1)m/\Delta}+1$ vertices intersects all of $A_1,\dots,A_h$. In particular, $h \geqslant 2$. Since C is connected and $|V(C)| > 1$, C does not contain isolated vertices. By Lemma 7 with $r = \sqrt{(h-1)m/\Delta}$, there exists a set $F \subseteq E(C)$ with

$$|F| \leqslant (h-1)m/\sqrt{(h-1)m/\Delta} = \sqrt{(h-1)\Delta m} \leqslant \sqrt{(t-3)\Delta m} < p$$

such that no component of $C - F$ intersects all of $A_1,\dots,A_h$. See Figure 1. Among all such sets F, choose a smallest one. Since C is connected and the sets $A_1,\dots,A_h$ are nonempty, $F \neq \emptyset$.

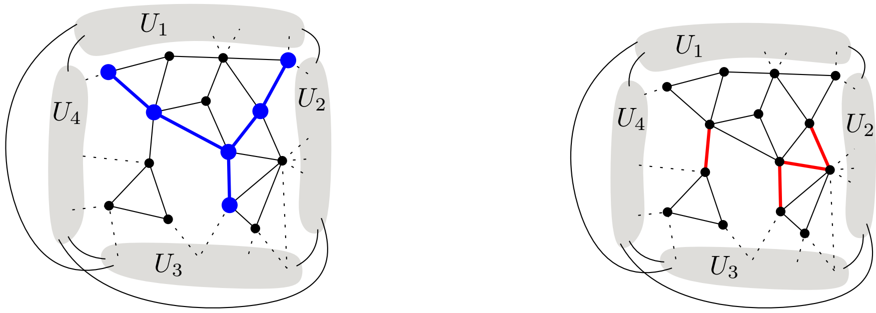

Figure 1: The component C is adjacent to each set U_i in the K_h-model. Every dotted edge with an end in a set U_i has the other end in A_i and belongs to E_i (the set E_i may contain other edges without any ends in C). One of the following holds: (1) there exists a tree $T \subseteq C$ which intersects all sets A_i and its vertices are incident to at most $\lfloor p \rfloor$ edges of C, or (2) there exists a set $F \subseteq E(C)$ with $|F| \leqslant \lfloor p \rfloor$ such that no component of $C - F$ intersects all sets A_i.

Let $C_1, \ldots, C_s$ be the components of $C - F$. Observe that $C_1, \ldots, C_s$ are induced subgraphs of C (and thus of G), by the minimality of F. Our goal now is to show that for each $j \in \{1, \ldots, s\}$, there exists a $(t-2, p)$-partition H_j of $L(G)[E(C_j) \cup F \cup E_1 \cup \cdots \cup E_h]$ rooted at $\{F, E_1, \ldots, E_h\}$. By Fact 5(iii), this will then imply that $L(G)[E(C) \cup E_1 \cup \cdots \cup E_h]$ admits a $(t-2, p)$-partition H rooted at $\{E_1, \ldots, E_h\}$, as desired.

Towards this goal, fix $j \in \{1, \ldots, s\}$, and let $i' \in \{1, \ldots, h\}$ be such that $V(C_j) \cap A_{i'} = \emptyset$. Consider the sets $E'_1, \ldots, E'_h$ where $E'_i = E_i$ for $i \neq i'$ and $E'_{i'} = F$.

Let X denote the set of all vertices of C that lie in some component of $C - F$ containing at least one vertex from $A_{i'}$. Since each component of $C[X]$ contains a vertex from $A_{i'}$, the set $U_{i'} \cup X$ is connected in G. Let $(U'_1, \ldots, U'_h)$ be the K_h-model where $U'_i = U_i$ for $i \neq i'$ and $U'_{i'} = U_{i'} \cup X$. By the minimality of F, each edge $e \in F$ with an end in C_j belongs to one component of $C - (F \setminus \{e\})$ with an element of A'_i, so e has an end in $U'_{i'}$. Hence, $N(C_j) \subseteq (U'_1 \cup \cdots \cup U'_h)$. Note also that $2|V(C_j)| + h < 2|V(C)| + h$. Therefore, we may apply the induction hypothesis to C_j and the sets $E'_1, \ldots, E'_h$ to obtain a $(t-2, p)$-partition H^0_j of $L(G)[E(C_j) \cup E'_1 \cup \cdots \cup E'_h]$ that is rooted at $\{E'_1, \ldots, E'_h\}$.

Since $V(C_j) \cap A_{i'} = \emptyset$, and since $E_{i'} \subseteq E(G) \setminus E(C)$, no edge of $E_{i'}$ is incident to an edge in $E(C_j)$. Thus, the line graph $L(G)$ does not contain any edges between $E_{i'}$ and $E(C_j)$. Since $E(V(C), U_{i'}) \subseteq E_{i'}$ and $h \leqslant t-2$, we can obtain a $(t-2, p)$-partition H_j of $L(G)[E(C_j) \cup F \cup E_1 \cup \cdots \cup E_h]$ rooted at $\{F, E_1, \ldots, E_h\}$ from H^0_j by adding $E_{i'}$ as a new vertex adjacent to $E'_1, \ldots, E'_h$.

By Fact 5(iii), $H := H_1 \cup \cdots \cup H_s$ is a $(t-2, p)$-partition of $L(G)[E(C) \cup E_1 \cup \cdots \cup E_h]$ rooted at $\{F, E_1, \ldots, E_h\}$ and in particular at $\{E_1, \ldots, E_h\}$. This concludes the proof of the lemma. $\square$

Proof of Theorem 2. Let G be a K_t-minor-free graph of maximum degree Δ with m edges, and let $G_1, \ldots, G_s$ be the components of G. For each $j \in \{1, \ldots, s\}$, we construct a $(t-2, p)$-partition H_j of $L(G_j)$. If G_j is an isolated vertex, then $L(G_j)$ is an empty graph and we can take the empty graph as H_j. If G_j is not an isolated vertex, then choose any vertex $x \in V(G_j)$. By Lemma 8 applied to $C = G_j - x$, $E_1 = E(\{x\}, V(G))$ and $U_1 = \{x\}$, $L(G_j)$ has a $(t-2,p)$-partition H_j. Hence, by Fact 5(iii), $H = H_1 \cup \cdots \cup H_s$ is a $(t-2,p)$-partition of $L(G)$, and therefore $L(G)$ is isomorphic to a subgraph of $H \boxtimes K_{\lfloor p \rfloor}$. □

Proof of Theorem 3. Let G be a K_t-minor-free graph of maximum degree Δ with m edges. By Theorem 2, there exists a graph H with $\mathrm{tw}(H) \leqslant t-2$ such that $L(G)$ is isomorphic to a subgraph of $H \boxtimes K_{\lfloor p \rfloor}$ where $p = \sqrt{(t-3)\Delta m} + \Delta$. If $(B_u)_{u \in V(T)}$ is a tree-decomposition of H of width at most $t-2$, then $(B_u \times V(K_{\lfloor p \rfloor}))_{u \in V(T)}$ is a tree-decomposition of $H \boxtimes K_{\lfloor p \rfloor}$ of width at most $(t-1)\lfloor p \rfloor - 1$, so

$$\mathrm{tw}(L(G)) \leqslant \mathrm{tw}(H \boxtimes K_{\lfloor p \rfloor}) \leqslant (t-1)\lfloor p \rfloor - 1 = (t-1)\left\lfloor \sqrt{(t-3)\Delta m} + \Delta \right\rfloor - 1. \qquad \square$$

Proof of Theorem 1. Let G be a K_t-minor-free graph of maximum degree Δ with n vertices and m edges, and let $w \colon V(G) \to [0, \frac{1}{2}]$ be a weight function such that $\sum_{x \in V(G)} w(x) = 1$.

By Theorem 3, we have $\mathrm{tw}(L(G)) \leqslant (t-1)\lfloor \sqrt{(t-3)\Delta m} + \Delta \rfloor - 1$. Let $(B_u)_{u \in V(T)}$ be a tree-decomposition of $L(G)$ of minimum width. For each $x \in V(G)$, the set $E(\{x\}, V(G))$ is a clique in $L(G)$, so by Fact 5(iii) we can choose a node $u(x) \in V(T)$ such that $E(\{x\}, V(G)) \subseteq B_{u(x)}$.

For a subtree T' of T, we define the weight $w(T')$ of T' as the sum of weights $w(x)$ of all vertices $x \in V(G)$ such that $u(x) \in V(T')$. Let us orient an edge $u_1u_2 \in E(T)$ from u_1 to u_2 when in $T - u_1u_2$ the component containing u_2 has weight greater than $\frac{1}{2}$ (and thus the component containing u_1 has weight smaller than $\frac{1}{2}$). We do not orient an edge $e \in E(T)$ in any direction if both components of $T - e$ have weight exactly $\frac{1}{2}$.

Choose a node v in T such that no edge incident with v in T is oriented away from v. We claim that $F = B_v$ satisfies the theorem. Since the tree-decomposition has minimum width, we have $|F| \leqslant (t-1)\lfloor \sqrt{(t-3)\Delta m} + \Delta \rfloor$. Consider a component C of $G - F$. If C consists of a single vertex, then $\sum_{x \in V(C)} w(x) \leqslant \frac{1}{2}$ clearly holds. If C has more than one vertex, then $E(C) \neq \emptyset$, and $L(G)[E(C)]$ is connected since C is connected. We have $E(C) \cap F = \emptyset$, so by Fact 5(ii), there must exist a component T' of $T - \{v\}$ such that every node $u \in V(T)$ with $B_u \cap E(C) \neq \emptyset$ belongs to T'. In particular, $u(x) \in V(T')$ for each $x \in V(C)$. By our choice of v, the weight of T' is at most $\frac{1}{2}$, so $\sum_{x \in V(C)} w(x) \leqslant \frac{1}{2}$.

Kostochka [5] and Thomason [9] showed that a K_t-minor-free graph on n vertices has at most $O(t\sqrt{\log t} \cdot n)$ edges, so $(t-1)\lfloor \sqrt{(t-3)\Delta m} + \Delta \rfloor \leqslant O\left(t^2 (\log t)^{1/4} \cdot \sqrt{\Delta n}\right)$. This completes the proof. □

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References

[1] Noga Alon, Paul Seymour, and Robin Thomas. A separator theorem for nonplanar graphs. *Journal of the American Mathematical Society*, 3(4):801–808, 1990.

[2] Punyashloka Biswal, James R. Lee, and Satish Rao. Eigenvalue bounds, spectral partitioning, and metrical deformations via flows. *J. ACM*, 57(3), mar 2010.

[3] Krzysztof Diks, Hristo N. Djidjev, Ondrej Sýkora, and Imrich Vrťo. Edge separators of planar and outerplanar graphs with applications. *Journal of Algorithms*, 14(2):258–279, 1993.

[4] Freddie Illingworth, Alex Scott, and David R. Wood. Product structure of graphs with an excluded minor. arXiv:2104.06627.

[5] Alexandr V. Kostochka. The minimum Hadwiger number for graphs with a given mean degree of vertices. *Metody Diskret. Analiz.*, 38:37–58, 1982.

[6] Michał Lasoń and Małgorzata Sulkowska. Modularity of minor-free graphs. *Journal of Graph Theory*, 102(4):728–736, 2023.

[7] Richard J. Lipton and Robert Endre Tarjan. A separator theorem for planar graphs. *SIAM Journal on Applied Mathematics*, 36(2):177–189, 1979.

[8] Ondrej Sýkora and Imrich Vrťo. Edge separators for graphs of bounded genus with applications. *Theoretical Computer Science*, 112(2):419–429, 1993.

[9] Andrew Thomason. An extremal function for contractions of graphs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 95(2):261–265, 1984.