Generalized Heawood numbers

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Abstract

This survey explains the origin and the further development of the Heawood inequalities, the Heawood number, and generalizations to higher dimensions with results and further conjectures.

Mathematics Subject Classifications: 05C15, 05C10, 52B70, 57Q15, 57Q35

1 The classical Heawood problem on surfaces

The original motivation for Heawood's[1] investigations [27] seems to be the problem of coloring maps on surfaces. A *surface* is interpreted as an abstract two-dimensional compact and connected manifold without boundary. By definition it is assumed that every point has an open neighborhood that is homeomorphic with an open planar disc, moreover any two points have to be separable by two disjoint open neighborhoods (Hausdorff axiom). A *map* on a surface is a decomposition into compact "countries" that meet along their common boundaries. A *coloring* of a map assigns a color to each of the parts such that adjacent "countries" get different colors. For the 2-sphere (or, equivalently, the euclidean plane) the celebrated 4-color theorem was incorrectly proved in the 19^{th} century, it remained unsolved until recently. In 1890 Heawood pointed out the error and proved a 5-color theorem [27]. Moreover, he observed the following:

Proposition 1. *The torus admits a decomposition into* 7 *countries (each being a topological disc, actually an abstract hexagon) that are mutually adjacent along pieces of their borders. So any coloring requires* 7 *colors.*

For a picture see [49, p.3].

Based on that observation the *map color problem* for closed surfaces was formulated. Heawood did not solve it; he just showed a necessary condition involving what we now call the *Heawood number* of a surface, see below. The solution to the problem covers the

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[1] Percy John Heawood, 1861–1955, Professor of Mathematics at the University of Durham, UK

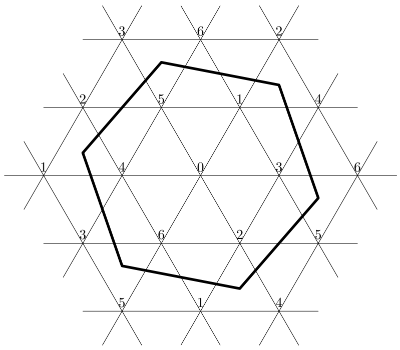

Figure 1: The unique 7-vertex triangulation of the torus

entire book [49], compare also [25, 44, 59]. Here we focus on the possible generalizations explained in the subsequent sections. Sarkarias's approach [51] to Heawood inequalities is somewhat different.

If we put a vertex into the interior of each country and if we join two such vertices if and only if the two countries are adjacent, we obtain an embedding of a graph into the same surface, in the torus case it is the *complete graph* K_7 on 7 vertices. In general this *duality principle* transforms the problem of coloring a map into the equivalent problem of coloring an embedded graph such that two adjacent vertices get distinct colors. In this setting it is quite plausible that attaching a "handle" in form of a "bridge" or a "cylinder" to a surface will change the situation and make it possible to add further edges, namely, such using this bridge. However, it turns out that any intuition is practically impossible how many bridges one needs for attaching a certain number of edges. Attaching one bridge to the plane or the sphere enables us to add 6 edges to the spherical situation since a triangulated 2-sphere with 7 vertices has 15 edges and since $\binom{7}{2} = 21$ which is realized by the torus. In fact, the possible genus of the surface grows quadratically with the number of vertices, see below.

At this point we can see that the usual standard model for surfaces in topology *"an orientable surface of genus g is represented as a $4g$-gon in the plane with certain identifications on the boundary"* does not allow to draw a complete graph of the appropriate maximum size within this $4g$-gon. Instead, many edges of the graph with have to cross the edges of this $4g$-gon which, in fact, are not used as edges. However, for the torus this is still possible: The graph K_7 can be drawn within a planar square or a hexagon, see Fig. 1. Any graph embeddable into the torus can be drawn by straight edges in such a planar fundamental domain [43], usually a quadrilateral or a hexagon. In the case of K_7 in the torus any embedding is *triangular* meaning that its complement consists of abstract triangles only, so it induces a triangulation of the torus, see Fig. 1.

Such a triangulation with a complete edge graph is also called 2-*neighborly* because any two vertices are direct neighbors to one another in the sense that they are joined by an edge. A 2-neighborly triangulation of a surface has the minimum possible number of vertices: if we want to remove one vertex then we would have to include more edges than pairs of vertices are avaiable. This is impossible for a (simplicial) triangulation since there can be at most one edge between two vertices.

In the torus case we see from Fig. 1 that in an appropriate quotient of the triangular tessellation $\{3,6\}$ of the euclidean plane (a *flat torus*) the full automorphism group of order 42 acts by euclidean symmetries: 7 translations and 6 rotations around each vertex, compare Fig. 1. The group can be interpreted as the affine group $A(1,\mathbb{Z}_7)$. This structure is unique, it has been known already to Cayley [15] and Möbius around 1850, and it is an important example in various branches of geometry and combinatorics like block designs, convex polytopes, tight embeddings of surfaces and regular maps [36]. In particular it separates the boundary complex of the cyclic 4-polytope [60] with 7 vertices into two solid tori that are isomorphic but not identical, a fact further exploited in [7] for obtaining a 4-dimensional triangulation. A projection of this boundary complex onto euclidean 3-space leads to *Császár's torus* [17], a polyhedron in 3-space without diagonals, compare [8].

Motivating classical problems:

1. Find the *coloring number* of a given surface, i.e., the smallest number of colors that is sufficient for coloring any given map on that surface.

2. Find the smallest number of vertices that is necessary for a (simplicial) triangulation of a given surface.

3. Find the maximal n such that the complete graph K_n can be embedded into a given surface.

It turns out that the solutions to all three problems lead to the same number, namely, the so-called *Heawood number* listed already in [27]. Before stating the results we describe the heart of the argument in an elementary way:

The simplest example of a complete graph in a surface is the K_4 in the sphere. Obviously one needs all $\binom{4}{2} = 6$ edges for obtaining a triangulation, the tetrahedron. Already in a 5-vertex triangulation we have one missing edge, and the number of missing edges grows with the number of vertices by the classical Euler formula $V - E + F = 2$ which is equivalent to $V - \frac{E}{3} = 2$ and to $\binom{V}{2} - E = \binom{V-3}{2}$. On a closed surface M with Euler characteristic $\chi(M)$ the same consideration reads as $V - \frac{E}{3} = \chi(M)$ and $\binom{V}{2} - E = \binom{V-3}{2} + 3\big(\chi(M) - 2\big)$. Since the number of missing edges cannot be negative this implies the solution to Problem 2:

Proposition 2. (First Heawood inequality)
For any n-vertex triangulation of a surface M the inequality

$$\binom{n-3}{2} \geqslant 3(2 - \chi(M))$$

holds with equality if and only if the are no missing edges, i.e., if the number E of edges satisfies $E = \binom{n}{2}$.

On the other hand, if we start with an embedding of the complete graph K_n into M, then the complement $M \setminus K_n$ consists of a number F of open countries (each connected but not necessarily simply connected). Each country F_i is bounded by $c_i \geqslant 3$ edges and has $\chi(F_i) \leqslant 1$ which in turn implies the inequality $\chi(M \setminus K_n) \leqslant F$. On the other hand we have $2\binom{n}{2} = \sum_i c_i \geqslant 3F$ since every edge occurs in the boundary of precisely two countries. This implies

$$\chi(M) = n - \binom{n}{2} + \chi(M \setminus K_n) \leqslant n - \binom{n}{2} + F \leqslant n - \frac{1}{3}\binom{n}{2} = \frac{n}{6}(7-n) = -\frac{1}{3}\binom{n-3}{2} + 2$$

or, equivalently, the solution to Problem 3:

Proposition 3. (Second Heawood inequality)
If a surface M admits an embedding of the complete graph K_n then the inequality

$$\binom{n-3}{2} \leqslant 3(2 - \chi(M))$$

holds with equality if and only if the embedding is triangular, i.e., its complement consists only of triangles.

The Heawood inequalities differ only by their direction, therefore the case of equality is most interesting: These are triangulations with a complete edge graph or, equivalently, triangular embeddings of the complete graph [25, 49]. These are also called the *regular cases* in the Heawood problem where the equation $\binom{n-3}{2} = 3(2 - \chi(M))$ is algebraically equivalent to

$$n = \frac{1}{2}\Big(7 + \sqrt{49 - 24\chi(M)}\Big) = \frac{1}{2}\Big(7 + \sqrt{1 + 24\beta_1(M)}\Big), \tag{1}$$

where $\beta_1(M)$ is the first $\mathbb{Z}_2$-Betti number of M, similarly for the two Heawood inequalities. The right hand expression of Equation (1) is called the *Heawood number of M* [2]. It grows like the square root of the genus. A necessary condition for the regular cases is, of course, the integrality of the Heawood number. The first regular cases are listed in the following table where $g = 2\beta_1$ refers to the genus in the orientable cases with $n \equiv 0, 3, 4, 7$ (12):

χ	2	1	0	-3	-5	-10	-13	-20	-24	-33	-38
g	0		1			6		11	13		20
n	4	6	7	9	10	12	13	15	16	18	19

Already with 19 vertices one can triangulate an orientable surface of genus 20, for the construction see Section 5. So it is not uninteresting that even for polyhedral surfaces

[2] Often the largest integer not greater than that expression is called the Heawood number; this difference is not important here.

embedded into ordinary 3-space the number n of vertices can be smaller than the genus of the surface: There is an example of genus 577 with $n = 576$ vertices [42].

The results by G.Ringel and many others [49, 48, 31] show the following (for the method compare also Section 6):

Theorem 4. *The first Heawood inequality is sharp for all topological types of surfaces except for the Klein bottle, the real projective plane with one handle, and the orientable surface of genus* 2.

The second Heawood inequality is sharp for all topological types of surfaces except for the Klein bottle.

In the exceptional case of the Klein bottle the complete graph K_6 can be embedded, and at least 8 vertices are required for a triangulation. Surprisingly, there is no 9-vertex triangulation of the orientable surface of genus 2 although the Heawood number $n = \frac{1}{2}\big(7 + \sqrt{49 + 24 \cdot 2}\big)$ lies between 8 and 9 and since there is a 9-vertex triangulation of a 2-dimensional complex which is homotopy equivalent to this surface [9]. An embedding of the complete graph K_8 appears as part of a 9-vertex triangulation of a pinched surface of the same genus [35, p.37], [36].

Open Problem. [36] *Decide in which cases the Heawood number can be realized by triangulations of pinched surfaces with given numbers and multiplicities of pinch points.*

In the case of the simply pinched surface of genus 2 we have $\chi = -3$, so this is a regular case with $n = 9$. The Heawood number depends only on the Euler characteristic.

Obviously, the realization of a triangulated surface by an embedded polyhedron in 3-space is much more restrictive: Depending on the positions of the vertices, edges and triangles can intersect in their interiors. So one has to be very careful to avoid that.

Open Problem. *How can one decide in a geometric or combinatorial way – besides using oriented matroids [52] – whether a given abstract triangulation of a surface is realizable in* 3*-space with straight edges and planar triangles?*

In particular, for triangulated surfaces with a small number of vertices near the Heawood number this is a difficult problem. Already the regular case $n = 12, g = 6$ turned out to be nonrealizable by a polyhedron with straight edges and planar triangles:

Theorem 5. (Császár, Bokowski, Brehm, Hougardy, Lutz, Zelke, Schewe)
1. *The torus with* 7 *vertices is realizable in* 3*-space [17, 8], especially in the Schlegel diagram of the cyclic* 4*-polytope with* 7 *vertices.*

2. *All vertex-minimal triangulations of orientable surfaces of genus* 3 *(with* 10 *vertices) and genus* 4 *(with* 11 *vertices) are realizable in* 3*-space [28].*

3. *Some* 12*-vertex triangulations of the orientable surface of genus* 5 *are realizable in* 3*-space [28].*

4. *None of the* 59 *combinatorially distinct* 12*-vertex triangulation of the orientable surface of genus* 6 *is realizable in* 3*-space [52].*

The missing case of genus 2 is not interesting since there is no (abstract) 9-vertex triangulation. With 10 vertices it is realizable because even genus 3 is realizable. It is a long-standing conjecture [19] that every triangulated torus is realizable in 3-space. Triangulations with a small number $n \leqslant 23$ of vertices have been enumerated by F.H.Lutz using a computer program. For surfaces with boundary this is completely different: It was shown by Brehm [10] that a certain triangulated Möbius band is not realizable as a polyhedron in 3-space with straight edges. In contrast, the (smallest) 5-vertex triangulation of the Möbius band is realizable in the Schlegel diagram of the 4-simplex.

Finally, the solution to Problem 1 turns out to be equivalent to the solution to Problem 3:

Theorem 6. (G.Ringel [49, Thm.2.4])
The coloring number of a surface M coincides with the maximum n such that the complete graph K_n admits an embedding into M.

The case of the 2-sphere is not included in Ringel's result, in fact, the celebrated 4-color problem on the sphere (or the plane) was solved much later with a huge effort. Nevertheless, K_4 admits an embedding into the plane, K_5 does not, and 4 happens to be also the coloring number.

2 The Heawood number of a 4-manifold

Higher dimensional generalizations can be considered for *pure hypergraphs* instead of graphs or higher dimensional manifolds instead of surfaces. Hypergraphs can also be considered as simplicial complexes. In higher dimensions Problem 2 makes sense as stated since it is plausible that any topologial type of a manifold defines the miminum number of vertices that is possible for a triangulation of that manifold. An a priori bound can be found in [11].

On the other hand: Any graph admits an embedding into any 3-manifold, and any 3-manifold admits a 2-neighborly triangulation with arbitrarily large number of vertices [50], so neither Problem 1 nor Problem 3 make sense as stated. Instead we have to find a higher dimensional analogue that admits generalized Heawood inequalities. A natural obstruction to a triangulation with a small number of vertices is the fact that few edges, few triangles and so on cannot realize a complicated topology or large Betti numbers. A natural obstruction to a small coloring number is the occurrence of many neighbors of many countries.

There is the similar concept of *balanced triangulations* of d-dimensional complexes admitting a $(d+1)$-coloring of the edge graph [56]. However, for odd-dimensional manifolds the Euler characteristic always vanishes, so it is impossible to obtain direct analogues of the Heawood inequalities. Therefore we start with even-dimensional manifolds.

Embedding the 1-skeleton of a simplex into manifolds is easy in higher dimensions, but embedding the 2-skeleton of a simplex into a 4-manifold will not be possible in general because two triangles can intersect in their interiors, see Theorem 21 below for the case of the 4-sphere. This indicates that we can expect a natural obstruction for embedding the k-skeleton of a simplex into a given $2k$-manifold, compare Section 4. This is related to the following question: How many triangles can an n-vertex triangulation of a 4-manifold have?

Notation. In the sequel we denote the number of i-dimensional faces of a simplicial complex by f_i. Altogether they form the *f-vector* where often formally $f_{-1} = 1$ is used in notations. We also use $n = f_0$ if this is convenient.

The 4-dimensional sphere can be triangulated with any number $n \geqslant 6$ of vertices and $f_1 = \binom{n}{2}$ edges. However, the number of triangles then behaves like the number of edges in a surface. The f-vector satisfies the so-called *Dehn-Sommerville equations*

$$\begin{aligned} n - f_1 + f_2 - f_3 + f_4 &= \chi(M) \\ 2f_1 - 3f_2 + 4f_3 - 5f_4 &= 0 \\ 2f_3 - 5f_4 &= 0 \end{aligned}$$

By eliminating f_3 and f_4 we see that the f-vector is completely determined by f_0, f_1, f_2, and these satisfy:

$$10n - 4f_1 + f_2 = 10\chi(M) = 20$$

Under the assumption $f_1 = \binom{n}{2}$ this in turn implies that the number of missing triangles is

$$0 \leqslant \binom{n}{3} - f_2 = \binom{n}{3} - 4\binom{n}{2} + 10n - 20 = \binom{n-4}{3}.$$

Consequently, for $n \geqslant 7$ there are necessarily missing triangles, just as we had to expect missing edges in the case of the 2-sphere. A refinement of this argument with $\chi(M)$ instead of $\chi(S^4) = 2$ leads to the following:

Proposition 7. (Kühnel [35, Thm.4.9])
For any n-vertex triangulation of a closed 4-manifold M the inequality

$$\binom{n-4}{3} \geqslant 10\big(\chi(M) - 2\big) \tag{2}$$

holds with equality if and only if $f_2 = \binom{n}{3}$. This absense of missing triangles is also called 3-neighborliness. *Equality is possible only if M is simply connected.*

3-neighborliness of a manifold of dimension $d \geqslant 4$ can also be interpreted as follows: In the dual map each vertex appears as a "country", each edge as a pair of adjacent countries, normally with a $(d-1)$-dimensional intersection. So 3-neighborliness means that any three countries have a non-empty intersection along their boundaries (normally $(d-2)$-dimensional). In this sense any three countries are simulteneously "neighbors" of each other. Especially in dimension 4 the dual complex has the same number of 2-faces as the original complex, and that is $\binom{n}{3}$ for 3-neighborly triangulations with n vertices.

Equality in Proposition 7 implies that M is simply connected: Any loop can be deformed into the 2-skeleton, and this space is simply connected. Conversely, if for a simply connected 4-manifold M we denote by $\beta_2(M) = \chi(M) - 2$ the second Betti number of M, then this number is a kind of a "genus" of the manifold. In this case the Heawood inequality transforms into $\binom{n-4}{3} \geqslant 10\beta_2$ or, equivalently,

$$(n-5)^3 - (n-5) - 60\beta_2 \geqslant 0.$$

By Cardano's formula for the roots of a cubic polynomial this is equivalent to

$$n \geqslant 5 + \sqrt[3]{z_0} + \frac{1}{3\sqrt[3]{z_0}} \tag{3}$$

with $z_0 = 30\beta_2 + \sqrt{900\beta_2^2 - \frac{1}{27}}$ satisfying the equation $z_0^2 - 60\beta_2 z_0 + \frac{1}{27} = 0$, so we can call the right hand side of inequality (3) the 4-*dimensional Heawood number*.

For $\beta_2 = 1$ we obtain $z_0 = 30 + \sqrt{\frac{24299}{27}}$, hence $\sqrt[3]{z_0} + \frac{1}{3\sqrt[3]{z_0}} = 4$ and $n \geqslant 9$. In other words: The 4-dimensional Heawood number of the complex projective plane is 9. This is the next case of equality after the boundary of a 5-simplex according to the following:

Corollary 8. *For a simply connected* 4*-manifold the* 4*-dimensional Heawood number* n *is an integer if and only if*

$$n \equiv 0,\ 1,\ 4,\ 5,\ 6,\ 9,\ 10,\ 14,\ 16\ (20).$$

This follows by considering divisors of $\binom{n-4}{3}$. The equality $\binom{n-4}{3} = 10(\chi(M) - 2) = 10\beta_2$ requires 10 to be a divisor. The corresponding second Betti number is the quotient:

$$\beta_2 = \frac{1}{10}\binom{n-4}{3}$$

Therefore $(n-4)(n-5)(n-6)$ must be divisible by 20. The sequence of possible integer values starts with $\beta_2 = 0,\ 1,\ 2,\ 12,\ 22,\ 56$ within the range $6 \leqslant n \leqslant 20$. ☐

Corollary 9. *The (theoretical) minimum number of vertices for a triangulation of* M *according to inequality (2) is given in the following table:*

M	n	β_2	equality (3-neighborly)	reference
S^4	6	0	yes, unique	$\partial\Delta^5$
$\mathbb{C}P^2$	9	1	yes, unique	[37]
$S^2 \times S^2$	10	2	no	[37]
$\mathbb{C}P^2\#(\pm\mathbb{C}P^2)$	10	2	no	[37]
$(S^2 \times S^2)^{\#6}$	14	12	?	Sect. 5
K3 surface	16	22	yes	[14]
$(S^2 \times S^2)^{\#11}$	16	22	?	
$(S^2 \times S^2)^{\#28}$	20	56	?	

A general reference for small triangulations is the computer aided enumeration in [40] including the cases of $S^2 \times S^2$ and $(S^2 \times S^2)\#(S^2 \times S^2)$. For a 16-vertex triangulation of $(S^2 \times S^2)^{\#7}$ containing all triangles of the 8-dimensional cross polytope (the generalized octahedron) see [21]. There are two natural 10-vertex triangulations of $\mathbb{C}P^2$, necessarily with missing edges and/or missing triangles [6, 7].

Open Problem. *Decide the cases indicated by a question mark in the table above.*

As in the case of surfaces, the opposite inequality can be conjectured to constitute a necessary condition for the embeddability of the complete 2-skeleton of an $(n-1)$-dimensional simplex into a simply connected 4-manifold M with second Betti number $\beta_2(M)$ (see Section 4 for details):

$$\binom{n-4}{3} \leqslant 10\big(\chi(M)-2\big) = 10\ \beta_2(M) \tag{4}$$

3 Lower bounds for the number of vertices

Definition 10. A simplicial complex with n vertices is called *k-neighborly* if the number of $(k-1)$-dimensional simplices equals $\binom{n}{k}$.

2-neighborliness means that any vertex is joined with any other by an edge. A triangulation of a closed 2-dimensional surface M is 2-neighborly if and only if $n = \frac{1}{2}\big(7+\sqrt{49-24\chi(M)}\big)$ with the Heawood number as above, and 3-neighborliness of a triangulated 4-manifold was discussed in Section 2.

Just as two edges in a 2-manifold can intersect each other, so two triangles in a 4-manifold can intersect each other, two k-dimesional simplices in a $(2k)$-manifold can intersect each other, therefore in general the higher dimensional analogue of 2-neighborliness for surfaces will be $(k+1)$-neighborliness for $2k$-manifolds. The idea is: If a triangulation is saturated in the sense that with these vertices no additional edges, triangles etc. can be inserted, then the number of vertices will be smallest possible. The following result is not hard to obtain:

Proposition 11. *Let M be a closed $(k-1)$-connected $2k$-manifold with a triangulation with n vertices which is k-neighborly. Then the* generalized Heawood inequality

$$\binom{n-k-2}{k+1} \geqslant (-1)^k \binom{2k+1}{k+1}\big(\chi(M)-2\big)$$

holds with equality if and only if the triangulation is $(k+1)$-neighborly.

With the notation $n_{k+1} = n(n-1)\cdots(n-k)$ this inequality can be written as

$$(n-k-2)_{k+1} \geqslant (2k+1)_{k+1} \cdot (-1)^k \big(\chi(M)-2\big).$$

Sketch of proof. The proof uses the Dehn-Sommerville equations and the Vandermonde convolution formula for binomial coefficients, see [35, p.65]. For a proof of the Dehn-Sommerville equations see [60]. In the case of a $2k$-manifold M these equations are the following:

$$\begin{aligned}
n - f_1 + f_2 - f_3 + f_4 - f_5 + f_6 - f_7 + f_8 - \cdots + f_{2k} &= \chi(M) \\
\tbinom{2}{1} f_1 - \tbinom{3}{1} f_2 + \tbinom{4}{1} f_3 - \tbinom{5}{1} f_4 + \tbinom{6}{1} f_5 - \tbinom{7}{1} f_6 + \cdots - \tbinom{2k+1}{1} f_{2k} &= 0 \\
\tbinom{4}{3} f_3 - \tbinom{5}{3} f_4 + \tbinom{6}{3} f_5 - \tbinom{7}{3} f_6 + \cdots - \tbinom{2k+1}{3} f_{2k} &= 0 \\
\tbinom{6}{5} f_5 - \tbinom{7}{5} f_6 + \cdots - \tbinom{2k+1}{5} f_{2k} &= 0 \\
&\vdots \\
\tbinom{2k-2}{2k-3} f_{2k-3} - \tbinom{2k-1}{2k-3} f_{2k-2} + \tbinom{2k}{2k-3} f_{2k-1} - \tbinom{2k+1}{2k-3} f_{2k} &= 0 \\
2 f_{2k-1} - (2k+1) f_{2k} &= 0
\end{aligned}$$

By eliminating the variables $f_{k+1}, \ldots, f_{2k}$ one obtains one equation between the other variables $n, f_1, \ldots, f_k$ and $\chi(M)$. If one puts in $f_1 = \binom{n}{2}, f_2 = \binom{n}{3}, \ldots, f_{k-1} = \binom{n}{k}$ then the assertion of Proposition 11 boils down to the obvious inequality $f_k \leqslant \binom{n}{k+1}$. $\square$

Example 12. We know the following cases of $(k+1)$-neighborly triangulations of $2k$-manifolds:

1. $k = 1$: Examples for any $n \not\equiv 2(3)$, see Section 1.

2. $k = 2$: The unique 9-vertex triangulation of $\mathbb{C}P^2$ [35, Sect.4B], [3], [53] and a 16-vertex triangulation of the K3-surface [14].

3. $k = 3$: Several 13-vertex triangulations of $S^3 \times S^3$ [40].

4. $k = 4$: Several 15-vertex triangulations of the quaternionic projective plane $\mathbb{H}P^2$ [12, 24].

5. $k = 8$: Many 27-vertex triangulations of a 16-manifold “like the octonionic plane”, see [22].

The hard question is: Does Proposition 11 remain valid without the assumption of k-neighborliness? This has been posed as Conjecture B in [35, p.61] and in [34]. A direct proof attempt runs into the problem of finding a sharp bound for an alternating sum. However, for $n > k^2 + 4k + 2$ it follows from a simple comparison of the number of missing k-simplices and missing $(k-1)$-simplices [35, p.66]. This is typical also for the Upper Bound Problem [60, Sect.8.4] for polytopes and spheres and its solution. The general case requires serious algebraic methods. These grew out of 50 years development of understanding face-vectors of simplicial polytopes and spheres:

Theorem 13. (Novik and Swartz [46, Thm.4.4])
Let M be a closed $2k$-manifold, triangulated with n vertices. Then the generalized Heawood inequality

$$\binom{n-k-2}{k+1} \geqslant (-1)^k \binom{2k+1}{k+1} \big(\chi(M) - 2\big) \tag{5}$$

holds with equality if and only if the triangulation is $(k+1)$-neighborly. The latter implies that M is $(k-1)$-connected and $(-1)^k(\chi(M) - 2) = \beta_k(M)$ (the k^{th} Betti number).

Here the notion $(k-1)$-connected is understood in the topological sense, i.e., that the homotopy groups $\pi_j(M)$ vanish for $1 \leqslant j \leqslant k-1$. Then β_k can be regarded as a kind of "genus" of the manifold.

Consequently, we can define a *generalized Heawood number for M* as the n realizing equality in Equation (5). By analogy with the 4-dimensional case in Section 2 we restrict ourselves to the case of $(k-1)$-connected $(2k)$-manifolds. For the other cases see Conjecture 3.9.

Definition 14. The *generalized Heawood number H_k* for a $(k-1)$-connected $2k$-manifold M with k^{th} Betti number β_k is the largest (real) root of the polynomial

$$P_k(n) = \binom{n-k-2}{k+1} - \binom{2k+1}{k+1}\beta_k.$$

There is always a positive real root since for $\beta_k = 0$ we have $P_k(2k+2) = 0$ and since $P_k(n)$ is monotonically increasing in n and decreasing in β_k.

Corollary 15. *Any n-vertex triangulation of a $(k-1)$-connected $2k$-manifold satisfies*

$$n \geqslant H_k$$

with equality if and only if the triangulation is $(k+1)$-neighborly.

Proof. The inequality is equivalent to the inequality (5) since $(-1)^k(\chi - 2) = \beta_k$, and the discussion of equality if the same as in Theorem 13. $\square$

In particular we have $H_k = 3k+3$ for $(-1)^k(\chi - 2) = \beta_k = 1$ and $H_k = 3k+4$ for $(-1)^k(\chi - 2) = \beta_k = 2$. These values coincide with the a priori bounds in [11]. This number H_k is an algebraic number although in general no explicit algebraic expression is possible. For $k = 1$ we have $H_1 = \frac{1}{2}\Big(7 + \sqrt{49 - 24\chi(M)}\Big) = \frac{1}{2}\Big(7 + \sqrt{1 + 24\beta_1(M)}\Big)$, for a simply connected 4-manifold with second Betti number β_2 we have

$$H_2 = 5 + \sqrt[3]{30\beta_2 + \sqrt{900\beta_2^2 - \tfrac{1}{27}}} + \left(3\sqrt[3]{30\beta_2 + \sqrt{900\beta_2^2 - \tfrac{1}{27}}}\right)^{-1},$$

see Sect.1 and Sect. 2.

For $k = 3$ the polynomial H_3 is biquadratic, and therefore for a 2-connected 6-manifold with third Betti number β_3 we have

$$(n-5)(n-6)(n-7)(n-8) \geqslant 840\beta_3$$

or with $x = n - \frac{13}{2}$

$$(x^2 - \tfrac{9}{4})(x^2 - \tfrac{1}{4}) \geqslant 840\beta_3$$

and

$$H_3 = \frac{1}{2}\Big(13 + \sqrt{5 + 4\sqrt{1 + 840\beta_3}}\Big).$$

Corollary 16. *For every k there are infinitely many possible integer pairs (n, β_k) such that $n = H_k$ for a candidate $(2k)$-manifold with middle Betti number β_k, possibly admitting a $(k+1)$-neighborly triangulation with n vertices. For fixed k the possible integers $n = H_k$ occur with a periodicity, and the associated β_k grows polynomially in n.*

Proof. The polynomial growth is obvious from the case of equality in the formulas above. The possible values for $n = H_k$ can be seen by considering the prime factors of the binomial coefficients $\binom{n-k-2}{k+1}$ and $\binom{2k+1}{k+1}$ that are involved. In particular we have:

$k = 1$: $n \not\equiv 2\ (3)$

$k = 2$: $n \not\equiv 3\ (4)$ and $n \not\equiv 2,\ 3\ (5)$, together $n \equiv 0,\ 1,\ 4,\ 5,\ 6,\ 9,\ 10,\ 14,\ 16\ (20)$.

$k = 3$: $n \not\equiv 4\ (5)$ and $n \not\equiv 2,\ 3,\ 4\ (7)$, together
$n \equiv 0,\ 1,\ 5,\ 6,\ 7,\ 8,\ 12,\ 13,\ 15,\ 20,\ 21,\ 22,\ 26,\ 27,\ 28,\ 33\ (35)$.

$k = 4$: $n \not\equiv 4,\ 5\ (7)$, $\quad n \not\equiv 3,\ 5\ (8)\quad$ and
$n \equiv 0,\ 1,\ 6,\ 7,\ 8,\ 9,\ 10,\ 15,\ 16,\ 18,\ 19,\ 24,\ 25\ (27)$, altogether with period 1512.

This continues in an obvious way, so for any k there is such a period. It gives only a necessary integer condition and does not mean anything further about existence of such a triangulation or the topology of such a manifold. ☐

The first relevant cases besides the standard cases $n = 3k + 3$ and $n = 3k + 4$ are the following:

$k = 2$: $n = 14,\ 16,\ 20,\ 21$ with $\beta_k = 12,\ 22,\ 56,\ 68$
$k = 3$: $n = 15,\ 20,\ 21,\ 22$ with $\beta_k = 6,\ 39,\ 52,\ 68$
$k = 4$: $n = 24,\ 28,\ 34,\ 36$ with $\beta_k = 68,\ 209,\ 780,\ 1131$.

However, the cases $\beta_3 = 1$ and $\beta_3 = 39$ cannot occur for topological reasons: There are no 2-connected 6-manifolds with an odd intersection form. Moreover, an odd number β_k can occur only in the dimensions $2k = 2, 4, 8, 16$ [57].

Open Problem. *Find more n-vertex triangulations of $2k$-manifolds satisfying the Heawood equality in (5), i.e., where n equals the generalized Heawood number. Such manifolds are necessarily $(k-1)$-connected and the triangulations are necessarily $(k+1)$-neighborly, i.e., they contain the k-skeleton of an $(n-1)$-dimensional simplex.*

A necessary condition is that the Heawood number is an integer. Then one has to look for manifolds realizing the appropriate topology. Natural candidates are the following:

14-vertex triangulation of $(S^2 \times S^2)^{\#6}$, $\binom{10}{3} = 12\binom{5}{3}$
(not possible with a vertex transitive automorphism group)

16-vertex triangulation of $(S^2 \times S^2)^{\#11}$, $\binom{12}{3} = 22\binom{5}{3}$

15-vertex triangulation of $(S^3 \times S^3)^{\#3}$, $\binom{10}{4} = 6\binom{7}{4}$

16-vertex triangulation of $S^4 \times S^4$, $\binom{10}{5} = 2\binom{9}{5}$

28-vertex triangulation of $S^8 \times S^8$, $\binom{18}{9} = 2\binom{17}{9}$

A further generalization: Although Theorem 13 holds for any $2k$-manifold, it is not useful if the Euler characteristic is too small. In this case one has to find other inequalities specifically adapted to the topology of M. An example is the lower bound $n \geqslant (k+1)^2$ for triangulatíons of $\mathbb{C}P^k$ [2] whereas the Euler characteristic is $\chi = k+1$ with the consequence that $(-1)^k\big(\chi - 2\big) = (-1)^k(k-1)$ leading to a rather weak inequality (5) (trivial for odd k).

For manifolds that are not highly connected or are odd-dimensional, there are similar inequalities, motivated by examples and by a certain interpolation between the parameters (at least as conjectures). These inequalities can be presented in a Pascal-like triangle depending on the dimension d and a parameter j with $1 \leqslant j \leqslant \frac{d}{2}$ as follows:

$$
\begin{array}{lllll}
d=2: & \binom{n-3}{2} \geqslant \frac{1}{2}\binom{4}{2}\beta_1 & & & \\
d=3: & \binom{n-4}{2} \geqslant \binom{5}{2}\beta_1 & & & \\
d=4: & \binom{n-5}{2} \geqslant \binom{6}{2}\beta_1 & \binom{n-4}{3} \geqslant \frac{1}{2}\binom{6}{3}\beta_2 & & \\
d=5: & \binom{n-6}{2} \geqslant \binom{7}{2}\beta_1 & \binom{n-5}{3} \geqslant \binom{7}{3}\beta_2 & & \\
d=6: & \binom{n-7}{2} \geqslant \binom{8}{2}\beta_1 & \binom{n-6}{3} \geqslant \binom{8}{3}\beta_2 & \binom{n-5}{4} \geqslant \frac{1}{2}\binom{8}{4}\beta_3 & \\
d=7: & \binom{n-8}{2} \geqslant \binom{9}{2}\beta_1 & \binom{n-7}{3} \geqslant \binom{9}{3}\beta_2 & \binom{n-6}{4} \geqslant \binom{9}{4}\beta_3 & \\
d=8: & \binom{n-9}{2} \geqslant \binom{10}{2}\beta_1 & \binom{n-8}{3} \geqslant \binom{10}{3}\beta_2 & \binom{n-7}{4} \geqslant \binom{10}{4}\beta_3 & \binom{n-6}{5} \geqslant \frac{1}{2}\binom{10}{5}\beta_4
\end{array}
$$

The factor $\frac{1}{2}$ reflects the fact that for even d the weight of the middle Betti number is only half of the weight of the others since $\beta_k = \beta_{d-k}$ in all other cases. This "triangle of conjectures" is motivated by the fact that it is actually true for $\beta_1 = 1$ and arbitrary d [11] and for $d = 3$ and arbitrary β_1 [41]. Furthermore in even dimensions the last inequality in each row is equivalent to the generalized Heawood inequality above if the manifold is $(k-1)$-connected. So the Pascal-like triangle fits in as a kind of very natural interpolation between these cases. For products of two spheres it also coincides with the a priori bound in [11].

Example 17. We know the following cases of equality:

1. $\beta_1 = 1$: $(d-1)$-sphere bundles over S^1 for any d [4, 16].
2. $d = 3$: Many examples, see [41], [13].
3. $d = 4, \beta_1 = 3$: See [5].
4. $d = 4$: See [13].
5. $d = 5, \beta_2 = 1$: A 12-vertex triangulation of $S^2 \times S^3$ [40].
6. $j = \frac{d}{2}$: See Example 12.
7. An infinite series with increasing d and
   $n = d^2 + 5d + 5, \quad \beta_1 = \binom{n-d-1}{2} / \binom{d+2}{2} = (d+3)(d+2) = n+1$ [18].
   So the first Betti number is larger than the number of vertices.

Conjecture 18. Let M be a closed d-manifold and let j be any integer between 1 and $\frac{d}{2}$, β_j denotes the corresponding Betti number of M with coefficients in a field ($\mathbb{Z}_2$ as the standard case). Then for any n-vertex triangulation of M the following inequalities are satisfied:

$$\binom{n-d+j-2}{j+1} \geqslant \binom{d+2}{j+1}\beta_j \quad \text{for } 1 \leqslant j < \frac{d}{2} \tag{6}$$

$$\binom{n-d+j-2}{j+1} \geqslant \frac{1}{2}\binom{d+2}{j+1}\beta_j \quad \text{for } j = \frac{d}{2} \tag{7}$$

Remarks: (a) In a recent article by Adiprasito [1] the inequality (6) is proved by sophisticated algebraic methods. These go far beyond the algebraic methods that were needed for the proof of Theorem 13.

(b) Inequality (7) coincides with inequality (5) if $d = 2k$ and if M is $(k-1)$-connected since $\frac{1}{2}\binom{2k+2}{k+1} = \binom{2k+1}{k+1}$ and $\beta_k = (-1)^k\big(\chi(M) - 2\big)$. If inequality (7) is true then Corollary 3.6 can be improved: One could omit the assumption about the $(k-1)$-connectivity in the definition of the Heawood number.

(c) For $\beta_j = 0$ (and fixed j) the inequalities are trivial, the case of equality is $n = d+2$, realized only by the boundary of a $(d+1)$-simplex.

(d) Coming back to the case of $\mathbb{C}P^k$ for $k \geqslant 3$ with $\beta_2 = 1$, the inequality (6) takes the form $\binom{n-2k}{3} \geqslant \binom{2k+2}{3}$ and, consequently, $n \geqslant 4k+2$. This bound is better than the bound from Theorem 13 for even k but it is still far from being sharp: It is known that $n \geqslant (k+1)^2$ [2].

The case of equality leads to further numbers of Heawood type and can be conjectured as follows:

Conjecture 19. For any fixed j equality in the inequality above implies that

1. the triangulation is $(j+1)$-neighborly,
2. $\beta_{j+1} = \beta_{j+2} = \cdots = \beta_{d-j-1} = 0$.

The $(j+1)$-neighborliness in the first line implies also $\beta_1 = \beta_2 = \cdots = \beta_{j-1} = 0$ and – by duality – $\beta_{d-j+1} = \beta_{d-j+2} = \cdots = \beta_{d-1} = 0$. In other words: β_j and β_{d-j} are the only non-vanishing Betti numbers besides β_0 and β_d. So for $1 \leqslant j < \frac{d}{2}$ M has the same $\mathbb{Z}_2$-homology as a connected sum of β_j copies of $S^j \times S^{d-j}$.

The converse of this conjecture is not true, as the example of a 3-neighborly 13-vertex triangulation of $M \cong SU(3)/SO(3)$ shows [40]. This has $\beta_2(M;\mathbb{Z}_2) = 1$ and thus the same $\mathbb{Z}_2$-homology as $S^2 \times S^3$ which does admit a 3-neighborly triangulation with $n = 12$ vertices, thus realizing equality in the inequality (6), see Examples 3.8. Equality for $d = 3$ and $j = 1$ is the case of *tight-neighborly triangulations* [13]. These have the minimum possible number of vertices and, in addition, they are *tight triangulations* [41]. Recently, the notion of a tight triangulation – introduced in [35] – has been shown to be equivalent to a purely algebraic property of the Stanley-Reisner ring [29]. So far all known triangulations satisfying equality in any of the inequalites in Conjecture 18 are tight triangulations.

Special cases where the conjectures were proved before [1] are the following:

1. For $d = 2$ the first conjecture coincides with the Heawood inequality (1) including the discussion of equality.
2. For $(k-1)$-connected $2k$-manifolds it coincides with the result of Theorem 13.
3. Both conjectures are true for $d = 3$ [41, Thm.5], [13, Sect.2.3].
4. For $(j-1)$-connected manifolds with $\beta_j = 1$ and arbitrary $d \geqslant 3$ the first conjecture follows from the a priori bound in [11].
5. Moreover, for $j = 1$, $\beta_1 = 1$ and arbitrary $d \geqslant 4$ also the second conjecture follows from the uniqueness of the examples of (twisted) sphere products of S^1 and S^{d-1} with the minimum number $2d+3$ of vertices shown in [4, Thm.4], [16, Thm.3.6]. These examples can be described as follows: Let $n = 2d+3$ and consider the $\mathbb{Z}_n$-orbit of the $(d+1)$-simplex $(0\ 1\ 2\ 3\ \ldots\ d+1)$ which is a 2-neighborly triangulation of a 1-handle, orientable if d is even, nonorientable otherwise. Then the example is the boundary of this 1-handle: a sphere product $S^1 \times S^{d-1}$ if d is even and a twisted product otherwise.

Corollary 20. *The conjectures above suggest that theoretically a vertex minimal triangulation of $S^k \times S^m$ with $2 \leqslant k \leqslant m$ should have $n = k + 2m + 4$ vertices and should be $(k+1)$-neighborly. At least in the case $k = m = 2$ this bound is not attained, see Corollary 9. It is attained for $k = 2, m = 3$ and for $k = m = 3$ [40]. The same bound (without neighborliness) follows also from [11].*

Open Problem. *Find more examples for equality in any of the inequalities* (6).

4 van Kampen – Flores problems

It is easy to see that the complete graph K_5 cannot be embedded into the 2-plane or 2-sphere: The embedding of a K_3 is unique: three edges forming a cycle. A fourth vertex can be inserted into any of the two resulting open triangles in the complement. But then a fifth vertex can be joined to at most three of the other vertices but not to the fourth one. This follows from the Jordan curve theorem: *Any embedded closed curve decomposes the plane or the sphere into two disjoint open components.* This embeddability problem can be formulated in another terminology as follows:

The 1*-skeleton of an* $(n-1)$*-dimensional simplex can be embedded into the* 2*-sphere if and only if* $n \leqslant 4$.

A direct generalization to higher dimensions is known as the *van Kampen - Flores theorem*:

Theorem 21. (van Kampen and Flores, [26, Sect.11.2])
The k*-skeleton of an* $(n-1)$*-dimensional simplex can be embedded into the* $2k$*-sphere if and only if* $n \leqslant 2k+2$.

As a common generalization of this theorem and the Heawood inequality (2) we proposed the following

Conjecture 22. (Kühnel [34])
If the k-skeleton of an $(n-1)$-dimensional simplex can be embedded into a $(k-1)$-connected $2k$-manifold M then the generalized Heawood inequality

$$\binom{n-k-2}{k+1} \leqslant (-1)^k \binom{2k+1}{k+1}\big(\chi(M)-2\big) = \binom{2k+1}{k+1}\beta_k(M) \tag{8}$$

holds. In terms of the Heawood number from Definition 14 this reads as $n \leqslant H_k$.

In other words: As in the case of surfaces in Section 1, the maximum possible n in comparison to a given "genus" β_k is given by the generalized Heawood number according to Theorem 13, and again the case of equality is most interesting. These are $(k+1)$-neighborly triangulations of $2k$-manifolds or n-vertex triangulations containing the complete k-skeleton of the $(n-1)$-dimensional simplex (the *regular cases*). For examples see Sect. 3. In particular the 2-skeleton of the 8-simplex embeds into the complex projective plane and the 4-skeleton of the 14-simplex embeds into the quaternionic projective plane. Moreover the 8-skeleton of the 26-simplex embeds into certain manifolds "like the Cayley plane" [22].

Recently new approaches were made for proving at least weaker versions of Conjecture 22 by cohomological methods, see [23, 47]. One can define a so-called *van Kampen obstruction* to embeddability in terms of a certain cohomology class which is induced by a mapping of a simplicial complex K into a $2k$-manifold or the induced mapping on the level of chain complexes. The vanishing of the obstruction is then shown to be a necessary condition for embeddability. The best bounds in [23, Thm.5] are $n \leqslant 2k+1+(k+1)\beta_k(M)$ and, if the intersection form is skew symmetric, $n \leqslant 2k+1+\frac{1}{2}(k+2)\beta_k(M)$. These coincide with Conjecture 22 in the cases of $\beta_k = 1$ and – in the skew-symmetric case – also $\beta_k = 2$.

Corollary 23. ([23, Thm.5])
Conjecture 22 is true for $k = 1$ (by Section 1) and for the projective planes over $\mathbb{C}, \mathbb{H}$ and the Cayley numbers and their "look-alikes" [33]. Moreover it is true for all sphere products $S^k \times S^k$ with odd k and all S^k-bundles over S^k with the same intersection form $\binom{0\ \ -1}{1\ \ \ \ 0}$.

Open Problems. *Can the 2-skeleton of the 8-simplex or of the 9-simplex be embedded into $S^2 \times S^2$? There is no 10-vertex triangulation with a complete 2-skeleton, compare Corollary 9 above.*

Can the 4-skeleton of the 15-dimensional simplex be embedded into $S^4 \times S^4$ or into any other S^4-bundle over S^4 ?

Theorem 24. (Adiprasito [1, Sect.4.6])
Conjecture 22 is true in general under the extra assumption that the embedding of the k-skeleton can be extended to a triangulation of M.

It seems to be an open question whether for $k \geqslant 2$ there are embeddings of the complete k-skeleton of the $(n-1)$-dimensional simplex which cannot be extended to a triangulation. In the regular cases one can also ask if there are such embeddings that cannot be the k-skeleton of an n-vertex triangulation of the same manifold. Possible candidates are embeddings containing knotted 2-spheres in a 4-manifold since the 2-skeleton of a simplex is always a wedge product of a number of 2-spheres.

5 Centrally-symmetric versions

There are centrally symmetric versions of the various inequalities of Heawood type. A triangulated manifold is regarded as *centrally symmetric* if it admits an involution without fixed points that preserves the triangulation. In this case the k-skeleton of the simplex is replaced by that of the cross polytope (generalized octahedron).

We first discuss the case of surfaces. For a centrally symmetric triangulation the number of vertices is always an even number $n = 2m$, and it has at most $\binom{n}{2} - m = 4\binom{m}{2}$ edges. Then the Euler formula $2m - f_1 + f_2 = \chi(M)$ is equivalent to $2m - \frac{f_1}{3} = \chi(M)$, and the number of missing edges satisfies

$$4\binom{m}{2} - f_1 = 2m(m-1) - 3(2m - \chi(M)) = 2(m-1)(m-3) + 3\big(\chi(M) - 2\big).$$

Since the number of missing edges cannot be negative this implies:

Proposition 25. (First centrally symmetric Heawood inequality)
For any centrally symmetric $2m$-vertex triangulation of a surface M the inequality

$$2(m-1)(m-3) \geqslant 3(2-\chi(M))$$

holds with equality if and only if the are no missing edges except for the m diagonals, i.e., if the number of edges satisfies $f_1 = \binom{2m}{2} - m$.

On the other hand, if we start with an embedding of the edge graph G of the m-dimensinal cross polytope with $\binom{2m}{2} - m = 4\binom{m}{2}$ edges into a surface M, then the complement $M \setminus G$ consists of a number F of countries (each connected but not necessarily simply connected). Each country F_i is bounded by $c_i \geqslant 3$ edges and has $\chi(F_i) \leqslant 1$. on the other hand we have $8\binom{m}{2} = \sum_i c_i \geqslant 3F$ since every edge occurs in the boundary of precisely two countries. This implies

$$\chi(M) = 2m-4\binom{m}{2}+\chi(M\setminus G) \leqslant 2m-4\binom{m}{2}+F \leqslant 2m-\frac{4}{3}\binom{m}{2} = -\frac{2}{3}(m-1)(m-3)+2.$$

Proposition 26. (Second centrally symmetric Heawood inequality)
If a surface M admits an embedding of the edge graph of the m-dimensional cross polytope then the inequality

$$2(m-1)(m-3) \leqslant 3(2-\chi(M))$$

holds with equality if and only if the embedding is triangular, i.e., its complement consists only of triangles. In this case of equality the triangulations can be embedded into the 2-skeleton of the m-dimensional cross polytope. However, it is not necessarily centrally symmetric but for any $m \not\equiv 2$ (3) there are centrally symmetric examples [30].

Again these Heawood inequalities differ only by their direction, therefore the case of equality is most interesting: These are $(2m)$-vertex triangulations whose edge graph coincides with that of the m-dimensional cross polytope. These are also called the *regular cases* where the equation $2(m-1)(m-3) = 3(2-\chi(M))$ is equivalent to

$$m = 2 + \tfrac{1}{2}\sqrt{16-6\chi(M)} = 2 + \tfrac{1}{2}\sqrt{4+6\beta_1(M)}, \tag{9}$$

where $\beta_1(M)$ is the first $\mathbb{Z}_2$-Betti number of M. This leads to the *centrally symmetric Heawood number of M*. It grows like the square root of the genus. One regular case is the unique 8-vertex centrally symmetric torus (for a figure see [36]). The regular cases cover all $m \not\equiv 2$ (3) with associated $\beta_1 = \frac{2}{3}(m-1)(m-3)$.

In higher dimensions we have the following quite natural generalization:

Definition 27. A centrally symmetric triangulation with $2m$ vertices is called *nearly k-neighborly* if it contains all possible simplices of dimension $k-1$ except those which would contain one of the m diagonals.

Theorem 28. (Klee and Novik [32, Prop.5.6], conjectured by Sparla [54])
For any centrally symmetric $2m$-vertex triangulation of a $(k-1)$-connected $2k$-manifold M the generalized Heawood inequality

$$2(m-1)(m-3)(m-5)\cdots(m-2k-1) \geqslant (2k+1)(2k-1)(2k-3)\cdot\dots\cdot 3\cdot\beta_k \tag{10}$$

holds with equality if and only if the triangulation is nearly $(k+1)$-neighborly.

Similarly, for an appropriate class of manifolds one can define a *centrally symmetric Heawood number* as the largest root of the polynomial

$$P_k^c(m) = 2(m-1)(m-3)(m-5)\cdots(m-2k-1) - (2k+1)(2k-1)(2k-3)\cdot\dots\cdot 3\cdot\beta_k,$$

or, with the notation $x_{k;2} := x(x-2)\cdots(x-2(k-1))$:

$$P_k^c(m) = 2(m-1)_{k+1;2} - (2k+1)_{k+1;2}\cdot\beta_k$$

Note, however, that the centrally symmetric Heawood number does not indicate the minimum number of vertices but only half of it. The case $k=1$ was discussed above.

For $k=2$ we have to consider the inequality

$$2(m-1)(m-3)(m-5) \geqslant 15\beta_2$$

or, after substitution $x = m-3$,

$$2x(x^2-4) \geqslant 15\beta_2.$$

Ths leads to the inequality

$$m \geqslant 3 + \sqrt[3]{\tfrac{15}{4}\beta_2 + \sqrt{\tfrac{225}{16}\beta_2^2 - \tfrac{64}{27}}} + \tfrac{4}{3}\left(\sqrt[3]{\tfrac{15}{4}\beta_2 + \sqrt{\tfrac{225}{16}\beta_2^2 - \tfrac{64}{27}}}\right)^{-1}$$

whenever $\beta_2 \geqslant 1$. For $\beta_2 = 2$ (this is the case of $S^2\times S^2$) we obtain $x=3$ and $m=6$. The special case $\beta_2 = 0$ directly implies $m \geqslant 5$, equality is obtained for the boundary of the 5-dimensional cross polytope.

For $k=3$ we have the inequality

$$2(m-1)(m-3)(m-5)(m-7) \geqslant 105\beta_3.$$

By introducing $x = m-4$ we obtain

$$(x^2-9)(x^2-1) \geqslant \frac{105}{2}\beta_3$$

with the explicit inequalities

$$x^2 \geqslant 5 + \sqrt{16 + \tfrac{105}{2}\beta_3} \quad \text{and} \quad m \geqslant 4 + \sqrt{5 + \sqrt{16 + \tfrac{105}{2}\beta_3}}.$$

Equality in the case $\beta_3 = 2$ is realized by two 16-vertex triangulations of $S^3\times S^3$ [40].

Example 29. We know the following cases of equality in the inequality (10):

1. Centrally symmetric surfaces [30].
2. The boundary of the $(2k+1)$-dimensional cross polytope with $\beta_k = 0$ (the $2k$-sphere).
3. Sparla's centrally symmetric 12-vertex triangulations of $S^2 \times S^2$, one of them with a symmetry group isomorphic to A_5 of order 60.
4. In general for any k a product of type $S^k \times S^k$ with $\beta_k = 2$ as a subcomplex of the $(2k+2)$-dimensional cross polytope constructed by Klee and Novik [32].
5. A 4-manifold of type $(S^2 \times S^2)^{\#7}$ with $\beta_2 = 14$ as a subcomplex of the 8-dimensional cross polytope [21].

Open Problem. *Find more centrally symmetric triangulations of $2k$-manifolds satisfying the Heawood equality, i.e., where m equals the centrally symmetric Heawood number.*

A necessary condition is that the Heawood number is an integer. Then one has to look for manifolds realizing the appropriate topology. Natural candidates are the following:

$k = 2, m = 10$: 20-vertex triangulation of $(S^2 \times S^2)^{\#21}$

$k = 2, m = 11$: 22-vertex triangulation of $(S^2 \times S^2)^{\#32}$

$k = 3, m = 10$: 20-vertex triangulation of $(S^3 \times S^3)^{\#9}$

$k = 3, m = 12$: 24-vertex triangulation of $(S^3 \times S^3)^{\#33}$

$k = 4, m = 28$: 56-vertex triangulation of $(S^4 \times S^4)^{\#230}$

By analogy with Conjecture 18 one can ask for similar inequalities involving single Betti numbers $\beta_1, \beta_2, \ldots$ that apply to centrally-symmetric triangulations. One might conjecture the following Pascal-like triangle of inequalities:

$d = 2: \ (m-1)_{2;2} \geqslant 3_{2;2} \cdot \frac{1}{2}\beta_1$

$d = 3: \ (m-1)_{2;2} \geqslant 4_{2;2} \cdot \beta_1$

$d = 4: \ (m-1)_{2;2} \geqslant 5_{2;2} \cdot \beta_1 \quad (m-1)_{3;2} \geqslant 5_{3;2} \cdot \frac{1}{2}\beta_2$

$d = 5: \ (m-1)_{2;2} \geqslant 6_{2;2} \cdot \beta_1 \quad (m-1)_{3;2} \geqslant 6_{3;2} \cdot \beta_2$

$d = 6: \ (m-1)_{2;2} \geqslant 7_{2;2} \cdot \beta_1 \quad (m-1)_{3;2} \geqslant 7_{3;2} \cdot \beta_2 \quad (m-1)_{4;2} \geqslant 7_{4;2} \cdot \frac{1}{2}\beta_3$

$d = 7: \ (m-1)_{2;2} \geqslant 8_{2;2} \cdot \beta_1 \quad (m-1)_{3;2} \geqslant 8_{3;2} \cdot \beta_2 \quad (m-1)_{4;2} \geqslant 8_{4;2} \cdot \beta_3$

$d = 8: \ (m-1)_{2;2} \geqslant 9_{2;2} \cdot \beta_1 \quad (m-1)_{3;2} \geqslant 9_{3;2} \cdot \beta_2 \quad (m-1)_{4;2} \geqslant 9_{4;2} \cdot \beta_3 \quad (m-1)_{5;2} \geqslant 9_{5;2} \cdot \frac{1}{2}\beta_4$

The examples by Klee and Novik [32] of type $S^j \times S^{d-j}$ realize equality in each of these inequalities with $\beta_j = 1$ for $j < \frac{d}{2}$ and $\beta_k = 2$ for $d = 2k$. For $j = 2$ also the examples of Wang and Zheng [58] realize equality. For $\beta_j = 0$ (and fixed j) the inequalities are trivial but equality cannot be attained except for $j = \frac{d}{2}$ and the boundary of the cross polytope itself.

6 Construction of examples

G.Ringel [49] described methods for constructing 2-neighborly triangulations of surfaces, in particular such with a vertex transitive cyclic automorphism group. If the vertices are regarded as the integers modulo n then the first key observation is that any difference $1, 2, 3, \dots$ modulo n has to occur precisely twice in the set of $\mathbb{Z}_n$-orbits of triples. The second key observation is that the link of the vertex 0 (or any other) must be a cycle of length $n-1$. Both conditions are combined in a graph satisfying *Kirchhoff's Current Law* where the link of 0 is called the *log* [49]. The simplest example is the 7-vertex torus with the *log* (1 3 2 -1 -3 -2) modulo 7, see Fig. 1.

Another interesting instance is the 19-vertex triangulation of an orientable surface of genus 20 given by the very simple scheme in [49, Fig.2.1]. In this case the link of 0 is the cycle (9 7 4 -2 -9 -1 5 -3 -7 2 6 1 -8 -5 -6 -4 3 8) modulo 19, see [49, 2.10]. Then the cyclic shift $x \mapsto x+1$ mod 19 will lead to the other vertex links and thus to the entire surface. This elegant solution (which works for any $n \equiv 7$ (12)) was one of the first cases solved by G.Ringel in 1961. Another presentation of the same object is given by the following table that shows a kind of "double partition" of the possible unoriented differences 1, .. 9 into six triples:

1				5	6			
1							8	9
	2		4		6			
	2					7		9
		3	4			7		
		3		5			8	

An oriented version determines such a *log*, the *log* above can be described by the following oriented triples:

1 5 -6 -1 -8 9 -4 -2 6 7 2 -9 4 3 -7 -5 -3 8

This scheme of oriented triples contains each number between 1 and 9 twice but with opposite signs. It generates all triangles in the vertex star of 0 if the numbers are interpreted as oriented differences modulo 19 along the three oriented edges of the triangle. So 7 2 -9 generates the oriented triangles 0 7 9, 0 2 -7 and 0 -9 -2, just by considering the $\mathbb{Z}_{19}$-orbit of a triangle realizing these three differences 7, 2, -9 in this order and picking those three items containing 0 as a vertex. The oriented edges 9 0 and 0 -9 force the next triple -1 -8 9 to the three oriented triangles 0 9 8, 0 -1 -9 and 0 -8 1. Inductively the *log* above follows.

However, another orientation can lead to another *log*. So the oriented triples

6 -5 -1 1 8 -9 2 4 -6 -2 -7 9 -4 -3 7 3 5 -8

lead to the log (7 9 1 6 2 -7 -4 -6 -5 -8 -9 -2 4 -3 5 -1 8 3) which is essentially different.

With these methods G.Ringel and others were able to show that both Heawood inequalities (i.e., in both directions) are actually sharp with very few exceptions, where the main exception is the Klein bottle which does not admit an embedding of K_7 and requires 8 vertices for a triangulation [35, Sect.2C]. See also Theorem 4 in Sect. 1.

A 3-dimensional analogue would be given by the following unoriented quadruples of differences 1 1 2 4, 1 2 1 4, 2 1 1 4 modulo 10 leading to the four generating oriented tetrahedra 0 1 2 4, 0 1 4 3, 0 2 3 4. Then the cyclic $\mathbb{Z}_{10}$-symmetry leads to 12 oriented triangles in the link of 0:

$$\begin{array}{rrrl} 1\ 2\ 4, & 1\ 4\ 3, & 2\ 3\ 4, \\ \text{-}1\ 1\ 3, & \text{-}1\ 3\ 2, & \text{-}2\ 2\ 1, \\ \text{-}2\ \text{-}1\ 2, & \text{-}4\ \text{-}3\ \text{-}1, & \text{-}3\ 1\ \text{-}1, \\ \text{-}4\ \text{-}2\ \text{-}3, & \text{-}3\ \text{-}2\ 1, & \text{-}4\ \text{-}1\ \text{-}2. \end{array}$$

This is a triangulated 2-sphere with 8 vertices as an analogue of the *log* above. The $\mathbb{Z}_{10}$-action produces all the other links. The same procedure is possible for any $n \geqslant 9$ but the manifold is nonorientable for odd n. The manifold is 2-neighborly for $n = 9$ and centrally symmetric and nearly 2-neighborly for $n = 10$.

In more generality, there are infinite families of similar cyclic triangulations (examples in [38]) also in higher dimensions but these are at most 2-neighborly, so they do not fit into Section 2 or 3. For 3-manifolds 2-neighborliness means that in an n-vertex triangulation the link of 0 has $n-1$ vertices and, therefore, $2n-6$ triangles. The analogue of Kirchhoff's current law would be that any triangle $(x\ y\ z)$ in the link of 0 implies the presence of $(-x\ \ y-x\ \ z-x)$, $(-y\ \ x-y\ \ z-y)$, $(-z\ \ x-z\ \ y-z)$ also, all numbers modulo n. If this is satisfied and if the link of 0 is a 2-sphere then the $\mathbb{Z}_n$-action leads to a 2-neighborly 3-manifold. So there is a method to construct a triangulated manifold from the possible link of one vertex. In principle, a 2-neighborly 3-sphere as link can lead to a 3-neighborly 4-manifold.

Unfortunately, in higher dimensions no systematic and efficient principles for constructing $(k+1)$-neighborly triangulations of $2k$-manifolds are known. One possibility is the assumption of a certain vertex-transitive automorphism group and a computer-aided check on the class of the orbits [40]. Nevertheless we know only a few sporadic examples, besides the 4-dimensional ones already mentioned in Corollary 9 and the centrally-symmetric versions in Section 5 these are:

1. Several asymmetric triangulations of $S^3 \times S^3$ with 13 vertices found by F.Lutz [40].

2. Three distinct triangulations of the quaternionic projective plane $\mathbb{H}P^2$ with 15 vertices [12, 24], among them one with a vertex transitive group action of A_5.

3. Many 27-vertex triangulations of a manifold "like the projective Cayley plane", among them such with a vertex transitive automorphism group of order 351 found by A.Gaifullin [22]. This is remarkable because of the huge f-vector with $f_{16} = 100386$.

Cyclic group actions can be often used for constructing examples of various kinds including neighborly and tight triangulations [13], but for $(k+1)$-neighborly $(2k)$-manifolds this does not seem to be the right approach. It has been checked that there is no 14-vertex 3-neighborly triangulation of any 4-manifold with $\chi = 14$ admitting a vertex transitive automorphism group $\mathbb{Z}_{14}$. However, there is a 4-dimensional pseudomanifold with the same properties otherwise and with singularities precisely along an embedded Klein bottle [39]. Besides the trivial case of the simplex itself, so far we know only one example with a twofold transitive group: the 16-vertex triangulation of the K3 surface [14]. For small n all n-vertex triangulations with a vertex transitive group action were enumerated in [40]. On the other hand it is quite plausible that many more such examples will exist (possibly in all even dimensions) but they will be complicated and will have a large number of vertices and – presumably – only a small automorphism group. In any case the inequalities in Theorem 13 and in Conjecture 22 can be expected to be sharp in principle in the sense that equality is possible and occurs not only rarely and sporadically.

Open Problem. *Find a method for constructing an infinite family of triangulations of $2k$-manifolds for some fixed $k \geqslant 2$ satisfying the Heawood equality in* (5) *and* (8)*, i.e., where the number of vertices always equals the generalized Heawood number of the manifold.*

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