Weak degeneracy of planar graphs and locally planar graphs

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Submitted: Dec 28, 2022; Accepted: Oct 2, 2023; Published: Nov 3, 2023

Abstract

Weak degeneracy is a variation of degeneracy which shares many nice properties of degeneracy. In particular, if a graph G is weakly d-degenerate, then for any $(d+1)$-list assignment L of G, one can construct an L-coloring of G by a modified greedy coloring algorithm. It is known that planar graphs of girth 5 are 3-choosable and locally planar graphs are 5-choosable. This paper strengthens these results and proves that planar graphs of girth 5 are weakly 2-degenerate and locally planar graphs are weakly 4-degenerate.

Mathematics Subject Classifications: 05C10, 05C15

1 Introduction

For a graph G, the greedy coloring algorithm colors vertices one by one in order $v_1, v_2, \ldots, v_n$, assigning v_i the least-indexed color not used on its colored neighbors. An upper bound for the number of colors used in such a coloring is captured in the notion of graph degeneracy. Let $\mathbb{Z}$ be the set of integers, and $\mathbb{Z}^G$ be the set of mappings $f : V(G) \to \mathbb{Z}$. For $f \in \mathbb{Z}^G$ and a subset U of $V(G)$, let $f|_U$ be the restriction of f to U, and let $f_{-U} : V(G) - U \to \mathbb{Z}$ be defined as $f_{-U}(x) = f(x) - |N_G(x) \cap U|$ for $x \in V(G) - U$. For convenience, we may use f for $f|_U$, and write f_{-v} for $f_{-\{v\}}$. We denote by $E[U]$ the set of edges in G with both end vertices in U.

Let $\mathcal{L}$ be the set of pairs (G, f), where G is a graph and $f \in \mathbb{Z}^G$.

Definition 1. The *deletion operation* $\mathsf{Delete}(u) : \mathcal{L} \to \mathcal{L}$ is defined as

$$\mathsf{Delete}(u)(G, f) = (G - u, f_{-u}).$$

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We say $\mathsf{Delete}(u)$ is *legal* for (G, f) if both f and f_{-u} are non-negative. A graph G is *f-degenerate* if, starting with (G, f), it is possible to remove all vertices from G by a sequence of legal deletion operations. For a positive integer d, we say that G is *d-degenerate* if it is degenerate with respect to the constant d function. The *degeneracy* of G, denoted by $\mathsf{d}(G)$, is the minimum d such that G is d-degenerate.

The quantity $\mathsf{d}(G) + 1$ is called the *coloring number* of G, and is an upper bound for many graph coloring parameters: the chromatic number $\chi(G)$, the choice number $\chi_\ell(G)$, the paint number $\chi_{\mathsf{P}}(G)$, the DP-chromatic number $\chi_{\mathsf{DP}}(G)$ and the DP-paint number $\chi_{\mathsf{DPP}}(G)$. The definitions of some of these parameters are complicated. As we shall not discuss these parameters, other than saying that they are bounded by the weak degeneracy defined below, we omit the definitions and refer the reader to [6] for the definitions and discussion about these parameters.

The coloring number $\mathsf{d}(G) + 1$ of G, as an upper bound for the above mentioned graph coloring parameters, is often not tight. It is therefore interesting to see if we can modify the greedy coloring algorithm to save some of the colors and get a better upper bound. Motivated by this, Bernshteyn and Lee [1] recently introduced the concept of weak degeneracy of a graph.

Assume L is a list assignment of G and we try to construct an L-coloring of G. Assume uw is an edge of G. In the greedy coloring algorithm, if we assign a color to u, then it is counted that $L(w)$ loses one color. However, if $|L(u)| > |L(w)|$, then one can assign to u a color from $L(u) - L(w)$, and hence $L(w)$ will not lose a color in this step. The concept of weak degeneracy deals with this situation.

Definition 2. The *deletion-save operation* $\mathsf{DeleteSave}(u, w) : \mathcal{L} \to \mathcal{L}$ is defined as

$$\mathsf{DeleteSave}(u, w)(G, f) = (G - u, f_{-u} + \delta_w),$$

where $\delta_w(v) = 1$ if $v = w$ and $\delta_w(v) = 0$ otherwise. We say $\mathsf{DeleteSave}(u, w)$ is *legal* for (G, f) if uw is an edge of G, $f(u) > f(w)$ and both f and $f_{-u} + \delta_w$ are non-negative.

Definition 3. A *removal scheme* $\Omega = \mathsf{Del}(\theta_1, \theta_2, \ldots, \theta_k) : \mathcal{L} \to \mathcal{L}$, where for each i, either $\theta_i = \langle u_i \rangle$ representing the deletion operation $\mathsf{Delete}(u_i)$, or $\theta_i = \langle u_i, w_i \rangle$ representing the deletion-save operation $\mathsf{DeleteSave}(u_i, w_i)$, is defined recursively as follows:

$$\mathsf{Del}(\langle u \rangle)(G, f) = \mathsf{Delete}(u)(G, f), \quad \mathsf{Del}(\langle u, w \rangle)(G, f) = \mathsf{DeleteSave}(u, w)(G, f)$$

and for $k \geqslant 2$,

$$\mathsf{Del}(\theta_1, \theta_2, \ldots, \theta_k)(G, f) = \mathsf{Del}(\theta_k)(\mathsf{Del}(\theta_1, \theta_2, \ldots, \theta_{k-1})(G, f)).$$

We say $\mathsf{Del}(\theta_1, \theta_2, \ldots, \theta_k)$ is legal for (G, f) if $\mathsf{Del}(\theta_1, \theta_2, \ldots, \theta_{k-1})$ is legal for (G, f) and $\mathsf{Del}(\theta_k)$ is legal for $\mathsf{Del}(\theta_1, \theta_2, \ldots, \theta_{k-1})(G, f)$. Each θ_i in a removal scheme is called a *move*. A move $\theta_i = \langle u \rangle$ or $\theta_i = \langle u, w \rangle$ *removes* u from G. A graph G is *weakly f-degenerate* if there is a removal scheme $\Omega = \mathsf{Del}(\theta_1, \theta_2, \ldots, \theta_n)$ which is legal for (G, f) and removes all vertices of G. For a positive integer d, we say that G is *weakly d-degenerate* if it is weakly degenerate with respect to the constant d function. The *weak degeneracy* of G, denoted by $\mathsf{wd}(G)$, is the minimum d such that G is weakly d-degenerate.

The following proposition was proved in [1].

Proposition 4. *For every graph G,*

$$\chi(G) \leqslant \chi_\ell(G) \leqslant \chi_{DP}(G) \leqslant \chi_{DPP}(G) \leqslant \mathsf{wd}(G) + 1.$$

Some well-known upper bounds for $\chi_{\mathsf{DP}}(G)$ for families of graphs turn out to be upper bounds for $\mathsf{wd}(G) + 1$. For example, Bernshteyn and Lee [1] proved that planar graphs are weakly 4-degenerate and Brooks' theorem remains true for weak degeneracy.

It was proved by Thomassen [8] that planar graphs of girth at least 5 are 3-choosable. Dvořák and Postle [4] observed that planar graphs with girth at least 5 are DP-3-colorable. This paper strengthens this result and show that planar graphs of girth at least 5 are weakly 2-degenerate. Indeed, we shall prove graphs in a slightly larger graph family are weakly 2-degenerate.

We write $P = v_1 v_2 \dots v_s$ to indicate that P is a path with vertices $v_1, v_2, \dots, v_s$ in this order, and write $K = (v_1 v_2 \dots v_k)$ to indicate that K is a cycle with vertices $v_1, v_2, \dots, v_k$ in this cyclic order. For convenience, we also denote by P and K the vertex sets of P and K, respectively. The *length* of a path or a cycle is the number of edges in the path or cycle. A k-cycle (respectively, a k^--cycle or a k^+-cycle) is a cycle of length k (respectively, at most k or at least k). Two cycles are *adjacent* if they share an edge, and we say they are *normally adjacent* if their intersection is isomorphic to K_2. Let $\mathcal{G}$ denote the class of triangle-free plane graphs in which no 4-cycle is normally adjacent to a 5^--cycle. Dvořák, Lidický and Škrekovski [3] proved that every graph in $\mathcal{G}$ is 3-choosable. In this paper, we prove that every graph in $\mathcal{G}$ is weakly 2-degenerate. The proof uses induction, and for this purpose, we prove a stronger and more technical result.

For a plane graph and a cycle K, we use $\text{int}(K)$ to denote the set of vertices in the interior of K, and $\text{ext}(K)$ to denote the set of vertices in the exterior of K. Denote by $\text{int}[K]$ and $\text{ext}[K]$ the subgraph of G induced by $\text{int}(K) \cup K$ and $\text{ext}(K) \cup K$, respectively. For the plane graph G, we denote by $B(G)$ the boundary walk of the infinite face of G. For a face F of G, let $V(F)$ be the set of vertices on the boundary of F, and use $F = [v_1 v_2 \dots v_k]$ to indicate that the boundary closed walk of F is $v_1 v_2 \dots v_k$ in this cyclic order.

Theorem 5. *Let $G \in \mathcal{G}$, and $P = p_1 p_2 \dots p_s$ be a path on $B(G)$ with at most four vertices. Let $f \in \mathbb{Z}^G$ be a function satisfying the following conditions:*

(i) $f(p_i) = 0$ for $1 \leqslant i \leqslant s$, $f(v) = 2$ for all $v \notin B(G)$, and $1 \leqslant f(v) \leqslant 2$ for all $v \in B(G) \setminus V(P)$;

(ii) $I = \{v \mid f(v) = 1\}$ is an independent set in G, and each vertex in I has at most one neighbor in P.

Then $G - E[P]$ is weakly f-degenerate.

The following is an easy consequence of Theorem 5.

Corollary 6. *Every graph in $\mathcal{G}$ is weakly 2-degenerate. In particular, every planar graph of girth at least 5 is weakly 2-degenerate.*

The proof of Theorem 5 uses induction, and follows a similar line as the proof of the 3-choosability of these graphs in [3]. Indeed, the idea of DeleteSave operation was used in some cases in [3] (as well as in many other papers on list coloring of graphs), although the term DeleteSave was not used explicitly. Nevertheless, the proof of Theorem 5 requires rather different treatments in some cases. The conclusion that these graphs are weakly 2-degenerate is intrinsically stronger. For example, it implies that these graphs are DP 3-paintable, and the proof in [3] does not apply to DP-coloring.

We now turn our attentions to our next main result. To that end, assume S is a surface and G is a graph embedded in S. A cycle C in G is *contractible* if, as a closed curve on S, it separates S into two parts, and one part is homeomorphic to the disc. We say C is *non-contractible* otherwise. The length of the shortest non-contractible cycle in G is called the *edge-width* of G and is denoted by $\mathrm{ew}(G)$. Note that if S is the sphere, then every closed curve in S is contractible, and hence $\mathrm{ew}(G) = \infty$ for any graph G embedded in S. We say a graph G embedded in a surface S is *"locally planar"* if $\mathrm{ew}(G)$ is "large". It was proved by Thomassen [7] that for any surface S, there is a constant w such that any graph G embedded in S with $\mathrm{ew}(G) \geqslant \mathsf{w}$ is 5-colorable. Roughly speaking, this result says that locally planar graphs are 5-colorable. This result was strengthened in a sequence of papers, where it was proved that locally planar graphs are 5-choosable [2], 5-paintable [5] and DP 5-paintable [6]. In this paper, we further strengthen this result by proving the following result.

Theorem 7. *For any surface S, there is a constant $\mathsf{w}(S)$ such that every graph G embedded in S with edge-width at least $\mathsf{w}(S)$ is weakly 4-degenerate.*

2 Some preliminaries

For $\Omega = \mathsf{Del}(\theta_1, \theta_2, \ldots, \theta_k)$ and $(G, f) \in \mathcal{L}$, let

$$(G_\Omega, f_\Omega) = \mathsf{Del}(\theta_1, \theta_2, \ldots, \theta_k)(G, f).$$

If $\Omega = \mathsf{Del}(\theta_1, \ldots, \theta_k)$ and for each i, either $\theta_i = \langle u_i \rangle$ or $\theta_i = \langle u_i, w_i \rangle$, then let $U_\Omega = \{u_1, u_2, \ldots, u_k\}$. Note that for any removal scheme Ω, we have $G_\Omega = G - U_\Omega$ and $f_\Omega \geqslant f_{-U_\Omega}$. If $G[U]$ is weakly f-degenerate, then there is a removal scheme Ω legal for $(G[U], f)$ with $U_\Omega = U$. Thus we have the following observation.

Observation 8. *If $G - U$ is weakly f_{-U}-degenerate, then G is weakly f-degenerate if and only if $G[U]$ is weakly f-degenerate. In particular, if $f(x) \geqslant \deg_G(x)$, then G is weakly f-degenerate if and only if $G - x$ is weakly f-degenerate.*

Observation 9. *The following follows from the definition.*

1. *If $uv \notin E(G)$ and $\mathsf{Del}(\langle u \rangle, \langle v \rangle)$ is legal for (G, f), then $\mathsf{Del}(\langle v \rangle, \langle u \rangle)$ is legal for (G, f), and $\mathsf{Del}(\langle u \rangle, \langle v \rangle)(G, f) = \mathsf{Del}(\langle v \rangle, \langle u \rangle)(G, f)$.*

2. *If $uv \notin E(G)$, and $\textsf{Del}(\langle u,w\rangle,\langle v\rangle)$ is legal for (G,f), then $\textsf{Del}(\langle v\rangle,\langle u,w\rangle)$ is legal for (G,f), and $\textsf{Del}(\langle u,w\rangle,\langle v\rangle)(G,f) = \textsf{Del}(\langle v\rangle,\langle u,w\rangle)(G,f)$.*

3. *If $\textsf{Del}(\langle u,v\rangle,\langle v\rangle)$ is legal for (G,f), then $\textsf{Del}(\langle v\rangle,\langle u\rangle)$ is legal for (G,f), $\textsf{Del}(\langle v\rangle,\langle u\rangle)(G,f) = \textsf{Del}(\langle u,v\rangle,\langle v\rangle)(G,f)$.*

Proposition 10. *If $f(v) = 0$, then G is weakly f-degenerate if and only if $G-v$ is weakly f_{-v}-degenerate.*

Proof. If $\Omega = \textsf{Del}(\theta_1,\ldots,\theta_n)$ is legal for $(G-v, f_{-v})$ and removes all the vertices of $G-v$, then $\Omega' = \textsf{Del}(\langle v\rangle,\theta_1,\ldots,\theta_n)$ is legal for (G,f) and removes all the vertices of G since $\textsf{Del}(\langle v\rangle)(G,f) = (G-v, f_{-v})$.

Conversely, assume that $\Omega = \textsf{Del}(\theta_1,\ldots,\theta_n)$ is legal for (G,f) and removes all the vertices of G. As $f(v) = 0$, v is removed by a deletion operation and so there is an index i such that $\theta_i = \langle v\rangle$. For $j < i$, let

$$\theta'_j = \begin{cases} \langle u\rangle, & \text{if } \theta_j = \langle u,v\rangle, \\ \theta_j, & \text{otherwise.} \end{cases}$$

Note that if $uv \in E(G)$, and u is removed in a move θ_j for some $j < i$, then since $f(v) = 0$, we must have $\theta_j = \langle u,v\rangle$. By repeatedly applying Observation 9, we conclude that $\Omega' = \textsf{Del}(\theta'_1,\ldots,\theta'_{i-1},\theta_{i+1},\ldots,\theta_n)$ is legal for $(G-v, f_{-v})$ and removes all vertices of $G-v$. □

3 Proof of Theorem 5

It follows from Proposition 10 that the conclusion of Theorem 5 is equivalent to the statement that $G-P$ is weakly f_{-P}-degenerate. In the proof below, for different cases, we shall prove either of these two statements.

Definition 11. Assume G is a plane graph, P is a boundary path and $f \in \mathbb{Z}^G$. We say $\Omega = \textsf{Del}(\theta_1,\theta_2,\ldots,\theta_k)$ is *legal* for (G,P,f) if Ω is legal for $(G-P, f_{-P})$.

It follows from the definition and Proposition 10 that if Ω is legal for (G,P,f), and $G_\Omega - E[P]$ is weakly f_Ω-degenerate, then $G - E[P]$ is weakly f-degenerate.

Assume Theorem 5 is not true, and (G,P,f) is a counterexample with minimum $|V(G)| + |E(G)|$, and subject to this, with minimum $\sum_{v\in V(G)\setminus V(P)} f(v)$. To derive a contradiction, it suffices to find a removal scheme Ω legal for (G,P,f), so that (G_Ω, P, f_Ω) satisfies the conditions of Theorem 5. Note that Ω is required to be legal for $(G-P, f_{-P})$, and is not required to be legal for (G,f). We first apply Ω to $(G-P, f_{-P})$. Then to apply the induction hypothesis to the resulting graph, we need to change $(G-P)_\Omega$ back to G_Ω (i.e., add back the path P) and change $(f_{-P})_\Omega$ back to f_Ω, i.e., let $f_\Omega(x) = (f_{-P})_\Omega(x) + |N_G(x) \cap P|$ for every vertex x.

Lemma 12. *The graph G is 2-connected.*

Proof. Suppose that G has a cut-vertex v. Let G_1 and G_2 be two induced subgraphs of G such that $V(G) = V(G_1) \cup V(G_2)$ and $V(G_1) \cap V(G_2) = \{v\}$ and $E(G) = E(G_1) \cup E(G_2)$. If $P \subseteq G_1$, then (G_1, P, f) satisfies the conditions of Theorem 5. Hence $G_1 - P$ is weakly f_{-P}-degenerate. Also $(G_2, \{v\}, f)$ satisfies the conditions of Theorem 5, and hence $G_2 - v$ is weakly f_{-v}-degenerate. Note that $G - V(G_1) = G_2 - v$ and the restriction of $f_{-V(G_1)}$ to $G_2 - v$ equals f_{-v}. So by Observation 8, $G - P$ is weakly f_{-P}-degenerate.

Assume $P \nsubseteq G_1$ and $P \nsubseteq G_2$. Let $P_1 = P \cap G_1$ and $P_2 = P \cap G_2$. Then $v \in V(P)$, and P_i is a path in G_i for each $i \in \{1, 2\}$. Then $G_1 - P_1$ and $G_2 - P_2$ are weakly f_{-P_i}-degenerate. Hence $G - P$ is weakly f_{-P}-degenerate. □

It follows from Lemma 12 that the boundary $B(G)$ of G is a cycle. A cycle K in G is *separating* if both $\mathrm{int}(K)$ and $\mathrm{ext}(K)$ are not empty.

Lemma 13. $|B(G)| \geqslant 8$ *and every separating cycle in* G *has length at least* 8.

Proof. Assume $B(G) = (v_1 v_2 \dots v_k)$ for some $k \leqslant 7$. As G is triangle-free and no 4-cycle is normally adjacent to a 5^--cycle, $B(G)$ is an induced cycle.

For convenience, assume $k = 7$ (the $k \leqslant 6$ can be treated similarly), and assume that $P = v_2 v_3 v_4 v_5$. Let $G' = G - v_7$, and let $f'(v) = f(v) - 1$ for $v \in N_G(v_7) - \{v_1, v_6\}$, $f'(v_1) = f'(v_6) = 1$ and $f'(v) = f(v)$ otherwise. As G is triangle-free and no 4-cycle is normally adjacent to a 5^--cycle, no neighbor of v_7 is adjacent to two vertices of P. Hence (G', P, f') satisfies the conditions of Theorem 5 and $G' - P$ is weakly f'_{-P}-degenerate. As $f'_{-P}(v_1) = f'_{-P}(v_6) = 0$, it follows from Proposition 10 that $G' - (P \cup \{v_1, v_6\})$ is weakly $f'_{-(P \cup \{v_1, v_6\})}$-degenerate. Since $G - B(G) = G' - (P \cup \{v_1, v_6\})$ and $f_{-B(G)} = f'_{-(P \cup \{v_1, v_6\})}$, and $B(G) - P$ is certainly weakly f_{-P}-degenerate, it follows from Observation 8 that $G - P$ is weakly f_{-P}-degenerate.

Next we assume that K is a separating 7^--cycle in G. Since $G \in \mathcal{G}$, K is an induced cycle. By the minimality of G, $\mathrm{ext}[K] - P$ is weakly f_{-P}-degenerate. By Observation 8, to show that $G - P$ is weakly f_{-P}-degenerate, it suffices to show that $G - \mathrm{ext}[K] = \mathrm{int}[K] - K$ is weakly f_{-K}-degenerate.

Assume $K = (v_1 v_2 \dots v_k)$, where $k \leqslant 7$. If $k = 4$, then let $P' = v_1 v_2 \dots v_k$. Then $f'_{-K} = f'_{-P'}$. By the minimality of G, $\mathrm{int}[K] - K$ is weakly f_{-K}-degenerate. Assume $k \geqslant 5$. Let $P' = v_1 v_2 \dots v_{k-3}$, $G' = \mathrm{int}[K]$ and $f'' \in \mathbb{Z}^{G'}$ be defined as $f''(x) = 0$ for $x \in P'$, $f''(v_{k-2}) = f''(v_k) = 1$, $f''(v_{k-1}) = 2$ and $f''(x) = f(x)$ for $x \notin K$. By the minimality of G, $G' - P'$ is weakly $f''_{-P'}$-degenerate. The same argument as above shows that $\mathrm{int}[K] - K$ is weakly f_{-K}-degenerate. □

Lemma 14. *There are no* 4*-cycles adjacent to* 4*- or* 5*-cycles.*

Proof. Suppose to the contrary that a 4-cycle C_1 is adjacent to a 5^--cycle C_2. Assume that $C_1 = (a_1 a_2 a_3 a_4)$. By assumption, C_1 and C_2 are not normally adjacent. So they intersect at three vertices. By symmetry, we may assume $C_2 = (a_1 a_2 a_3 b_4)$ or $C_2 = (a_1 a_2 a_3 b_4 b_5)$. By Lemma 13, each of C_1 and C_2 bounds a face. Thus a_2 is a 2-vertex which must be on the outer face. This implies that either $a_4 \in \mathrm{int}(C_2)$ or $b_4 \in \mathrm{ext}(C_1)$. As vertices not in $B(G)$ has degree at least 3 by Observation 8, it follows that $(a_1 a_4 a_3 b_4)$ or $(a_1 a_4 a_3 b_4 b_5)$ is a separating cycle of length at most 5, contradicting Lemma 13. □

A k-*chord* of $B(G)$ is a path Q of length k such that only its two ends are on $B(G)$. A 1-chord is also called a chord of $B(G)$. Assume Q is a k-chord of G. Let G_1, G_2 be the two subgraphs with $V(G_1) \cap V(G_2) = V(Q)$ and $V(G_1) \cup V(G_2) = V(G)$ and $E(G) = E(G_1) \cup E(G_2)$. We say G_1 and G_2 are the subgraphs of G *separated* by Q. We index the subgraphs so that $|E(P \cap G_1)| \geqslant |E(P \cap G_2)|$. Hence $|E(P \cap G_2)| \leqslant 1$.

Observation 15. *Let $P_2 = Q \cup (P \cap G_2)$. It is obvious that (G_1, P, f) satisfies the conditions of Theorem 5, and hence $G_1 - P$ is weakly f_{-P}-degenerate. If P_2 is an induced path and (G_2, P_2, f) also satisfies the conditions of Theorem 5, then $G_2 - P_2$ is weakly f_{-P_2}-degenerate, and it follows from Observation 8 that $G - P$ is weakly f_{-P}-degenerate, a contradiction. Thus we may assume that (G_2, P_2, f) does not satisfy the conditions of Theorem 5.*

Lemma 16. *$B(G)$ has no chords.*

Proof. Assume to the contrary that $B(G)$ has a chord uw. Let G_1, G_2 be the two subgraphs of G separated by uw.

Assume $P \subseteq G_1$. Since G is triangle-free, each vertex in G_2 is adjacent to at most one vertex in $\{u, w\}$. Thus (G_2, uw, f) satisfies the conditions of Theorem 5, in contrary to Observation 15.

Assume $P \nsubseteq G_1$ and $P \nsubseteq G_2$. Without loss of generality, assume that $w \in V(P)$. Then $|E[P] \cap E(B(G_i))| < |E[P]| \leqslant 3$. Since G is triangle-free, $u \notin V(P)$ and P is an induced path.

We may assume that $|E(P \cap G_2)| = 1$. Let $P_2 = Q \cup (P \cap G_2)$. If P_2 is not contained in a 4-cycle in G_2, then (G_2, P_2, f) satisfies the conditions of Theorem 5, a contradiction.

Assume P_2 is contained in a 4-cycle in G_2. Since no 4-cycle in G is adjacent to a 5^--cycle, uw is not contained in a 5^--cycle in G_1. Let $P_1 = uw \cup (P \cap G_1)$. It is easy to verify that (G_2, P, f) and (G_1, P_1, f) satisfy the conditions of Theorem 5. By the minimality of G, $G_2 - P$ is weakly f_{-P}-degenerate, and $G_1 - P_1$ is weakly f_{-P_1}-degenerate. It follows from Observation 8 that $G - P$ is weakly f_{-P}-degenerate, a contradiction. □

Since $B(G)$ is an induced cycle of length at least 8 and (G, P, f) is a counterexample with minimum $\sum_{v \in V(G) \setminus V(P)} f(v)$, we may assume that $P = p_1p_2p_3p_4$ is an induced path of length three. Assume $B(G) = (p_1p_2p_3p_4x_1x_2 \ldots x_m)$, where $m \geqslant 4$. We say a k-chord Q of $B(G)$ *splits off* a face F from G if one of the two subgraphs separated by Q is the boundary cycle of F.

Lemma 17. *Let uvw be a 2-chord of $B(G)$. Then $\{u, w\} \nsubseteq V(P)$, and uvw splits off a 5^--face F such that $|V(F) \cap V(P)| \leqslant 2$. Moreover, if $|V(F) \cap V(P)| \leqslant 1$, then F is a 4-face. Consequently, every internal vertex is adjacent to at most two vertices in $B(G)$ and adjacent to at most one vertex in $V(P)$.*

Proof. Assume uvw is a 2-chord and G_1 and G_2 are subgraphs separated by uvw. Assume $|E(P \cap G_1)| > |E(P \cap G_2)|$. Let $P_2 = uvw \cup (P \cap G_2)$. As $|E(P \cap G_1)| > |E(P \cap G_2)|$, we know that $|E(P \cap G_2)| \leqslant 1$ and hence P_2 has length 2 or 3.

If P_2 is not induced, then since G is triangle-free and $B(G)$ has no chord, $G[P_2]$ is a 4-cycle that bounds a 4-face by Lemma 13. So uvw splits off a 4-face. Moreover, $\{u, w\} \not\subseteq V(P)$, for otherwise, the boundary of G_1 is a 5-cycle, contradicting to Lemma 14.

Assume P_2 is an induced path. By Observation 15, (G_2, P_2, f) does not satisfy the conditions of Theorem 5. This means that G_2 has a vertex y with $f(y) = 1$ and y is adjacent to two vertices of P_2.

If P_2 has length 2, then G_2 is a 4-cycle, and hence uvw splits off a 4-face. Moreover, $\{u, w\} \not\subseteq V(P)$, for otherwise, G_1 is a 5^--cycle, in contrary to Lemma 14.

Assume P_2 has length 3. We may assume $w = p_3$ and $P_2 = uvp_3p_4$. As $B(G)$ has no chord, we know that either y is adjacent to p_4 and v, or y is adjacent to p_4 and u. Note that since y is a boundary vertex with $f(y) = 1$ and G has no chord, y cannot be adjacent to w.

If y is adjacent to p_4 and v, then $u \notin P$ (for otherwise G_1 is a 4-cycle and G contains two adjacent 4-cycles). Then yvu is a 2-chord which separates G into G'_1 and G'_2 with $P \subseteq V(G'_1)$. Let $P'_2 = yvu$. Then (G'_1, P, f) and (G'_2, P'_2, f) satisfy the conditions of Theorem 5, and hence $G'_1 - P$ is weakly f_{-P}-degenerate, and $G'_2 - P'_2$ is weakly $f_{-P'_2}$-degenerate (note that in this case, G'_2 is not a 4-cycle as G contains no two adjacent 4-cycles). By Observation 8, $G - P$ is weakly f_{-P}-degenerate.

Assume y is adjacent to p_4 and u. Then G_2 is a facial 5-cycle by Lemma 13. If $u \in P$, then $u = p_1$. By Lemma 13, G_1 is a facial 4-cycle. It implies that $\deg_G(v) = 2$, which contradicts Observation 8. Thus $u \notin P$ and uvw splits off a 5-face. □

Lemma 18. *If uvw is a 2-chord and $f(u) = 1$, then $w \in \{p_2, p_3\}$.*

Proof. Assume uvw is a 2-chord with $f(u) = 1$. By Lemma 17, the 2-chord uvw splits off a 5^--face $F = [uvw \dots x]$ with $|V(F) \cap V(P)| \leqslant 2$. Thus $\deg(x) = 2$. Since $f(u) = 1$, and $f(x) \leqslant \deg(x) - 1 = 1$, we conclude that $x \in P$ (as I is an independent set). Since F is a 5^--face, we have $x \in \{p_1, p_4\}$ and $w \in \{p_2, p_3\}$. □

Lemma 19. *If $Q = uvwz$ is a 3-chord with $\{u, z\} \cap \{p_2, p_3\} = \emptyset$, then Q splits off a 5^--face F' such that $V(F') \cap V(P) \subseteq \{u, z\}$.*

Proof. Let G_1 and G_2 be the two subgraphs of G separated by Q. Since $\{u, z\} \cap \{p_2, p_3\} = \emptyset$, we may assume that $P \subset G_1$.

If $uz \in E(G)$, then since $B(G)$ has no chord, G_2 is a facial 4-cycle and Q splits off a 4-face. Assume $uz \notin E(G)$.

By Observation 15, (G_2, Q, f) does not satisfy the conditions of Theorem 5. Then there exists a vertex x with $f(x) = 1$ adjacent to two vertices in $\{u, v, w, z\}$. If x is adjacent to u and z, the Q splits off the 5-face F' bounded by $(uvwzx)$. If x is adjacent to u and w, then by Lemma 17, the 2-chord xwz splits off a 4-face $[xwzz']$. Then $(uvwx)$ and $(xwzz')$ are adjacent 4-cycles, which contradicts Lemma 14. The case that x is adjacent to z and v is symmetric. □

We may assume that $f(x_1) = 1$ or $f(x_2) = 1$, for otherwise, we let $f' = f$ except that $f'(x_1) = 1$. Since G is chordless by Lemma 16, $I \cup \{x_1\}$ is an independent set of G. So

(G, P, f') satisfies the conditions of Theorem 5. By the minimality of $\sum_{v \in V(G) \setminus V(P)} f(v)$, $G - P$ is weakly f'_{-P}-degenerate, and hence $G - P$ is weakly f_{-P}-degenerate.

We write $f(x_1, x_2, \ldots, x_i) = (a_1, a_2, \ldots, a_i)$ to mean that $f(x_j) = a_j$ for $j = 1, 2, \ldots, i$. Let X be the set of boundary vertices defined as follows:

$$X = \begin{cases} \{x_1\}, & \text{if } f(x_1, x_2, x_3) = (1,2,2), \\ \{x_2, x_3, x_4\}, & \text{if } f(x_1, x_2, x_3, x_4) = (1,2,1,2) \text{ and either } m = 4 \text{ or } f(x_5) = 1, \\ \{x_2, x_3, x_4, x_5\}, & \text{if } f(x_1, x_2, x_3, x_4, x_5) = (1,2,1,2,2) \text{ and either } m = 5 \text{ or } f(x_6) \\ & = 1, \text{ and there is a 2-chord connecting } x_2 \text{ and } x_4, \\ \{x_2, x_3\}, & \text{if } f(x_1, x_2, x_3, x_4, x_5) = (1,2,1,2,2) \text{ and either } m = 5 \text{ or } f(x_6) \\ & = 1, \text{ and there is no 2-chord connecting } x_2 \text{ and } x_4, \\ \{x_2, x_3, x_4\}, & \text{if } f(x_1, x_2, x_3, x_4, x_5, x_6) = (1,2,1,2,2,2), \\ \{x_1, x_2, x_3\}, & \text{if } f(x_1, x_2, x_3, x_4) = (2,1,2,1), \\ \{x_2\}, & \text{if } f(x_1, x_2, x_3, x_4) = (2,1,2,2). \end{cases}$$

Let

$$Y = \{u : u \text{ is an interior vertex of a } 3^{-}\text{-chord connecting two vertices of } X\}.$$

Observe that if $x_j u x_{j'}$ is a 2-chord with $j < j'$, then $x_j u x_{j'}$ splits off a 4-face by Lemma 17. Hence $j' = j + 2$, $\deg(x_{j+1}) = 2$ and hence $f(x_{j+1}) = 1$ by Observation 8. Similarly, if $x_j u v x_{j'}$ is a 3-chord with $j < j'$, then $x_j u v x_{j'}$ splits off a 5^{-}-face by Lemma 19. So $j' \leqslant j + 2$ and if $j' = j + 2$, then $\deg(x_{j+1}) = 2$ and $f(x_{j+1}) = 1$ by Lemma 13 and Observation 8.

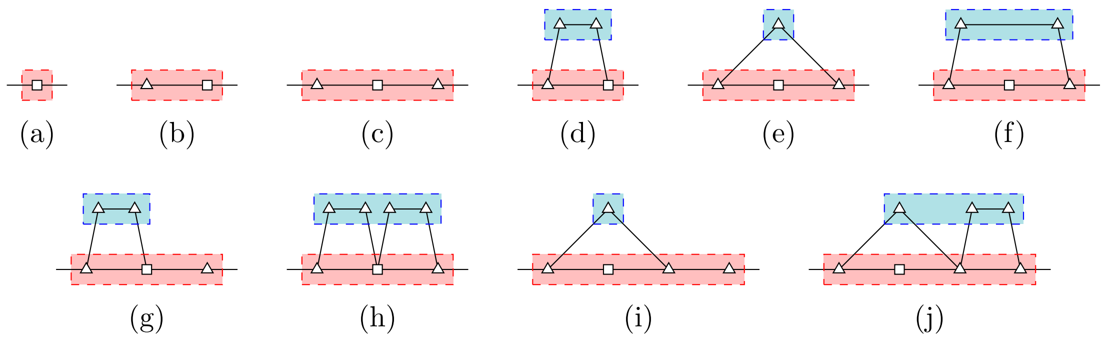


Figure 1: The possible local structures of $X \cup Y$, where the set X is in the dashed red box and Y is in the dashed blue box. Here and in figures below, a square indicates a vertex v with $f(v) = 1$, a triangle indicates a vertex v with $f(v) = 2$.

Lemma 20. *Assume X and Y are defined as above. The following hold:*

(1) No two vertices of $X \cup Y$ are connected by a 3^{-}-path with interior vertices in $V(G) - (X \cup Y)$, where a 3^{-}-path means a path with length at most 3.

(2) If $x_j uvx_{j'}$ is a 3-chord connecting two vertices of X, then at most one of u and v has a neighbor in P.

(3) Each vertex in Y has degree 2 in $G[X \cup Y]$. Hence $G[Y]$ consists of some isolated vertices and at most two copies of K_2.

(4) There is no 2-chord xuz with $x \in X$ and $z \notin X$ and $f(z) = 1$.

Proof. (1)-(3) can be easily checked in each case. We omit the details, but note that we may need to use the fact that G is triangle-free, no 4-cycle is adjacent to a 5^--cycle (Lemma 14), and there is no separating 7^--cycle (Lemma 13). (4) follows from Lemma 18. $\square$

Note that X is a set of at most 3 consecutive vertices in $B(G)$, with one exception $X = \{x_2, x_3, x_4, x_5\}$ and there is a 2-chord connecting x_2 and x_4. By Lemma 17 and 19, there are at most two 3^--chords connecting two vertices of X. Hence $|Y| \leqslant 4$. Assume that $Y = \{y_1, \ldots, y_t\}$, where $0 \leqslant t \leqslant 4$ ($Y = \emptyset$ if $t = 0$). If $y_i y_j$ is an edge and y_i is adjacent to a vertex in P, then by Lemma 20 (2), y_j is not adjacent vertices in P and we index the vertices of Y so that $i < j$.

In the following, for convenience, we let $x_{m+1} = p_1$.

Let

$$\Omega = \begin{cases} \mathsf{Del}(\langle x_1 \rangle), & \text{if } f(x_1, x_2, x_3) = (1, 2, 2), \\ \mathsf{Del}(\langle x_4, x_5 \rangle, \langle x_3 \rangle, \langle x_2, x_1 \rangle, \langle y_1 \rangle, \ldots, \langle y_t \rangle), & \text{if } f(x_1, x_2, x_3, x_4) = (1, 2, 1, 2) \\ & \ \text{and either } m = 4 \text{ or } f(x_5) = 1, \\ \mathsf{Del}(\langle x_5, x_6 \rangle, \langle x_4 \rangle, \langle x_3 \rangle, \langle x_2, x_1 \rangle, \langle y_1 \rangle, \ldots, \langle y_t \rangle), & \text{if } f(x_1, x_2, x_3, x_4, x_5) = (1, 2, 1, 2, 2) \\ & \ \text{and either } m = 5 \text{ or } f(x_6) = 1, \text{ and} \\ & \ \text{there is a 2-chord connecting } x_2 \text{ and } x_4, \\ \mathsf{Del}(\langle x_3 \rangle, \langle x_2, x_1 \rangle, \langle y_1 \rangle, \ldots, \langle y_t \rangle), & \text{if } f(x_1, x_2, x_3, x_4, x_5) = (1, 2, 1, 2, 2) \\ & \ \text{and either } m = 5 \text{ or } f(x_6) = 1, \text{ and} \\ & \ \text{there is no 2-chord connecting } x_2 \text{ and } x_4, \\ \mathsf{Del}(\langle x_4 \rangle, \langle x_3 \rangle, \langle x_2, x_1 \rangle, \langle y_1 \rangle, \ldots, \langle y_t \rangle), & \text{if } f(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 2, 1, 2, 2, 2), \\ \mathsf{Del}(\langle x_3, x_4 \rangle, \langle x_2 \rangle, \langle x_1 \rangle, \langle y_1 \rangle, \ldots, \langle y_t \rangle), & \text{if } f(x_1, x_2, x_3, x_4) = (2, 1, 2, 1), \\ \mathsf{Del}(\langle x_2 \rangle), & \text{if } f(x_1, x_2, x_3, x_4) = (2, 1, 2, 2). \end{cases}$$

It is straightforward to verify that Ω is legal for $(G - P, f_{-P})$ by Lemma 20 (2). In particular, $f_{-P}(x_1) = f(x_1) - 1$. Hence the operation $\langle x_2, x_1 \rangle$ in Cases 2–5 are legal.

To finish the proof of Theorem 5, it suffices to prove that (G_Ω, P, f_Ω) satisfies the conditions of Theorem 5, and hence $G_\Omega - E[P]$ is weakly f_Ω-degenerate.

Assume $v \in B(G_\Omega)$. If $f(v) = 2$, then by Lemma 20 (1), $f_\Omega(v) \geqslant 1$. If $f(v) = 1$, then since G has no chord (Lemma 16) and there is no 2-chord xuv with $x \in X$ and $v \notin X$ and $f(v) = 1$ (Lemma 20 (4)), then $f_\Omega(v) = f(v) = 1$. So (G_Ω, P, f_Ω) satisfies the condition (i) of Theorem 5.

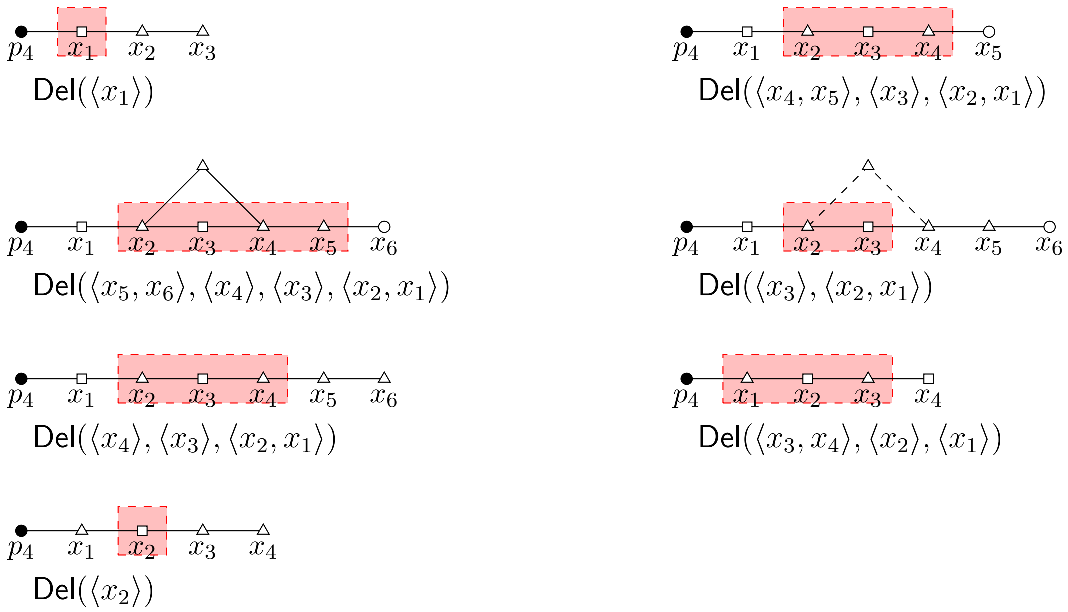


Figure 2: The subset X and the corresponding removal scheme.

Let $I' = \{v \in B(G_\Omega) : f_\Omega(v) = 1\}$. Assume there exists $uv \in E(G_\Omega)$ with $u, v \in I'$. As I is independent in G, we may assume that $f(u) = 2$. By Lemma 20 (1), $f(v) = 1$ and $v \in B(G)$. By Lemma 20 (4), u is adjacent to a vertex $y \in Y$. Assume $yx \in E(G)$ and $x \in X$, then $vuyx$ is a 3-chord in G with $\{v, x\} \cap \{p_2, p_3\} = \emptyset$. By Lemma 19, the 3-chord $vuyx$ splits off a 5^--face F with $V(F) \cap V(P) = \emptyset$. If F is a 5-face $[vuyxz]$, then $\deg_G(z) = 2$ and by Observation 8, $f(z) = 1$, a contradiction (as $f(v) = 1$ and I is an independent set). So F is a 4-face. On the other hand, by definition, y is contained in a 3^--chord Q connecting two vertices of X. By Lemma 17 and Lemma 19, Q splits off a 5^--face F' with $v \notin F'$, which is distinct from the 4-face F. So G has a 4-cycle adjacent to a 5^--cycle, a contradiction. Thus I' is an independent set. By Lemma 17, every internal vertex of G is adjacent to at most one vertex in P. So (G_Ω, P, f_Ω) satisfies the condition (ii) of Theorem 5. So (G_Ω, P, f_Ω) satisfies the conditions of Theorem 5. This completes the proof of Theorem 5.

4 Proof of Theorem 7

The *face-width* $\mathrm{fw}(G)$ of a graph G embedded in a surface S is the largest integer k such that every non-contractible closed curve in S intersects G in at least k points. It is obvious that for any graph G embedded in S, $\mathrm{fw}(G) \leqslant \mathrm{ew}(G)$. Theorem 7 will follow from the following Theorem 21.

Theorem 21. *For every surface S there exists a constant $\mathsf{w}(S)$ such that every 5-connected graph that can be embedded in S with face-width at least $\mathsf{w}(S)$ is weakly 4-degenerate.*

First, we show that Theorem 21 implies Theorem 7. The following result was proved in [1] and is used in our proof.

Lemma 22. *Let G be a plane graph with at least 3 vertices, and P be a set of 2 consecutive vertices on $B(G)$. Let $f : V(G) \to \mathbb{Z}$ be defined by*

$$f(u) = \begin{cases} 0, & \text{if } u \in P; \\ 2, & \text{if } u \in B(G) - P; \\ 4, & \text{otherwise.} \end{cases}$$

Then $G - E[P]$ is weakly f-degenerate, or equivalently, $G - P$ is weakly f_{-P}-degenerate.

Proof of Theorem 7. Assume Theorem 21 is true. Let $\mathsf{w}(S)$ be the constant in Theorem 21. We shall prove that every graph embedded in S with edge-width at least $3\mathsf{w}(S)$ is weakly 4-degenerate.

Assume this is not true and G_0 is a counterexample with minimum $|V(G_0)|$. We construct a triangulation G of S as follows: For each 6^--face of G_0, add edges to triangulate it. For each 7^+-face, we add a *chimney* as follows: Assume the boundary of the face is $C_0 = (v_{0,1}v_{0,2} \dots v_{0,k})$. Note that C_0 is not necessarily a cycle. Add k new cycles $C_i = (v_{i,1}v_{i,2} \dots v_{i,k})$ $(i = 1, 2, \dots, k)$, add edges $v_{i,j}v_{i+1,j}, v_{i,j}v_{i+1,j+1}$ $(i = 0, 1, \dots, k-1$, $j = 1, 2, \dots, k$, where the additions are carried out modulo $k)$, and finally add a new vertex adjacent to all vertices of C_k. The construction of a chimney in 7-face is illustrated in Figure 3.

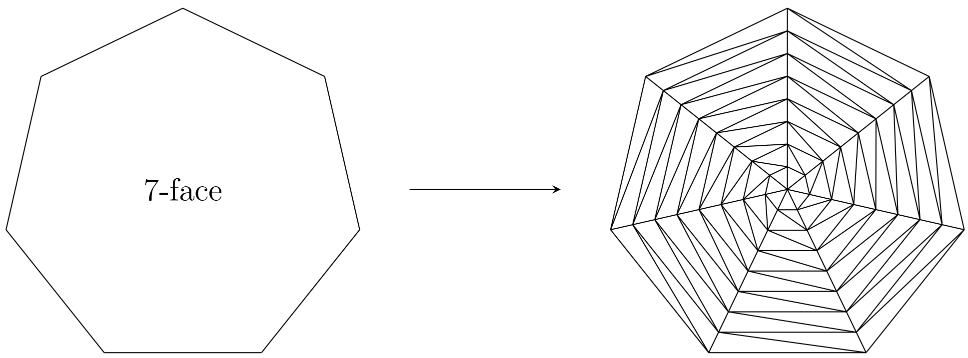


Figure 3: Adding a chimney in a 7-face.

It is easy to see (cf. [2]) that $\mathrm{ew}(G) = \mathrm{fw}(G) \geqslant \frac{1}{3}\mathrm{ew}(G_0) \geqslant \mathsf{w}(S)$. As G_0 is a subgraph of G and G_0 is not weakly 4-degenerate, G is not weakly 4-degenerate. Thus, by Theorem 21, G is not 5-connected. Thus, G has a vertex-cut of size at most 4. As G is a triangulation of S and $\mathrm{ew}(G)$ is large, there is a contractible separating 4^--cycle C. Let $D = \mathrm{int}[C]$ and $D^o = D - C$ be the interior of D. We choose C so that D^o contains the minimum number of vertices (subject to the condition that $D^o \neq \emptyset$). This implies that either D^o contains a single vertex, or each vertex in D^o is adjacent to at most two vertices of C. Observe that in constructing G from G_0, new vertices are added only when chimneys are added. As chimneys are added to 7^+-faces, the interior of D cannot contain added vertices only. Therefore, $D' := G_0 \cap D^o \neq \emptyset$.

Let $G_0' = G_0 - D'$ and $f(v) = 4$ for $v \in G_0$. As $\mathrm{ew}(G_0') \geqslant \mathrm{ew}(G_0) \geqslant 3\mathsf{w}(S)$, by the minimality of G_0, the graph G_0' is weakly f-degenerate.

Next, we show that $G_0 - G_0' = D'$ is weakly $f_{-G_0'}$-degenerate. If D' has a single vertex v, then $f_{-G_0'}(v) \geqslant 0$ since $N_{G_0'}(v) \subseteq V(C)$ and $|C| \leqslant 4$. Assume that $|D'| \geqslant 2$. Let $B(D')$ be the boundary of D'. Then each vertex $v \in B(D')$ is adjacent to at most two neighbors in G_0', and so $f_{-G_0'}(v) \geqslant 2$. Every interior vertex $v \in V(D' - B(D'))$ is not adjacent to any vertex of G_0', and hence $f_{-G_0'}(v) = 4$. By Lemma 22, D' is weakly $f_{-G_0'}$-degenerate.

By Observation 8, G_0 is weakly f-degenerate, a contradiction. □

Now it suffices to prove Theorem 21. We may assume that G is a triangulation of the surface S because adding edges does not decrease the face-width or the connectivity of a graph. To prove that G is weakly 4-degenerate, we cut the surface along some closed curves that partition the vertices of G into pieces. Each piece induces a planar graph, and we shall apply removal schemes to these pieces in order. When all pieces are processed, we eventually remove all the vertices of G. As each piece induces a planar graph, we may apply Lemma 22 to construct a removal scheme for each piece. The pieces are ordered. The central problem one needs to consider is how the removal scheme for one piece affects later pieces. The removal schemes for earlier pieces only affect the boundary vertices of the later pieces. The partition of the vertices of G is chosen carefully so that these impacts are controlled in such a way that Lemma 22 can still be applied to later pieces.

Such partitions of the vertices of G are used and modified in a sequence of papers [2, 5, 6, 7]. We use the definition given in [5].

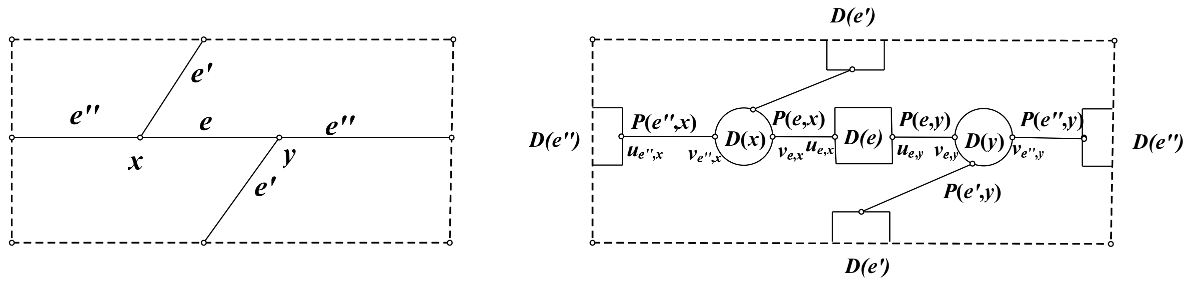
Definition 23. Assume G is a graph embedded in S and H is a cubic graph (probably with parallel edges). An *H-scheme* in G is a family $\mathcal{F}$ of induced subgraphs of G together with a labeling which associates subgraphs in $\mathcal{F}$ to vertices and edges of H such that the following hold:

A1 $\mathcal{F} = \{D(x) : x \in V(H)\} \cup \{D(e) : e \in E(H)\} \cup \{P(e, x) : e \in E(H), x \in e\}$ consists of a family of subgraphs of G, each embedded in a disk in S, and for $e \in E(H)$ and $x \in e$, $P(e, x)$ is a path connecting a vertex $v_{e,x}$ on the boundary of $D(x)$ to a vertex $u_{e,x}$ on the boundary of $D(e)$.

A2 The subgraphs in $\mathcal{F}$ are pairwise disjoint, except that $v_{e,x}$ belongs to both $P(e, x)$ and $D(x)$, and $u_{e,x}$ belongs to both $P(e, x)$ and $D(e)$. Also no edge of G connects vertices of distinct subgraphs in $\mathcal{F}$, except that $v_{e,x}$ has neighbors in both $D(x)$ and $P(e, x)$, and $u_{e,x}$ has neighbors in both $D(e)$ and $P(e, x)$.

A3 By contracting each $D(x)$ into a single vertex for each $x \in V(H)$, and replacing each $P(e, x) \cup D(e) \cup P(e, y)$ by an edge joining x and y for each edge $xy \in E(H)$, we obtain a 2-cell embedding of H in S.

Intuitively, the union of all the subgraphs in $\mathcal{F}$ resembles the cubic graph H: Each vertex v is replaced by $D(v)$, and each edge $e = xy$ is replaced by $P(e, x) \cup D(e) \cup P(e, y)$, which is a link between $D(x)$ and $D(y)$. Figure 4 shows a cubic graph H embedded in the torus and an H-scheme in G.



Cubic graph H

H-scheme

Figure 4: A cubic graph H embedded in the torus and an H-scheme.

For $e = xy \in E(H)$, let $v^-_{e,x}$ and $v^+_{e,x}$ be the two neighbors of $v_{e,x}$ on the boundary of the outer face of $D(x)$, $u^-_{e,x}$ and $u^+_{e,x}$ be the two neighbors of $u_{e,x}$ on the boundary of the outer face of $D(e)$, and $u'_{e,x}$ be the unique common neighbor of $u_{e,x}, u^+_{e,x}, u^-_{e,x}$ in $D(e)$ if such a vertex exists. Let

$$D'(e) = D(e) - \{u_{e,x}, u^-_{e,x}, u^+_{e,x}, u'_{e,x}, u_{e,y}, u^-_{e,y}, u^+_{e,y}, u'_{e,y}\}.$$

If the vertices $u'_{e,x}$ and/or $u'_{e,y}$ do not exist, then ignore them in the above formula. Let $P'(e,x) = P(e,x) \cup \{v^-_{e,x}, u^-_{e,x}\}$. A *segment* of a path P is a subset of its vertices that induces a subpath of P.

Assume $\mathcal{F}$ is an H-scheme in a graph G embedded in S. Let $U = \bigcup_{F\in\mathcal{F}} V(F)$, $U' = V(G) \setminus U$ and $G' = G[U']$. By A3, each component of G' is a plane graph embedded in a disk on S. Let R be the bipartite subgraph of G induced by edges between U and U'. For an orientation on R, if u is oriented to v for an edge $uv \in E(G)$, then we say uv is an out-edge of u and v is an out-neighbor of u. For $v \in U \cup U'$, let $N^+_R(v)$ be the set of the out-neighbors of v and $deg^+_R(v) = |N^+_R(v)|$.

The following lemma is a combination of Lemma 2.4 and Lemma 3.2 in [5].

Lemma 24. *For any surface S, there is a constant $\mathsf{w}(S)$ such that the following holds: If a 5-connected triangulation G of S has face-width at least $\mathsf{w}(S)$, then there is a cubic graph H such that G has an H-scheme $\mathcal{F}$ satisfying the following: for each edge $e = xy$ of H, for any vertex $u \in D'(e)$,*

(a) $N_G(u) \cap U \subseteq D(e)$;

(b) $|N_G(u) \cap \{u_{e,x}, u^+_{e,x}, u^-_{e,x}, u'_{e,x}, u_{e,y}, u^+_{e,y}, u^-_{e,y}, u'_{e,y}\}| \leqslant 2$;

(c) $dist_G(u_{e,x}, u_{e,y}) \geqslant 5$.

Moreover, the associated bipartite graph R has an orientation for which the following holds:

(1) For $v \in U$, $\deg^+_R(v) \leqslant 1$. Moreover, for $e \in E(H)$ and $x \in e$, if $v \in D(x)$ or $v \in D(e) - \{u_{e,x}, u^-_{e,x}, u_{e,y}, u^-_{e,y}\}$, then $\deg^+_R(v) = 0$;

(2) For $v \in U'$, $\deg^+_R(v) \leqslant 2$.

(3) For $e \in E(H)$, $x \in e$ and $v \in U'$, if $N_R(v) \cap P'(e,x) \neq \emptyset$, then $N_R(v) \cap V(G)$ is a segment of $P'(e,x)$, and if av is an in-edge of v, then v has two out-edges vb, vc such that b, c lies between $v_{e,x}$ and a on $P'(e,x)$.

An H-scheme $\mathcal{F}$ satisfying (a)–(c) and moreover part of Lemma 24 is called a *nice H-scheme.*

Proof of Theorem 21. Assume S is a surface, $\mathsf{w}(S)$ is the constant in Lemma 24, and G is a 5-connected triangulation of S with face-width at least $\mathsf{w}(S)$. Let $\mathcal{F}$ be a nice H-scheme in G, and R be the associated bipartite graph oriented as in Lemma 24. Let $f(u) = 4$ for each vertex $u \in V(G)$, and let

$$\begin{aligned}
G_1 &= \bigcup_{x \in V(H)} D(x),\\
G_2 &= \bigcup_{e \in E(H),\ x \in e} (P(e,x) \cup \{u^-_{e,x}\}) \setminus \{v_{e,x}\},\\
G_3 &= \bigcup_{e \in E(H),\ x \in e} \{u'_{e,x}, u^+_{e,x}\},\\
G_4 &= \bigcup_{e \in E(H)} D'(e),\\
G_5 &= G[U'].
\end{aligned}$$

Observe that $V(G_1), V(G_2), V(G_3), V(G_4), V(G_5)$ form a partition of $V(G)$. As each component of G_1 is a plane graph $D(x)$ for some $x \in V(H)$, G_1 is a plane graph. By Lemma 22, G_1 is weakly f-degenerate.

Note that each component of G_2 is a path $(P(e,x) \cup \{u^-_{e,x}\}) \setminus \{v_{e,x}\}$ for some $e \in E(H)$, $x \in e$. We order the vertices of G_2 as $x_1, x_2, \ldots, x_t$ so that if $i < j$ and $x_i, x_j \in (P(e,x) \cup \{u^-_{e,x}\}) \setminus \{v_{e,x}\}$, then x_i lies between $v_{e,x}$ and x_j on $P'(e,x)$. By (A2) of Definition 23, $f_{-G_1}(x_i) = 3$ if x_i is the unique vertex of $P(e,x)$ adjacent to $v_{e,x}$ and $f_{-G_1}(x_i) = 4$ otherwise. By Lemma 24, $\deg^+_R(x_i) \leqslant 1$ for $i \in [t]$. Let $w_i \in N^+_R(x_i)$ if $\deg^+_R(x_i) = 1$ for $i \in [t]$. We define a removal scheme $\Omega = \mathsf{Del}(\theta_1, \theta_2, \ldots, \theta_t)$ as follows: for $i \in [t]$,

$$\theta_i = \begin{cases} \langle x_i \rangle, & \text{if } N^+_R(x_i) = \emptyset,\\ \langle x_i, w_i \rangle, & \text{if } N^+_R(x_i) = \{w_i\} \end{cases}$$

It suffices to show that θ_i is legal for $i \in [t]$. Let $A_i = \{x_1, x_2, \ldots, x_i\}$ for $i \in [t]$. If $\deg^+_R(x_i) = 0$, then $f_{-(G_1 \cup A_{i-1})}(x_i) \geqslant 3 \geqslant 0$, and so $\theta_i = \langle x_i \rangle$ is legal. Otherwise, $\deg^+_R(x_i) = 1$. By (3) of Lemma 24, w_i has two removed out-neighbors. Thus, $f_{-(G_1 \cup A_{i-1})}(w_i) \leqslant 2 < 3 \leqslant f_{-(G_1 \cup A_{i-1})}(x_i)$, and so $\theta_i = \langle x_i, w_i \rangle$ is also legal.

Now we consider the vertices in G_3. By definition and Lemma 24, each component of G_3 has at most two vertices, $u^+_{e,x}$ and $u'_{e,x}$ (if $u'_{e,x}$ does not exist, then $u^+_{e,x}$ is an isolated vertex of G_3). For each $e \in E(H)$ and $x \in e$, the vertex $u^+_{e,x}$ has only one neighbor in $G_1 \cup G_2$, and the vertex $u'_{e,x}$ (if it exists) has at most two removed neighbors in $G_1 \cup G_2$. Therefore $f_{-(G_1 \cup G_2)}(v) \geqslant 2$ for $v \in G_3$. Then for each component, we can remove its vertices by the deletion operation in the order $u^+_{e,x}$, $u'_{e,x}$. With the same operation, we can remove all the vertices of G_3 legally since the components of G_3 do not affect each other.

In the subgraph G_4, as each component of G_4 is a plane graph $D'(e)$ for some $e \in E(H)$, G_4 is a plane graph. By Lemma 24, for each edge $e = xy$ of H, for any vertex $u \in D'(e)$,

$$N_G(u) \cap \left(\bigcup_{i=1}^{3} V(G_i) \right) \subseteq \{u_{e,x}, u^+_{e,x}, u^-_{e,x}, u'_{e,x}, u_{e,y}, u^+_{e,y}, u^-_{e,y}, u'_{e,y}\},$$

and $|N_G(u) \cap \{u_{e,x}, u^+_{e,x}, u^-_{e,x}, u'_{e,x}, u_{e,y}, u^+_{e,y}, u^-_{e,y}, u'_{e,y}\}| \leqslant 2$. Therefore, $f_{-(\bigcup_{i=1}^3 G_i)}(u) \geqslant 2$ if u is on the boundary of G_4, and $f_{-(\bigcup_{i=1}^3 G_i)}(u) = 4$ otherwise. By Lemma 22, G_4 is weakly $f_{-(\bigcup_{i=1}^3 G_i)}$-degenerate.

Let Ω be the removal scheme constructed above that removes all the vertices in $\bigcup_{i=1}^4 G_i$. Then $G_\Omega = G_5$. Let $h = f_\Omega$. For each vertex u on the boundary of G_5, $h(u) \geqslant 2$ since the value of $f(u)$ only decreases when its out-neighbor is removed and u has at most two out-neighbors in $\bigcup_{i=1}^4 G_i$. For the interior vertex u, $h(u) = f(u) = 4$. Similarly, by Lemma 22, G_5 is weakly h-degenerate. Therefore, G is weakly f-degenerate. □

As a consequence of Theorem 7, the condition in Theorem 21 on the connectivity is redundant.

Corollary 25. *For any surface S there is a constant $\mathsf{w}(S)$ such that every graph G embedded in S with face-width at least $\mathsf{w}(S)$ is weakly 4-degenerate.*

Remark After the submission of this paper, it is proved recently in [9] that if a graph G is weakly f-degenerate, then G is $(f+1)$-AT. Combined with Theorem 5, we know triangle-free planar graphs with no 4-cycle normally adjacent to a 5^--cycle are 3-AT. This strengthens a classical result of Thomassen [8] that planar graphs of girth at least 5 are 3-choosable. Combined with Theorem 7, we know that locally planar graphs are 5-AT. This generalizes the result in [5] that locally planar graphs are 5-paintable, and the result in [10] that planar graphs are 5-AT.

Acknowledgements

Xuding Zhu received support from Grants NSFC 12371359, U20A2068.

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