Weak $(2,3)$-Decomposition of Planar Graphs

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Abstract

This paper introduces the concept of weak (d,h)-decomposition of a graph G, which is defined as a partition of $E(G)$ into two subsets E_1, E_2, such that E_1 induces a d-degenerate graph H_1 and E_2 induces a subgraph H_2 with $\alpha(H_1[N_{H_2}(v)]) \leqslant h$ for any vertex v. We prove that each planar graph admits a weak $(2,3)$-decomposition. As a consequence, every planar graph G has a subgraph H such that $G - E(H)$ is 3-paintable and any proper coloring of $G - E(H)$ is a 3-defective coloring of G. This improves the result in [G. Gutowski, M. Han, T. Krawczyk, and X. Zhu, *Defective 3-paintability of planar graphs*, Electron. J. Combin., 25(2):2.34, 2018] that every planar graph is 3-defective 3-paintable.

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1 Introduction

A *decomposition* of a graph G is a collection of spanning subgraphs whose edge sets form a partition of $E(G)$. A graph is *d-degenerate* if every subgraph has a vertex of degree at most d. The *degeneracy* $d(G)$ of G is the minimum d such that G is d-degenerate. Given non-negative integers d and h, a *(d,h)-decomposition* of a graph G is a decomposition H_1, H_2 of G such that H_1 is d-degenerate and H_2 has maximum degree at most h. We say G is *(d,h)-decomposable* if there exists a (d,h)-decomposition of G.

The concept of (d,h)-decomposition raises naturally in the study of the defective coloring of graphs. Assume f is a (not necessarily proper) coloring of the vertices of G. An edge $e = xy$ is *defected* if $f(x) = f(y)$. A *d-defective coloring* of a graph G is a coloring of the vertices of G in which each vertex v is incident to at most d defected edges. Thus a 0-defective coloring is a proper coloring. Defective coloring of graphs was first studied by Cowen, Cowen and Woodall [2], who proved that every planar graph is 2-defective 3-colorable. We say a graph G is d-defective k-choosable if for any k-list assignment L, G has a d-defective L-coloring (i.e., a coloring f with $f(v) \in L(v)$ for each vertex v). Škrekovski [8] and Eaton and Hull [5] independently extended the result in [2] to the list

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version and proved that every planar graph is 2-defective 3-choosable. They both asked the question whether every planar graph is 1-defective 4-choosable, and the question was answered by Cushing and Kierstead [3] in the affirmative.

The online version of defective list coloring of a graph is defined through a two-player game. Assume f is a function that assigns to each vertex v a non-negative integer $f(v)$. The *d-defective f-painting game on G* is played by two players: Lister and Painter. Initially, each vertex v has $f(v)$ tokens and is uncolored. In each round, Lister chooses a set M of uncolored vertices and removes one token from each chosen vertex. Painter colors a subset X of M which induces a subgraph $G[X]$ of maximum degree at most d, and colors vertices in X. Lister wins if at the end of some round, there is an uncolored vertex with no more tokens left. Otherwise, at some round, all vertices are colored and Painter wins. We say G is *d-defective f-paintable* if Painter has a winning strategy in this game. If $f(v) = k$ is a constant function, then d-defective f-paintable is called d-defective k-paintable. A graph is *f-paintable* if it is 0-defective f-paintable, and the *paint number* $\chi_P(G)$ of G is the minimum k for which G is k-paintable. The d-defective f-painting game is an online version of the d-defective f-list coloring, in which Painter needs to color vertices before the full information of the list assignment is revealed. Thus d-defective f-paintable graphs are d-defective f-choosable, and $ch(G) \leqslant \chi_P(G)$ for any graph G. Moreover, it is known [4] that $\chi_P(G) - ch(G)$ can be arbitrarily large.

A natural question is whether the results in [2, 8, 3] can be extended to online list coloring.

(1) Is it true that every planar graph is 2-defective 3-paintable?

(2) Is it true that every planar graph is 1-defective 4-paintable?

The answer to (1) is negative. It was proved in [7] that there are planar graphs that are not 2-defective 3-paintable. However, the following positive result was proved in [7].

Theorem 1. *Every planar graph is 3-defective 3-paintable.*

The answer to (2) is "strongly" positive. It was proved in [6] that every planar graph G can be decomposed into a matching M (i.e., a graph of maximum degree at most 1) and a graph H of Alon-Tarsi number $AT(H)$ at most 4. We omit the definition of the Alon-Tarsi number (we shall not discuss this parameter further) and just mention that $\chi_P(G) \leqslant AT(G)$ for any graph G. Thus the following statement is a consequence of the mentioned result in [6].

Every planar graph G has a subgraph H of maximum degree at most 1 such that $G - E(H)$ is 4-paintable.

A natural question is whether Theorem 1 can be strengthened in the same manner.

(3). Is it true that every planar graph G has a subgraph H maximum degree at most 3 such that $G - E(H)$ is 3-paintable?

The answer to (3) is negative. It was proved in [1] that there are planar graphs G such that for any subgraph H with maximum degree 3, $G - E(H)$ is not 3-choosable.

Observe that if $\{H_1, H_2\}$ is a (d, h)-decomposition of a graph G, where H_1 is d-degenerate and H_2 has maximum degree at most h, then H_1 is $(d + 1)$-paintable, and any proper coloring ϕ of H_1 is an h-defective coloring of G, as all the defected edges with respect to ϕ are contained in H_2. Motivated by the above questions, (d, h)-decomposability of planar graphs was studied in [1]. For $d = 1, 2, 3, 4$, let h_d be the minimum integer h such that every planar graph admits a (d, h)-decomposition. It was shown in [1] that $h_1 = \infty$, $4 \leqslant h_2 \leqslant 6$, $h_3 = 2$ and $h_4 = 1$. In particular, it was shown in [1] that there are planar graphs that are not $(2, 3)$-decomposable.

In this paper, also motivated by the above questions, we introduce a concept of weak (d, h)-decomposition. For a graph H, a vertex v of H and a subset X of vertices, $N_H(v)$ is the set of neighbours of v in H, $H[X]$ is the subgraph of H induced by X, and $\alpha(H)$ is the independence number of H.

Definition 2. A weak (d, h)-decomposition of a graph G is a decomposition of G into two subgraphs H_1 and H_2 such that H_1 is d-degenerate, and for each vertex v of G, $\alpha(H_1[N_{H_2}(v)]) \leqslant h$.

The difference between (d, h)-decomposition and weak (d, h)-decomposition is that in a (d, h)-decomposition H_1, H_2 of G, for every vertex v, instead of $\alpha(H_1[N_{H_2}(v)]) \leqslant h$, we require $d_{H_2}(v) = |N_{H_2}(v)| \leqslant h$. Thus a (d, h)-decomposition of G is a weak (d, h)-decomposition of G, but the converse is not true.

Nevertheless, if H_1, H_2 is a weak (d, h)-decomposition, then H_1 is $(d+1)$-paintable, and any proper coloring of H_1 is an h-defective coloring of G: If v has s neighbors $u_1, u_2, \ldots, u_s$ that are colored the same colors as v, then $\{u_1, u_2, \ldots, u_s\} \subseteq N_{H_2}(v)$, and $\{u_1, u_2, \ldots, u_s\}$ is an independent set in H_1 (as they are colored the same color). So $s \leqslant \alpha(H_1[N_{H_2}(v)])$.

In this paper, we prove the following results:

Theorem 3. *Every planar graph is weakly* $(2, 3)$*-decomposable.*

For a directed graph D, the *out-degree* $d^+_D(v)$ of a vertex v in D is the number of out-neighbours of v in D. Let$\Delta^+(D) = \max\{d^+_D(v) : v \in V(G)\}$. It is well-known and easy to see that a graph G is d-degenerate if and only if it has an acyclic orientation D with $\Delta^+(D) \leqslant d$.

For a subgraph H of G and a vertex v of G, let $d^*_H(v) = \alpha(G[N_H(v)] - E(H))$ and let $\Delta^*(H) = \max\{d^*_H(v) : v \in V(G)\}$. Thus a weak (d, h)-decomposition can be expressed as a pair (D, H) such that H is a subgraph of G with $\Delta^*(H) \leqslant h$ and D is an acyclic orientation of $G - E(H)$ with $\Delta^+(D) \leqslant d$.

Let G be a plane graph. A plane subgraph of G is a subgraph of G whose plane embedding is inherited. We say G is a near triangulation if every face of G except the outer face is a triangle.

A directed edge is represented by an ordered pair of vertices: (u, v) is an arc from u to v. For a graph G and a set E of unordered pairs on $V(G)$, let $G + E$ (resp. $G - E$) denote the graph obtained from G by adding (resp. deleting) the elements of E to (resp.

from) the edge set of G. If $|E| = 1$, say $E = ww'$, then denote $G + E$ (resp. $G - E$) by $G + ww'$ (resp. $G - ww'$). For a digraph D and a set A of ordered pairs on $V(D)$, define $D + A, D - A, D + (w, w')$, and $D - (w, w')$ similarly. Moreover, for a digraph D and vertices $x, y{\in}V(D)$, let $D - xy$ denote the subdigraph $D - \{(x, y), (y, x)\}$. We often drop the parentheses to improve the readability. For instance, for a digraph D and sets A_1, A_2, A_3 of ordered pairs on $V(D)$, both $D - A_1 + A_2 + A_3$ and $D - A_1 + (A_2 + A_3)$ denote $((D - A_1) + A_2) + A_3$. For two (di)graphs G_1 and G_2, let $G_1 \cup G_2$ be the (di)graph such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. For a subset X of vertices of G, $E_G(X)$ is the set of edges of G with both end vertices in X.

For a plane graph G and a cycle K, we use int(K) to denote the set of vertices in the interior of K, and ext(K) to denote the set of vertices in the exterior of K. Denote by int[K] and ext[K] the subgraph of G induced by int(K) $\cup$ K and ext(K) $\cup$ K, respectively. For a simple path P and two distinct vertices u, v on that path, let $P[u, v]$ denote the subpath of P that traverses P from vertex u to vertex v, and let $P(u, v)$, $P[u, v)$, and $P(u, v]$ to denote $P[u, v] - \{u, v\}, P[u, v] - v$, and $P[u, v] - u$, respectively. Similarly, for a simple cycle D in G and two distinct vertices u, v on that cycle, let $D[u, v]$ denote the subpath of D that traverses D in the clockwise direction from vertex u to vertex v.

2 Some preliminaries

In this section, we fix some terminology and prove some propositions needed in our proof of Theorem 3. Let G be a connected non-empty plane graph. By a plane graph, we mean a graph with a fixed planar drawing. Let B be the boundary walk of the outer face of G. For a vertex v on B, the set of B-neighbors of v is the set of vertices u such that $uv \in E(B)$. Observe that there may be more than two B-neighbors for a single vertex as B is not necessarily a simple walk.

Definition 4. A *target* is a triple (G, A, b), where G is a near triangulation of the plane, A is a set of at most two (possibly zero) vertices that appear consecutively on B, and $\boldsymbol{b}$ is a vertex of $B - A$. Vertices in $A \cup \{b\}$ are called *special vertices* of G.

We write $A = \{a_1\}$ if $|A| = 1$ and $A = \{a_1, a_2\}$ if $|A| = 2$. We say that a vertex v is an (A, b)*-cut* if $v \notin A \cup \{b\}$ and v is on every path between A and $\boldsymbol{b}$ in G. A vertex on B that is neither a special vertex nor an (A, b)-cut is called *a regular boundary vertex.* Instead of proving Theorem 3 directly, we prove the following more technical result, which is easily seen to imply Theorem 3.

Theorem 5. *Let (G, A, b) be a target and $E'(A, b) = E(A \cup \{b\}) \cap E(B)$, where B is the boundary of G. Then there exist a subgraph H of $G - E'(A, b)$ and an acyclic orientation D of $G - E(H) - E'(A, b)$ satisfying the following:*

(i) For every interior vertex $w \in G - B$, $d^+_D(w) \leqslant 2$ and $d^*_H(w) \leqslant 3$.

(ii) For every regular boundary vertex w, $d^+_D(w) \leqslant 1$ and $d^*_H(w) \leqslant 2$.

(iii) *For every* (A,b)*-cut vertex* w, $d_D^+(w)=0$ *and* $d_H^*(w)\leqslant 2$.

(iv) *For every special vertex* $w\in A\cup\{b\}$, $d_D^+(w)=0$ *and* $d_H^*(w)\leqslant 1$. *Moreover, if* $ab\in E(B)$ *for some* $a\in A$, *then* $d_H^*(a)=0, d_H^*(b)\leqslant 1$ *and* $N_H(b)\subseteq N_G(a)$.

We call such a pair (D,H) a valid decomposition *of* (G,A,b).

For any two vertices $u,v\in B$, let $N_G(u,v)=N_G(u)\cap N_G(v)$ be the set of common neighbors of u and v. Note that by (iv), if $A=\{a_1,a_2\}$ and $a_ib\in E(B)$ for $i\in[2]$, then $N_H(b)\subseteq N_G(a_1,a_2)$.

If Theorem 5 is true, then for any planar graph G, we can choose any b on the boundary of the outer face and set $A=\emptyset$. By Theorem 5, $(G,\emptyset,b)$ has a valid decomposition (D,H), which is a weak $(2,3)$-decomposition of G. The remainder of the paper is devoted to the proof of Theorem 5.

First, we have two easy propositions.

Proposition 6. *Assume* H_1 *and* H_2 *are subgraphs of* G.

1. *If* $E_{H_1}(N_{H_2}(v))=\emptyset$ *and* $E_{H_2}(N_{H_1}(v))=\emptyset$, *then*

$$d^*_{H_1\cup H_2}(v)\leqslant d^*_{H_1}(v)+d^*_{H_2}(v).$$

2. *If* $N_{H_1}(v)\neq\emptyset$, $vw\in E(G)-E(H_1)$, $N_{H_1}(v)\subseteq N_G(w)$ *and* H_2 *is obtained from* H_1 *by adding the edge* vw, *then* $d^*_{H_2}(v)=d^*_{H_1}(v)$.

Proof. (1) Let X be a maximum independent set in $G[N_{H_1\cup H_2}(v)-E(H_1\cup H_2)]$. Since $E_{H_1}(N_{H_2}(v))=\emptyset$ and $E_{H_2}(N_{H_1}(v))=\emptyset$, $X\cap N_{H_1}(v)$ is an independent set in $G[N_{H_1}(v)-E(H_1)]$ and $X\cap N_{H_2}(v)$ is an independent set in $G[N_{H_2}(v)-E(H_2)]$. Hence $d^*_{H_1\cup H_1}(v)=|X|\leqslant|X\cap N_{H_1}(v)|+|X\cap N_{H_2}(v)|\leqslant d^*_{H_1}(v)+d^*_{H_2}(v)$.

(2) By definition, $d^*_{H_1}(v)=\alpha(G[N_{H_1}(v)]-E(H_1))$ and $d^*_{H_2}(v)=\alpha(G[N_{H_2}(v)]-E(H_2))$. As $N_{H_2}(v)=N_{H_1}(v)\cup\{w\}$, and w is adjacent to all vertices in $N_{H_1}(v)$, $\alpha(G[N_{H_2}(v)]-E(H_2))=\alpha(G[N_{H_1}(v)]-E(H_1))$. ☐

Proposition 7. *Let* G *be a graph and* D_1,D_2 *be the acyclic orientation on some subgraphs of* G.

1. *If* $d^+_{D_1}(v)=0$, *then* $D'=D_1+(u,v)$ *is acyclic.*
2. *If* $d^+_{D_1}(v)=0$ *for every* $v\in V(D_1)\cap V(D_2)$, *then* D_1+D_2 *is acyclic.*

The result of Proposition 7 is trivial and the proof is omitted.

3 Proof of Theorem 5

Assume G is a counterexample of Theorem 5 with minimum $|V(G)|$. It is obvious that G is connected. We shall prove a sequence of properties of G that lead to a final contradiction.

Lemma 8. $|V(G)| \geqslant 4$.

Proof. If $E(G) = E'(A,b)$, then $(\emptyset, \emptyset)$ is a valid decomposition of (G,A,b). Thus $E(G) \neq E'(A,b)$, and hence $|V(G)| \geqslant 2$. If $|V(G)| = 2$, say $V(G) = \{b,c\}$, then $A = \emptyset$ and $((c,b),\emptyset)$ is a valid decomposition of (G,A,b). Assume that $V(G) = \{a,b,c\}$. As $E(G) \setminus E'(A,b) \neq \emptyset$, we have $|A| \leqslant 1$. If $A = \emptyset$, then set $H = \{ac\} \cap E(G)$ and $D = \{(v,b) : v \in N_G(b)\}$. We have $d^*_H(b) = d^+_D(b) = 0$ and $d^+_D(v), d^*_H(v) \leqslant 1$ for $v \in \{a,c\}$. Therefore, (D,H) is valid decomposition of $(G,\emptyset,b)$. Otherwise, assume that $A = \{a\}$. If c is an (A,b)-cut, then G is a path acb. Let $H = \{bc, ac\}$ and $D = \emptyset$. Thus, $d^*_H(c) = 2$, $d^*_H(v) = 1$ for $v \in \{a,b\}$, and so (D,H) is valid decomposition of $(G,\{a\},b)$. Otherwise, we have $ab \in E'(A,b)$ and $E(G) \setminus E'(A,b) \subseteq \{bc, ac\}$. Set $H = \{ac\} \cap E(G)$ and $D = \{(v,b) : v \in N(b) - a\}$. It is easy to check that (D,H) is a valid decomposition of $(G,\{a\},b)$. □

Lemma 9. *G is 2-connected.*

Proof. Assume to the contrary that G has a cut-vertex w. Let G_1, G_2 be two subgraphs of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{w\}$. Let B_i be the boundary of G_i for $i \in [2]$. Without loss of generality, we may assume that $b \in G_1$. We choose w such that $|A \cap G_1|$ is maximal, and G_2 is 2-connected. Depending on the position of A relative to w, we consider three subcases.

Case 1 $A \subseteq G_1$.

Let $A_1 = A, A_2 = \{w\}, b_1 = b$ and $b_2 = y$, where y is a B-neighbour of w in G_2. Then there is a valid decomposition (D_i, H_i) of (G, A_i, b_i) for $i \in [2]$. Let $H = H_1 \cup H_2$ and $D = D_1 \cup D_2 + (y,w)$. As $V(G_1) \cap V(G_2) = \{w\}$ and $d^+_{D_2}(w) = 0$, $D_2 + (y,w)$ is acyclic, and D is acyclic. Observe that $\{w\} = G_1 \cap G_2$. Condition (i-iv) hold since $d^+_D(y) = d^+_{D_2}(y) + 1 = 0 + 1 = 1$, $d^+_D(w) = d^+_{D_1}(w)$ and $d^*_H(w) = d^*_{H_1}(w) + d^*_{H_2}(w) = d^*_{H_1}(w)$.

Case 2 $A \subseteq G_2$ and $w \in A$.

We may assume that $A = \{a_1, a_2\}$ and $w = a_1$. Let $A_1 = A_2 = \{a_1\}, b_1 = b$ and $b_2 = a_2$. Note that $E'(A_1, b_1) = E'(\{a_1\}, b)$ and $E'(A_2, b_2) = \{a_1 a_2\}$. By the minimality of G, there is a valid decomposition (D_i, H_i) of (G, A_i, b_i) for $i \in [2]$. Let $H = H_1 \cup H_2$ and $D = D_1 \cup D_2$. As $d^+_{D_2}(a_1) = d^*_{H_2}(a_1) = 0$, (D,H) is a valid decomposition of (G,A,b).

Case 3 $A \subseteq G_2$ and $w \notin A$.

We may assume that $w \neq b$, for otherwise we can apply Case 1. Thus, w is an (A,b)-cut. Let $A_1 = \{w\}, A_2 = A, b_1 = b$ and $b_2 = w$. Then there is a valid decomposition (D_i, H_i) of (G, A_i, b_i) for $i \in [2]$. Observe that $d^+_{D_1}(w) = d^+_{D_1}(b) = 0$ and $d^*_{H_1}(b), d^*_{H_1}(w) \leqslant 1$. If $wb \in E(B_1)$, then $d^*_{H_1}(w) = 0, d^*_{H_1}(b) \leqslant 1$ and $N_{H_1}(b) \subseteq N_{G_1}(w)$. In this case, we add wb to H_1, and so we still have $d^*_{H_1}(w) = 1, d^*_{H_1}(b) \leqslant 1$ by Proposition 6. Similarly, if $aw \in E(B_2)$ for $a \in A$, then $d^*_{H_2}(a) = 0, d^*_{H_2}(w) \leqslant 1$ and $N_{H_2}(w) \subseteq N_{G_2}(a)$. Again, we add

wa to H_2, and so we still have $d^*_{H_2}(a) = 1, d^*_{H_2}(w) \leqslant 1$. Let $H = H_1 \cup H_2$ and $D = D_1 \cup D_2$. Then, $d^+_D(w) = d^+_{D_1}(w) + d^+_{D_2}(w) = 0 + 0 = 0$ and $d^*_H(w) = d^*_{H_1}(w) + d^*_{H_2}(w) \leqslant 1 + 1 = 2$. Therefore, (D, H) is a valid decomposition of (G, A, b). □

By Lemma 9, the boundary walk B is a simple cycle. Furthermore, we assume that A has exactly two elements. If $A = \emptyset$, then we can choose any vertex $a_1 \in B - b$ and set $A = \{a_1\}$. Thus, if (D, H) is a valid decomposition of $(G, \{a_1\}, b)$, then (D', H) is a valid decomposition of $(G, \emptyset, b)$, where $D' = H + (a_1, b)$ if $a_1 b \in E(B)$ and $D' = D$ otherwise. If $A = \{a_1\}$, then choose a B-neighbour $a_2 \neq b$ of a_1, and set $A = \{a_1, a_2\}$. Note that a_2 is not a_1, b-cut. Thus, if (D, H) is a valid decomposition of $(G, \{a_1, a_2\}, b)$, then $(D + (a_2, a_1), H')$ is a valid decomposition of $(G, \{a_1\}, b)$, where $H' = H + a_2 b$ if $a_2 b \in E(B)$ and $H' = H$ otherwise.

For a vertex $v \in B$, we define the path $NP(v)$ that traverses neighbors of v from u to w, where u, w are two B-neighbours of v. Assume that uv is a boundary edge. The *minimum common neighbor* of u and v, denoted $\mathrm{minn(u, v)}$, is a vertex $w \in N(u, v)$ such that $\mathrm{int(uvwu)}$ contains no common neighbor of u and v. The *maximum common neighbor* of u and v, denoted $\mathrm{maxn(u, v)}$, is a vertex $w \in N(u, v)$ such that $\mathrm{int[uvwu]}$ contains all common neighbors of u and v. As u and v are B-neighbors, any two common neighbors x_1, x_2 of u and v are on the same side of the edge uv. Thus, one of the sets $\mathrm{int(x_1uvx_1)}$, $\mathrm{int(x_2uvx_2)}$ is contained in the other and that both $\mathrm{minn(u, v)}$ and $\mathrm{maxn(u, v)}$ exist.

Lemma 10. $|B| \geqslant 4$.

Proof. Assume to the contrary that $|B| = 3$. Depending on whether a_1, a_2 and b have a common neighbor, we consider two subcases.

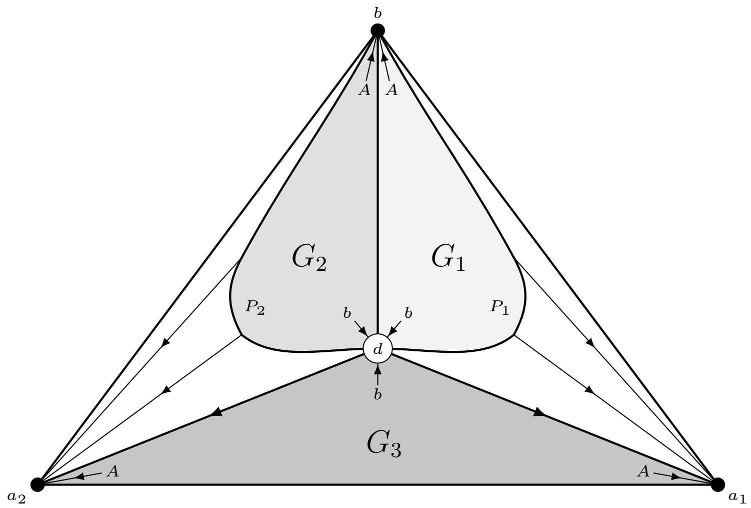

Figure 1: Graph division in Case 1.

Case 1 a_1, a_2 and b have a common neighbor d.

As shown in Figure 1, let P_1 be the path $NP(a_1)[d, b)$ and P_2 be the path $NP(a_2)(b, d]$, and

$$G_1 = \mathrm{int[bP_1db], G_2 = int[bdP_2b], G_3 = int[a_1a_2da_1]}.$$

By the minimality of G, there is a valid decomposition (D_i, H_i) of (G_i, A_i, d), where $A_1 = A_2 = \{b\}, A_3 = A$. Let $H_4 = db$ and $D_4 = \{(v, u) : v \in P_1 \cup P_2, u \in N_A(v)\}$.

Then for $v \in P_1 \cup P_2 - d$, $d^+_{D_4}(v) = 1, d^*_{H_4}(v) = 0$ and $d^+_{D_4}(b) = 0, d^+_{D_4}(d) = 2, d^*_{H_4}(d) = d^*_{H_4}(b) = 1$. Set $D = \cup_{i=1}^4 D_i, H = \cup_{i=1}^4 H_i$. Then (D, H) is a decomposition of $G - E'(A, b)$. As $d^+_{D_i}(d) = 0$ for $i \in [3]$, $\cup_{i=1}^3 D_i$ is acyclic by (2) of Proposition 7. Then D is acyclic since $d^+_D(a_i) = 0$ for $i \in [2]$. As $bd \in E(B_1)$, $N_{H_1}(d) \subseteq N_{G_1}(b)$, and so $d^*_{H_1+H_4}(d) = d^*_{H_1}(d)$ by Proposition 6 (2), where B_1 is the boundary of G_1. For $v \in G - B - P_1 \cup P_2$, v is contained in exactly one subgraph of G_1, G_2, G_3. For $i \in [2]$ and $v \in P_i - d$, $d^+_D(v) = d^+_{D_i}(v) + d^+_{D_4}(v) \leqslant 1 + 1 = 2$ and $d^*_H(v) = d^*_{H_i}(v) \leqslant 2$. Thus, condition (i) holds since $d^+_D(d) = d^+_{D_4}(d) = 2$ and $d^*_H(d) = \sum_{i=1}^3 d^*_{H_i}(d) \leqslant 1 + 1 + 1 = 3$ (by Proposition 6 (1)). Condition (iv) holds since $N_H(b) = \{d\} \subseteq N_G(a_1, a_2)$,

$$d^+_D(a_i) = d^+_{D_3}(a_i) = 0, d^*_H(a_i) = d^*_{H_3}(a_i) = 0 \text{ for } i \in [2];$$
$$d^+_D(b) = d^+_{D_1}(b) + d^+_{D_2}(b) = 0, d^*_H(b) = d^*_{H_1}(b) + d^*_{H_2}(b) + d^*_{H_4}(b) = 0 + 0 + 1 = 1.$$

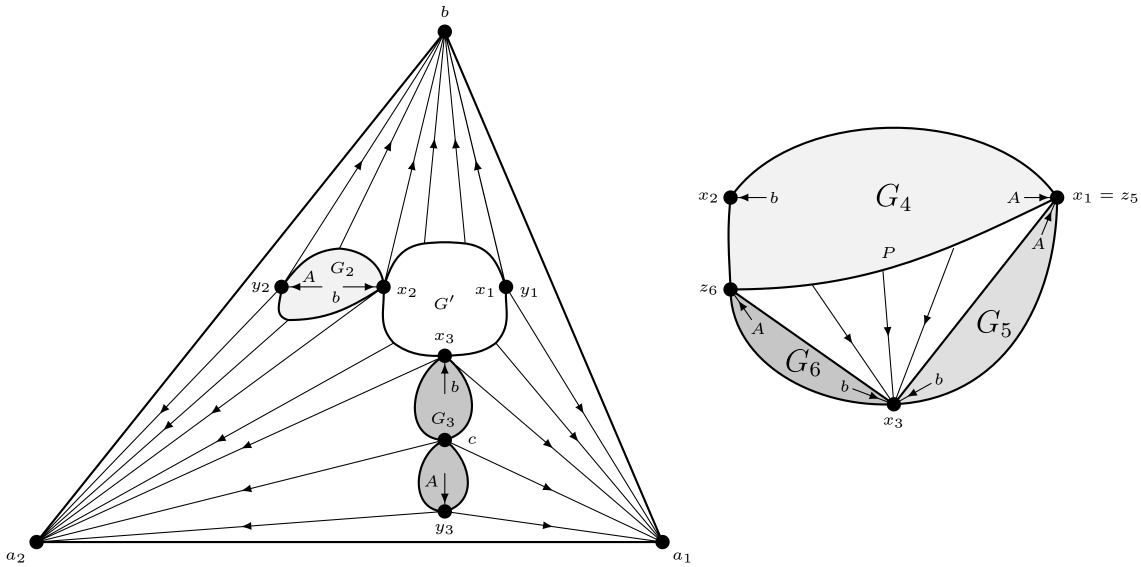

Figure 2: Graph division in Case 2. On the left: graph G. Since $x_1 = y_1$, G_1 consists only of x_1. Vertex c is an (A_3, b_3)-cut in G_3. On the right: graph G' with boundary B'.

Case 2 a_1, a_2 and b have no common neighbor.

As shown in Figure 2, let $x_i = \mathrm{maxn(a_i, b)}$, $y_i = \mathrm{minn(a_i, b)}$ for $i \in [2]$ and $x_3 = \mathrm{maxn(a_1, a_2)}$, $y_3 = \mathrm{minn(a_1, a_2)}$. Then we define G'_i be the connected component of $G - \{a_1, a_2, b, x_1, x_2, x_3\}$ that contains y_i for $i \in [3]$. Then let $G_i = G'_i + x_i$ for $i \in [3]$ and $G' = G - B - \cup_{i=1}^3 G'_i$. Observe that the choice of x_1, x_2 and x_3 guarantees that the boundary walk B' of G' is a simple cycle, and each vertex $v \in V(B') - \{x_1, x_2, x_3\}$ is a neighbor of exactly one of the vertices a_1, a_2, b. We further divide G' into three small subgraphs as follows: If $x_1x_3 \in E(G)$, then let $z_5 = x_1$. Otherwise let $z_5 \in N_{B'}(x_3)$ such that $N(x_3) \cap B'(x_1, z_5) = \emptyset$. Similarly, if $x_2x_3 \in E(G)$, then let $z_6 = x_2$. Otherwise let

$z_6 \in N_{B'}(x_3)$ such that $N(x_3) \cap B'(z_6, x_2) = \emptyset$. As shown in Figure 2, let P be the path $NP(x_3)(z_6, z_5)$ and let

$$G_4 = \mathrm{int}[z_5PB'[z_6, z_5]], G_5 = \mathrm{int}[B'[z_5, x_3]z_5], G_6 = \mathrm{int}[B'[x_3, z_6]x_3].$$

Clearly, each G_i is a near triangulation. For $i \in [3], j \in \{5, 6\}$, let $A_i = \{y_i\}, b_i = x_i$, $A_4 = \{x_1\}$, $b_4 = x_2$ and $A_j = \{z_j\}, b_j = x_3$. By the minimality of G, there is a valid decomposition (D_i, H_i) of (G_i, A_i, b_i) for $i \in [6]$.

Now we will make some modifications on (D_i, H_i) and combine them to obtain a valid decomposition of (G, A, b). Let W be the boundary of the graph $G - V(B)$. Observe that $E(G) - E'(A, b) - \bigcup_{i\in[6]}(E(D_i) \cup E(H_i)) = \{vw : v \in W, w \in N_B(v)\} \bigcup \{vx_3 : v \in V(P) - \{z_5, z_6\}\} \bigcup (\cup_{i\in[6]} E'(A_i, b_i))$. Let

$$D_7 = \{(v, w) : v \in W, w \in N_B(v)\} \cup \{(v, x_3) : v \in V(P) - \{z_5, z_6\}\}$$

and

$$H_7 = \cup_{i\in[6]} E'(A_i, b_i).$$

Let C_i be the set of (A_i, b_i)-cut for $i \in [3]$. Then $d^+_{D_7}(v) = 2$ for $v \in \cup_{i\in[3]}(\{x_i, y_i\} \cup C_i)$. For other $v \in W \cup P$, $d^+_{D_7}(v) = 1$. For $i \in [6]$ and $a \in A_i$, if $ab_i \in E(B_i)$, then $d^+_{D_i}(a) = d^+_{D_i}(b_i) = 0, d^*_{H_i}(a) = 0, d^*_{H_i}(b_i) \leqslant 1$ and $N_{H_i}(b_i) \subseteq N_{G_i}(a)$. In this case, we have $d^*_{H_i\cup(H_7\cap G_i)}(a) = d^*_{H_7}(a) \leqslant 1$ and $d^*_{H_i\cup(H_7\cap G_i)}(b_i) = d^*_{H_i}(b_i) \leqslant 1$ by Proposition 6.

Set $H = \cup_{i=1}^7 H_i$ and $D = \cup_{i=1}^7 D_i$. As $d^+(D)(v) = 0$ for $v \in B$, D is acyclic. Note that $d^+_D(v) = d^+_{D_i}(v)$ for $v \in G_i - P \cup W$ and $i \in [6]$. For $v \in P \cup W$, we have

$$d^+_D(v) = d^+_{D_7}(v) \leqslant 2 \text{ for } v \in \bigcup_{i\in[3]} (\{x_i, y_i\} \cup C_i);$$

and for $i \in [4]$, $v \in (P \cup W - \bigcup_{i\in[3]}(\{x_i, y_i\} \cup C_i)) \cap G_i$,

$$d^+_D(v) = d^+_{D_i}(v) + d^+_{D_7}(v) \leqslant 1 + 1 = 2.$$

Thus, condition (i-iv) about the out-degree hold. It suffices to check condition (i-iv) about the d^*_H. For $v \in V(G) - B - \{x_1, x_2, x_3, z_5, z_6\}$, v is contained in one of subgraphs, say G_i, and so $d^*_H(v) = d^*_{H_i}(v) \leqslant 3$. Besides, by Proposition 6 (1), we have

$$d^*_H(x_3) = d^*_{H_3}(x_3) + d^*_{H_5}(x_3) + d^*_{H_6}(x_3) \leqslant 1 + 1 + 1 = 3;$$

for $i \in [2]$,$j \in \{5, 6\}$

$$d^*_H(x_i) = \begin{cases} d^*_{H_i}(x_i) + d^*_{H_4}(x_i) \leqslant 1 + 1 = 2 & \text{if } x_i \notin \{z_5, z_6\}; \\ d^*_{H_i}(x_i) + d^*_{H_4}(x_i) + d^*_{H_j}(x_i) \leqslant 1 + 1 + 1 = 3 & \text{otherwise, say } x_i = z_j; \end{cases}$$

$$d^*_H(z_j) = d^*_{H_4}(z_j) + d^*_{H_j}(z_j) \leqslant 2 + 1 = 3,$$

and so condition (i) holds. For $v \in V(B)$, we have $d^+_D(v) = d^*_H(v) = 0$, and so condition (ii-iv) holds. □

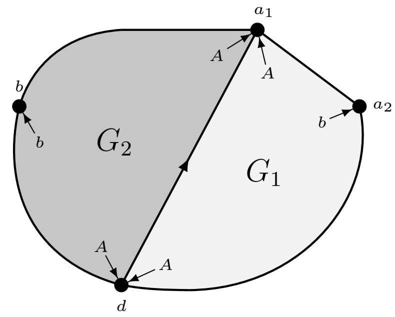

Figure 3: Graph division in Lemma 11.

Lemma 11. *There is no chord of B incident to a_1 which separates a_2 from b, and no chord of B incident to a_2 that separates a_1 from b.*

Proof. Assume a_1d is a chord of B that separates a_2 and b as illustrated in Figure 3. Depending on whether $a_1b \in E(B)$ or not, we consider two cases.

Case 1 $a_1b \notin E(B)$.

Let $G_1 = \mathrm{int[B[a_1, d]a_1]}$, $A_1 = \{a_1, d\}, b_1 = a_2$ and $G_2 = \mathrm{int[B[d, a_1]d], A_2 = \{a_1, d\}, b_2 = b}$. By the minimality of G, there is a valid decomposition (D_i, H_i) of (G_i, A_i, b_i) for $i \in [2]$.

Observe that $d \in A_1 \cap A_2$, $a_1a_2 \in E(B_1)$. Thus, $d^*_{H_1}(a_1) = 0$ and $d^*_{H_2}(a_1) \leqslant 1$, $d^*_{H_2}(b), d^*_{H_1}(a_2) \leqslant 1$ and $d^*_{H_i}(d) \leqslant 1$ for $i \in [2]$. If $da_2 \in E(B_1)$, then $d^*_{H_1}(d) = 0$ and $N_{H_1}(a_2) \subseteq N_{G_1}(a_1, d)$. Let

$$H'_1 = \begin{cases} H_1 + a_2d, & \text{if } a_2d \in E(B_1), \\ H_1, & \text{Otherwise.} \end{cases}$$

Then we have $d^*_{H'_1}(d), d^*_{H'_1}(a_2) \leqslant 1$. Similarly, let

$$H'_2 = \begin{cases} H_2 + bd, & \text{if } bd \in E(B_2), \\ H_2, & \text{Otherwise.} \end{cases}$$

We have $d^*_{H'_2}(d), d^*_{H'_2}(b) \leqslant 1$. Set $D = D_1 \cup D_2 + (d, a_1)$ and $H = H'_1 \cup H'_2$. As $d \in N(a_1)$, $N_{H'_2}(b) \subseteq N_G(a_1)$. So $N_H(b) \subseteq N_G(a_1)$. As $d^+_{D_1}(a_1) = d^+_{D_1}(d) = 0$, $D_1 \cup D_2$ is acyclic. Since $d^+_D(a_1) = 0$, D is still acyclic.

Observe that $d^*_H(a_1) = d^*_{H'_1}(a_1) + d^*_{H'_2}(a_1) \leqslant 1$. It suffices to check the degree bound of d in D and H. Since

$$d^+_D(d) = 1; d^*_H(d) = d^*_{H'_1}(d) + d^*_{H'_2}(d) \leqslant 1 + 1 = 2,$$

condition (ii) holds.

Case 2 $a_1b \in E(B)$.

In this case, we consider the same decomposition (D, H) as shown in Case 1. Then as a_1, b play the role of A and b in G_2, we have $N_{H_2}(b) \subseteq N(a_1)$ and $d^*_{H_2}(a_1) = 0$. So $d^*_H(a_1) = d^*_{H'_1}(a_1) + d^*_{H'_2}(a_1) \leqslant 1$. Other conditions of Theorem 5 also hold by the same argument in Case 1. □

Lemma 12. *b is not adjacent to A.*

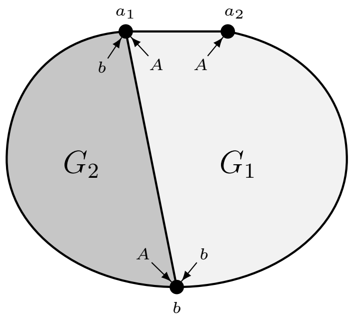

Figure 4: Graph division in Case 1.

Proof. Assume to the contrary that $a_1b \in E(G)$. Depending on whether a_1b is an edge of B or not, we consider two cases.

Case 1 a_1b is a chord of B.

Let $G_1 = \mathrm{int[B[a_1, b]a_1]}$, $A_1 = A, b_1 = b$ and $G_2 = \mathrm{int[B[b, a_1]b], A_2 = \{b\}, b_2 = a_1}$ as illustrated in Figure 4. By the minimality of G, there is a valid decomposition (D_i, H_i) of (G_i, A_i, b_i) for $i \in [2]$. Observe that $a_1b \in E(B_1) \cap E(B_2)$. Thus, $d^*_{H_1}(a_1) = d^*_{H_2}(b) = 0$, $d^*_{H_2}(a_1), d^*_{H_1}(b) \leqslant 1$ and $N_{H_2}(a_1) \subseteq N_{G_2}(b), N_{H_1}(b) \subseteq N_{G_1}(a_1)$. We add a_1b to H_i for $i \in [2]$. By Proposition 6, we have $d^*_{H_2}(a_1), d^*_{H_1}(b) \leqslant 1$. Set $D = D_1 \cup D_2$ and $H = H_1 \cup H_2$. As $d^+_D(a_1) = d^+_D(b) = 0$, D is acyclic and condition (i-iv) about the out-degree hold. For $v \in V(G) - \{a_1, b\}$, $d^*_H(v) = d^*_{H_i}(v)$ for some $i \in [2]$ and so condition (i-iii) hold. Condition (iv) holds since $d^*_H(a_1) = d^*_{H_2}(a_1) \leqslant 1$ and $d^*_H(b) = d^*_{H_1}(b) \leqslant 1$.

Case 2 $a_1b \in E(B)$. By Lemma 10, $a_2b \notin E(B)$. Let $d = \mathrm{maxn(a_1, b)}$. By Lemma 11,

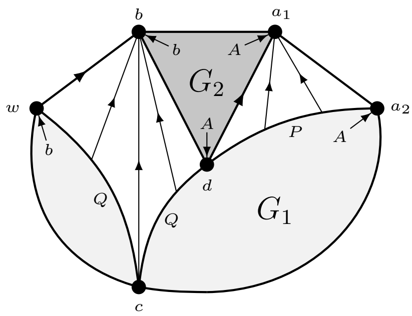

Figure 5: Graph division in Cases 2. Vertex c is an (A, b)-cut in G_1.

$d \notin B$. Let P be the path $NP(a_1)(a_2, d]$ and Q be the path $NP(b)(d, w]$, where w is the B-neighbour of b other than a_1. Let $G_1 = \mathrm{int[QPB[a_2, w]], A_1 = \{a_2\}, b_1 = w}$ and $G_2 = \mathrm{int[ba_1db], A_2 = \{a_1, d\}, b_2 = b}$, as illustrated in Figure 5. By the minimality of G, there is a valid decomposition (D_i, H_i) of (G_i, A_i, b_i) for $i \in [2]$. Let $D_3 = \{(v, w) : v \in P \cup Q - d, w \in N(v) \cap \{a_1, b\}\} \cup \{(d, a_1)\}$. Thus, D_3 is acyclic and $d^+_{D_3}(v) = 1$ for $v \in P \cup Q$. As $d^+_{D_2}(d) = 0$, $D_1 \cup D_2$ is acyclic.

Now we add bd to H_2. Then $d^*_{H_2}(d) = 1$ and $d^*_{H_2}(b) \leqslant 1$ by Proposition 6. Let $D = \cup_{i=1}^3 D_i$ and $H = H_1 \cup H_2$. As D_1, D_2 are acyclic and $d^+_D(a_1) = d^+_D(b) = 0$, D is acyclic. For $v \in V(G_i) - P - Q$ and $i \in [2]$, $d^+_D(v) = d^+_{D_i}(v)$ and $d^*_H(v) = d^*_{H_i}(v)$. Besides, for $v \in P \cup Q - d$, $d^*_H(v) = d^*_{H_1}(v) \leqslant 2$. If v is a (A_1, b_1)-cut, then $d^+_D(v) = d^+_{D_1}(v) + d^+_{D_3}(v) \leqslant 0 + 1 = 1$; otherwise $d^+_D(v) = d^+_{D_1}(v) + d^+_{D_3}(v) \leqslant 1 + 1 = 2$. Meanwhile, $d^+_D(d) = d^+_{D_1}(d) + d^+_{D_2}(d) \leqslant 1+1 = 2$ and $d^*_H(d) = d^*_{H_1}(d) + d^*_{H_2}(d) \leqslant 2+1 = 3$. Therefore, condition (i-iv) of Theorem 5 hold. □

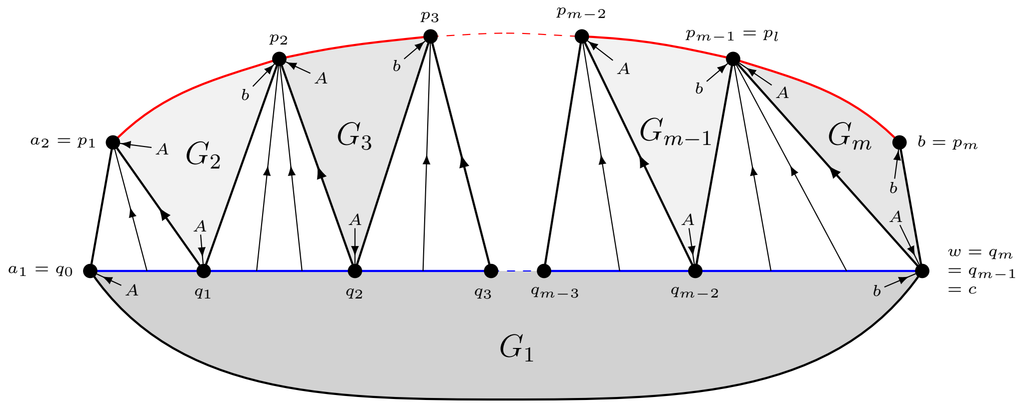

Figure 6: Graph division of final case. In this figure $q_{m-1} = q_m$, $l = m-1$, and $c = w$. Path P is depicted in red, and path Q is depicted in blue.

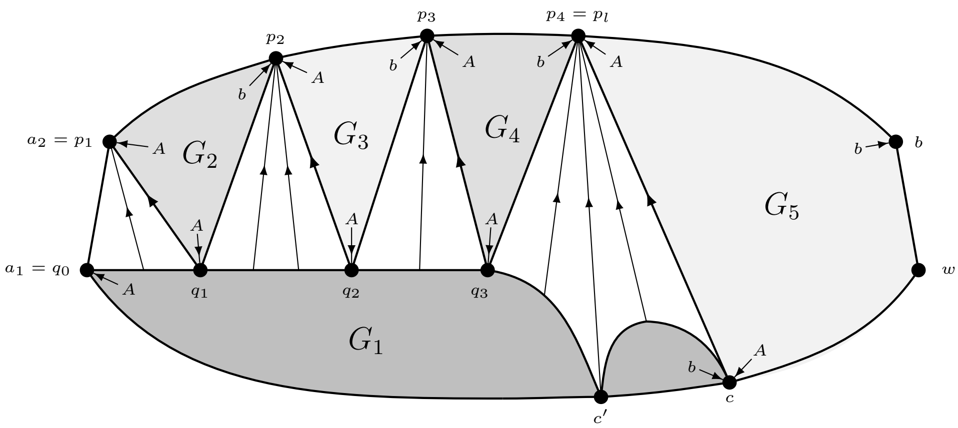

Figure 7: Graph division of final case. In this figure $c \neq w$.

Now we are ready to derive the final contradiction.

Let $P = B[a_2, b]$ and Q denote the unique longest simple path from a_1 to w in the subgraph induced by $V(G) - V(P)$ that traverses only vertices adjacent to P in G, where $w \in B - P$ is the unique B-neighbour of b. Let $p_1 = a_2$, and let $p_2, p_3, \ldots, p_{m-1}$ be the set of all interior vertices of path P that have at least two neighbors in Q, and occur in this order in P, and let $p_m = b$. As G is a near triangulation, for $i \in \{1, \ldots, m-1\}$,

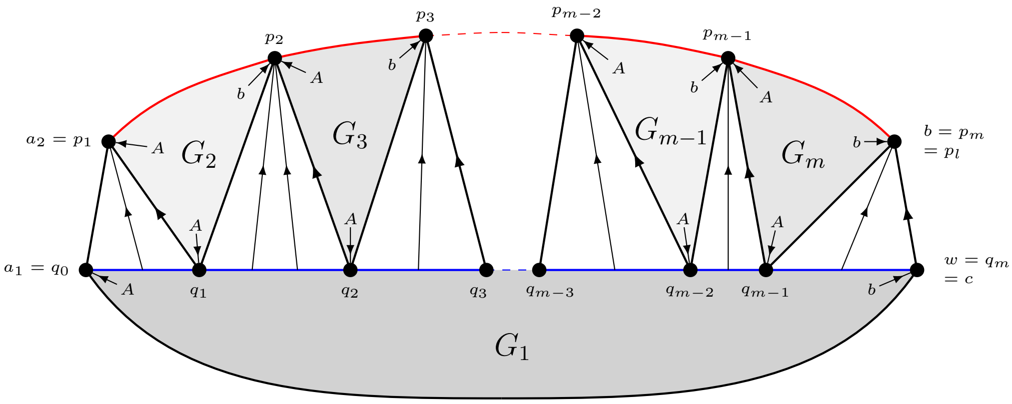

Figure 8: Graph division of final case. In this figure $q_{m-1} \neq q_m$, $l = m$, and $c = w$. Path P is depicted in red, and path Q is depicted in blue.

vertices p_i and p_{i+1} have a unique common neighbor in Q. Let $q_0 = a_1, q_m = w$, and for $i \in \{1, \ldots, m-1\}$, let $q_i \in N(p_i, p_{i+1})$. Observe that for $i, j \in \{0, 1, \ldots, m-1\}$ with $i \neq j$, $q_i \neq q_j$ by the choice of $p_1, p_2, \ldots, p_{m-1}$. Moreover, we have the following observation.

Observation 13. *Based on the choice of p_i, q_i for $i \in [m]$, we have*

1. *for $i \in [m-1]$, $N(q_i) \cap \{p_1, p_2, \ldots, p_m\} = \{p_i, p_{i+1}\}$;*
2. *For $i \in [m]$ and any $v \in Q(q_{i-1}, q_i)$, $N(v) \cap P = \{p_i\}$;*
3. *For $i \in [m]$, $V(\mathrm{int}[\mathrm{Q}[\mathrm{q}_{\mathrm{i}-1}, \mathrm{q}_\mathrm{i}]\mathrm{p}_\mathrm{i}]) = \mathrm{Q}[\mathrm{q}_{\mathrm{i}-1}, \mathrm{q}_\mathrm{i}] \cup \{\mathrm{p}_\mathrm{i}\}$.*

Note that it is possible that $q_{m-1} = q_m$ as shown in Figure 6. We divided the graph into several subgraphs depending on whether $Q \cap B(w, a_1) = \emptyset$ or not. Let $p \in P$ such that $N(p) \cap B[w, a_1] \neq \emptyset$ and for $v \in P[p_1, p)$, $N(v) \cap B[w, a_1] = \emptyset$. The vertex p exists since $p_m = b$ is adjacent to $w \in B[w, a_1)$. Let $c \in N(p) \cap B[w, a_1)$ such that $N(p) \cap B[w, c) = \emptyset$ and observe that $c \in Q$. Let l be the minimal l such that $c \in Q(q_{l-1}, q_l]$ and observe that $p = p_l$. By Remark 11, we have $p_l \neq p_1$ and so $l \in \{2, 3, \ldots, m\}$. Thus, we have the following observation:

Observation 14. *By the choice of p and c, we have*

1. *If $l = m$, then $c = w$; if $l = m-1$, then $q_m = q_{m-1} = w = c$; Otherwise, $c \neq w$.*
2. *$Q(a_1, q_{l-1}) \cap B = \emptyset$ and $Q(q_{l-1}, c) \cap B \subseteq N_B(p_l)$. If $Q(q_{l-1}, c) \cap B \neq \emptyset$, then any $v \in Q(q_{l-1}, c) \cap B$ is an $(\{a_1\}, c)$-cut.*

Let $G_1 = \mathrm{int}[\mathrm{B}[\mathrm{c}, \mathrm{a}_1]\mathrm{Q}(\mathrm{a}_1, \mathrm{c}]], \mathrm{A}_1 = \{\mathrm{a}_1\}, \mathrm{b}_1 = \mathrm{c}$ and $G_i = \mathrm{int}[\mathrm{P}[\mathrm{p}_{\mathrm{i}-1}, \mathrm{p}_\mathrm{i}]\mathrm{q}_{\mathrm{i}-1}\mathrm{p}_{\mathrm{i}-1}]$, $A_i = \{p_{i-1}, q_{i-1}\}, b_i = p_i$ for $i \in \{2, 3, \ldots, l\}$. Additionally, if $l < m$, let $G_{l+1} = \mathrm{int}[\mathrm{B}[\mathrm{p}_\mathrm{l}, \mathrm{c}]\mathrm{p}_\mathrm{l}]$, $A_{l+1} = \{p_l, c\}$ and $b_{l+1} = b$, as illustrated in Figure 7. If $l = m$ (as shown in Figure 8), then let $G_{l+1} = \emptyset$. By the minimality of G, there is a valid decomposition (D_i, H_i) of (G_i, A_i, H_i) for $i \in [l+1]$. Set

$$D_{l+2} = \cup_{i=1}^{l-1}\{(v, p_i) : v \in Q(q_{i-1}, q_i]\} \bigcup \{(v, p_l) : v \in Q(q_{l-1}, c]\}.$$

Then for $v \in Q(a_1, c]$, $d^+_{D_{l+2}}(v) \leqslant 1$. First, we will modify H_i, D_i for $i \in [l+1]$. For $i \in [l+1]$, if $ab_i \in E(B_i)$ for $a \in A_i$, then $d^*_{H_i}(a) = 0, d^*_{H_i}(b_i) \leqslant 1$ and $N_{H_i}(b_i) \subseteq N_{G_i}(a)$. Thus, we add ab_i to H_i, and so (D_i, H_i) is a valid $(2,3)$-decomposition of $G_i - E_{B_i}(A_i)$ such that $d^+_{D_i}(a) = d^+_{D_i}(b_i) = 0$ and $d^*_{H_i}(a), d^*_{H_i}(b_i) \leqslant 1$.

Let $D = \cup_{i=1}^{l+2} D_i$ and $H = \cup_{i=1}^{l+1} H_i$. Note that for $i \in [l]$, $d^+_{D_{i+1}}(q_i) = d^+_{D_{i+1}}(p_i) = 0$. Thus, $\cup_{i=1}^{l+1} D_i$ is acyclic by (2) of Proposition 7. Then as $d^+_D(p_i) = 0$ for $i \in [l]$, D is acyclic by (1) of Proposition 7. For $v \in G - \{p_i : i \in [l]\} - Q(a_1, c]$, v is contained in exactly one subgraph of $G_1, G_2, \ldots, G_{l+1}$, and so condition about $d^+_D(v), d^*_H(v)$ holds. If $v \in Q(a_1, c]$, then $v \in Q(q_{i-1}, q_i]$ for some $i \in [l-1]$ or $v \in Q(q_{l-1}, c]$. Condition (i) holds since

$$d^+_D(v) = \begin{cases} d^+_{D_1}(v) + d^+_{D_{l+2}}(v) \leqslant 0 + 1 = 1, & \text{if } v \text{ is a } (a_1, c)\text{-cut in } G_1, \\ d^+_{D_1}(v) + d^+_{D_{l+2}}(v) \leqslant 1 + 1 = 2, & \text{otherwise;} \end{cases}$$

$$d^*_H(v) = \begin{cases} d^*_{H_1}(v) + d^*_{H_{l+1}}(v) \leqslant 1 + 1 = 2, & \text{if } v = c, \\ d^*_{H_1}(v) + d^*_{H_{i+1}}(v) \leqslant 2 + 1 = 3, & \text{if } v = q_i \text{ and } v \neq c, \\ d^*_{H_1}(v) \leqslant 2, & \text{if } v \in Q(q_{i-1}, q_i) \text{ or } v \in Q(q_{l-1}, c). \end{cases}$$

For $i \in \{2, 3, \ldots, l\}$, vertex p_i satisfies

$$d^+_D(p_i) = d^+_{D_i}(p_i) + d^+_{D_{i+1}}(p_i) = 0 + 0 = 0, d^*_H(p_i) = d^*_{H_i}(p_i) + d^*_{H_{i+1}}(p_i) \leqslant 1 + 1 = 2,$$

and so condition (ii) holds. For vertex $a_i \in A$, observe that a_i plays the role of A in G_i and a_i only appear in G_i for $i \in [2]$. Thus, for $i \in [2]$, $d^+_D(a_i) = 0$ and $d^*_H(a_i) \leqslant 1$. Vertex b plays the role of b in G_l or G_{l+1}, and so $d^+_D(b) = 0$ and $d^*_H(b) \leqslant 1$. Therefore, condition (iv) also holds.

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