On sequences without short zero-sum subsequences

Xiangneng Zeng[a]   Pingzhi Yuan[b]

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Abstract

Let G be a finite abelian group. It is well known that every sequence S over G of length at least $|G|$ contains a zero-sum subsequence of length at most $\mathsf{h}(S)$, where $\mathsf{h}(S)$ is the maximal multiplicity of elements occurring in S. It is interesting to study the corresponding inverse problem, that is to find information on the structure of the sequence S which does not contain zero-sum subsequences of length at most $\mathsf{h}(S)$. Under the assumption that $|\sum(S)| < \min\{|G|, 2|S|-1\}$, Gao, Peng and Wang showed that such a sequence S must be strictly behaving. In the present paper, we explicitly give the structure of such a sequence S under the assumption that $|\sum(S)| = 2|S|-1 < |G|$.

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1 Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of integers, positive integers, and nonnegative integers, respectively. For $a, b \in \mathbb{Z}$, let $[a, b] = \{x \in \mathbb{Z} : a \leqslant x \leqslant b\}$.

Let G be an abelian group (written additively) and $\mathcal{F}(G)$ be the free abelian (multiplicative) monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. We write a sequence $S \in \mathcal{F}(G)$ in the form

$$S = g_1 \cdot \ldots \cdot g_r = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $r \in \mathbb{N}_0$, $g_1, \ldots, g_r \in G$ and $\mathsf{v}_g(S) \in \mathbb{N}_0$. We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S and $|S| = r = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length* of S. The identity of $\mathcal{F}(G)$, denoted by $\mathbb{1}$, is called the *empty sequence*, which is simply the sequence having no terms. Denote by $\operatorname{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\}$ the *support* of S and by $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) : g \in G\}$ the *height* of S.

[a]Sino-French Institute of Nuclear Engineering and Technology, Sun Yat-Sen University, Guangzhou 510275, P.R. China (zengxneng@mail.sysu.edu.cn).
[b]School of Mathematical Science, South China Normal University, Guangzhou 510631, P.R. China (yuanpz@scnu.edu.cn).

A sequence S_1 is called a *subsequence* of S if $S_1|S$ in $\mathcal{F}(G)$ (i.e. $\mathsf{v}_g(S_1) \leqslant \mathsf{v}_g(S)$ for all $g \in G$), and called a *proper subsequence* of S if S_1 is a nonempty subsequence of S with $S_1 \neq S$. If S_1 is a subsequence of S, we use $S(S_1)^{-1}$ to denote the sequence obtained by deleting the terms of S_1 from S (equivalently, $S = (S(S_1)^{-1}) \cdot S_1$).

For a sequence S as above and $k \in \mathbb{N}$, define

- $\sigma(S) = \sum\limits_{i=1}^{r} g_i = \sum\limits_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S;
- $\Sigma(S) = \{\sigma(T) : T|S, T \neq \mathbb{1}\}$ the *subsum set* of S;
- $\Sigma_0(S) = \Sigma(S) \cup \{0\}$;
- $\Sigma_k(S) = \{\sigma(T) : T|S, |T| = k\}$ the set of *k-term subsums* of S;
- $\Sigma_{\leqslant k}(S) = \bigcup_{i=1}^{k} \Sigma_i(S)$.

A sequence S is called

- *zero-sum* if $\sigma(S) = 0$;
- *minimal zero-sum* if S is not empty, $\sigma(S) = 0$ and $\sigma(T) \neq 0$ for every proper subsequence $T|S$;
- *zero-sum free* if $0 \notin \Sigma(S)$.

As a fundamental result in zero-sum theory, the following theorem has been used in many papers, see e.g. [4, 5, 7, 12].

Theorem 1 ([1, 3, 15])**.** *Let $S \in \mathcal{F}(G)$ be a sequence of length $\geqslant |G|$, then S contains a zero-sum subsequence of length in $[1, \mathsf{h}(S)]$, that is, $0 \in \sum_{\leqslant \mathsf{h}(S)}(S)$.*

The example $S = \prod\limits_{g \in G \setminus \{0\}} g$ shows that the lower bound on $|S|$ can not be relaxed.

It is natural to ask what can we say about the structure of S when $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$. In [6], Gao et al. proved a result on the structure of such S under some additional condition. To state the main theorem of [6], they introduced a definition which can be viewed as the modification of [14, Proposition 4] and [8, Definition 5.1.3].

Definition 2 ([6])**.** Let $S \in \mathcal{F}(G)$ be a sequence over an abelian group G. S is called strictly g-behaving (strictly behaving for short) for some $g \in G$ if $S = (n_1 g)(n_2 g) \cdots (n_r g)$, where $|S| = r \in \mathbb{N}$, $1 = n_1 \leqslant \cdots \leqslant n_r \leqslant \mathrm{ord}(g)$ and $n_t \leqslant \sum_{i=1}^{t-1} n_i$ for all $t \in [2, r]$.

Clearly, if $S \in \mathcal{F}(G)$ is strictly g-behaving, then $\sum(S) = \{g, 2g, \ldots, ng\}$ where $n = \min\{\mathrm{ord}(g), \sum_{i=1}^{r} n_i\}$. Also, if $|S| \geqslant 2$, then $\mathsf{h}(S) \geqslant \mathsf{v}_g(S) \geqslant 2$.

Theorem 3 ([6])**.** *Let G be an abelian group and $S \in \mathcal{F}(G)$ a sequence such that $\langle \mathrm{supp}(S) \rangle = G$, where $\langle \mathrm{supp}(S) \rangle$ denotes the subgroup of G generated by* $\mathrm{supp}(S)$*. If $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$, then either S is strictly g-behaving for some $g \in G$ or $|\sum(S)| \geqslant \min\{|G|, 2|S| - 1\}$.*

As shown in [6], a lot of well-known results, including those in [2, 11, 13, 14, 16, 17], are special cases or corollaries of the theorem.

In the present paper, we take a step forward and give the structure of S such that the equality in Theorem 3 holds.

Theorem 4. *Let G be an abelian group and $S \in \mathcal{F}(G)$ a sequence with $|S| = r$ and $\langle \mathrm{supp}(S) \rangle = G$. Suppose that $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$ and $|\sum(S)| = 2r - 1 \leqslant |G| - 1$. Then one of the following holds.*

(I) $S = (n_1 a)(n_2 a) \cdots (n_r a)$, *where S is strictly a-behaving and* $n_1 + \cdots + n_r = 2r - 1 \leqslant \mathrm{ord}(a) - 1$.

(II) $S = a^{r-1} b$, *where* $\mathrm{ord}(a) \geqslant r$ *and* $b \notin \{-(r-2)a, \ldots, -a, 0, a, 2a, \ldots, (r-1)a\}$.

(III) $S = a^u (a + e)^v e$, *where* $\mathrm{ord}(e) = 2$, $u \geqslant v \geqslant 0$ *and* $r = u + v + 1 \leqslant \mathrm{ord}(a + \langle e \rangle)$ *in* $G/\langle e \rangle$.

(IV) $S = a^u (a + e)^v$, *where* $\mathrm{ord}(e) = 2$, $u \geqslant v \geqslant 1$ *and* $r = u + v \leqslant \mathrm{ord}(a + \langle e \rangle)$ *in* $G/\langle e \rangle$.

(V) $S = a^{r-2} bc$, *where* $\mathrm{ord}(a) = r \geqslant 3$, $b, c \notin \langle a \rangle$, $b - c \in \langle a \rangle \setminus \{0\}$ *and* $b + c = a$.

(VI) $S = a^{r-2} b^2$, *where* $r \geqslant 4$, $\mathrm{ord}(a) = r$, $b \notin \langle a \rangle$ *and* $2b = 3a$.

(VII) $r = 3$ *and* $S = (-a)a(2a)$, *where* $\mathrm{ord}(a) \geqslant 6$.

(VIII) $r = 4$ *and* $S = (-2a)^2 a^2$ *or* $S = (-2a)(-a)a(2a)$, *where* $\mathrm{ord}(a) \geqslant 8$.

(IX) $r = 4$ *and* $S = (-a)a(-b)b$, *where* $\mathrm{ord}(a) = 4$, $b \notin \langle a \rangle$ *and* $2b \in \langle a \rangle \setminus \{0\}$.

(X) $r = 6$ *and* $S = a^3 (5a)^3$, *where* $\mathrm{ord}(a) = 12$.

If S is a zero-sum free sequence or a subset of $G \setminus \{0\}$, then it is clear that $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$. Applying Theorems 3 and 4, it is easy to deduce the following two corollaries, which generalize Theorem 1.2 of [17] and the main result of [10] respectively.

Corollary 5. *Let G be an abelian group and $S \in \mathcal{F}(G)$ a zero-sum free sequence with $|S| = r$ and $\langle \mathrm{supp}(S) \rangle = G$. Suppose that $|\sum(S)| \leqslant 2r - 1$. Then one of the following holds.*

(i) $S = (n_1 a)(n_2 a) \cdots (n_r a)$, *where S is strictly a-behaving and* $n_1 + \cdots + n_r \leqslant \min\{2r - 1, \mathrm{ord}(a) - 1\}$.

(ii) $S = a^{r-1} b$, *where* $\mathrm{ord}(a) \geqslant r$ *and* $b \notin \{-(r-1)a, \ldots, -a, 0, a, 2a, \ldots, (r-1)a\}$.

(iii) $S = a^u (a + e)^v e$, *where* $\mathrm{ord}(e) = 2$, $u \geqslant v \geqslant 0$ *and* $r = u + v + 1 \leqslant \mathrm{ord}(a + \langle e \rangle)$ *in* $G/\langle e \rangle$.

(iv) $S = a^u(a+e)^v$, *where* $\mathrm{ord}(e) = 2$, $u \geqslant v \geqslant 1$, $r = u+v \leqslant \mathrm{ord}(a+\langle e\rangle)$ *in* $G/\langle e\rangle$ *and* $\sigma(S) = (u+v)a + ve \neq 0$.

(v) $S = a^{r-2}bc$, *where* $\mathrm{ord}(a) = r \geqslant 3$, $b, c \notin \langle a\rangle$, $b - c \in \langle a\rangle \setminus \{0\}$ *and* $b + c = a$.

Corollary 6. *Let S be a generating subset of an abelian group G such that $0 \notin S$ and $|S| \geqslant 5$. Then $|\sum(S)| \geqslant \min\{|G|, 2|S|\}$.*

2 Some tools

In this section, we collect some useful tools and some technical lemmas, which will be frequently used in the subsequent sections.

Lemma 7. *Let G be an abelian group and $S \in \mathcal{F}(G)$ such that $G = \langle \mathrm{supp}(S)\rangle$ and $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$. Then $|\sum(S)| \geqslant |S|$, and the equality holds if and only if $S = a^{|S|}$ for some $a \in G$ with $|S| < \mathrm{ord}(a)$.*

Proof. The case when $|S| = 1$ is trivial. From now on, assume that $|S| \geqslant 2$. We will show that $|\sum(S)| \leqslant |S|$ if and only if $S = a^{|S|}$ with $|S| < \mathrm{ord}(a)$.

First suppose that $S = a^{|S|}$ with $|S| < \mathrm{ord}(a)$. Then $\sum(S) = \{a, 2a, \ldots, |S|a\}$ and hence $|\sum(S)| = |S|$.

Next suppose that $|\sum(S)| \leqslant |S|$. Since $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$, Theorem 1 implies $|S| < |G|$. Then we have $|\sum(S)| \leqslant |S| < \min\{|G|, 2|S| - 1\}$. Now Theorem 3 implies that S is strictly a-behaving for some $a \in G$. Let $S = (n_1a)(n_2a)\cdots(n_ra)$, where $|S| = r \in \mathbb{N}$ and $n_i \in [1, \mathrm{ord}(a) - 1]$ for all $i \in [1, r]$. Hence $|\sum(S)| = n_1 + n_2 + \cdots + n_r \geqslant r$. Recall that $|\sum(S)| \leqslant r$. Hence $|\sum(S)| = r$, which implies that $n_i = 1$ for all $i \in [1, r]$. Therefore $S = a^r$ with $r < \mathrm{ord}(a)$. □

Lemma 8 ([6, Lemma 2.4])**.** *Let $S = S_1 \cdot S_2 \in \mathcal{F}(G)$ be a sequence such that* $\mathrm{Stab}(\sum(S)) = \{0\}$, *where* $\mathrm{Stab}(C) := \{g \in G : g + C = C\}$ *denotes the* stabilizer *of C for $C \subset G$. Then $|\sum(S)| \geqslant |\sum(S_1)| + |\sum_0(S_2)| - 1$. In particular, if $0 \in \sum(S)$, then $|\sum(S)| \geqslant |\sum_0(S_1)| + |\sum_0(S_2)| - 1$.*

Lemma 9. (i) *Let $S = a_1a_2a_3$ be a zero-sum free sequence, where a_1, a_2, a_3 are pairwise distinct. Suppose that $\mathrm{ord}(a_i) \neq 2$ for all $i \in [1, 3]$. Then $|\sum(S)| \geqslant 6$ and $|\sum_0(S)| \geqslant 7$.*

(ii) *Let $S = a_1a_2a_3 \in \mathcal{F}(G \setminus \{0\})$, where a_1, a_2, a_3 are pairwise distinct and $a_i + a_j \neq 0$ for all distinct $i, j \in [1, 3]$. Then $|\sum_0(S)| \leqslant 7$ if and only if $a_3 = \pm a_1 \pm a_2$.*

Proof. Part (i) is the second result of Proposition 5.3.2 of [9].

Now we prove Part (ii). All possible elements of $\sum_0(S)$ are listed as follows:

$$0, a_1, a_2, a_3, a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3.$$

Then $|\sum_0(S)| \leqslant 7$ if and only if these eight elements are not pairwise distinct. Recall that a_1, a_2, a_3 are pairwise distinct and $a_i + a_j \neq 0$ for all distinct $i, j \in [1, 3]$. It follows that $|\sum_0(S)| \leqslant 7$ if and only if one of the following equalities holds:

$$a_3 = a_1 + a_2, a_2 = a_1 + a_3, a_1 = a_2 + a_3, 0 = a_1 + a_2 + a_3.$$

Therefore $|\sum_0(S)| \leqslant 7$ if and only if $a_3 = \pm a_1 \pm a_2$. □

Lemma 10. (1) *Let* $S = a_1a_2a_3a_4 \in \mathcal{F}(G \setminus \{0\})$*, where* a_1, a_2, a_3, a_4 *are pairwise distinct and* $a_i + a_j \neq 0$ *for all distinct* $i, j \in [1, 4]$*. Let* $T_i = Sa_i^{-1}$ *for all* $i \in [1, 4]$*. Suppose that* $|\sum_0(T_i)| \leqslant 7$ *for all* $i \in [2, 4]$*. Then* $\mathrm{ord}(a_1) = 3$*. Suppose further that* $|\sum_0(T_1)| \leqslant 7$*. Then* $|\sum(S)| = |\langle \mathrm{supp}(S)\rangle| - 1 = 8$*.*

(2) *Let* $S = a_1a_2a_3a_4a_5 \in \mathcal{F}(G \setminus \{0\})$*, where* a_1, a_2, a_3, a_4, a_5 *are pairwise distinct and* $a_i + a_j \neq 0$ *for all distinct* $i, j \in [1, 5]$*. Then there exists a subsequence* $T|S$ *of length* $|T| = 3$ *such that* $|\sum(T) \setminus \{0\}| = 7$*.*

Proof. (1) By Lemma 9 (ii), we have

$$a_1 = \lambda_1a_2 + \mu_1a_3, a_1 = \lambda_2a_2 + \mu_2a_4 \text{ and } a_1 = \lambda_3a_3 + \mu_3a_4,$$

where $\lambda_i, \mu_j \in \{1, -1\}$ for all $i, j \in [1, 3]$.

If $\lambda_1 = \lambda_2$, then $\mu_1a_3 = \mu_2a_4$, that is $a_3 = a_4$ or $a_3 + a_4 = 0$, a contradiction. Hence $\lambda_1 = -\lambda_2$. By the same argument, we have $\mu_1 = -\lambda_3$ and $\mu_2 = -\mu_3$. It follows that

$$3a_1 = (\lambda_1 + \lambda_2)a_2 + (\mu_1 + \lambda_3)a_3 + (\mu_2 + \mu_3)a_4 = 0.$$

Thus $\mathrm{ord}(a_1) = 3$. This completes the proof of the first part of (1).

Now suppose further that $|\sum_0(T_1)| \leqslant 7$ and let $H = \langle \mathrm{supp}(S)\rangle$. By the proof above, we have $\mathrm{ord}(a_i) = 3$ for all $i \in [1, 4]$. Since a_1, a_2, a_3, a_4 are pairwise distinct and $a_i + a_j \neq 0$ for all distinct $i, j \in [1, 4]$, we infer that any two elements of S are independent. Applying Lemma 9 (ii) to T_3 and T_4, we have $a_4 = \pm a_1 \pm a_2$ and $a_3 = \pm a_1 \pm a_2$. Then $H = \langle a_1, a_2\rangle$ and hence $|H| = 9$.

Finally we calculate the value of $|\sum(S)|$. Since $a_3 \neq a_4$, $a_3 + a_4 \neq 0$ and the roles of a_3 and a_4 are the same, we only need to consider the following four cases:

$$(a_3, a_4) = (a_1 + a_2, a_1 - a_2), (a_1 + a_2, -a_1 + a_2), (-a_1 - a_2, a_1 - a_2), (-a_1 - a_2, -a_1 + a_2).$$

Direct calculation shows that

$$\sum\nolimits_0(S) = H \setminus \{2a_2\}, H \setminus \{2a_1\}, H \setminus \{2a_1 + a_2\}, H \setminus \{a_1 + 2a_2\}$$

in these four cases, respectively. Hence $|\sum_0(S)| = |H| - 1 = 8$.

(2) Suppose to the contrary that such subsequence T does not exist. By Lemma 9 (ii), we have $a_3, a_4, a_5 \in \{\pm a_1 \pm a_2\}$. Hence two elements among a_3, a_4, a_5 are equal or have sum zero, a contradiction. ☐

Lemma 11. *Let* $S = a^2b^2$ *be a zero-sum free sequence. Suppose that* S *is not strictly behaving and* $2a \neq 2b$*. Then* $|\sum(S)| = 8$*.*

Proof. All possible elements of $\sum(S)$ are

$$a, b, a + b, 2a, 2b, 2a + b, a + 2b, 2a + 2b.$$

The assumption implies that above elements are pairwise distinct. Therefore $|\sum(S)| = 8$. ☐

Lemma 12. *Let $T \in \mathcal{F}(G)$ be a strictly a-behaving sequence with $|T| = k \geqslant 2$, and let $S = T \cdot b$ be such that S is not strictly a-behaving, $|\sum(S)| < |\langle \mathrm{supp}(S) \rangle|$ and $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$. Suppose that $|\sum(S)| \leqslant 2|S| - 1 = 2k + 1$. Then $\mathrm{ord}(a) > k$, $T = a^k$ and $b \notin \{-(k-1)a, \ldots, -a, 0, a, \ldots, ka\}$. In this case, $|\sum(S)| = 2|S| - 1$.*

Proof. Since $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$, we have $0 \notin \sum_{\leqslant \mathsf{h}(T)}(T)$. Then Theorem 1 implies that $|T| = k < \mathrm{ord}(a)$.

Let $T = a^u(n_1 a) \cdot \cdots \cdot (n_v a)$ where $2 \leqslant n_1 \leqslant \cdots \leqslant n_v$, $u + v = k$, $u \geqslant 2$ and $v \geqslant 0$. Let $t = u + n_1 + \cdots + n_v$.

First suppose that $t \geqslant \mathrm{ord}(a)$. Since S is not strictly a-behaving, we have $b \notin \langle a \rangle$. Then $\sum(S) = \langle a \rangle \cup (b + \langle a \rangle)$ and hence $|\sum(S)| = 2\mathrm{ord}(a) \geqslant 2(k+1) > 2k+1$, a contradiction.

Next suppose that $t < \mathrm{ord}(a)$. If $b \notin \{-(t-1)a, \ldots, ta\}$, then $\sum(S) = \{a, \ldots, ta\} \cup \{b, b+a, \ldots, b+ta\}$ and $|\sum(S)| = 2t + 1$. Recall that $|\sum(S)| \leqslant 2k+1$. Then $k \geqslant t \geqslant u + 2v = k + v$. Hence $v = 0$, $u = k$, $t = k$ and $T = a^k$, as desired. If $b \in \{a, \ldots, ta\}$, then S is strictly a-behaving, a contradiction. If $b \in \{-(t-1)a, \ldots, -a, 0\}$, that is $b = -\gamma a$ for some $\gamma \in [0, t-1]$, then $\sum(S) = \{-\gamma a, \ldots, ta\}$. Since $|\sum(S)| < |\langle \mathrm{supp}(S) \rangle|$, we infer that $|\sum(S)| = t + \gamma + 1 < \mathrm{ord}(a)$. Then we have $2k + 1 \geqslant t + \gamma + 1 \geqslant u + 2v + \gamma + 1 = 2k - u + \gamma + 1$ and so $\gamma \leqslant u$. Note that ba^γ is a zero-sum subsequence of length $\gamma + 1$. Since $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$, we have $\gamma + 1 > \mathsf{h}(S) \geqslant u$ and hence $\gamma = u$. Since $\gamma \leqslant t - 1$, we have $v \geqslant 1$. Since T is strictly a-behaving, we have $n_1 \leqslant u$. Therefore $b(n_1 a)a^{u-n_1}$ is a zero-sum subsequence of length $u - n_1 + 2 \leqslant u \leqslant \mathsf{h}(S)$, a contradiction. $\square$

Lemma 13 ([13, Theorem 4.5] and [18, Lemma 4.4])**.** *Let G be an abelian group, $Y \subset X \subset G \setminus \{0\}$, $G = \langle X \rangle$ and $H = \langle Y \rangle$. Suppose $|H| > m$ and $|G/H| > m$, where $m \in \mathbb{N}$, and suppose $A \subset G$ satisfies $|(A+x) \setminus A| \leqslant m$ for all $x \in X$. Then* $\min\{|A|, |G \setminus A|\} \leqslant m^2$.

The following notation will be used.

For $S \in \mathcal{F}(G)$ and $g \in G$, define

$$\lambda_S(g) = \left| \sum(S \cdot g) \setminus \sum(S) \right|.$$

Lemma 14 ([6, Lemma 2.8])**.** $\lambda_S(g) \leqslant \lambda_{Sg^{-1}}(g)$ *for every $g | S$.*

3 Some special cases

In this section, we prove some special cases of Theorem 4. Throughout the section, we assume that G is an abelian group, $S \in \mathcal{F}(G)$ with $G = \langle \mathrm{supp}(S) \rangle$, $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$ and $|\sum(S)| = 2|S| - 1 \leqslant |G| - 1$.

Lemma 15. *If $|S| \leqslant 3$, then Theorem 4 holds.*

Proof. Suppose $|S| = 1$. Then S is strictly behaving and we are done.

Suppose $|S| = 2$ and let $S = a_1 a_2$ for some $a_1, a_2 \in G$. If $a_1 = a_2$, then $\sum(S) = \{a_1, 2a_1\}$ and $|\sum(S)| = 2 < 2|S| - 1$, a contradiction. If $a_1 \neq a_2$, then S satisfies (II).

Suppose $|S| = 3$ and let $S = a_1a_2a_3$ for some $a_1, a_2, a_3 \in G$. If $a_1 = a_2 = a_3$, then $|\sum(S)| = 3 < 2|S| - 1$, a contradiction. If $a_1 = a_2$ and $a_3 \neq a_1$, then

$$\sum(S) = \{a_1, 2a_1\} \cup \{a_3, a_3 + a_1, a_3 + 2a_1\}.$$

Since $|\sum(S)| = 2|S| - 1 = 5$, we have $a_3 \notin \{-a_1, 0, a_1, 2a_1\}$ and so S satisfies (II). If a_1, a_2, a_3 are pairwise distinct, then all possible elements of $\sum(S)$ are

$$a_1, a_2, a_3, a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3.$$

Note that $a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3$ are pairwise distinct. Hence

$$\sum(S) = \{a_1, a_2, a_3\} \cup \{a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3\}.$$

Since $|\sum(S)| = 5$, there are exactly two elements of $\{a_1, a_2, a_3\}$, say a_2 and a_3, contained in $\{a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3\}$. Then $a_2 \in \{a_1 + a_3, a_1 + a_2 + a_3\}$ and $a_3 \in \{a_1 + a_2, a_1 + a_2 + a_3\}$. If $a_2 = a_1 + a_3$ and $a_3 = a_1 + a_2$, then $a_1 = a_2 - a_3$ and $2a_1 = 0$. Hence $S = a_3a_2a_1 = a_3(a_3 + a_1)a_1$, satisfying (III). If $a_2 = a_1 + a_3$ and $a_3 = a_1 + a_2 + a_3$, then $a_1 = -a_2$ and $a_3 = a_2 - a_1 = 2a_2$. Hence $S = (-a_2)a_2(2a_2)$, satisfying (VII). If $a_2 = a_1 + a_2 + a_3$ and $a_3 = a_1 + a_2$, then $a_1 = -a_3$ and $a_2 = 2a_3$. Hence $S = a_1a_3a_2 = (-a_3)a_3(2a_3)$, satisfying (VII). □

Lemma 16. *Let $S = a^ub^2$, where $u \geqslant 2$ and $a \neq b$. Then Theorem 4 holds.*

Proof. Since $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$, we have $\mathrm{ord}(a) > u$ and $b \notin \{-(u-1)a, \dots, -a, 0\}$. If $b \in \{a, \dots, ua\}$, then S is strictly a-behaving and so S satisfies (I). If $b = -ua$, then $\sum(S) = \{-2ua, (-2u+1)a, \dots, -a, 0, a, \dots, ua\}$. Since $\sum(S) \neq G$, we infer that $|\sum(S)| = 3u + 1 < \mathrm{ord}(a)$. Then $2u + 3 = 2|S| - 1 = |\sum(S)| = 3u + 1$ and so $u = 2$, which shows that S satisfies (VIII).

It remains to consider the case when $b \notin \{-ua, \dots, -a, 0, a, \dots, ua\}$. Since

$$\sum(S) = \{a, \dots, ua\} \cup (b + \{0, a, \dots, ua\}) \cup (2b + \{0, a, \dots, ua\}),$$

where $(b + \{0, a, \dots, ua\}) \cap \{a, \dots, ua\} = \emptyset$ and $(b + \{0, a, \dots, ua\}) \cap (2b + \{0, a, \dots, ua\}) = \emptyset$, we have

$$|\{a, \dots, ua\} \cup (2b + \{0, a, \dots, ua\})| = \left|\sum(S)\right| - |b + \{0, a, \dots, ua\}| = u + 2,$$

and hence

$$|\{a, \dots, ua\} \setminus (2b + \{0, a, \dots, ua\})| = 1,$$

which implies that $2b \in \langle a \rangle$. First suppose $\mathrm{ord}(a) = u + 1$. Then $\{a, \dots, ua\} \subset (2b + \{0, a, \dots, ua\})$, a contradiction. Next suppose $\mathrm{ord}(a) = u + 2$. Then $2b \in \{a, 2a, 3a\}$, for otherwise there is some $t \in [0, u-2]$ such that a^tb^2 is a zero-sum subsequence of length $t + 2 \leqslant u \leqslant \mathsf{h}(S)$. If $2b = a$, then S is strictly b-behaving and so S satisfies (I). If $2b = 2a$, then S satisfies (IV). If $2b = 3a$, then S satisfies (VI). Finally suppose $\mathrm{ord}(a) \geqslant u + 3$. Then $2b = 2a$ or $2b = -a$. If $2b = 2a$, then S satisfies (IV). If $2b = -a$, then $u = 2$ and $S = (-2b)^2b^2$, satisfying (VIII). □

Lemma 17. *Let $S = a^u bc$, where $u \geqslant 2$ and a, b, c are pairwise distinct. Then Theorem 4 holds.*

Proof. Note that $\sum(S)$ is the union of the following sets:

$$\{a, 2a, \ldots, ua\},\ b + \{0, a, 2a, \ldots, ua\},\ c + \{0, a, 2a, \ldots, ua\},\ b + c + \{0, a, 2a, \ldots, ua\}.$$

If $b \in \{a, 2a, \ldots, ua\}$, then $a^u b$ is strictly a-behaving. Applying Lemma 12, we have S is strictly a-behaving. If $b \in \{-(u-1)a, \ldots, -a, 0\}$, then $0 \in \sum_{\leqslant \mathsf{h}(S)}(S)$, a contradiction. If $b = -ua$ and $\mathrm{ord}(a) \leqslant 2u+1$, then $\sum(a^u b) = \langle a \rangle$. Since $\sum(S) \neq G$, we have $c \notin \langle a \rangle$ and so $|\sum(S)| = 2\mathrm{ord}(a) \neq 2|S| - 1$, a contradiction. If $b = -ua$ and $\mathrm{ord}(a) > 2u+1$, then $\sum(a^u b) = \{-ua, \ldots, -a, 0, a, \ldots, ua\}$ and $\lambda_{Sc^{-1}}(c) = 2$. Hence $c = \pm 2a$. Note that $c = 2a$ implies that $a^{u-1}bc$ is a zero-sum subsequence of length $u = \mathsf{h}(S)$, a contradiction. Also note that $c = -2a$ implies that $u = 2$ and $b = -2a = c$, a contradiction. Therefore we may assume $b \notin \{-ua, \ldots, -a, 0, a, \ldots, ua\}$. Also we may assume $c \notin \{-ua, \ldots, -a, 0, a, \ldots, ua\}$ for the same reason.

Set $A = (b + \{0, a, \ldots, ua\}) \cup (c + \{0, a, \ldots, ua\})$ and $B = \{a, \ldots, ua\} \cup (b + c + \{0, a, \ldots, ua\})$. The assumptions on b and c show that $A \cap B = \emptyset$. Since $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$, we have $\mathrm{ord}(a) \geqslant u+1$. If $\mathrm{ord}(a) = u+1$, then $|A| \in \{u+1, 2u+2\}$ and $|B| \in \{u+1, 2u+1\}$. So $2u+3 = |\sum(S)| = |A| + |B| \in \{2u+2, 3u+2, 3u+3, 4u+3\}$, a contradiction. Now suppose that $\mathrm{ord}(a) \geqslant u+2$. Clearly, $|A| \geqslant u+2$ and $|B| \geqslant u+1$. Since $2u+3 \leqslant |A| + |B| = |\sum(S)| = 2|S| - 1 = 2u+3$, we obtain $|A| = u+2$ and $|B| = u+1$. That $|B| = u+1$ implies $b + c = a$. If $\mathrm{ord}(a) = u+2$, then S satisfies (V). If $\mathrm{ord}(a) \geqslant u+3$, it follows from $|A| = u+2$ that $b = c + a$ or $c = b + a$. Together with $b + c = a$, we have $2c = 0$ or $2b = 0$, which show that S satisfies (III). ☐

4 Proof of Theorem 4

Now we are in the position to complete the proof of Theorem 4.

Proof of Theorem 4. Assume that the theorem is false and let S be a counterexample with minimal length. Recall the assumptions on S that $G = \langle \mathrm{supp}(S) \rangle$, $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$ and $|\sum(S)| = 2r - 1 \leqslant |G| - 1$. Note that $|S| = r \geqslant 4$ by Lemma 15 and that $|\mathrm{supp}(S)| \geqslant 2$.

Claim A. Sg^{-1} is not strictly behaving for all $g | S$.

Proof of Claim A. Suppose to the contrary that Sg^{-1} is strictly a-behaving for some $g | S$ and $a \in G$. Since S is not strictly a-behaving, by Lemma 12 S satisfies (II), a contradiction.

Claim B. $\langle \mathrm{supp}(Sg^{-1}) \rangle = G$ for all $g | S$.

Proof of Claim B. Suppose to the contrary that $\langle \mathrm{supp}(Sg^{-1}) \rangle \neq G$ for some $g | S$. Clearly, $\sum(S) = \sum(Sg^{-1}) \cup (g + \sum_0(Sg^{-1}))$ and

$$2r - 1 = \left| \sum(S) \right| = \left| \sum(Sg^{-1}) \right| + \left| g + \sum\nolimits_0 (Sg^{-1}) \right| \geqslant 2 \left| \sum(Sg^{-1}) \right|,$$

hence $|\sum(Sg^{-1})| \leqslant r - 1 = |Sg^{-1}|$. By Lemma 7, Sg^{-1} is strictly behaving, a contradiction with Claim A. This proves Claim B.

Claim C. $\lambda_{Sg^{-1}}(g) \leqslant 1$ for all $g|S$.

Proof of Claim C. Suppose to the contrary that $\lambda_{Sg^{-1}}(g) \geqslant 2$ for some $g|S$. Let $T = Sg^{-1}$. By Claim B, $\langle \mathrm{supp}(T) \rangle = G$. Since $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$, we have $0 \notin \sum_{\leqslant \mathsf{h}(T)}(T)$. Note that $|\sum(T)| \leqslant |\sum(S)| - 2 = 2r - 3 = 2|T| - 1 \leqslant |G| - 3$. If $|\sum(T)| < 2|T| - 1$, then we can apply Theorem 3 to T and obtain that T is strictly behaving, a contradiction with Claim A. Thus $|\sum(T)| = 2|T| - 1$ and $\lambda_T(g) = 2$. By the minimality of $|S|$, the main theorem holds for T. Thus we just need to check all possible cases given in the main theorem. Since T is not strictly behaving, T does not satisfy (I). Since $|\sum(T)| \leqslant |G| - 3$, T does not satisfy (V), (VI), (IX) or (X). If T satisfies (II), then $S = a^{r-1}b$, $S = a^{r-2}b^2$ or $S = a^{r-2}bc$, which have been proven in Lemmas 12, 16 and 17.

Suppose T satisfies (III), that is, $T = a^{u'}(a+e)^{v'}e$. Note that

$$\sum(T) = \{e, a, a+e, 2a, 2a+e, \dots, (u'+v')a, (u'+v')a+e\},$$

where $2 \leqslant u' + v' \leqslant \mathrm{ord}(a + \langle e \rangle) - 2$. Since $\lambda_T(g) = 2$, we have $|(g + \sum_0(T)) \setminus \sum(T)| = 2$. Then we infer that $g = a$ or $g = a + e$, both of which imply that S satisfies (III), a contradiction.

Suppose T satisfies (IV), that is, $T = a^{u'}(a+e)^{v'}$. Note that

$$\sum(T) = \{a, a+e, 2a, 2a+e, \dots, (u'+v'-1)a, (u'+v'-1)a+e, \sigma(T)\},$$

where $3 \leqslant u' + v' \leqslant \mathrm{ord}(a + \langle e \rangle) - 1$. Again we have $|(g + \sum_0(T)) \setminus \sum(T)| = 2$ and so $g \in \{e, a, a+e\}$ or $g = \sigma(T) \in \{-a, -a+e\}$. If $g = e$, then S satisfies (III), a contradiction. If $g \in \{a, a+e\}$, then S satisfies (IV), a contradiction. If $g = \sigma(T) \in \{-a, -a+e\}$, then $0 \in \sum_{\leqslant 2}(S) \subset \sum_{\leqslant \mathsf{h}(S)}(S)$, a contradiction.

Finally suppose that T satisfies (VII) or (VIII). Clearly, $\sum(T)$ is an arithmetic progression with difference a and $0 \in \sum(T)$. Recall that $|\sum(T)| \leqslant |G| - 3 = \mathrm{ord}(a) - 3$. Since $\lambda_T(g) = 2$, we have $g \in \{2a, -2a\}$. If $T = (-a)a(2a)$, then $g = -2a$ and $S = (-2a)(-a)a(2a)$, satisfying (VIII), a contradiction. If $T = (-2a)^2a^2$ or $T = (-2a)(-a)a(2a)$, then $0 \in \sum_{\leqslant \mathsf{h}(S)}(S)$, a contradiction.

This completes the proof of Claim C.

Claim C, together with Lemma 14, implies that $\lambda_S(g) \leqslant \lambda_{Sg^{-1}}(g) \leqslant 1$ for all $g|S$.

Claim D. $|\sum(S)| = |G| - 1$.

Proof of Claim D. First suppose that there exists some $g|S$ such that $G \neq \langle g \rangle$. Then applying Lemma 13 with $m = 1$, $X = \mathrm{supp}(S)$, $Y = \{g\}$ and $A = \sum(S)$, we obtain that $|\sum(S)| \geqslant |G| - 1$ and hence the equality holds. Therefore we may assume that $G = \langle g \rangle$ for all $g|S$.

Suppose to the contrary that Claim D does not hold, that is, $|\sum(S)| \leqslant |G| - 2$. Choose $g|S$. If $\lambda_S(g) = 0$, then $\sum(S) = \sum(S) + \langle g \rangle = G$, a contradiction. Hence $\lambda_S(g) = 1$ and so $\sum(S)$ is an arithmetic progression with difference g. Choose another $g'|S$ such that $g' \neq g$. By Claim C, $|(\sum(S) + g') \setminus (\sum(S))| \leqslant \lambda_S(g') \leqslant 1$. Since $|\sum(S)| \leqslant |G| - 2$ and $\sum(S)$ is an arithmetic progression with difference g, we have $g' = -g$. By the arbitrariness of g', we infer that $\mathrm{supp}(S) = \{g, -g\}$. Note that $g(-g)$ is a zero-sum subsequence of S. Hence $\mathsf{h}(S) \leqslant 1$ and so $|S| = 2$, a contradiction.

This completes the proof of Claim D.

From Claim D, it follows that

$$|S| = r = \frac{|G|}{2} \text{ and } \mathrm{Stab}\left(\sum(S)\right) = \{0\}. \tag{1}$$

Also, we have

$$\lambda_S(g) = \lambda_{Sg^{-1}}(g) = 1 \text{ for all } g|S. \tag{2}$$

Claim E. $|\sum_0(T)| < |\langle \mathrm{supp}(T)\rangle|$ for any nonempty proper subsequence $T|S$.

Proof of Claim E. Suppose to the contrary that $\sum_0(T) = \langle \mathrm{supp}(T)\rangle$ for some nonempty proper subsequence $T|S$. First assume that $\sum(ST^{-1}) \cap \langle \mathrm{supp}(T)\rangle \neq \emptyset$. Then $0 \in \sum(S)$ and so

$$\sum(S) = \sum_0(T) + \sum_0(ST^{-1}) = \langle \mathrm{supp}(T)\rangle + \sum_0(ST^{-1}),$$

a contradiction with that $\mathrm{Stab}\left(\sum(S)\right) = \{0\}$.

Now let $\sum(ST^{-1}) \cap \langle \mathrm{supp}(T)\rangle = \emptyset$. Let $H = \langle \mathrm{supp}(T)\rangle$ and $\Phi : G \to G/H$ denote the natural homomorphism modulo H. Consider the sequence $\Phi(ST^{-1}) \in \mathcal{F}(G/H)$. Since $\sum(ST^{-1}) \cap \langle \mathrm{supp}(T)\rangle = \emptyset$, $\Phi(ST^{-1})$ is a zero-sum free sequence in $\mathcal{F}(G/H)$. Thus $|\sum(\Phi(ST^{-1}))| \geqslant |\Phi(ST^{-1})| = |S| - |T|$, where the equality holds if and only if $|\mathrm{supp}(\Phi(ST^{-1}))| = 1$. By Lemma 7, $|\sum_0(T)| \geqslant |T| + 1$. We have

$$2|S| - 1 = |\sum(S)| \geqslant |\sum(\Phi(ST^{-1}))||\sum_0(T)| + |\sum(T)| \geqslant (|S| - |T|)(|T| + 1) + |T|,$$

and hence $|T| = |S| - 1$ or $|T| = 1$. If $|T| = |S| - 1$, it contradicts Claim B. If $|T| = 1$, then $|H| = |\sum_0(T)| = |T| + 1 = 2$ and $\mathrm{supp}(\Phi(ST^{-1})) = 1$, which implies that S satisfies (III), a contradiction.

This completes the proof of Claim E.

From Claim E, we may assume that $\mathrm{ord}(g) \geqslant \mathsf{v}_g(S) + 2 \geqslant 3$ for all $g|S$.

Now we use a case-by-case method in the subsequent part of the proof.

Case 1. $|\mathrm{supp}(S)| = 2$.

Let $S = a^u b^v$ with $u \geqslant v \geqslant 1$. The cases when $v = 1, 2$ are showed in Lemma 12 and 16. Thus we may assume that $v \geqslant 3$. If $2a = 2b$, then S satisfies (IV), a contradiction. Thus we may assume $2a \neq 2b$.

First suppose that $v = 3$. By (2), $|\sum(a^u b^2)| = |\sum(S)| - 1 = |G| - 2 = 2u + 4$. Also note that $\mathrm{ord}(a) \geqslant u + 1$ and that $\sum(a^u b^2)$ is the union of the following sets:

$$\{a, 2a, \ldots, ua\},\ b + \{0, a, \ldots, ua\},\ 2b + \{0, a, \ldots, ua\}.$$

If $b \in \{a, 2a, \ldots, ua\}$, then S is strictly behaving, a contradiction. If $b \in \{-(u-1)a, \ldots, -a, 0\}$, then $b + ta = 0$ for some $t \in [0, u-1]$ and so $0 \in \sum_{\leqslant \mathsf{h}(S)}(S)$, a contradiction. If $b = -ua$, then $\sum(a^u b^2) = \{-2ua, \ldots, -a, 0, a, \ldots, ua\}$ is an arithmetic progression. By (2), we obtain $b \in \{a, -a\}$, a contradiction. If $b \notin \{-ua, \ldots, -a, 0, a, \ldots, ua\}$, then $(b+\{0, a, \ldots, ua\}) \cap \{a, 2a, \ldots, ua\} = \emptyset$ and $(b+\{0, a, \ldots, ua\}) \cap (2b+\{0, a, \ldots, ua\}) = \emptyset$. Thus $2b \in \langle a\rangle$, $|(2b + \{0, a, \ldots, ua\}) \setminus \{a, 2a, \ldots, ua\}| = 3$ and $\mathrm{ord}(a) \geqslant u + 3$. Since $\mathrm{ord}(a)|2u + 6$, we have $\mathrm{ord}(a) \in \{u + 3, 2u + 6\}$. If $\mathrm{ord}(a) = u + 3$, then $\sum(a^u b^2) =$

$\langle a\rangle \cup (b + \{0, a, \ldots, ua\})$ and $\sum(S) = G$, a contradiction. If $\mathrm{ord}(a) = 2u + 6$, then $2b \in \{3a, -2a\}$. Since $|G|$ is even, we infer that $2b \neq 3a$. Hence $2b = -2a$, which implies $u = 3$, $\mathrm{ord}(a) = 12$, $b = 6a - a = 5a$ and S satisfies (X), a contradiction.

Next suppose that $v \geqslant 4$. By Lemma 8 and 11, we have

$$|\sum(a^{u-2}b^{v-2})| \leqslant |\sum(S)| + 1 - |\sum_0(a^2b^2)| = 2|S| - 9 = 2|a^{u-2}b^{v-2}| - 1,$$

in particular, $\sum(a^{u-2}b^{v-2}) < |G| - 1$. If $a^{u-2}b^{v-2}$ is strictly behaving, then so is S, a contradiction. By Theorem 3, $|\sum(a^{u-2}b^{v-2})| = 2|a^{u-2}b^{v-2}| - 1$. By the minimality of $|S|$, we can apply the main theorem to $a^{u-2}b^{v-2}$. It is clear that $a^{u-2}b^{v-2}$ does not satisfy (II), (III) and (V). Since $a^{u-2}b^{v-2}$ is not strictly behaving, it does not satisfy (I). Since $2a \neq 2b$, it does not satisfy (IV). Since $\sum(a^{u-2}b^{v-2}) < |G| - 1$, it does not satisfy (VI) and (X). Since $a^{u-2}b^{v-2}$ does not contain zero-sum subsequences of length at most 4, it does not satisfy (VII), (VIII) and (IX). This completes the proof of this case.

Case 2. $\mathsf{h}(S) = 1$.

Let $S = a_1a_2 \cdots a_r$ with $r \geqslant 4$. Suppose there exists a subsequence $T|S$ such that $|T| = 3$ and $|\sum_0(T)| \geqslant 8$, then by Lemma 8,

$$1 \leqslant |\sum(ST^{-1})| \leqslant |\sum(S)| - |\sum_0(T)| + 1 \leqslant 2r - 8 = 2|ST^{-1}| - 2.$$

In particular, we have $|ST^{-1}| \geqslant 2$. By Claim E,

$$|\sum(ST^{-1})| \leqslant |\sum_0(ST^{-1})| < |\langle \mathrm{supp}(ST^{-1})\rangle|.$$

Hence by Theorem 3, ST^{-1} is strictly behaving of length at least 2. Then $\mathsf{h}(ST^{-1}) \geqslant 2$, a contradiction. Therefore we may assume that such a subsequence does not exist.

Now suppose that S contains a subsequence $U = b_1b_2b_3b_4$ such that $b_i + b_j \neq 0$ for all distinct $i, j \in [1, 4]$. By Lemma 10 (1), $|\sum(U)| = |\langle \mathrm{supp}(U)\rangle| - 1 = 8 > 2|U| - 1$. Thus U is a proper subsequence of S. Take $g|SU^{-1}$. By Lemma 10 (2), $g = -b_i$ for some $i \in [1, 4]$. Hence $\sum(Ug) = \langle \mathrm{supp}(Ug)\rangle$, a contradiction to Claim E.

It remains to consider the following cases:

$$S = a(-a)b(-b)c(-c),\ a(-a)b(-b)c,\ a(-a)bc \ \text{ or } \ a(-a)b(-b),$$

where $a, b, c, -a, -b, -c$ are pairwise distinct. Firstly assume that $S = a(-a)b(-b)c(-c)$. Recall that the orders of a, b and c are all greater than 2. Without loss of generality, $a+b+c \neq 0$, for otherwise we may change the roles of a and $-a$. Hence abc and $(-a)(-b)(-c)$ are both zero-sum free. By Lemma 9 (i), we have $|\sum_0(abc)| = |\sum_0((-a)(-b)(-c))| \geqslant 7$. By Lemma 8, $|\sum(S)| \geqslant |\sum_0(abc)| + |\sum_0((-a)(-b)(-c))| - 1 \geqslant 13 > 2|S| - 1$, a contradiction. Secondly assume that $S = a(-a)b(-b)c$. The same as above, we have $|\sum_0(abc)| \geqslant 7$. Note that $|\sum_0((-a)(-b))| = 4$. By Lemma 8 again, $|\sum(S)| \geqslant |\sum_0(abc)| + |\sum_0((-a)(-b))| - 1 \geqslant 10 > 2|S| - 1$, a contradiction. Thirdly assume that $S = a(-a)bc$. The same as above, we have $|\sum_0(abc)| \geqslant 7$. Note that $|\sum_0(-a)| = 2$. By Lemma 8 again, $|\sum(S)| \geqslant |\sum_0(abc)| + |\sum_0(-a)| - 1 \geqslant 8 > 2|S| - 1$, a contradiction. Finally assume that $S = a(-a)b(-b)$. Now $|G| = 8$. If $\mathrm{ord}(a) = \mathrm{ord}(b) = 8$, then $S = a(3a)(5a)(7a)$ and so $\sum(S) = G$, a contradiction. Therefore we may assume that $\mathrm{ord}(a) = 4$. Then $b \notin \langle a\rangle$ and so S satisfies (IX), a contradiction.

Case 3. $\mathsf{h}(S) \geqslant 2$ and $|\mathrm{supp}(S)| \geqslant 4$.

Let $a|S$ be a term such that $\mathsf{v}_a(S) = \mathsf{h}(S) \geqslant 2$. Choose arbitrarily $bcd|S$ such that $\mathsf{h}(abcd) = 1$. Claim E shows that $\mathrm{ord}(a) \geqslant 4$. By Lemma 10 (1), there exist two terms in b, c, d, say b, c, such that $|\sum_0(abc)| \geqslant 8$. By Lemma 8, $|\sum(S(abc)^{-1})| \leqslant |\sum(S)| - |\sum_0(abc)| + 1 = 2|S| - 1 - 8 + 1 = 2|S(abc)^{-1}| - 2$. By Claim E, $|\sum(S(abc)^{-1})| \leqslant |\sum_0(S(abc)^{-1})| < |\langle \mathrm{supp}(S(abc)^{-1})\rangle|$. Hence it follows from Theorem 3 that $S(abc)^{-1}$ is strictly behaving. Since $a|S(abc)^{-1}$, we have $S(bc)^{-1}$ is strictly behaving. Let $S(bc)^{-1} = (n_1h)(n_2h)\cdots(n_{r-2}h)$ be strictly h-behaving, where $h \in G$ and $1 = n_1 \leqslant \cdots \leqslant n_{r-2}$. Since $|\mathrm{supp}(S)| \geqslant 4$, we have $|\mathrm{supp}(S(bc)^{-1})| \geqslant 2$ and so

$$m := n_1 + \cdots + n_{r-2} \geqslant r - 1 = \frac{|G|}{2} - 1. \tag{3}$$

Note that $|\sum_0(S(bc)^{-1})| = \{0, h, \ldots, mh\}$. By Claim E, $\mathrm{ord}(h) > m + 1 \geqslant |G|/2$, which implies $\mathrm{ord}(h) = |G|$. Let $b = \alpha h$ and $c = \beta h$, where $\alpha, \beta \in [1, |G| - 1]$. By Claim A, Sc^{-1} is not strictly behaving and so $\alpha \notin [1, m]$. If $\alpha = m + 1$, then $\sum(Sc^{-1}) = \{h, 2h, \ldots, (2m + 1)h\}$ and so

$$|\sum(Sc^{-1})| \geqslant \min\{|G|, 2m + 1\} \geqslant \min\{2r, 2(r - 1) + 1\} = 2r - 1,$$

a contradiction with the equality (2). Hence $\alpha \in [m + 2, |G| - 1]$. Since $\alpha + m \geqslant m + 2 + m \geqslant |G|$, we have

$$\sum(Sc^{-1}) = \{h, 2h, \ldots, mh, \alpha h, (\alpha + 1)h, \ldots, |G|h\}$$

and so $|\sum(Sc^{-1})| = m + |G| - \alpha + 1 = 2r - (\alpha - m - 1)$. By the equality (2), $|\sum(Sc^{-1})| = |\sum(S)| - 1 = 2r - 2$. Thus $\alpha - m - 1 = 2$ and so $\alpha = m + 3$. By the same argument, we also infer $\beta = m + 3$. It follows that $b = c$, a contradiction.

Case 4. $\mathsf{h}(S) \geqslant 2$ and $|\mathrm{supp}(S)| = 3$.

Let $S = a^u b^v c^t$ and $T = S(abc)^{-1} = a^{u-1}b^{v-1}c^{t-1}$, where $u \geqslant v \geqslant t \geqslant 1$. By Lemma 17, we may assume that $v \geqslant 2$.

First suppose that $u \geqslant 3$. Then abc is zero-sum free. By Claim E, every term of S has order at least 3. Hence $|\sum_0(abc)| \geqslant 7$ by Lemma 9 (i). Then by Lemma 8, $|\sum(T)| \leqslant |\sum(S)| - |\sum_0(abc)| + 1 \leqslant 2r - 7 = 2|T| - 1$. By Claim E, $|\sum(T)| < |\langle \mathrm{supp}(T)\rangle|$. Hence we may apply Theorem 3 or the main theorem to T by the minimality of $|S|$. If T is strictly behaving, then Sc^{-1} is strictly behaving, a contradiction to Claim A. If T satisfies (III), then S contains some term of order 2, a contradiction to Claim E. If T satisfies (IV), then $\mathrm{ord}(a - b) = 2$ and $S = a^u b^v c$. Hence $|\sum(Sc^{-1})| = |\sum(a^u b^v)| \leqslant 2(u + v) - 1 = 2r - 3 < |\sum(S)| - 1$, a contradiction to the equality (2). If T satisfies (V), (VI), (IX) or (X), then $|\sum(T)| = |\langle \mathrm{supp}(T)\rangle| - 1$. So $\sum(Sc^{-1}) = \sum(Tab) = \langle \mathrm{supp}(T)\rangle$, a contradiction to Claim E. If T satisfies (VII) or (VIII), then $0 \in \sum_{\leqslant 3}(T) \subset \sum_{\leqslant \mathsf{h}(S)}(S)$, a contradiction.

Now let T satisfy (II), then $u = r - 3$, $T = a^{u-1}b$ and $S = a^u b^2 c$. Since $Sc^{-1} = a^u b^2$ is not strictly behaving by Claim A, we have $b \notin \{a, 2a, \ldots, ua\}$. Since $0 \notin \sum_{\leqslant \mathsf{h}(S)}(S)$, we have $b \notin \{-(u-1)a, -(u-2)a, \ldots, -a, 0\}$. Note that $|G| = 2|S| = 2u+6$ by the equality (1), $|\sum(a^u b^2)| = |\sum(S)| - 1 = 2u+4$ by the equality (2) and $\mathrm{ord}(a) \geqslant u+2$ by Claim E. Since $\mathrm{ord}(a)$ is a divisor of $2u+6$, we have $\mathrm{ord}(a) \in \{u+3, 2u+6\}$. If $\mathrm{ord}(a) = u+3$, then $b \notin \langle a \rangle$ and $\sum(a^u b^2) = \langle a \rangle \cup (b + \{0, a, \ldots, ua\})$. Since $|\sum(S)| = |\sum(a^u b^2)| + 1$ by the equality (2), we have $c \in \{a, -a\}$, a contradiction. Hence we may assume that $\mathrm{ord}(a) = 2u+6$. Since $b \notin \{-(u-1)a, \ldots, 0, \ldots, ua\}$, we have $b = (u+\gamma)a$ for some $\gamma \in [1, 6]$. If $\gamma = 1$, then $\sum(a^u b^2) = \{a, 2a, \ldots, (3u+2)a\}$ and so $|\sum(a^u b^2)| = \min\{\mathrm{ord}(a), 3u+2\} > 2u+4$, a contradiction. If $\gamma = 4$, then $2b = 2a$ and $\sum(a^u b^2) = \{a, \ldots, (u+2)a\} \cup \{(u+4)a, \ldots, (2u+4)a\}$. So $|\sum(a^u b^2)| = u+2+u+1 = 2u+3 < 2u+4$, a contradiction. If $\gamma = 5$, then $2b = 4a$ and $\sum(a^u b^2) = \{a, \ldots, (2u+5)a\}$. So $|\sum(a^u b^2)| = 2u+5 > 2u+4$, a contradiction. If $\gamma \in \{2, 3, 6\}$, then $2b + 2a = 0$, $2b = 0$ or $b + ua = 0$. Hence $0 \in \sum(S)$. By Lemma 8, $|\sum_0(T)| \leqslant |\sum(S)| - |\sum_0(abc)| + 1 \leqslant 2r - 1 - 7 + 1 = 2|T| - 1 = |\sum(T)|$, which implies $\sum(T) = \sum_0(T)$. Then $0 \in \sum(T)$ and so $b \in \{-(u-1)a, \ldots, -a, 0\}$, a contradiction.

Finally it remains to consider the case when $u = 2$, that is $S = a^2 b^2 c^2$ or $S = a^2 b^2 c$. We claim that $|\sum_0(abc)| \geqslant 7$. If $a+b+c \neq 0$, then abc is zero-sum free. Note that every term of S has order at least 3 by Claim E. So $|\sum_0(abc)| \geqslant 7$ by Lemma 9 (i). If $a+b+c = 0$, then $a \neq b+c$, $b \neq a+c$ and $c \neq a+b$, for otherwise S contains some term of order 2. Hence $a, b, c, a+b, a+c, b+c, 0$ are pairwise distinct and so $|\sum_0(abc)| \geqslant 7$. This completes the proof of the claim. First let $S = a^2 b^2 c^2$. By Lemma 8, $|\sum(S)| \geqslant |\sum(abc)| + |\sum_0(abc)| - 1 \geqslant 6 + 7 - 1 = 12 > 2|S| - 1$, a contradiction. Now let $S = a^2 b^2 c$. If $0 \in \sum(S)$, then $|\sum(S)| \geqslant |\sum_0(abc)| + |\sum_0(ab)| - 1 \geqslant 7 + 4 - 1 = 10 > 2|S| - 1$, a contradiction. Hence we may assume that S is zero-sum free. If $\mathrm{ord}(a) = \mathrm{ord}(b) = |G| = 10$, then $b = 3a$ or $b = -3a$. Hence $\sum(S) = \{a, 2a, \ldots, 8a\}$ or $\sum(S) = \{-6a, \ldots, -a\} \cup \{a, 2a\}$. Since $0 \notin \sum(S)$, we have $c = a$ or $c = -3a = b$, a contradiction. Hence we may assume a or b, say a, has order less than 10. Since $\mathrm{ord}(a) \geqslant \mathsf{v}_a(S) + 2 = 4$ by Claim E, we have $\mathrm{ord}(a) = 5$. Then $b \notin \langle a \rangle$ and $\sum(a^2 b^2) \subset (\{0, b, 2b\} + \{0, a, 2a\}) \subset \langle a \rangle \cup (b + \{0, a, 2a\})$. Since $|\sum(a^2 b^2)| = |\sum(S)| - 1 = 8$, we have $\sum(a^2 b^2) = \langle a \rangle \cup (b + \{0, a, 2a\})$ and so $0 \in \sum(a^2 b^2)$, a contradiction.

Now all the possibilities have been considered. Therefore we have completed the proof of the theorem. ☐

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