

Separating the Online and Offline DP-Chromatic Numbers

Peter Bradshaw^a

Submitted: Sep 19, 2022; Accepted: Sep 2, 2023; Published: Nov 17, 2023

© The author. Released under the CC BY-ND license (International 4.0).

Abstract

The DP-coloring problem is a generalization of the list-coloring problem in which the goal is to find an independent transversal in a certain topological cover of a graph G . In the online DP-coloring problem, the cover of G is revealed one component at a time, and the independent transversal of the cover is constructed in parts based on incomplete information. Kim, Kostochka, Li, and Zhu asked whether the chromatic numbers corresponding to these two graph coloring problems can have an arbitrarily large difference in a single graph. We answer this question in the affirmative by constructing graphs for which the gap between the online DP-chromatic number and the offline DP-chromatic number is arbitrarily large.

Mathematics Subject Classifications: 05C15

1 Introduction

We consider several graph coloring problems. In the *list coloring problem*, we have a graph G and a list $L(v)$ of colors at each vertex $v \in V(G)$. In this setting, we say that an L -coloring of G is a proper coloring $\varphi : V(G) \rightarrow \bigcup_{v \in V(G)} L(v)$ of G in which $\varphi(v) \in L(v)$ for every vertex $v \in V(G)$. If G always has an L -coloring whenever $|L(v)| = f(v)$ for each vertex $v \in V(G)$, then we say that G is f -choosable. If f is a constant function $f(v) = k$, then we say that the *list-chromatic number* (or *choosability*) of G is at most k , and we write $\chi_\ell(G) \leq k$.

The *DP-coloring problem* is a generalization of the list coloring problem introduced by Dvořák and Postle [3], defined as follows. Given a graph G and a function $f : V(G) \rightarrow \mathbb{N}$, an f -fold cover of G is a graph H obtained by the following process:

- For each vertex $v \in V(G)$, add a clique $K_{f(v)}$ to H , and write $L(v)$ for the vertex set of this clique.
- For each edge $uv \in E(G)$, add a matching between $L(u)$ and $L(v)$.

Then, we say that an independent set in H of size $|V(G)|$ is a *DP-coloring* of G with respect to H . If G always has a DP-coloring for every f -fold cover H of G , then we say that G is DP- f -colorable. If f is a constant function $f(v) = k$, then we say that H is a

^aDepartment of Mathematics, Simon Fraser University, Burnaby, Canada (pb38@illinois.edu).

k -fold cover of G , and if G always has a DP-coloring for every k -fold cover H of G , then we say that the DP-chromatic number of G is at most k , and we write $\chi_{DP}(G) \leq k$. Given a cover H of G , we often refer to the vertices of H as *colors*, and if $c \in L(v)$ for a vertex $v \in V(G)$, then we say that the color c is *above* v . Note that when $f(v) = k$ is a constant function, if the cliques in H corresponding to vertices in G are replaced with independent sets, and if each matching between sets $L(u)$ and $L(v)$ is a perfect matching, then H is a k -sheeted covering space of G , and a DP-coloring of G is equivalent to an independent transversal of the fibers in H above the vertices of G (see [4] for an introduction to graphs as topological spaces).

Every list-coloring problem can be transformed into a DP-coloring problem as follows. Given a graph G with a color list $L'(v)$ at every vertex $v \in V(G)$, we construct a cover H of G by adding a clique with vertex set $L(v)$ for every vertex $v \in V(G)$, with elements of $L(v)$ corresponding to colors in $L'(v)$. Then, we consider each edge $uv \in E(G)$, and we add an edge in H between each pair $(c, c') \in L(u) \times L(v)$ for which c and c' both correspond to a common color from $L'(u) \cap L'(v)$. When H is constructed this way, a DP-coloring of G with respect to H is equivalent to an L' -coloring of G . Therefore, it holds that $\chi_\ell(G) \leq \chi_{DP}(G)$.

We also consider two online graph coloring problems. The *online DP-coloring* problem takes place in the form of a DP-coloring game between two players, called Lister and Painter. The game is played on a graph G with a function $f : V(G) \rightarrow \mathbb{N}$. At the beginning of the game, each vertex $v \in V(G)$ has $f(v)$ tokens. On each turn i , Lister removes some number $m_i(v)$ (possibly zero) of tokens from each vertex $v \in V(G)$ and then reveals a clique $K_{m_i(v)}$ above each vertex v . Furthermore, for each edge $uv \in E(G)$, Lister reveals a matching between the revealed cliques above u and v . The cliques and matchings revealed on this turn i form a cover H_i of G . After H_i is revealed, Painter chooses an independent set from the vertices of H_i . The game ends when G has no more tokens for Lister to remove. Painter wins the game if she manages to choose at least one color above each vertex of G before the game is over; otherwise, Lister wins. If Painter always has a winning strategy in the DP-coloring game on a graph G when each vertex $v \in V(G)$ begins with k tokens, then we say that the *online DP-chromatic number* (or *DP-paintability*) of G is at most k , and we write $\chi_{DPP}(G) \leq k$. Observe that if each vertex of G begins with k tokens, then Lister has the option of revealing a k -fold cover H of G on the first turn and asking Painter to find a DP-coloring of G with respect to H , and therefore $\chi_{DP}(G) \leq \chi_{DPP}(G)$.

If, in the DP-coloring game, Lister is only allowed to remove at most one token from each vertex of G during each turn and must always reveal edges wherever possible, then we call this variant of the game the *list-coloring game*. For the list-coloring game, we may equivalently imagine that on each turn i , Lister reveals a single color c_i above each vertex of some induced subgraph G'_i of G , and Painter must choose some independent set I_i of G'_i and color each vertex in I_i with c_i . In this equivalent setting, each vertex $v \in V(G)$ still begins with $f(v)$ tokens, and a single token is removed from v whenever Lister reveals a color above v . In this setting, Painter wins the game if and only if she can color every vertex of G before the game ends. If Painter always has a strategy to win the list-coloring game on a graph G when each vertex $v \in V(G)$ begins with $f(v)$ tokens, then we say that G is *f -paintable*. If f is a constant function $f(v) = k$, then we say that G is *k -paintable*, and we write $\chi_P(G) \leq k$. The online list-coloring game was originally

invented using this framework of revealing colors above vertices independently by Schauz [7] and Zhu [8].

At the end of the list-coloring game on G with a constant function $f(v) = k$, the colors revealed at each vertex v form a set $L(v)$ of k colors, and if Painter wins the game, then Painter completes a proper L -coloring of G . Since Lister is free to form any list assignment L on the vertices of G , it follows that if G is k -paintable, then G is also k -choosable, and hence $\chi_\ell(G) \leq \chi_P(G)$. Also, since the online list-coloring game is at least as difficult for Lister as the DP-coloring game, it also follows that $\chi_P(G) \leq \chi_{DPP}(G)$.

After putting all of the inequalities between these parameters together, we obtain two inequality chains:

$$\begin{aligned} \chi_\ell(G) &\leq \chi_{DP}(G) \leq \chi_{DPP}(G); \\ \chi_\ell(G) &\leq \chi_P(G) \leq \chi_{DPP}(G). \end{aligned}$$

Given these inequality chains, it is natural to ask whether the differences between adjacent parameters can be arbitrarily large. For three out of these four differences, we find an affirmative answer by letting $G = K_{n,n}$. Indeed, Bernshteyn [1] showed that a graph G of average degree d satisfies $\chi_{DP}(G) = \Omega\left(\frac{d}{\log d}\right)$, implying that $\chi_{DP}(K_{n,n}) = \Omega\left(\frac{n}{\log n}\right)$. Since it is known that $\chi_\ell(K_{n,n}) \leq \chi_P(K_{n,n}) = \log_2 n + O(1)$ [2], this shows that

$$\chi_{DP}(K_{n,n}) - \chi_\ell(K_{n,n}) = \Omega\left(\frac{n}{\log n}\right) \quad \text{and} \quad \chi_{DPP}(K_{n,n}) - \chi_P(K_{n,n}) = \Omega\left(\frac{n}{\log n}\right).$$

Duraj, Gutowski, and Kozik [2] also showed that

$$\chi_P(K_{n,n}) - \chi_\ell(K_{n,n}) = \Omega(\log \log n).$$

Therefore, by letting $G = K_{n,n}$, we achieve an arbitrarily large difference for each adjacent pair of parameters except for $\chi_{DPP}(G) - \chi_{DP}(G)$. For this final difference, Kim, Kostochka, Li, and Zhu [5] showed there exist graphs G for which $\chi_{DPP}(G) - \chi_{DP}(G) \geq \chi_P(G) - \chi_{DP}(G) \geq 1$, and they asked whether the difference $\chi_{DPP}(G) - \chi_{DP}(G)$ can be arbitrarily large.

In this note, we generalize the ideas from the construction of [5] to construct a graph G_t for each $t \geq 2$ that satisfies $\chi_{DPP}(G_t) - \chi_{DP}(G_t) \geq \chi_P(G_t) - \chi_{DP}(G_t) \geq t$. Thus, we give an affirmative answer to the question of Kim, Kostochka, Li, and Zhu [5]. By combining this equality with the fact that $\chi_{DP}(K_{n,n}) - \chi_P(K_{n,n}) = \Omega\left(\frac{n}{\log n}\right)$, we also see that the difference $\chi_{DP}(G) - \chi_P(G)$ can achieve both positive and negative values of arbitrarily large magnitude.

2 The construction

For each integer $t \geq 2$, we construct a graph G_t that satisfies $\chi_P(G_t) - \chi_{DP}(G_t) \geq t$. As we are concerned with showing that the paintability of each graph G_t is large enough, we begin with an observation about the online list-coloring game. If Lister and Painter play the online list-coloring game on a graph G with some initial token assignment, then Lister wins if and only if he can reach a position in which each uncolored vertex $v \in V(G)$

has some $g(v)$ remaining tokens, and the uncolored subgraph of G is not g -choosable. In the original paper of Kim, Kostochka, Li, and Zhu [5], the authors take advantage of this idea in order to construct a graph G satisfying $\chi_P(G) \geq \chi_{DP}(G) + 1$. In order to show that their graph G has a large enough paintability, they show that in the online list-coloring game on their graph G , Lister always has a strategy to create an uncolored $K_{1,k}$ subgraph of G in which each leaf ℓ has $g(\ell) = 1$ token and the center vertex v has $g(v) = k$ tokens. Since $K_{1,k}$ is not g -choosable, it follows that Lister has a strategy to win the online list-coloring game on their graph G .

In our construction, we use a similar idea. We first fix an integer $t \geq 2$ and a value $k = 2^{8t^3}$. (With more careful calculation, our proof works with a smaller value of k , but we use this larger value for clearer presentation.) In our graph G_t , we will show that Lister can always create an uncolored K_{t,k^t} subgraph in which each t -degree vertex u has $g(u) = t$ tokens and each k^t -degree vertex v has $g(v) = k$ tokens. The following lemma shows that if Lister manages to create such an uncolored subgraph of G_t , then Lister wins the online list-coloring game.

Lemma 1. *Given the function $g : V(K_{t,k^t}) \rightarrow \mathbb{N}$ defined above, K_{t,k^t} is not g -choosable.*

Proof. We let the t vertices v_1, \dots, v_t of degree k^t have pairwise disjoint color lists $L(v_1), \dots, L(v_t)$ of size k . Then, for each of the k^t elements $L \in L(v_1) \times \dots \times L(v_t)$, we assign L as the color list of a vertex of degree t . Then, for each L -coloring of v_1, \dots, v_t , some vertex of degree t has no available color in its list, and hence K_{t,k^t} is not g -choosable. \square

The most important piece of our construction of G_t is the following gadget H_t . We construct our gadget H_t along with a function $h : V(H_t) \rightarrow \mathbb{N}$ as follows. We let H_t contain $(t+1)k^t$ copies $K^1, \dots, K^{(t+1)k^t}$ of the clique K_{t+1} , and we write $u_1^\ell, \dots, u_{t+1}^\ell$ for the vertices of each clique K^ℓ . We write U for the set of all of these vertices of the form u_j^ℓ ; in other words, we let U consist of all vertices that we have introduced so far. Then, for each value $1 \leq j \leq t+1$, we add t independent vertices x_j^1, \dots, x_j^t , and we make each of these vertices adjacent to $u_j^1, u_j^2, \dots, u_j^{(t+1)k^t}$. We write X for the set consisting of all of these vertices of the form x_j^i . For each vertex $u_j^\ell \in U$, we let $h(u_j^\ell) = t+1$, and for each vertex $x_j^i \in X$, we let $h(x_j^i) = k-t+1$. We now prove two lemmas that show that under appropriate circumstances, winning the online list-coloring game on H_t as Painter is harder than finding a DP-coloring on H_t .

Lemma 2. *H_t is not $(h+t-1)$ -paintable.*

Proof. We give each vertex $v \in V(H_t)$ exactly $h(v) + t - 1$ tokens, and we show that Painter cannot win the list-coloring game on H_t .

On each of the first t turns, Lister reveals a color at each vertex of each clique K^ℓ . After these first t turns, each clique K^ℓ has an uncolored vertex u_j^ℓ with exactly $h(u_j^\ell) - 1 = t$ tokens. Furthermore, since we have $(t+1)k^t$ cliques K^ℓ , each with at least one uncolored vertex, there exists some value $1 \leq j^* \leq t+1$ for which at least k^t vertices of the form $u_{j^*}^\ell$ are uncolored. We let A be a set of k^t uncolored vertices of the form $u_{j^*}^\ell$. Then, the set A along with the t vertices $x_{j^*}^1, \dots, x_{j^*}^t$ induce an uncolored K_{t,k^t} subgraph of H_t in which each t -degree vertex v has only $g(v) = t$ remaining tokens, and each k^t -degree vertex v has only $g(v) = k$ remaining tokens. Lemma 1 shows that this K_{t,k^t} subgraph of H_t is not g -choosable, so Lister has a strategy to win the game. \square

Lemma 3. H_t is DP- h -colorable.

Proof. Consider an h -fold cover H' of H_t . Recall that given a vertex $v \in V(H_t)$, we say that v corresponds to a clique $K_{h(v)}$ in H' with a vertex set $L(v)$, and we say that $L(v)$ contains $h(v)$ colors. Using this terminology, we say that each color in H' appears above only one vertex of H_t , as the sets $L(v)$ forming the cliques of H' are pairwise disjoint.

We first color the vertices $x_j^i \in X$. For each vertex $x_j^i \in X$ and color $c \in L(x_j^i)$, we define a set $S_c \subseteq [(t+1)k^t]$ consisting of those indices ℓ for which $L(u_j^\ell)$ contains a color adjacent to c . We will color the $t(t+1)$ vertices of X using $t(t+1)$ colors $c_1, \dots, c_{t(t+1)}$ that correspond to a family $\mathcal{S} = \{S_{c_1}, \dots, S_{c_{t(t+1)}}\}$ such that for each value $1 \leq q \leq t+1$, the following property holds:

$$\text{The intersection of any } q \text{ sets of } \mathcal{S} \text{ contains at most } \lfloor k^{t-\frac{qt}{t+1}} \rfloor - 1 \text{ elements.} \quad (\star)$$

In particular, the intersection of any $t+1$ sets of \mathcal{S} is empty.

We show that we may greedily color each vertex of X while satisfying (\star) . Suppose we wish to color some vertex $x \in X$ and that we have already colored some subset $Y \subseteq X$ while satisfying (\star) . To color x in a way that satisfies (\star) , we must choose a color $c \in L(x)$ such that for each subset $A \subseteq Y$ of some size $q \in [0, t]$ whose vertices are colored with colors a_1, \dots, a_q ,

$$|S_c \cap S_{a_1} \cap \dots \cap S_{a_q}| \leq \lfloor k^{t-\frac{(q+1)t}{t+1}} \rfloor - 1. \quad (\bullet)$$

We aim to show that a color $c \in L(x)$ exists which satisfies (\bullet) for each subset $A \subseteq Y$ of size at most t . To this end, we fix a subset $A \subseteq Y$ of some size $q \in [0, t]$ whose vertices have colors a_1, \dots, a_q , and we construct a bipartite graph B with partite sets $L(x)$ and $S_{a_1} \cap \dots \cap S_{a_q}$. If A is empty, so that $q = 0$, then we consider this intersection to be equal to $[(t+1)k^t]$. For each color $c \in L(x)$ and value $\ell \in S_{a_1} \cap \dots \cap S_{a_q}$, we add an edge in B from c to ℓ if and only if $\ell \in S_c$. We observe that as $h(u_j^\ell) = t+1$ for each vertex $u_j^\ell \in U$, each value $\ell \in S_{a_1} \cap \dots \cap S_{a_q}$ has degree at most $(t+1)$ in B . Furthermore, $|S_{a_1} \cap \dots \cap S_{a_q}| \leq \lfloor k^{t-\frac{qt}{t+1}} \rfloor$ by (\star) when $q \geq 1$, and $|S_{a_1} \cap \dots \cap S_{a_q}| = (t+1)k^t$ when $q = 0$. Thus, it follows that in all cases, $|E(B)| \leq (t+1)^2 k^{t-\frac{qt}{t+1}}$. Therefore, the number of colors $c \in L(x)$ with degree at least $\lfloor k^{t-\frac{(q+1)t}{t+1}} \rfloor$ in B is at most

$$\frac{(t+1)^2 k^{t-\frac{qt}{t+1}}}{\lfloor k^{t-\frac{(q+1)t}{t+1}} \rfloor} < 2(t+1)^2 k^{\frac{t}{t+1}}.$$

Now, a color $c \in L(x)$ violates (\bullet) for our fixed subset A if and only if c has degree at least $\lfloor k^{t-\frac{(q+1)t}{t+1}} \rfloor$ in B , so at most $2(t+1)^2 k^{\frac{t}{t+1}}$ colors $c \in L(x)$ violate (\bullet) for the subset A . As fewer than $2^{t(t+1)}$ subsets $A \subseteq Y$ exist, the number of colors $c \in L(x)$ that violate (\bullet) for some subset $A \subseteq Y$ is less than

$$\begin{aligned} 2^{t(t+1)+1} (t+1)^2 k^{\frac{t}{t+1}} &= 2^{t(t+1)+1+\frac{8t^4}{t+1}} (t+1)^2 \\ &= 2^{t^2+t+1+8(t^3-t^2+t-1+\frac{1}{t+1})} (t+1)^2 \\ &< 2^{8t^3-t} \\ &< k - t + 1 = h(x). \end{aligned}$$

Therefore, some color $c \in L(x)$ satisfies (\bullet) for every subset $A \subseteq Y$ of size $q \in [0, t]$, and hence we can color x with c while satisfying (\star) .

Now, with every vertex $x_j^i \in X$ colored, and with (\star) satisfied, no index $\ell \in [(t+1)k^t]$ belongs to the intersection of more than t sets $S_c \in \mathcal{S}$. Hence, after coloring the vertices in X , at most t colors are unavailable at each clique K^ℓ . Therefore, the vertices of each clique K^ℓ can be ordered so that the first vertex has at least one available color, the second vertex has at least two available colors, and so forth, until the last vertex has $t+1$ available colors. Therefore, each remaining clique K^ℓ can be DP -colored with its available colors, and the lemma is proven. \square

Now, we construct our graph G_t . First, we make k^{k-2t} copies of the graph H_t , and we index these copies by the $(k-2t)$ -tuples in $[k]^{k-2t}$. We also add $k-2t$ vertices y_1, \dots, y_{k-2t} that are adjacent to all vertices of U in each copy of H_t . We write \tilde{U} for this set of neighbors of y_1, \dots, y_{k-2t} , that is, the set of vertices belonging to a set U in some copy of H_t . The following two theorems show that $\chi_P(G_t) - \chi_{DP}(G_t) \geq t$.

Theorem 4. $\chi_{DP}(G_t) \leq k - t + 1$.

Proof. We give G_t a DP -coloring with lists of size $(k-t+1)$ as follows. First, we arbitrarily color the vertices y_1, \dots, y_{k-2t} . Next, we observe that the vertices in \tilde{U} have lost at most $k-2t$ available colors, so for each vertex v in a copy of H_t , v has at least $h(v)$ available colors remaining. Therefore, by Lemma 3, we finish our DP -coloring of G_t by giving each remaining copy of H_t a DP - h -coloring. \square

Theorem 5. $\chi_P(G_t) > k$.

Proof. Suppose that the online list-coloring game is played on G_t with k tokens at each vertex. We show that the following strategy is winning for Lister. For each pair (i, j) satisfying $1 \leq i \leq k$ and $1 \leq j \leq k-2t$, Lister executes the following command. When Lister executes this command for a given pair (i, j) , we say that this takes place on *turn* (i, j) .

Reveal a color $c_{i,j}$ above y_j and above each vertex of \tilde{U} that belongs to a copy of H_t indexed by a $(k-2t)$ -tuple with the value i in the j th coordinate.

For each value $j \in [k-2t]$, we write $L(y_j) = \{c_{1,j}, \dots, c_{k,j}\}$ for the set of colors revealed above y_j .

For each $j \in [k-2t]$, we may assume that for some value $i_j \in [k]$, Painter colors y_j during turn (i_j, j) and hence does not color any vertex of \tilde{U} during turn (i_j, j) . Indeed, if this is not the case, then y_j is never colored, and Painter will not have another chance to color y_j . Therefore, for each value $j \in [k-2t]$, we assume that no vertex in a copy of H_t indexed by a $(k-2t)$ -tuple with an i_j entry in the j th coordinate is colored using a color in $L(y_j)$.

Now, let H be the copy of H_t indexed by the $(k-2t)$ -tuple (i_1, \dots, i_{k-2t}) . By our observation above, no vertex of H has been colored by a color in $L(y_1) \cup \dots \cup L(y_{k-2t})$, and hence no vertex of H has been colored. Furthermore, since $k-2t$ tokens have been removed from each vertex in $\tilde{U} \cap V(H)$, it follows that for each vertex $v \in V(H)$, only $h(v) + t - 1$ tokens remain at v . Therefore, Lister can follow the strategy in Lemma 2 on H in order to win the game, and thus $\chi_P(G_t) > k$. \square

3 Conclusion

While we have shown for each positive integer t the existence of a graph G_t for which $\chi_P(G_t) - \chi_{DP}(G_t) \geq t$, it is still open whether there exists a sequence $\{G_t\}_{t \geq 1}$ of graphs for which

$$\lim_{t \rightarrow \infty} \frac{\chi_{DPP}(G_t)}{\chi_{DP}(G_t)} > 1 \quad \text{or} \quad \lim_{t \rightarrow \infty} \frac{\chi_P(G_t)}{\chi_\ell(G_t)} > 1.$$

On the other hand, it is unknown whether $\chi_P(G)$ can be bounded above by a linear or even polynomial function of $\chi_\ell(G)$, and it is unknown whether $\chi_{DPP}(G)$ can be bounded above by a linear function of $\chi_{DP}(G)$. Duraj, Gutowski, and Kozik [2] have pointed out that currently, the best known bound for $\chi_P(G)$ in terms of $\chi_\ell(G)$ comes from the relationship between a graph's choosability and minimum degree. Namely, a result of Saxton and Thomason [6] states that a graph G of minimum degree δ satisfies $\chi_\ell(G) \geq (1+o(1)) \log_2 \delta$. Writing d for the degeneracy of a graph G , we observe that G has a subgraph of minimum degree d , and hence we use this result to observe that

$$\chi_P(G) \leq d + 1 \leq 2^{(1+o(1))\chi_\ell(G)}.$$

For $\chi_{DPP}(G)$, we may use a result of of Bernshteyn showing that a graph G of minimum degree δ satisfies $\chi_{DP}(G) \geq \frac{\delta/2}{\log(\delta/2)}$ in order to bound $\chi_{DPP}(G)$ in terms of $\chi_{DP}(G)$ in a similar way. Using the same observation as above, if d is the degeneracy of G , then

$$\chi_{DPP}(G) \leq d + 1 \leq (2 + o(1))\chi_{DP}(G) \log \chi_{DP}(G).$$

It is likely that a deeper understanding of these coloring parameters is necessary to determine tight bounds between them.

Acknowledgments

I am grateful to Alexandr Kostochka for offering helpful feedback on an earlier version of this paper, and I am grateful to the referees for their helpful comments.

References

- [1] Anton Bernshteyn. The asymptotic behavior of the correspondence chromatic number. *Discrete Math.*, 339(11):2680–2692, 2016.
- [2] Lech Duraj, Grzegorz Gutowski, and Jakub Kozik. Chip games and paintability. *Electron. J. Combin.*, 23(3):#P3.3, 12, 2016.
- [3] Zdeněk Dvořák and Luke Postle. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *J. Combin. Theory Ser. B*, 129:38–54, 2018.
- [4] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [5] Seog-Jin Kim, Alexandr Kostochka, Xuer Li, and Xuding Zhu. On-line DP-coloring of graphs. *Discrete Appl. Math.*, 285:443–453, 2020.
- [6] David Saxton and Andrew Thomason. Hypergraph containers. *Invent. Math.*, 201(3):925–992, 2015.

- [7] Uwe Schauz. Mr. Paint and Mrs. Correct. *Electron. J. Combin.*, 16(1):#R77, 18, 2009.
- [8] Xuding Zhu. On-line list colouring of graphs. *Electron. J. Combin.*, 16(1):#R127, 16, 2009.