Counterexamples to the Characterisation of Graphs with Equal Independence and Annihilation Number

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Abstract

In this paper, we disprove the claimed characterisation of graphs with equal independence and annihilation number as proposed by Larson and Pepper [*Electron. J. Comb.* 2011]. The annihilation number of a graph is defined as the largest integer p such that the sum of its smallest p degrees is greater than or equal to its size, i.e., its number of edges. Larson and Pepper claimed that for a given graph $G = (V, E)$, its independence number $\alpha(G)$ equals its annihilation number $a(G)$ if and only if

$$\begin{array}{lll} (1) & a(G) \geqslant \frac{n}{2}: & \alpha'(G) = a(G) \\ (2) & a(G) = \frac{n-1}{2}: & \alpha'(G - v) = a(G) \quad \text{for some } v \in V. \end{array}$$

This paper provides series of counterexamples with an arbitrarily large number of vertices, an arbitrarily large number of components, an arbitrarily large independence number, and an arbitrarily large difference between the critical and the regular independence number. Furthermore, we identify the error in the proof of Larson and Pepper's theorem. Yet, we show that the theorem still holds for bipartite graphs and connected claw-free graphs.

Mathematics Subject Classifications: 05C69

1 Introduction

Let $G = (V, E)$ be a finite and simple graph with vertex set V and edge set E. The order of G is defined as $|V|$ and denoted by n. The degree sequence of a given graph G is defined as the decreasingly ordered sequence of the degrees in G, denoted by $\pi(G) = \pi = (d_1 \geqslant \cdots \geqslant d_n)$.

For a vertex $v \in V$, we denote its neighbourhood in G by $\mathcal{N}_G(v)$ and define its degree in G by $\deg_G(v) \coloneqq |\mathcal{N}_G(v)|$. If a neighbourhood set clearly refers to a graph G, we omit the subscript G. Additionally, for $V' \subseteq V$, we denote $\mathcal{N}(V') = \bigcup_{v \in V'} \mathcal{N}(v)$.

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A subset $I \subseteq V$ of pairwise non-adjacent vertices is called an independent set. We call an independent set of maximum size I in G a maximum independent set and define the independence number of G as $\alpha(G) \coloneqq |I|$. Further, a critical independent set is defined as an independent set $I^c \subseteq V$ that maximises $|I^c| - |\mathcal{N}(I^c)|$ among all independent sets in G. If a critical independent set $|I^c|$ is of maximum cardinality, we call it a maximum critical independent set and the critical independence number of G is then defined as $\alpha'(G) \coloneqq |I^c|$. Consequently, it follows that $\alpha'(G) \leqslant \alpha(G)$.

In 2004, Pepper introduced the annihilation number of a graph as an upper bound on its independence number [6, 7]. Given a degree sequence $\pi = (d_1 \geqslant \cdots \geqslant d_n)$, its annihilation number $a(\pi)$ is defined as

$$a(\pi) \coloneqq \max_{p \in \mathbb{N}} \left\{ \sum_{i=1}^{n-p} d_i \geqslant \sum_{i=n-p+1}^{n} d_i \right\}.$$

For any realisation G of π, the annihilation number of G is $a(G) \coloneqq a(\pi)$. Therefore, $a(G)$ can equivalently be defined as the largest integer p such that $\sum_{i=n-p+1}^{n} d_i \leqslant |E|$. Again, if the independence, critical independence, or annihilation number clearly refers to a certain graph, we simplify the notation by omitting the graph and using only α, α', and a, respectively.

In [3], Larson and Pepper present the following characterisation of graphs with equal independence number α and annihilation number a using the critical independence number α'.

Theorem 1. [3] *Let $G = (V, E)$ be a graph on n vertices. Then*

$$\alpha(G) = a(G) \quad \text{if and only if} \quad \begin{array}{ll} (1)\ a(G) \geqslant \frac{n}{2}: & \alpha'(G) = a(G) \\ (2)\ a(G) = \frac{n-1}{2}: & \alpha'(G - v) = a(G) \quad \text{for some } v \in V. \end{array}$$

Since the critical independence number and the annihilation number can both be calculated in polynomial time, this characterisation would yield a polynomial-time algorithm to verify whether a graph meets the upper bound on the independence number.

We disprove Theorem 1 by creating various series of counterexamples in Section 2, and identify the error in the proof in Section 3. However, in Section 4, we show that the theorem holds true for restricted graph classes.
Before that, we note that the "if"-direction is still true.

Lemma 2. *Let $G = (V, E)$ be a graph on n vertices. Then*

$$\alpha(G) = a(G) \quad \text{if} \quad \begin{array}{ll} (1)\ a(G) \geqslant \frac{n}{2}: & \alpha'(G) = a(G) \\ (2)\ a(G) = \frac{n-1}{2}: & \alpha'(G - v) = a(G) \quad \text{for some } v \in V. \end{array}$$

Proof. Since $\alpha'(G) \leqslant \alpha(G) \leqslant a(G)$ for all graphs G, in case (1), we directly obtain $\alpha(G) = a(G)$. In case (2), we have $a(G) = \alpha'(G - v) \leqslant \alpha(G - v) \leqslant \alpha(G) \leqslant a(G)$. Thus, all inequalities hold with equality and it follows that $\alpha(G) = a(G)$. □

2 Counterexamples

In the following, we provide various series of counterexamples with an arbitrarily large number of vertices, an arbitrarily large number of components, an arbitrarily large independence number, and an arbitrarily large difference between the critical and the regular independence number. The smallest counterexample consists of a cycle of length three accompanied by a singleton, i.e., an isolated vertex, as shown in Figure 1. Filled vertices indicate a maximum independence set.

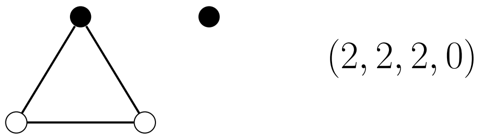

Figure 1: C_3 and a singleton with $\alpha = a = 2 \geqslant \frac{n}{2}$ while $\alpha' = 1$.

Expanding on this construction, we can now generate counterexamples with an arbitrary number of components by adding further singletons: For graphs consisting of a cycle of length three and t singletons, we obtain $n = t + 3$ and $\alpha = a = t + 1 \geqslant \frac{n}{2}$ since each singleton yields an additional vertex in every maximum independent and increases the annihilation number by one. However, we have $\alpha' = t$.

A further counterexample is the graph consisting of a cycle of length five with two chords and a singleton, as in Figure 2.

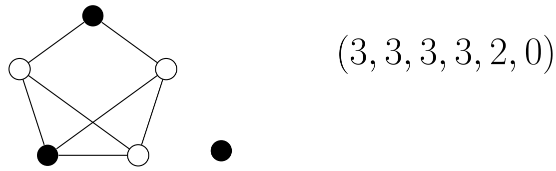

Figure 2: For a C_5 with two chords and a singleton, we obtain $\alpha = a = 3 \geqslant \frac{n}{2}$ but $\alpha' = 1$.

Counterexamples do not have to contain singletons: Consider the degree sequence of an odd cycle C_{2k+1}, where $k \in \mathbb{N}$, combined with a path of odd length P_{2l+1}, where $l \in \mathbb{N}$, is $\left(2^{2(k+l-1)}, 1^2\right)$. Consequently, the independence number equals the annihilation number $\alpha = a = k + l + 1 = \frac{n}{2}$, whereas the critical independence number $\alpha' = l + 1$ can be arbitrarily smaller.

Furthermore, we can also provide connected counterexamples, e.g., the graph shown in Figure 3.

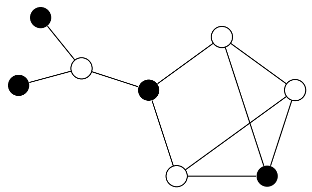

Figure 3: This graph with $n = 8$ fulfils $\alpha = a = 4 \geqslant \frac{n}{2}$, while $\alpha' = 2$.

To obtain counterexamples in which the difference between the critical independence number and the annihilation number becomes arbitrarily large, we can generalise the example above as follows: Starting with $C_{2k+1} = \{v_1, \ldots, v_{2k+1}\}$ for $k \geqslant 2$, we add chords $\{v_i, v_{i+k}\}$ for $i \in \{1, \ldots, k\}$ and a P_3 whose central vertex is connected to v_{2k+1}. For $k = 2$, the constructed graph corresponds to the one displayed in Figure 3; for $k = 3$ and $k = 4$, the graphs are shown in Figure 4 and Figure 5, respectively. By design, all vertices have degree three except the two end vertices of the P_3, which we denote by x_1 and x_2. Thus, the degree sequence of this connected graph with $n = 2k+4$ is $(3^{2k+2}, 1, 1)$, yielding $a = k + 2$.

Clearly, any maximum independent set contains x_1, x_2 and at most k vertices on the cycle, i.e., $\alpha \leqslant k + 2$. To prove $\alpha \geqslant k + 2$, we consider the cases $k \equiv_2 0$ and $k \equiv_2 1$.

In the first case, we construct the independent set

$$I \coloneqq \{x_1, x_2, v_{2k+1}, \underbrace{v_2, v_4, \ldots, v_k}_{\frac{k}{2}}, \underbrace{v_{k+3}, v_{k+5}, \ldots, v_{2k-1}}_{\frac{k-2}{2}}\}.$$

And in the second case, we construct the independent set

$$I \coloneqq \{x_1, x_2, v_{2k+1}, \underbrace{v_2, v_4, \ldots, v_{k-1}}_{\frac{k-1}{2}}, \underbrace{v_{k+1}, v_{k+3}, \ldots, v_{2k-2}}_{\frac{k-1}{2}}\}.$$

In both cases, $|I| = k + 2 \leqslant \alpha$; hence, we have $\alpha = k + 2 = a$, whereas $\alpha' = 2$. Now, $\alpha - \alpha' = k$ can become arbitrarily large.

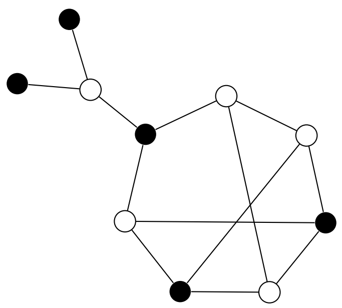

Figure 4: For $k = 3$, we get $n = 10$ and $\alpha = a = 5 \geqslant \frac{n}{2}$, while $\alpha' = 2$.

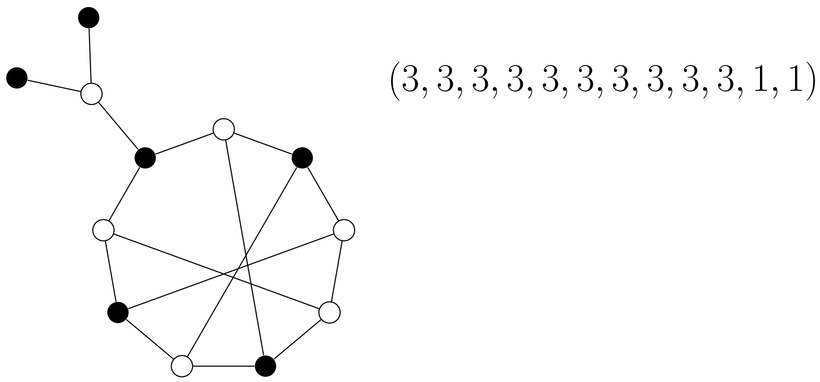

Figure 5: For $k = 4$, we get $n = 12$ and $\alpha = a = 6 \geqslant \frac{n}{2}$, while $\alpha' = 2$.

3 Error in the proof

The error in the proof in [3] occurs in the specific case, where G is not empty, $a(G) \geqslant \frac{n}{2}$, the neighbourhood of the maximum critical independent set J is empty and $a(G-J) < \frac{n(G-J)}{2}$. Larson and Pepper use the inductive assumption on $G-J+u$ for a vertex $u \in J$. However, for $J = \{u\}$, meaning $|J| = 1$, we obtain $G - J + u = G$ and the inductive assumption cannot be applied. Since the theorem is proven through induction, it remains unclear whether the given proof is salvageable even for restricted graph classes. As a consequence, the proof of the following corollary (Theorem 3.3 in [3]) is invalid as well: Larson and Pepper use the equality of independence and annihilation number to characterise Kőnig-Egerváry graphs, i.e., graphs fulfilling $\alpha(G)+\mu(G) = n$, where $\mu(G)$ denotes the maximum matching number of G.

Corollary 3. [3] *For a graph G with $a(G) \geqslant \frac{n}{2}$, $\alpha(G) = a(G)$ if and only if G is a Kőnig-Egerváry graph and every maximum independent set of G is a maximum annihilating set.*

Consider for example the graphs constructed in Section 2 (see Figure 3, 4, 5) with $n = 2k+4$ and $\alpha = a = k+2 \geqslant \frac{n}{2}$. The matching number of such a graph is $\mu = k+1$. Thus, $\alpha + \mu < n$ and the graph is not Kőnig-Egerváry. This also implies that the "only if"-part of Conjecture 3.4 in [3] does not hold true.

Conjecture 4. [3] For a graph G with $a \geqslant \frac{n}{2}$, $\alpha = a$ if and only if G is a Kőnig-Egerváry graph and every maximum independent set of G is a maximal annihilating set.

In [4] and [5], the authors provide counterexamples for the "if"-direction, but show the "only if"-direction using the just disproved results by Larson and Pepper.

It remains to retrace in which papers the theorems are used beyond the above mentioned and to review whether the subsequent results still hold.

4 Theorem for bipartite graphs and connected claw-free graphs

It is striking that all counterexamples mentioned above are either non-connected or contain a claw, i.e., an induced $K_{1,3}$, and an odd cycle. In fact, it turns out that the theorem still holds for bipartite graphs and connected claw-free graphs.

Note that for bipartite graphs the case $a = \frac{n-1}{2}$ cannot occur. Thus, for this graph class the following theorem is equivalent to Theorem 1.

Theorem 5. *Let G be a bipartite graph. Then,*

$$\alpha(G) = a(G) \quad \textit{if and only if} \quad \alpha'(G) = a(G).$$

Proof. As seen in Lemma 2, the "if"-direction holds true. Hence, it remains to prove that if G is bipartite, the equality $\alpha = a$ implies $\alpha' = a$. For bipartite graphs, the critical independence number equals the independence number [2], and therefore, the implication is true. $\square$

To prove the theorem for claw-free graphs, we need the following lemma.

Lemma 6. *Let $G = (V, E)$ be connected with $a(G) = \frac{n-1}{2}$. Then there exists a vertex $v \in V$ that does not occur in every maximum independent sets while $G-v$ is still connected.*

Proof. Let $I \subseteq V$ be a maximum independent set in G. Then $V = I \cup \mathcal{N}(I)$. Consider a path $P = \{v_0, \dots, v_k\}$ of maximum length in G. If $v_0 \notin I$, the removal of v_0 preserves the connectedness of G since all neighbours of v_0 have to be in P; otherwise, P was not a path of maximum length. If $v_0 \in I$, then $v_1 \notin I$. Note that $a = \frac{n-1}{2}$ implies $\sum_{v \in X} \deg(v) < \sum_{v \in Y} \deg(v)$ for all $X, Y \subseteq V$, $X \cap Y = \emptyset$ with $|X| < |Y|$. Therefore, the minimum degree is at least two and thus v_0 has at least one additional neighbour (apart from v_1), which has to be in P, as seen above. Now assume that $G - v_1$ is not connected. Then there exists a neighbour w of v_1 that is not adjacent to any other vertex of P. Since the minimum degree is at least two, it follows that w has another neighbour $z \notin P$. But this contradicts the assumption that P is a path of maximum length since we obtain a longer path by replacing v_0 with w and z. $\square$

Theorem 7. *Let G be a connected claw-free graph. Then*

$$\alpha(G) = a(G) \quad \textit{if and only if} \quad \begin{array}{ll} (1)\ a(G) \geqslant \frac{n}{2}: & \alpha'(G) = a(G) \\ (2)\ a(G) = \frac{n-1}{2}: & \alpha'(G - v) = a(G) \quad \textit{for some } v \in V. \end{array}$$

Proof. By Lemma 2, it suffices to consider the "only if"-direction for claw-free graphs $G = (V, E)$.

First, let $\alpha(G) = a(G) \geqslant \frac{n}{2}$ and suppose $\alpha'(G) < a(G)$. Then there exists a maximum critical independent set $I^c \subseteq V$ with $|I^c| - |\mathcal{N}(I^c)| > 0$ as well as $|I^c| < |I|$ and $|I^c| - |\mathcal{N}(I^c)| > |I| - |\mathcal{N}(I)|$ for all maximum independent sets $I \subseteq V$. This implies $|I^c| \geqslant 2$, $|\mathcal{N}(I^c)| \geqslant 1$ and for $R := V \setminus (I^c \cup \mathcal{N}(I^c))$, we get $|R| \geqslant 3$. By assumption, G is claw-free and I^c is an independent set. Thus, each vertex in $\mathcal{N}(I^c)$ can have at most two neighbours in I^c.

We consider the bipartite graph $\tilde{G} := (I^c \cup \mathcal{N}(I^c), \tilde{E})$ with $\tilde{E} := \{uv \in E \mid u \in \mathcal{N}(I^c), v \in I^c\}$. Any connected component K of $\tilde{G}$ falls naturally into one of two types; that is $|K \cap I^c| \leqslant |K \cap \mathcal{N}(I^c)|$ or $|K \cap I^c| > |K \cap \mathcal{N}(I^c)|$. We call the former Type I components and the latter Type II components.

Note that there exists at least one component of each type: Since $|I^c| - |\mathcal{N}(I^c)| > 0$, there exists a vertex in $\mathcal{N}(I^c)$ that is adjacent to exactly two vertices in I^c. And as G is claw-free, this vertex cannot have a neighbour in R. Additionally, since G is connected, there is a vertex in $\mathcal{N}(I^c)$ that is adjacent to exactly one vertex in I^c and at least one vertex in R.

In a Type II component K, we have at least $|K \cap I^c| + |K \cap \mathcal{N}(I^c)| - 1 \geqslant 2|K \cap \mathcal{N}(I^c)|$ edges. Therefore, all vertices of K in $\mathcal{N}(I^c)$ have degree two in $\tilde{G}$; otherwise, there would exist a vertex in $\mathcal{N}(I^c)$ of degree three, contradicting the claw-freeness. Furthermore, Type II components cannot contain vertices adjacent to R in G; otherwise, G would not be claw-free. Therefore, at least one Type II component K_2 has to be connected to a Type I component K_1 by an edge between two vertices in $\mathcal{N}(I^c)$. Let $w_1 \in K_1 \cap \mathcal{N}(I^c)$ and $w_2 \in K_2 \cap \mathcal{N}(I^c)$ be these vertices with $w_1 w_2 \in E$. However, since w_2 has already two non-connected neighbours within K_2, this contradicts the claw-freeness of G. Hence, the claim is proven to be true for $a(G) \geqslant \frac{n}{2}$.

Now, let $\alpha(G) = a(G) = \frac{n-1}{2}$. By Lemma 6, there exists a vertex $v \in V$ that does not occur in every maximum independent set, while $G - v$ is still connected. Hence, we have $\alpha(G - v) = \alpha(G) = a(G) = \frac{n-1}{2}$. As $\alpha(G - v) \leqslant a(G - v) \leqslant a(G)$, it follows that $\alpha(G-v) = a(G-v) = \frac{n-1}{2}$. Since G is assumed to be claw-free, $G-v$ is claw-free as well. Further, $a(G - v) \geqslant \frac{n(G-v)}{2}$ and therefore, we can apply the first case to $G - v$ and obtain $\alpha'(G - v) = a(G - v) = \frac{n-1}{2} = a(G)$. This completes the proof for claw-free graphs. $\square$

Note that Sbihi [8] has already shown in 1980 that maximum independent sets can be found in claw-free graphs in polynomial time. The proof uses the blossom algorithm by Edmonds [1] from 1965, which yields maximum matchings in polynomial time for any graph: Any maximum matching in a graph translates to a maximum independent set in the corresponding line graph and all claw-free graphs can be considered as a line graph.

It remains open whether the theorem holds for other restricted graph classes or for arbitrary graphs with $a = \frac{n-1}{2}$. Of particular interest in this regard are graph classes for which maximum independent sets cannot be found in polynomial time. Rauch and Rautenbach have recently shown in [9] that for a given graph G it can be determined in polynomial time whether $\alpha(G) = a(G)$.

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