A Spanning Tree with at Most k Leaves in a $K_{1,p}$-Free Graph

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Abstract

For an integer $k \geqslant 2$, a tree is called a k-ended tree if it has at most k leaves. It was shown that some $\sigma_{k+1}(G)$ conditions assure the existence of a spanning k-ended tree in a connected $K_{1,p}$-free graph G for the pairs (p, k) with $p \leqslant 4$, or $p = 5$ and $k = 4, 6$, where $\sigma_{k+1}(G)$ is the minimum degree sum of pairwise non-adjacent $k+1$ vertices of G. In this paper, we extend those results to the case with any integer $p \geqslant 3$ by proving that for any $k \geqslant 2$ and $p \geqslant 3$, there exists a constant $f(p, k)$ depending only on k and p such that if a connected $K_{1,p}$-free graph satisfies $\sigma_{k+1}(G) \geqslant |G| + f(p, k)$, then G has a spanning k-ended tree. The coefficient 1 of $|G|$ in the $\sigma_{k+1}(G)$ condition is best possible.

Mathematics Subject Classifications: 05C05, 05C35

1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For a vertex v of G, let $N_G(v)$ and $d_G(v)$ denote the neighborhood and the degree of v in G, respectively. Let $\alpha(G)$ be the independence number of G. For an integer $k \geqslant 2$, a tree is called a *k-ended tree* if it has at most k leaves. For an integer k with $\alpha(G) \geqslant k$, we define

$$\sigma_k(G) = \min\left\{\sum_{x \in S} d_G(x) : S \text{ is an independent set of } G \text{ of order } k\right\}$$

and $\sigma_k(G) = \infty$ if $\alpha(G) < k$.

A Hamiltonian path of a graph is a path which passes through all vertices of the graph. By adding the $K_{1,3}$-free condition, Matthews and Sumner [10] weakened the condition of Dirac's theorem [4] which says that if a connected graph G satisfies $\delta(G) \geqslant \frac{|G|}{2}$ (resp. $\delta(G) \geqslant \frac{|G|-1}{2}$), then G has a Hamiltonian cycle (resp. a Hamiltonian path).

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Theorem 1 (Matthews and Sumner). *Let G be a connected $K_{1,3}$-free graph. If $\delta(G) \geqslant \frac{|G|-2}{3}$, then G has a Hamiltonian path. In addition, if G is 2-connected, then G has a Hamiltonian cycle.*

Since a spanning 2-ended tree is equivalent to a Hamiltonian path, we can consider the existence of a spanning k-ended tree as an extension of Hamiltonian properties. In this point of view, the following series of studies have been done.

Theorem 2. *Let k and p be integers with $k \geqslant 2$ and $p \geqslant 3$, and let G be a connected $K_{1,p}$-free graph. If G satisfies one of the following conditions, then G has a spanning k-ended tree.*

- $p = 3$, *and* $\sigma_{k+1}(G) \geqslant |G| - k$ [Kano, Kyaw, Matsuda, Ozeki, Saito and Yamashita [5], Salamon [12]],
- $p = 4$, $k = 3$, *and* $\sigma_4(G) \geqslant |G| - 1$ [Kyaw [6]],
- $p = 4$, $k \neq 3$, *and* $\sigma_{k+1}(G) \geqslant |G| - \frac{k}{2}$ [Kyaw [7]],
- $p = 5$, $k = 4$, *and* $\sigma_5(G) \geqslant |G| - \frac{4}{3}$ [Chen, Ha and Hanh [3]],
- $p = 5$, $k = 6$, *and* $\sigma_7(G) \geqslant |G| - 2$ [Sun and Liu [13]].

Note that each of the condition on $\sigma_{k+1}(G)$ in Theorem 2 is best possible, see below. As we can see in Theorem 2, many researchers are interested in the following statement

> "for every connected $K_{1,p}$-free graph G, if $\sigma_{k+1}(G) \geqslant |G| + f(p, k)$, where $f(p, k)$ is a constant depending only on k and p, then G has a spanning k-ended tree",

and proved the statement for some pairs (k, p). However, all results in Theorem 2 assume p to be at most 5, and $k = 4, 6$ in the case $p = 5$. With this situation in mind, the purpose of this paper is to prove the statement above for all pairs (k, p) with $k \geqslant 2$ and $p \geqslant 3$, as follows.

Theorem 3. *Let k and p be integers with $k \geqslant 2$ and $p \geqslant 3$, and let G be a connected $K_{1,p}$-free graph. If $\sigma_{k+1}(G) \geqslant |G| + (k + 1)(p - 1) \min\{p - 2, k\}$, then G has a spanning k-ended tree.*

Las Vergnas [8] proved the following: For an integer k with $k \geqslant 2$, if a connected graph G satisfies $\sigma_2(G) \geqslant |G| - k + 1$, then G has a spanning k-ended tree. (See also [2].) In addition, considering bipartite graphs $K_{m,m+k}$ for $m \geqslant 1$, we see that the $\sigma_2(G)$ condition is best possible. Compared with this, Theorem 3 guarantees the existence of a spanning k-ended tree in a connected $K_{1,p}$-free graph by a much weaker degree sum condition.

The following example shows that we cannot replace $\sigma_{k+1}(G)$ with $\sigma_t(G)$ with $t \geqslant k+2$, and the coefficient 1 on $|G|$ in the $\sigma_{k+1}(G)$ condition with a constant smaller than 1.

At first, we show that in the case $k \geqslant p - 1$. Let k, l, m and p be integers such that $l \geqslant 1$, $m \geqslant 2$, $p \geqslant 3$ and $k = (p-2)m - 1$. For $1 \leqslant i \leqslant m$, let H_i be a graph consisting of

Table 1: Sufficient conditions on $\sigma_{k+1}(G)$ for a connected $K_{1,p}$-free graph G to have a spanning k-ended tree. The values with "?" are conjectured as in Conjecture 4.

$k \setminus p$	3	4	5	6
2	$\lvert G\rvert - k$ (Kano et al. [5], Salamon [12])	$\lvert G\rvert$ (Kyaw [7])	$\lvert G\rvert$ (this paper)	$\lvert G\rvert$?
3		$\lvert G\rvert - 1$ (Kyaw [6])		
4		$\lvert G\rvert - \frac{k}{2}$ (Kyaw [7])	$\lvert G\rvert - \frac{k}{3}$ (Chen et al. [3])	
5			$\lvert G\rvert - \frac{k}{3}$ (this paper)	$\lvert G\rvert - \frac{k}{4}$?
6			$\lvert G\rvert - \frac{k}{3}$ (Sun and Liu [13])	
$\geqslant 7$			$\lvert G\rvert - \frac{k}{3}$ (this paper)	

$p-2$ complete graphs of order $l+1$ that share only one vertex, say u_i. Let G be a graph such that $V(G) = \bigcup_{1 \leqslant i \leqslant m} V(H_i)$ and $E(G) = (\bigcup_{1 \leqslant i \leqslant m} E(H_i)) \cup \{u_i u_j : 1 \leqslant i < j \leqslant m\}$. Then G is a connected $K_{1,p}$-free graph which satisfies $\sigma_{k+1}(G) = l(k+1) = |G| - \frac{k+1}{p-2}$, and $\sigma_{k+2}(G) = \infty$, but G does not have a spanning k-ended tree. Next, we show the case $k \leqslant p-2$. Let k, l and p be integers such that $k, l \geqslant 1$ and $p \geqslant 3$, and let G be a graph consisting of $k+1$ complete graphs of order $l+1$ that share only one vertex. Then G is a connected $K_{1,p}$-free graph which satisfies $\sigma_{k+1}(G) = l(k+1) = |G| - 1$ and $\sigma_{k+2}(G) = \infty$, but G does not have a spanning k-ended tree.

By the examples above, the coefficient 1 at $|G|$ in the $\sigma_{k+1}(G)$ condition of Theorem 3 is best possible. However, the constant may be able to be reduced. Considering the above examples again, we pose the following conjecture.

Conjecture 4. Let k and p be integers with $k \geqslant 2$ and $p \geqslant 3$. Then there exists an integer $N_0 = N_0(k,p)$ such that for a connected $K_{1,p}$-free graph G with $|G| \geqslant N_0$, if G satisfies the following conditions, then G has a spanning k-ended tree.

$$\sigma_{k+1}(G) \geqslant \begin{cases} |G| & \text{if } k \leqslant p-2, \\ |G| - \dfrac{k}{p-2} & \text{if } k \geqslant p-1. \end{cases}$$

For the convenience, we display known results in Theorem 2 and Conjecture 4 in Table 1.

There are some small connected $K_{1,p}$-free graphs G satisfying the $\sigma_{k+1}(G)$ condition in Conjecture 4 but having no spanning k-ended tree. For example, consider $k = 2$, $p = 5$ and $G = K_{2,4}$, which is a connected $K_{1,5}$-free graph satisfying $\sigma_3(G) = 6 = |G|$, but having no spanning 2-ended tree (i.e. Hamiltonian path). This is the reason why we need the assumption on N_0 in Conjecture 4.

For the Hamiltonicity, Markus [9] investigated a minimum degree condition for the existence of a Hamiltonian cycle in a $K_{1,p}$-free graph for $p \geqslant 4$, which is an extension of the second half of Theorem 1.

Theorem 5 (Markus)**.** *Let $p \geqslant 4$ be an integer, and let G be a 2-connected $K_{1,p}$-free graph. If G satisfies one of the following conditions, then G has a Hamiltonian cycle.*

- $p = 4$ *and* $\delta(G) \geqslant \frac{|G|+2}{3}$,
- $p \geqslant 5$, $|G| \geqslant 2p-2$ *and* $\delta(G) \geqslant \frac{|G|+p-3}{3}$.

Therefore, following Theorem 5 one may think that it is easy to generalize Theorems 1 and 2 to $K_{1,p}$-free graphs. However, we point out that the proof strategy of Theorem 5 may not be used to our purpose. The key idea of the proof in [9] is to use the result by Nash-Williams [11] that for every 2-connected graph G if $\delta(G) \geqslant \frac{|G|+2}{3}$ and $\delta(G) \geqslant \alpha(G)$, then G has a Hamiltonian cycle. Under the assumption that G is $K_{1,p}$-free, $|G| \geqslant 2p-2$ and $\delta(G) \geqslant \frac{|G|+p-3}{3}$, we can show that $\delta(G) \geqslant \alpha(G)$, and hence Nash-Williams' theorem immediately implies the existence of a Hamiltonian cycle.

In order to use this strategy to prove Theorem 3, we need an analog of Nash-Williams' theorem for a spanning k-ended tree. In particular, if we would have the following statement, then we may use the same strategy: "for every connected graph G, if $\sigma_{k+1}(G) \geqslant |G| + A(k)$ and $\delta(G) \geqslant \alpha(G) + B(k)$ for some constants $A(k)$ and $B(k)$, then G has a spanning k-ended tree". However, we do not know whether such a statement holds.

Therefore, we employ a different proof strategy in this paper, using so-called a *system* of a graph. In Section 2, we give its definition together with basic properties, and in Section 3, we prove Theorem 3.

Before proceeding to the next section, we give some basic terminology used in this paper. Throughout this paper, we regard a connected subgraph with one or two vertices also as a (reduced) cycle. Thus, when we say a cycle, it means either a connected 2-regular subgraph in an ordinary sense, or a connected subgraph isomorphic to K_1 or K_2.

To each cycle and each path, we give an arbitrary orientation. (For K_2, we give both directions to the unique edge.) Let P be a path. For each pair of vertices x, y in P, we denote by $P[x, y]$ the subpath from x to y. For a vertex x in P, we denote the successor and the predecessor of x on P by x^+ and x^-, respectively. Similarly, for a pair of vertices x, y in a cycle C, we define $C[x, y]$ as the subpath from x to y along C, and x^+ and x^- are the successor and the predecessor of x on C, respectively. Note that when a cycle C is isomorphic to K_2, then both x^+ and x^- are the vertex in C other than x; when C is isomorphic to K_1, then $x^+ = x^- = x$. In either case, both $C[x^+, x]$ and $C[x, x^-]$ are Hamiltonian paths of C.

For two paths P and Q in a graph G, if the terminal vertex of P is adjacent to the initial vertex of Q, then we let PQ be the path obtained by the concatenation of P and Q.

2 A minimal system

In our proof, a system of a graph, which was defined by Win [14], plays a central role. Thus, we give the definitions of a system and show their properties in this section.

A *system* of a graph G is a set of pairwise vertex-disjoint paths and cycles such that their union contains all vertices in G. We often view a system as the subgraph formed by the union of its members. Let $\mathcal{S}$ be a system of a graph G. We denote by $|\mathcal{S}|$ the number of members in $\mathcal{S}$. We denote $\mathrm{P}ath(\mathcal{S}) = \{S \in \mathcal{S} : S \text{ is a path of length at least } 2\}$ and $\mathrm{C}yc(\mathcal{S}) = \{S \in \mathcal{S} : S \text{ is a cycle}\}$. Recall that we regard a path of length at most 1 as a cycle. Thus, $\mathrm{P}ath(\mathcal{S})$ and $\mathrm{C}yc(\mathcal{S})$ form a partition of $\mathcal{S}$. For $S \in \mathcal{S}$, let

$$f(S) = \begin{cases} 2 & \text{if } S \in \mathrm{P}ath(\mathcal{S}), \\ 1 & \text{if } S \in \mathrm{C}yc(\mathcal{S}). \end{cases}$$

We give each path $P \in \mathrm{P}ath(\mathcal{S})$ an orientation, and then let a_P and b_P be the initial and terminal vertex of P, respectively. We define $V(\mathcal{S}) = \bigcup_{S\in\mathcal{S}} V(S)$ and $f(\mathcal{S}) = \sum_{S\in\mathcal{S}} f(S)$. For each $S \in \mathcal{S}$, define

$$\mathrm{E}nd(S) = \begin{cases} \{a_P, b_P\} & \text{if } S \in \mathrm{P}ath(\mathcal{S}), \\ V(S) & \text{if } S \in \mathrm{C}yc(\mathcal{S}). \end{cases}$$

In [14], Win gave the following lemma which shows that a system is useful in the study of spanning k-ended trees.

Lemma 6 (Win [14])**.** *Let $k \geqslant 2$ be an integer, and G be a connected graph. If G has a system $\mathcal{S}$ such that $f(\mathcal{S}) \leqslant k$, then G has a spanning k-ended tree.*

By Lemma 6, to prove Theorem 3, it suffices to find a system $\mathcal{S}$ with $f(\mathcal{S}) \leqslant k$, instead of a spanning k-ended tree.

Let k be an integer with $k \geqslant 2$, and let G be a connected graph. We call a system $\mathcal{S}$ of G which satisfies the following conditions (S1) and (S2) a *minimal system*.

(S1) $f(\mathcal{S})$ is as small as possible, and

(S2) $|\mathrm{P}ath(\mathcal{S})|$ is as large as possible, subject to (S1).

In this section, we fix a minimal system $\mathcal{S}$ of a graph G, and show its properties. The following lemma is a direct consequence of the conditions (S1) and (S2), but useful for the later arguments.

Lemma 7. *For any $\mathcal{T} \subseteq \mathcal{S}$, there exists no system $\mathcal{T}'$ of G with $V(\mathcal{T}') = V(\mathcal{T})$ such that it satisfies one of the following conditions.*

(i) *$f(\mathcal{T}') < f(\mathcal{T})$, or*

(ii) *$f(\mathcal{T}') = f(\mathcal{T})$ and $|\mathrm{P}ath(\mathcal{T}')| > |\mathrm{P}ath(\mathcal{T})|$.*

We use the following basic properties on a system. Since the proof is not difficult and can be found in the textbook [1, Lemma 8.24], we omit it.

Lemma 8. *The following hold.*

(i) *For any $S_1, S_2 \in \mathcal{S}$ with $S_1 \neq S_2$, no vertex in $\mathrm{E}nd(S_1)$ is adjacent with a vertex in $\mathrm{E}nd(S_2)$.*

(ii) *For any $P \in \mathrm{P}ath(\mathcal{S})$, the vertices a_P and b_P are not adjacent.*

For any $C \in \mathrm{C}yc(\mathcal{S})$, a vertex v in C is said to be *insertible with respect to* $\mathcal{S}$, or simply *insertible* if v satisfies the following condition:

There exists $P \in \mathrm{P}ath(\mathcal{S})$ and $x \in V(P) \backslash \{b_P\}$ such that $x, x^+ \in N_G(v)$.

Lemma 9. *Any $C \in \mathrm{C}yc(\mathcal{S})$ contains a non-insertible vertex.*

Proof. Suppose that there exists an element $C \in \mathrm{C}yc(\mathcal{S})$ such that all vertices in C are insertible. Let $C = v_1 v_2 \dots v_t v_1$, where $t = |V(C)|$. Since v_1 is insertible, there exist $P \in \mathrm{P}ath(\mathcal{S})$ and $x \in V(P)$ such that $x, x^+ \in N_G(v_1)$. Let i_x be the maximum index with $1 \leqslant i_x \leqslant t$ such that $x, x^+ \in N_G(v_{i_x})$. Such an index i_x exists. Then we can insert the subpath $C[v_1, v_{i_x}]$ into P by replacing the edge xx^+ with the path $xC[v_1, v_{i_x}]x^+$. If $i_x = t$, then by replacing C, P in $\mathcal{S}$ with the above new path, we obtain a system whose value on f is smaller by one than $f(\mathcal{S})$, contradicting the condition (S1).

Thus, we have $i_x < t$, but even in this case, we can continue the similar procedure: Since v_{i_x+1} is insertible with respect to $\mathcal{S}$, there exist $P' \in \mathrm{P}ath(\mathcal{S})$ and $x' \in V(P')$ such that $x', x'^+ \in N_G(v_{i_x+1})$. By the maximality of i_x, we have $x' \neq x$. Then we let $i_{x'}$ be the maximum index with $i_x + 1 \leqslant i_{x'} \leqslant t$ such that $x', x'^+ \in N_G(v_{i_{x'}})$, and we can insert the subpath $C[v_{i_x+1}, v_{i_{x'}}]$ into P' by replacing the edge $x'x'^+$. Since all vertices in C are insertible, we can keep this procedure until all vertices in C are inserted. However, this contradicts the condition (S1). $\square$

Let d_C be a non-insertible vertex in C for each $C \in \mathrm{C}yc(\mathcal{S})$, which exists by Lemma 9, and let

$$LM(\mathcal{S}) = \bigcup_{P \in \mathrm{P}ath(\mathcal{S})} \{a_P, b_P\} \cup \bigcup_{C \in \mathrm{C}yc(\mathcal{S})} \{d_C\}.$$

By Lemma 8, we can obtain the following lemma.

Lemma 10. *$LM(\mathcal{S})$ is an independent set.*

For a minimal system $\mathcal{S}$ of G, let

$$X^{\mathcal{S}} = \{x \in V(G) : x \in N_G(d_1) \cap N_G(d_2) \text{ for some } d_1, d_2 \in LM(\mathcal{S}) \text{ with } d_1 \neq d_2\}.$$

If there is no confusion, we abbreviate $X^{\mathcal{S}}$ as X. By Lemma 8 (i), we can also obtain the following lemma.

Lemma 11. $X \subseteq V(\mathrm{P}ath(\mathcal{S}))$.

For each $x \in X$, let $P_x \in \mathrm{P}ath(\mathcal{S})$ be the path such that $x \in V(P_x)$.

Lemma 12. *For any $x \in X$ and $z \in LM(\mathcal{S})$, the following statements hold.*

(i) *There exists no minimal system $\mathcal{S}'$ of G such that $LM(\mathcal{S}') = (LM(\mathcal{S}) \setminus \{z\}) \cup \{x\}$.*

(ii) $x^- \notin N_G(b_{P_x})$ *and* $x^+ \notin N_G(a_{P_x})$.

Proof. Suppose that there exists a minimal system $\mathcal{S}'$ of G such that $LM(\mathcal{S}') = (LM(\mathcal{S}) \setminus \{z\}) \cup \{x\}$. By Lemma 10, $LM(\mathcal{S}')$ is an independent set. Thus, $x \notin N_G(z')$ for any $z' \in LM(\mathcal{S}) \setminus \{z\}$. However, this contradicts the definition of X. Hence statement (i) holds.

Suppose that $x^- \in N_G(b_{P_x})$. Let $P' = P_x[a_{P_x}, x^-]P_x[b_{P_x}, x]$ and $\mathcal{S}' = (\mathcal{S} \setminus \{P_x\}) \cup \{P'\}$. Then $\mathcal{S}'$ is a minimal system such that $LM(\mathcal{S}') = (LM(\mathcal{S}) \setminus \{b_{P_x}\}) \cup \{x\}$, which contradicts statement (i). Hence $x^- \notin N_G(b_{P_x})$. In the same way, we can also see $x^+ \notin N_G(a_{P_x})$. $\square$

Lemma 13. $x^-x^+ \notin E(G)$ *holds for any* $x \in X$.

Proof. Suppose that there exists $x \in X$ such that $x^-x^+ \in E(G)$. Then there exist $d_1, d_2 \in LM(\mathcal{S})$ with $d_1 \neq d_2$ such that $x \in N_G(d_1) \cap N_G(d_2)$. Let $Q_1, Q_2 \in \mathcal{S}$ be such that $d_1 \in V(Q_1) \cap LM(\mathcal{S})$ and $d_2 \in V(Q_2) \cap LM(\mathcal{S})$. Suppose $d_1 = b_{P_x}$. Let $P' = P_x[a_{P_x}, x^-]P_x[x^+, b_{P_x}]x$ and $\mathcal{S}' = (\mathcal{S} \setminus \{P_x\}) \cup \{P'\}$. Then $\mathcal{S}'$ is a minimal system such that $LM(\mathcal{S}') = (LM(\mathcal{S}) \setminus \{b_{P_x}\}) \cup \{x\}$, which contradicts Lemma 12 (i). Thus, we have $d_1 \neq b_{P_x}$. By symmetry, we further obtain $\{d_1, d_2\} \cap \{a_{P_x}, b_{P_x}\} = \emptyset$. Hence $Q_1, Q_2 \neq P_x$. Let $P'' = P_x[a_{P_x}, x^-]P_x[x^+, b_{P_x}]$. Suppose that $Q_1 \in \mathrm{P}ath(\mathcal{S})$. We may assume that $b_{Q_1} = d_1$. Let $Q_1' = Q_1[a_{Q_1}, b_{Q_1}]x$ and $\mathcal{S}'' = (\mathcal{S} \setminus \{P_x, Q_1\}) \cup \{P'', Q_1'\}$. Then $\mathcal{S}''$ is a minimal system such that $LM(\mathcal{S}'') = (LM(\mathcal{S}) \setminus \{b_{Q_1}\}) \cup \{x\}$, which contradicts Lemma 12 (i). Hence, by symmetry, we may assume that $Q_1, Q_2 \in \mathrm{C}yc(\mathcal{S})$. Then, for the path $R = Q_1[d_1^+, d_1]xQ_2[d_2, d_2^-]$, we have $V(P'') \cup V(R) = V(P_x) \cup V(Q_1) \cup V(Q_2)$, $f(\{P'', R\}) = 4 = f(\{P_x, Q_1, Q_2\})$ and $|\mathrm{P}ath(\{P'', R\})| = 2 > 1 = |\mathrm{P}ath(\{P_x, Q_1, Q_2\})|$, which contradicts Lemma 7 (ii). $\square$

Lemma 14. *For any* $x \in X$, $x^+ \notin N_G(z)$ *holds for any* $z \in LM(\mathcal{S}) \setminus \{b_{P_x}\}$ *and* $x^- \notin N_G(z)$ *holds for any* $z \in LM(\mathcal{S}) \setminus \{a_{P_x}\}$. *In particular, no two vertices in* X *are consecutive on any path in* $\mathrm{P}ath(\mathcal{S})$.

Proof. Let $x \in X$ and suppose that $x^+ \in N_G(z)$ for some $z \in LM(\mathcal{S}) \setminus \{b_{P_x}\}$. By Lemma 12 (ii), we have $z \neq a_{P_x}$. Thus, there exists $Q_z \in \mathcal{S}$ with $Q_z \neq P_x$ and $z \in V(Q_z)$. We obtain a contradiction considering the following two cases:

- Suppose that $Q_z \in \mathrm{P}ath(\mathcal{S})$. By the symmetry, we may assume that $z = b_{Q_z}$. Let $P' = P_x[a_{P_x}, x]$, $Q' = Q_z P_x[x^+, b_{P_x}]$ and $\mathcal{S}' = (\mathcal{S} \setminus \{P_x, Q_z\}) \cup \{P', Q'\}$. Then $\mathcal{S}'$ is a minimal system such that $LM(\mathcal{S}') = (LM(\mathcal{S}) \setminus \{b_{Q_z}\}) \cup \{x\}$, which contradicts Lemma 12 (i).

- Suppose that $Q_z \in \mathrm{C}yc(\mathcal{S})$. Since $x \in X$, there exists $z' \in LM(\mathcal{S})$ with $z' \neq a_{P_x}$ and $x \in N_G(z')$. Recall that z is not insertible by the choice of $LM(\mathcal{S})$. Thus, we have $z' \neq z$. If $z' = b_{P_x}$, then $P' = P[a_{P_x}, x]P[b_{P_x}, x^+]Q_z[z, z^-]$ is a path with $V(P') = V(P_x) \cup V(Q_z)$ and $f(\{P'\}) = 2 < 3 = f(\{P_x, Q_z\})$, contradicting Lemma 7 (i). Thus, $z' \neq b_{P_x}$, and there exists $R_{z'} \in \mathcal{S}$ with $R_{z'} \neq P_x$ and $z' \in V(R_{z'})$. Since $z' \neq z$ and Q_z is a cycle, we see $R_{z'} \neq Q_z$. Suppose that $R_{z'} \in \mathrm{P}ath(\mathcal{S})$, say $z' = a_{R_{z'}}$ by symmetry. Then for the two paths $P' = P_x[a_{P_x}, x]R_{z'}$

and $Q' = Q_z[z^+, z]P_x[x^+, b_{P_x}]$, we have $f(\{P', Q'\}) = 4 < 5 = f(\{P, Q_z, R_{z'}\})$, which contradicts Lemma 7 (i). Hence $R_{z'} \in \mathrm{C}yc(\mathcal{S})$. Then for the two paths $P' = P_x[a_{P_x}, x]R_{z'}[z', z'^-]$ and $Q' = Q_z[z^+, z]P_x[x^+, b_{P_x}]$, we have $f(\{P', Q'\}) = 4 = f(\{P, Q_z, R_{z'}\})$, and $|\mathrm{P}ath(\{P', Q'\})| = 2 > 1 = |\mathrm{P}ath(\{P, Q_z, R_{z'}\})|$, which contradicts Lemma 7 (ii).

This completes the first part of the proof of Lemma 14.

Suppose that there are two consecutive vertices $x, y \in X$ on some $P \in \mathrm{P}ath(\mathcal{S})$. By symmetry, we may assume that $y = x^+$. Since $y \in X$, there exists $z \in LM(\mathcal{S}) \setminus \{b_P\}$ such that $y \in N_G(z)$, but this is a contradiction. □

3 Proof of Theorem 3

In order to prove Theorem 3, we need the following theorem.

Theorem 15 (Win [14]). *Let k and n be integers with $n \geqslant 1$ and $k \geqslant 2$, and let G be an n-connected graph with $\alpha(G) \leqslant n + k - 1$. Then G has a spanning k-ended tree.*

Proof of Theorem 3

Let $f(k, p) = (k+1)(p-1)\min\{p-2, k\}$, and let G be a connected $K_{1,p}$-free graph which satisfies the assumptions of Theorem 3. Suppose that G does not have a spanning k-ended tree. By Theorem 15, we have $\alpha(G) \geqslant k + 1$. We first prove that G does not have a large independent set.

Claim 16. $\alpha(G) < (k+1)(p-1)$.

Proof. Let S be a maximum independent set of G, i.e. $|S| = \alpha(G) \geqslant k + 1$. We count $e_G(S, G \setminus S)$, which is the number of edges between S and $G \setminus S$, in two ways. First, by the $\sigma_{k+1}(G)$ condition for every $(k+1)$-element subset of S, we have

$$\binom{|S|}{k+1}\sigma_{k+1}(G) \leqslant \binom{|S|-1}{k}e_G(S, G \setminus S),$$

or
$$e_G(S, G \setminus S) \geqslant \frac{\sigma_{k+1}(G)}{k+1}|S| \geqslant \frac{|G| + f(k,p)}{k+1}\alpha(G) > \frac{|G|}{k+1}\alpha(G).$$

On the other hand, since G is $K_{1,p}$-free,

$$e_G(G \setminus S, S) \leqslant (p-1)|G \setminus S| = (p-1)(|G| - \alpha(G)) < (p-1)|G|.$$

Hence we obtain $\alpha(G) < (k+1)(p-1)$. □

We use the same terminology as in Section 2. Let $\mathcal{S}$ be a minimal system of G. Recall that

$$X = \{x \in V(G) : x \in N_G(d_1) \cap N_G(d_2) \text{ for some } d_1, d_2 \in LM(\mathcal{S}) \text{ with } d_1 \neq d_2\}.$$

Let

$$Y = \Big\{ x \in X : \text{there is a vertex } u \text{ in } V\big(P_x[x^+, b_{P_x}^-]\big) \text{ with } u \notin N_G(b_{P_x}) \Big\} \text{ and } \overline{Y} = X \setminus Y.$$

By the definition, any $x \in \overline{Y}$ satisfies $V\big(P_x[x^+, b_{P_x}]\big) \subseteq N_G(b_{P_x}) \cup \{b_{P_x}\}$, see Figure 1.

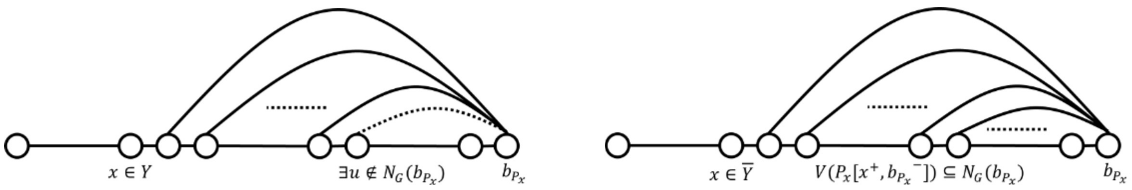

Figure 1: The definition of Y and $\overline{Y}$.

Claim 17. *For any $P \in \mathrm{P}ath(\mathcal{S})$, $|V(P) \cap \overline{Y}| \leqslant 1$. In particular, $|\overline{Y}| \leqslant \frac{1}{2}|LM(\mathcal{S})|$.*

Proof. Suppose that $|V(P) \cap \overline{Y}| \geqslant 2$ holds for some $P \in \mathrm{P}ath(\mathcal{S})$. Then let x, y be two distinct vertices in $V(P) \cap \overline{Y}$ that appear in P in that order. By Lemma 14, we have $y \neq x^+$. This implies that $y^- \in V\big(P[x^+, b_P]\big)$. Since $x \in \overline{Y}$, we have $y^- \in N_G(b_P)$, which contradicts Lemma 12 (ii). Therefore, $|V(P) \cap \overline{Y}| \leqslant 1$ for any $P \in \mathrm{P}ath(\mathcal{S})$. This implies $|\overline{Y}| \leqslant |\mathrm{P}ath(\mathcal{S})| \leqslant \frac{1}{2}|LM(\mathcal{S})|$. □

For each $x \in Y$, let u_x be the first vertex in $P_x[x^+, b_{P_x}^-]$ with $u_x \notin N_G(b_{P_x})$. Let $Y' = \{u_x : x \in Y\}$.

Claim 18. *$u_x \neq u_y$ for any $x, y \in Y$ with $x \neq y$. In particular, $|Y| = |Y'|$.*

Proof. Suppose that $u_x = u_y$ for some $x, y \in Y$ with $x \neq y$. We may assume that x and y are arranged in this order along P_x. By Lemma 14, we have $y \neq x^+$. By the definition of u_x, we have $y^- \in V(P_x[x^+, u_x^-]) \subseteq N_G(b_{P_x})$. However, this contradicts Lemma 12 (ii). □

Claim 19. *For any $x \in Y$ and $z \in LM(\mathcal{S})$, $u_x \notin N_G(z)$.*

Proof. Suppose that there exists a vertex $x \in Y$ such that $u_x \in N_G(z)$ for some $z \in LM(\mathcal{S})$. By the definition of u_x, we have $z \neq b_{P_x}$. By Lemma 14, we have $u_x \neq x^+$. Since u_x is the first vertex in $P_x[x^+, b_{P_x}^-]$ with $u_x \notin N_G(b_{P_x})$, we have $u_x^- \in N_G(b_{P_x})$. If $z = a_{P_x}$, then $C' = P_x[a_{P_x}, u_x^-]P_x[b_{P_x}, u_x]a_{P_x}$ is a cycle with $V(C') = V(P_x)$, and hence $f(\{C'\}) = 1 < f(\{P_x\})$, which contradicts Lemma 7(i). Otherwise, the path $P_x[a_{P_x}, u_x^-]P_x[b_{P_x}, u_x]$ can be extended by connecting Q_z through the edge $u_x z$, where $Q_z \in \mathcal{S}$ with $z \in V(Q_z)$. Let P' be the obtained path. Then $f(\{P'\}) = 2 < f(\{P_x, Q_z\})$, which contradicts Lemma 7(i). In either case, we obtain a contradiciton, and this completes the proof of Claim 19. □

The following claim is directly obtained from the definition of Y'.

Claim 20. *$Y' \cap LM(\mathcal{S}) = \emptyset$.*

For $x \in Y$, we say that x is *exceptional* if x has exactly two neighbors in $LM(\mathcal{S})$ and satisfies one of the following properties:

(i) $x \in N_G(a_{P_x}) \cap N_G(b_{P_x})$, and $x \notin N_G(z)$ for any $z \in LM(\mathcal{S})\backslash\{a_{P_x}, b_{P_x}\}$,

(ii) $x \in N_G(a_{P_x}) \cap N_G(d_C)$ for some $C \in \mathrm{Cyc}(\mathcal{S})$, and $x \notin N_G(z)$ for any $z \in LM(\mathcal{S})\backslash\{a_{P_x}, d_C\}$,

(iii) $x \in N_G(b_{P_x}) \cap N_G(d_C)$ for some $C \in \mathrm{Cyc}(\mathcal{S})$, and $x \notin N_G(z)$ for any $z \in LM(\mathcal{S})\backslash\{b_{P_x}, d_C\}$.

An exceptional vertex $x \in Y$ is said to be *of Type* (i), (ii) and (iii), if x satisfies (i), (ii) and (iii), respectively. Let

$$\begin{aligned} Y_{\text{ex}} &= \{x \in Y : x \text{ is exceptional}\},\ Y_{\text{non-ex}} = Y\backslash Y_{\text{ex}}, \\ Y'_{\text{ex}} &= \{u_x : x \in Y_{\text{ex}}\} \text{ and } Y'_{\text{non-ex}} = \{u_x : x \in Y_{\text{non-ex}}\}. \end{aligned}$$

By Claim 18, we obtain the following claim.

Claim 21. $|Y'_{\text{ex}}| = |Y_{\text{ex}}|$ *and* $|Y'_{\text{non-ex}}| = |Y_{\text{non-ex}}|$.

Claim 22. $Y'_{\text{non-ex}} \cup LM(\mathcal{S})$ *is an independent set.*

Proof. Suppose not. By Lemma 10 and Claim 19, there exist $u, v \in Y'_{\text{non-ex}}$ with $u \neq v$ and $uv \in E(G)$. Choose $x, y \in Y_{\text{non-ex}}$ so that $u_x = u$ and $u_y = v$. We divide the proof into two cases according as whether $P_x = P_y$ or not.

Case 1. $P_x \neq P_y$.

Recall that either $u_x^- \in N_G(b_{P_x})$ or $u_x^- = x$ by the definition of u_x, and the same holds for u_y. We first claim that $u_x^- = x$ and $u_y^- = y$. In fact, the other cases lead to a contradiction as follows.

- Suppose that $u_x^- \in N_G(b_{P_x})$ and $u_y^- \in N_G(b_{P_y})$. Let

  $$P' = P_x[a_{P_x}, u_x^-]P_x[b_{P_x}, u_x]P_y[u_y, b_{P_y}]P_y[u_y^-, a_{P_y}].$$

  Then $V(P') = V(\{P_x, P_y\})$ and $f(\{P'\}) = 2 < 4 = f(\{P_x, P_y\})$, which contradicts Lemma 7 (i).

- Suppose that $u_x^- = x$ and $u_y^- \in N_G(b_{P_y})$. Let

  $$P' = P_x[a_{P_x}, x], \quad Q' = P_y[a_{P_y}, u_y^-]P_y[b_{P_y}, u_y]P_x[u_x, b_{P_x}]$$

  and $\mathcal{S}' = (\mathcal{S}\backslash\{P_x, P_y\}) \cup \{P', Q'\}$. Then $\mathcal{S}'$ is a minimal system such that $LM(\mathcal{S}') = (LM(\mathcal{S})\backslash\{b_{P_y}\}) \cup \{x\}$, which contradicts Lemma 12 (i).

- The case that $u_x^- \in N_G(b_{P_x})$ and $u_y^- = y$ is symmetric to the previous case.

Therefore, we have $u_x^- = x$ and $u_y^- = y$, as claimed.

Since $x \in X$, we have $x \in N_G(z)$ for some $z \in LM(\mathcal{S})\backslash\{a_{P_x}\}$. We divide the proof as follows:

- Suppose that $z = b_{P_x}$. Let $P' = P_x[a_{P_x}, x]P_x[b_{P_x}, u_x]P_y[u_y, b_{P_y}]$, $Q' = P_y[a_{P_y}, y]$ and $\mathcal{S}' = (\mathcal{S}\backslash\{P_x, P_y\}) \cup \{P', Q'\}$. Then $\mathcal{S}'$ is a minimal system such that $LM(\mathcal{S}') = (LM(\mathcal{S})\backslash\{b_{P_x}\}) \cup \{y\}$, which contradicts Lemma 12 (i).

- Suppose that $z = a_{P_y}$. Let $P' = P_x[a_{P_x}, x]P_y[a_{P_y}, y]$, $Q' = P_x[b_{P_x}, u_x]P_y[u_y, b_{P_y}]$ and $\mathcal{S}' = (\mathcal{S}\backslash\{P_x, P_y\}) \cup \{P', Q'\}$. Then $\mathcal{S}'$ is a minimal system such that $LM(\mathcal{S}') = (LM(\mathcal{S})\backslash\{a_{P_y}\}) \cup \{y\}$, which contradicts Lemma 12 (i).

- Suppose that $z = b_{P_y}$. Let $P' = P_x[a_{P_x}, x]P_y[b_{P_y}, u_y]P_x[u_x, b_{P_x}]$, $Q' = P_y[a_{P_y}, y]$ and $\mathcal{S}' = (\mathcal{S}\backslash\{P_x, P_y\}) \cup \{P', Q'\}$. Then $\mathcal{S}'$ is a minimal system such that $LM(\mathcal{S}') = (LM(\mathcal{S})\backslash\{b_{P_y}\}) \cup \{y\}$, which contradicts Lemma 12 (i).

- Suppose that $z = a_R$ or b_R for some $R \in \mathrm{P}ath(\mathcal{S})\backslash\{P_x, P_y\}$. By symmetry, we may assume that $z = a_R$. Let $R_1 = P_x[a_{P_x}, x]R[a_R, b_R]$, $R_2 = P_y[a_{P_y}, y]$, $R_3 = P_x[b_{P_x}, u_x]P_y[u_y, b_{P_y}]$ and $\mathcal{S}' = (\mathcal{S}\backslash\{P_x, P_y, R\}) \cup \{R_1, R_2, R_3\}$. Then $\mathcal{S}'$ is a minimal system such that $LM(\mathcal{S}') = (LM(\mathcal{S})\backslash\{a_R\}) \cup \{y\}$, which contradicts Lemma 12 (i).

Therefore, we have $z \in V(C)$ for some $C \in \mathrm{C}yc(\mathcal{S})$. By the symmetry of x and y, we have $y \in N_G(z')$ for some $z' \in V(C') \cap LM(\mathcal{S})$ with $C' \in \mathrm{C}yc(\mathcal{S})$. Since $x \notin Y_{\mathrm{ex}}$, there are two choices for such a cycle C, and hence we can choose C and C' with $C \neq C'$. Then there exist three paths S_1, S_2, S_3 such that $S_1 = P_x[a_{P_x}, x]C[z, z^-]$, $S_2 = P_y[a_{P_y}, y]C'[z', z'^-]$ and $S_3 = P_x[b_{P_x}, u_x]P_y[u_y, b_{P_y}]$. Note that $f(\{S_1, S_2, S_3\}) = 6 = f(\{P_x, P_y, C, C'\})$, and $|\mathrm{P}ath(\{S_1, S_2, S_3\})| = 3 > 2 = |\mathrm{P}ath(\{P_x, P_y, C_1, C_2\})|$, which contradicts Lemma 7 (ii). This complets the proof of Case 1.

Case 2. $P_x = P_y$.

For convenience, let $P = P_x = P_y$. We may assume that x and y are arranged in this order along P. For the proof, we first claim the following.

> ($*$) There exists a subgraph Q with $V(Q) = V(P[x^+, b_P])$ such that either Q is a cycle or a path connecting y and b_P.

Recall that either $x^+ = u_x$ or $x^+ \in N_G(b_P)$ by the definition of u_x. Similarly, either $u_y^- \in N_G(b_P)$ or $u_y^- = y$. The statement ($*$) can be shown by considering the following cases:

- Suppose $x^+ \in N_G(b_P)$. In this case, $Q = P[x^+, b_P]x^+$ is a cycle desired in ($*$).

- Suppose $x^+ = u_x$ and $u_y^- \in N_G(b_P)$. Then, $Q = P[x^+, u_y^-]P[b_P, u_y]x^+$ is a cycle desired in ($*$).

- Suppose $x^+ = u_x$ and $y^+ = u_y$. Then, $Q = P[y, x^+]P[u_y, b_P]$ is a path connecting y and b_P, as desired in ($*$).

This shows the statement ($*$), as claimed.

Since $x \notin Y_{\mathrm{ex}}$, there exists $z_1 \in LM(\mathcal{S})$ with $x \in N_G(z_1)$ and $z_1 \notin V(P)$. Suppose first that $z_1 = a_{P_1}$ or b_{P_1} for some $P_1 \in \mathrm{P}ath(\mathcal{S})\backslash\{P\}$. We may assume that $z_1 = a_{P_1}$. Let

$P' = P[a_P, x]P_1[a_{P_1}, b_{P_1}]$ and let $\mathcal{S}' = (\mathcal{S}\backslash\{P, P_1\}) \cup \{P', Q\}$. If Q is a cycle, then $f(\mathcal{S}') = f(\mathcal{S}) - 1$, contradicting the condition (S1). On the other hand, if Q is a path connecting y and b_P, then $\mathcal{S}'$ is a minimal system such that $LM(\mathcal{S}') = (LM(\mathcal{S})\backslash\{a_{P_1}\}) \cup \{y\}$, which contradicts Lemma 12 (i).

Thus, we have $z_1 \in V(C)$ for some $C \in \mathrm{C}yc(\mathcal{S})$. Note that $P' = P[a_P, x]C[z_1, z_1^-]$ is a path such that one of its end vertices is a_P. Since $y \notin Y_{\mathrm{ex}}$, y is not exceptional of type (iii), which means that there exists $z_2 \in LM(\mathcal{S})$ with $y \in N_G(z_2)$ and $z_2 \neq b_P, z_1$. We divide the remaining proof according as $z_2 = a_P$ or not.

- Suppose $z_2 = a_P$. Then by concatenating P' and Q through the edge $a_P y$, and deleting an edge incident with y in the case when Q is a cycle, we obtain a path, say R, with $V(R) = V(P) \cup V(C)$. Then $f(\{R\}) = 2 < 3 = f(\{P, C\})$, which contradicts Lemma 7 (i).

- Suppose $z_2 \neq a_P$. Let $R \in \mathcal{S}$ with $z_2 \in V(R)$. Then by connecting Q and R through the edge yz_2, and deleting an edge incident with y and/or an edge incident with z_2 if necessary, we obtain a path, say Q', with $V(Q') = V(Q) \cup V(R)$. Then $f(\{P', Q'\}) = 4 \leqslant f(\{P, C, R\})$. Thus, if $f(R) = 2$, then this contradicts Lemma 7 (i). Otherwise, that is, if $f(R) = 1$, then $f(\{P', Q'\}) = 4 = f(\{P, C, R\})$ and $|\mathrm{P}ath(\{P', Q'\})| = 2 > 1 = |\mathrm{P}ath(\{P, C, R\})|$, contradicting Lemma 7 (ii).

This completes the proof of Claim 22. $\square$

Claim 23. $|X| - |Y_{\mathrm{ex}}| > (k + 1)(p - 1)$.

Proof. Let $I \subseteq LM(\mathcal{S})$ with $|I| = k + 1$. Since G is $K_{1,p}$-free and I is an independent set of order $k + 1$, we have $|\{z \in I : x \in N_G(z)\}| \leqslant \min\{p - 1, k + 1\}$ for each $x \in X \setminus Y_{\mathrm{ex}}$. Note that $|\{z \in I : x \in N_G(z)\}| \leqslant 2$ for each $x \in Y_{\mathrm{ex}}$ and by Lemma 10 and Claim 19, we have $\bigcup_{z\in I} N_G(z) \subseteq V(G) \setminus (LM(\mathcal{S}) \cup Y'_{\mathrm{ex}})$. Hence by Claims 20, 21 and 22, we obtain

$$\begin{aligned}
|G| + f(k, p) &\leqslant \sigma_{k+1}(G) \\
&\leqslant \sum_{z\in I} d_G(z) \\
&\leqslant (|G| - |LM(\mathcal{S})| - |Y'_{\mathrm{ex}}|) - |X| + (2|Y_{\mathrm{ex}}| + \min\{p - 1, k + 1\}\big(|X| - |Y_{\mathrm{ex}}|\big)) \\
&= |G| - |LM(\mathcal{S})| - (|X| - |Y_{\mathrm{ex}}|) + \min\{p - 1, k + 1\}\big(|X| - |Y_{\mathrm{ex}}|\big) \\
&< |G| + \min\{p - 2, k\}\big(|X| - |Y_{\mathrm{ex}}|\big),
\end{aligned}$$

that is, $|X| - |Y_{\mathrm{ex}}| > \dfrac{f(k, p)}{\min\{p - 2, k\}} = (k + 1)(p - 1)$. $\square$

By Claims 16, 17, 20, 21, 22 and 23, we obtain

$$\begin{aligned}(k+1)(p-1) &> \alpha(G)\\ &\geqslant |Y'_{\text{non-ex}}| + |LM(\mathcal{S})|\\ &= (|Y| - |Y_{\text{ex}}|) + |LM(\mathcal{S})|\\ &= (|X| - |Y_{\text{ex}}| - |\overline{Y}|) + |LM(\mathcal{S})|\\ &> \left((k+1)(p-1) - \frac{1}{2}|LM(\mathcal{S})|\right) + |LM(\mathcal{S})|\\ &> (k+1)(p-1),\end{aligned}$$

a contradiction. This completes the proof of Theorem 3. □

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