The degree and codegree threshold for linear triangle covering in 3-graphs

Yuxuan Tang[a] Yue Ma[b] Xinmin Hou[a,c]

Submitted: Dec 7, 2022; Accepted: Jul 19, 2023; Published: Nov 3, 2023

Abstract

Given two k-uniform hypergraphs F and G, we say that G has an F-covering if every vertex in G is contained in a copy of F. For $1 \leqslant i \leqslant k-1$, let $c_i(n, F)$ be the least integer such that every n-vertex k-uniform hypergraph G with $\delta_i(G) > c_i(n, F)$ has an F-covering. The covering problem has been systematically studied by Falgas-Ravry and Zhao [Codegree thresholds for covering 3-uniform hypergraphs, SIAM J. Discrete Math., 2016]. In 2021, Falgas-Ravry, Markström, and Zhao [Triangle-degrees in graphs and tetrahedron coverings in 3-graphs, Combinatorics, Probability and Computing, 2021] asymptotically determined $c_1(n, F)$ when F is the generalized triangle. In this note, we give the exact value of $c_2(n, F)$ and asymptotically determine $c_1(n, F)$ when F is the linear triangle C_6^3, where C_6^3 is the 3-uniform hypergraph with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\{v_1v_2v_3, v_3v_4v_5, v_5v_6v_1\}$.

Mathematics Subject Classifications: 05C35, 05C07, 05C65

1 Introduction

Given a positive integer $k \geqslant 2$, a *k-uniform hypergraph* (or a k-graph) $G = (V, E)$ consists of a vertex set $V = V(G)$ and an edge set $E = E(G) \subset \binom{V}{k}$, where $\binom{V}{k}$ denotes the set of all k-element subsets of V. We write graph for 2-graph for short. Let $G = (V, E)$ be a k-graph. For any $S \subseteq V(G)$, let $N_G(S) = \{T \subseteq V(G)\backslash S : T \cup S \in E(G)\}$ and the degree $d_G(S) = |N_G(S)|$. For $1 \leqslant i \leqslant k-1$, the *minimum i-degree* of G, denoted by $\delta_i(G)$, is the minimum of $d_G(S)$ over all $S \in \binom{V(G)}{i}$. We also call $\delta_{k-1}(G)$ the minimum codegree of G and $\delta_1(G)$ the minimum vertex-degree of G. The *link graph* of a vertex x in V, denoted by G_x, is a $(k-1)$-graph $G_x = \{V(G)\backslash\{x\}, N_G(x)\}$.

[a]School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China (yx0714@mail.ustc.edu.cn).
[b]School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing, Jiangsu 210094, China (yma@niust.edu.cn).
[c]CAS Key Laboratory of Wu Wen-Tsun Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China. Hefei National Laboratory, University of Science and Technology of China, Hefei, Anhui 230088, China. (xmhou@ustc.edu.cn).

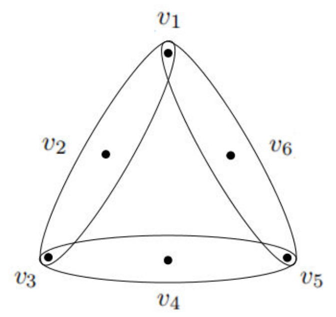

Figure 1: C_6^3

Given a k-graph F, we say a k-graph G has an F-covering if each vertex of G is contained in a subgraph of G isomorphic to F. For $1 \leqslant i \leqslant k-1$, define

$$c_i(n, F) = \max\{\delta_i(G) : G \text{ is a } k\text{-graph on } n \text{ vertices with no } F\text{-covering}\}.$$

For graphs F, in the concluding remarks [10], the authors noted that the covering problem is essentially equivalent to the Turán problem, and Falgas-Ravry and Zhao gave a roughly proof of $c_1(n, F) = \left(\frac{\chi(F)-2}{\chi(F)-1} + o(1)\right) n$ in [3], where $\chi(F)$ is the chromatic number of F. The authors in [3] also initiated the study of the covering problem in 3-graphs, and determined the exact value of $c_2(n, K_4^3)$ (where K_r^3 denotes the complete 3-graph on $r \geqslant 3$ vertices) for $n > 98$ and gave bounds on $c_2(n, F)$ which are apart by at most 2 in the cases where F is K_4^{3-} (K_4^3 with one edge removed, also called a *generalized triangle*), $K_5^{3?}$, and the tight cycle C_5^3 on 5 vertices. Yu, et al [9] showed that $c_2(n, K_4^{3-}) = \lfloor \frac{n}{3} \rfloor$, and $c_2(n, K_5^{3-}) = \lfloor \frac{2n-2}{3} \rfloor$. Last year, Falgas-Ravry, Markström, and Zhao [2] gave close to optimal bounds of $c_1(n, K_4^{(3)})$ and asymptotically determined $c_1(n, K_4^{(3)-})$. There are some other related results in literature, for example in [4, 5].

A *linear triangle* C_6^3 based on the triple $v_1v_3v_5$ is a 3-graph with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\{v_1v_2v_3, v_3v_4v_5, v_5v_6v_1\}$ (as shown in Fig.1). Gao and Han showed in [6] that, when $n \in 6\mathbb{Z}$ is sufficiently large, and H is a 3-graph on n vertices with $\delta_2(H) \geqslant n/3$, then H has a C_6^3-covering such that every vertex H is covered by exactly one C_6^3 (also called a C_6^3-factor or a perfect C_6^3-tiling). In this article, we determine the exact value of $c_2(n, C_6^3)$ and an asymptotic optimal value of $c_1(n, C_6^3)$. The main results are listed as follows.

Theorem 1. *For $n \geqslant 6$, $c_2(n, C_6^3) = 1$.*

Theorem 2. *For $n \geqslant 6$, $\frac{3-2\sqrt{2}}{4}n^2 - n < c_1(n, C_6^3) < \frac{3-2\sqrt{2}}{4}n^2 + 3n^{\frac{3}{2}}$.*

The rest of the article is arranged as follows. In Section 2, we construct extremal graphs with minimum codegree one that have no C_6^3-covering, and minimum degree greater than $\frac{3-2\sqrt{2}}{4}n^2 - n$ that have no C_6^3-covering, respectively. The proofs of Theorems 1 and 2 are given in Sections 3 and 4, respectively.

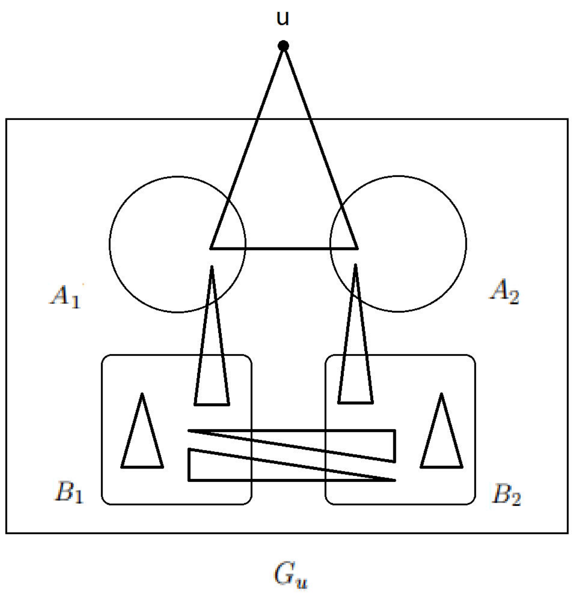

Figure 2: Construction 2

2 Constructions

We introduce two constructions involving our result. For two families of sets $\mathcal{A}$ and $\mathcal{B}$, define $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$.
Construction 1: Let $G_1 = (V_1, E_1)$ be a 3-graph with $V_1 = \{x\} \cup V'$ and

$$E_1 = \{\{x\}\} \vee \binom{V'}{2}.$$

The following observation can be checked directly.

Observation 3. *$\delta_2(G_1) = 1$ and G_1 has no C_6^3-covering (Indeed, the vertex x is not covered in any copy of C_6^3).*

Construction 2: Let $G_2 = (V_2, E_2)$ be a 3-graph with $V_2 = \{u\} \cup A_1 \cup A_2 \cup B_1 \cup B_2$, and

$$E_2 = \left(\{\{u\}\} \vee \binom{A_1}{1} \vee \binom{A_2}{1}\right) \cup \left(\binom{A_1}{1} \vee \binom{B_1}{2}\right) \cup \left(\binom{A_2}{1} \vee \binom{B_2}{2}\right) \cup \binom{B_1 \cup B_2}{3},$$

where $||A_1| - |A_2|| \leqslant 1$, $|B_1| = |B_2| = \lfloor (1 - \frac{\sqrt{2}}{2})n \rfloor$.

Observation 4. *$\delta_1(G_2) > \frac{3-2\sqrt{2}}{4}n^2 - n$ and G_2 has no C_6^3 covering u.*

Proof. It is easy to observe that G_2 has no C_6^3 covering u. Let $b = |B_1| = |B_2| = \lfloor (1 - \frac{\sqrt{2}}{2})n \rfloor$ and $a = \frac{n-1-2b}{2} \geqslant \frac{(\sqrt{2}-1)n-1}{2}$. Without loss of generality, assume $|A_1| = \lfloor a \rfloor$

and $|A_2| = \lceil a \rceil$. Now let us calculate $\delta_1(G_2)$. When $n \leqslant 23$, we have $\frac{3-2\sqrt{2}}{4}n^2 - n < 0$. So we may suppose $n \geqslant 24$. Therefore, $b \geqslant 2$. We have $|A_1| + |A_2| = n - 1 - 2b$. Choose $v \in V(G_2)$.

If $v = u$, then

$$d_{G_2}(v) = \lfloor a \rfloor \cdot \lceil a \rceil \geqslant a^2 - \frac{1}{4} \geqslant \frac{3-2\sqrt{2}}{4}n^2 - \frac{(\sqrt{2}-1)n}{2} > \frac{3-2\sqrt{2}}{4}n^2 - n.$$

If $v \in A_1 \cup A_2$, then

$$d_{G_2}(v) \geqslant \lfloor a \rfloor + \binom{b}{2} \geqslant \frac{(\sqrt{2}-1)n - 3}{2} + \frac{(\frac{2-\sqrt{2}}{2}n - 1)(\frac{2-\sqrt{2}}{2}n - 2)}{2} > \frac{3-2\sqrt{2}}{4}n^2 - n.$$

If $v \in B_1 \cup B_2$, then

$$d_{G_2}(v) \geqslant (b-1) \cdot \lfloor a \rfloor + \binom{2b-1}{2} \geqslant \lfloor a \rfloor + \binom{b}{2} > \frac{3-2\sqrt{2}}{4}n^2 - n.$$

Therefore, $\delta(G_2) > \frac{3-2\sqrt{2}}{4}n^2 - n$.

□

3 Proof of Theorem 1

We first introduce a lemma, which is of great importance to our proof. Let P_k (resp. C_k) denote a path (resp. a cycle) with k vertices.

Lemma 5. *Let G be a 3-graph with $\delta_2(G) \geqslant 2$ and $v \in V(G)$. If v is not covered by any C_6^3, then G_v must be P_5-free and $2P_3$-free.*

Proof. Indeed, let us first suppose that G_v contains a $2P_3$, say, $w_1u_1w_2$ and $w_3u_2w_4$. Since $\delta_2(G) \geqslant 2$, there exists a vertex $x \neq v$, $u_1u_2x \in E(G)$. Since $\{w_1, u_1, w_2\} \cap \{w_3, u_2, w_4\} = \emptyset$, we may assume without loss of generality that w_2 and w_3 are different from x. Then the subgraph induced by $\{v, w_2, u_1, x, u_2, w_3\}$ contains a C_6^3 based on v, u_1 and u_2. This gives the conclusion that G_v does not contain a $2P_3$.

Now suppose that G_v contains a P_5, say, $w_1u_1wu_2w_2$. Then similarly, there exists a vertex $x \neq v$, $u_1u_2x \in E(G)$. If $x \notin \{w_1, w_2\}$, then the subgraph induced by $\{v, w_1, u_1, x, u_2, w_3\}$ contains a C_6^3 based on v, u_1 and u_2. If $x \in \{w_1, w_2\}$, without loss of generality, assume $x = w_1$, then the subgraph induced by $\{v, w, u_1, w_1, u_2, w_2\}$ contains a C_6^3 based on v, u_1 and u_2. This gives the conclusion that G_v is P_5-free. □

Now we are ready to finish the proof of Theorem 1.

Proof of Theorem 1. Suppose to the contrary that there is a 3-graph G with $\delta_2(G) \geqslant 2$ and a vertex $v \in V(G)$ that is not covered by C_6^3. Since $\delta_2(G) \geqslant 2$, we have $\delta(G_v) \geqslant 2$.

By Lemma 5, G_v is P_5-free and $2P_3$-free. Therefore, the longest path in G_v must be P_4 or P_3 (otherwise, G_v must be a matching, which contradicts the fact that $\delta(G_v) \geqslant 2$).

Note that every component of G_v contains a cycle since $\delta(G_v) \geqslant 2$. If the longest path is P_3 in G_v, then every component of G_v must be a K_3. Since $|V(G_v)| \geqslant 5$, we have $G_v \cong tK_3$ for some $t \geqslant 2$. However, this contradicts the fact that G_v is $2P_3$-free.

Now suppose the longest path is P_4 in G_v. Consider the component H that contains $P_4 = u_1w_1w_2u_2$. Since P_4 is a longest path and $d_H(u_1) \geqslant 2$ and $d_H(u_2) \geqslant 2$, it can be easily checked that $C_4 \subseteq H$, and thus, $V(H) = V(P_4)$ (otherwise, H contains a P_5, a contradiction). Since $|V(G_v)| \geqslant n-1 \geqslant 5$, G_v has at least another component H'. Because every component of G_v contains a cycle, we have $|V(H')| \geqslant 3$, a contradiction to the fact that G_v is $2P_3$-free. $\square$

4 Proof of Theorem 2

We need the fundamental result in extremal graph theory due to Turán [8].

Theorem 6 (Turán, 1941)**.** *Let $r \geqslant 2$ and $T_{n,r}$ be the Turán graph. If G is a graph on n vertices containing no K_{r+1} as a subgraph, then $e(G) \leqslant e(T_{n,r})$.*

Now we give the proof of Theorem 2.

Proof of Theorem 2: The lower bound of $c_1(n, C_6^3)$ is a direct corollary of Observation 4. Therefore, it is sufficient to show that every 3-graph G on n vertices with $\delta_1(G) \geqslant \frac{3-2\sqrt{2}}{4}n^2 + 3n^{\frac{3}{2}}$ has a C_6^3-covering.

Suppose to the contrary that there is a 3-graph G on n vertices with $\delta_1(G) \geqslant \frac{3-2\sqrt{2}}{4}n^2 + 3n^{\frac{3}{2}}$ and a vertex $u \in V(G)$ that is not contained in a copy of C_6^3. Recall that the link graph G_u is the graph with vertex set $V' = V \backslash \{u\}$ and edge set $E = \{vw : uvw \in E(G)\}$. Denote M_0 as the set of components isomorphic to K_2 of G_u and let I_0 be the set of components isomorphic to K_1 in G_u. For any $v \in V(G_u)$, let $M(v)$ denote the set of components isomorphic to K_2 in $G_u - \{v\} - V(M_0)$, and let $I(v)$ denote the set of components isomorphic to K_1 in $G_u - \{v\} - I_0$. Note that $M_0 \cap M(v) = \emptyset$ and $I_0 \cap I(v) = \emptyset$ by the definitions.

Claim 7. *Let v be a vertex in G_u with $d_{G_u}(v) \geqslant 4$. Then*

$$E(G_v - u) \subseteq M_0 \cup M(v) \cup \binom{I_0 \cup I(v)}{2}.$$

Proof. Choose $v_1v_2 \in E(G_v - u)$. If $v_1v_2 \notin M_0 \cup M(v) \cup \binom{I_0 \cup I(v)}{2}$, then there is at least one of v_1, v_2, say v_1, that is not in $I_0 \cup I(v)$.

If $v_1 \in V(M_0) \cup V(M(v))$, since $v_1v_2 \notin M_0 \cup M(v)$, there exists v_1' different from v_2, such that $v_1v_1' \in M_0 \cup M(v)$. According to $d_{G_u}(v) \geqslant 4$, there exists v' different from v_1, v_1' and v_2, such that $vv' \in E(G_u)$. In this case, $uv_1'v_1$, v_1v_2v and $vv'u$ form a C_6^3 based on u, v_1 and v covering u, a contradiction. Otherwise, $v_1 \notin I_0 \cup I(v) \cup V(M_0) \cup V(M(v))$. Then $d_{G_u}(v_1) \geqslant 2$. Thus there exists v_1'' different from v_2 such that $v_1v_1'' \in E(G_u)$. Since $d_{G_u}(v) \geqslant 4$, there exists v'' different from v_1, v_1'' and v_2, such that $vv'' \in E(G_u)$. Again we have a copy of C_6^3 formed $uv_1''v_1$, v_1v_2v and $vv''u$ based on u, v_1 and v covering u, a contradiction.

This finishes the proof of our claim.

$\square$

Now, define a vertex $v \in V(G_u)$ to be a *good* vertex if $|I(v)| < n^{\frac{1}{2}}$; a *bad* vertex, otherwise. For a good vertex v, a vertex w in G_v is called *a private* vertex of v if $w \in I_0 \cup I(v)$ and $d_{G_v-u}(w) \geqslant 2$. Let $X(v)$ denote the set of all private vertices of v, i. e.

$$X(v) = \{w \in V(G_v) : w \in I_0 \cup I(v) \text{ and } d_{G_v-u}(w) \geqslant 2\}.$$

For a good vertex v with $d_{G_u}(v) \geqslant 4$, let $x = |X(v)|$ and let $J(v) = (I_0 \cup I(v)) \backslash X(v)$. Denote $H = G_v - u$. By Claim 7, we have

$$d_G(v) \leqslant d_{G_u}(v) + |M_0| + |M(v)| + \left| E(H) \cap \binom{I_0 \cup I(v)}{2} \right|.$$

By the definition of $X(v)$, $d_H(w) \leqslant 1$ for any $w \in J(v)$. Thus $\left| E(H) \cap \binom{I_0 \cup I(v)}{2} \right| \leqslant \sum_{w \in J(v)} d_H(w) + |\binom{X(v)}{2}| \leqslant |I_0| + |I(v)| - x + \binom{x}{2}$. Clearly, $d_{G_u}(v) \leqslant n$ and $|M_0| + |M(v)| \leqslant \frac{n-1-|I_0|-|I(v)|}{2}$. Therefore,

$$d_{G_u}(v) < 2n + \binom{x}{2}.$$

Note that

$$\frac{3-2\sqrt{2}}{4} n^2 + 3n^{\frac{3}{2}} \leqslant \delta_1(G) \leqslant d_G(v) < 2n + \binom{x}{2}.$$

Therefore, we have

$$|X(v)| = x \geqslant \frac{1 + \sqrt{(2-\sqrt{2})^2 n^2 + 24n^{\frac{3}{2}} - 12n + 1}}{2} \geqslant \left(1 - \frac{\sqrt{2}}{2}\right) n + n^{\frac{1}{2}}. \qquad (1)$$

Now, let us compute the number of edges of G_u. Define a *bad edge* in G_u as an edge that is adjacent to at least one bad vertex, and a *good edge* if its two ends are good. Let E_1 denote the set of bad edges in G_u, let E_2 be the set of good edges with one end of degree at most 3, and let E_3 denote the remaining edges in G_u. Then $|E(G_u)| = |E_1| + |E_2| + |E_3|$. By the definition of E_2, we have $|E_2| \leqslant 3n$. Note that $I(v_1) \cap I(v_2) = \emptyset$ for different vertices $v_1, v_2 \in V(G_u)$. Since $|I(v)| \geqslant n^{\frac{1}{2}}$ for a bad vertex $v \in V(G_u)$, the number of the bad vertices is at most $n/n^{\frac{1}{2}} = n^{\frac{1}{2}}$. Therefore, they will contribute at most $n^{\frac{3}{2}}$ edges. That is, $|E_1| \leqslant n^{\frac{3}{2}}$. Since $n \geqslant 6$, and

$$n^{\frac{3}{2}} + 3n + |E_3| = |E_1| + |E_2| + |E_3| = d_1(u) \geqslant \delta_1(G) \geqslant \frac{3-2\sqrt{2}}{4} n^2 + 3n^{\frac{3}{2}}, \qquad (2)$$

we have $E_3 \neq \emptyset$.

Claim 8. *For any $e = v_1v_2 \in E_3$, $X(v_1) \cap X(v_2) = \emptyset$.*

Proof. Suppose to the contrary that there exists $w \in X(v_1) \cap X(v_2)$. Since $d_{G_{v_i}-u}(w) \geqslant 2$ for $i = 1, 2$, there are two different vertices y_1, y_2 with $y_1w \in E(G_{v_1} - u)$ and $y_2w \in E(G_{v_2} - u)$. Therefore, v_1y_1w, wy_2v_2 and v_2uv_1 form a copy of C_6^3 based on v_1, w and v_2, a contradiction. □

Let $e = v_1v_2 \in E_3$ and $x_i = |X(v_i)|$ for $i = 1, 2$. By (1) and Claim 8, we have

$$2\left(1 - \frac{\sqrt{2}}{2}\right)n + 2n^{\frac{1}{2}} \leqslant x_1 + x_2 = |X(v_1) \cup X(v_2)| \leqslant |I_0 \cup I(v_1) \cup I(v_2)| < |I_0| + 2n^{\frac{1}{2}}, \quad (3)$$

the last inequality holds since $|I(v)| < n^{\frac{1}{2}}$ for any good vertex $v \in V(G_u)$. Therefore, $|I_0| > 2(1 - \frac{\sqrt{2}}{2})n$.

If $G_u[E_3] := G_u \cap E_3$ contains no K_3, then, by Theorem 6 and (3),

$$|E_3| \leqslant e(T_{n-1-|I_0|,2}) \leqslant \frac{(n-1-|I_0|)^2}{4} < \frac{\left((\sqrt{2}-1)n-1\right)^2}{4}.$$

Therefore, by (2),

$$d_1(u) = |E(G_u)| = \sum_{i=1}^{3} |E_i| < \frac{\left((\sqrt{2}-1)n-1\right)^2}{4} + 3n + n^{\frac{3}{2}} \leqslant \frac{3-2\sqrt{2}}{4}n^2 + 3n^{\frac{3}{2}},$$

a contradiction.

Now assume $G_u[E_3]$ contains a copy of K_3, say $v_1v_2v_3$. Let $x_i = |X(v_i)|$ for $i = 1, 2, 3$. Again by (1) and Claim 8, we have

$$3\left(1 - \frac{\sqrt{2}}{2}\right)n + 3n^{\frac{1}{2}} \leqslant x_1 + x_2 + x_3 \leqslant |I_0 \cup I(v_1) \cup I(v_2) \cup I(v_3)| < |I_0| + 3n^{\frac{1}{2}}.$$

Therefore, $|I_0| > 3(1 - \frac{\sqrt{2}}{2})n$. Thus we have

$$d_1(u) = |E(G_u)| \leqslant \binom{n-1-|I_0|}{2} < \binom{\frac{3\sqrt{2}-4}{2}n-1}{2} < \frac{3-2\sqrt{2}}{4}n^2 + 3n^{\frac{3}{2}},$$

a contradiction.

This completes the proof of Theorem 2. □

5 Discussion and Remarks

In this note, we show that $\frac{3-2\sqrt{2}}{4}n^2 - n < c_1(n, C_6^3) < \frac{3-2\sqrt{2}}{4}n^2 + 3n^{\frac{3}{2}}$ for $n \geqslant 6$ gave an optimal extremal construction (see Construction 2) with minimum degree greater than $\frac{3-2\sqrt{2}}{4}n^2 - n$. It will be interesting to show that $c_1(n, C_6^3) = \frac{3-2\sqrt{2}}{4}n^2 - O(n)$, we leave this as a problem.

Acknowledgements

Thanks to the anonymous referee for their many helpful comments to improve the paper. This work was supported by the National Natural Science Foundation of China (No. 12071453), the National Key R and D Program of China (2020YFA0713100), Anhui Initiative in Quantum Information Technologies (AHY150200), and the Innovation Program for Quantum Science and Technology (2021ZD0302904).

References

[1] B. Ergemlidze, E. Győri and A. Methuku, 3-uniform hypergraphs and linear cycles. *SIAM J. Discrete Math.*, 32(2), 933—950, 2018.

[2] V. Falgas-Ravry, K. Markström, and Y. Zhao, Triangle-degrees in graphs and tetrahedron coverings in 3-graphs. *Combinatorics, Probability and Computing*, 30(2), 175—199, 2021.

[3] V. Falgas-Ravry and Y. Zhao, Codegree thresholds for covering 3-uniform hypergraphs. *SIAM J. Discrete Math.*, 30(4) (2016), 1899-1917.

[4] A. Freschi, and A. Treglown, Dirac-type results for tilings and coverings in ordered graphs. *Forum of Mathematics, Sigma*, Vol. 10, E104, 2022. doi:10.1017/fms.2022.92

[5] Z. Füredi and Y. Zhao, Shadows of 3-Uniform Hypergraphs under a Minimum Degree Condition. *SIAM J. Discrete Math.*, 36(4), 2523—2533, (2022).

[6] W. Gao and J. Han, Minimum codegree threshold for C_6^3-factors in 3-uniform hypergraphs, *Combinatorics, Probability and Computing*, 26, 536—559, 2017.

[7] W. Mantel, Problem 28. *Wiskundige Opgaven*, 10, 60—61, 1907.

[8] P. Turán, On an extremal problem in graph theory. *Math. Fiz. Lapok*, 48, 436—452, 1941.

[9] L. Yu, X. Hou, Y. Ma and B. Liu, Exact minimum codegree thresholds for K_4^--covering and K_5^--covering. *The Electronic Journal of Combinatorics*, 27(3):#P3.22, 2020.

[10] C. Zang, Matchings and tilings in hypergraphs. PhD thesis, Georgia State University, 2016.