

An action of the cactus group on shifted tableau crystals

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Abstract

Recently, Gillespie, Levinson and Purbhoo introduced a crystal-like structure for shifted tableaux, called the shifted tableau crystal. We introduce, on this structure, a shifted version of the crystal reflection operators, which coincide with the restrictions of the shifted Schützenberger involution to any primed interval of two adjacent letters. Unlike type A Young tableau crystals, these operators do not realize an action of the symmetric group on the shifted tableau crystal since the braid relations do not need to hold. Following a similar approach as Halacheva, we exhibit a natural internal action of the cactus group on this crystal, realized by the restrictions of the shifted Schützenberger involution to all primed intervals of the underlying crystal alphabet, containing, in particular, the aforesaid action of the shifted crystal reflection operator analogues.

Mathematics Subject Classifications: 05E10, 05E18

1 Introduction

Young tableaux, as well as shifted tableaux, arise in many areas of mathematics, such as algebraic combinatorics, geometry or representation theory [9, 28, 31, 34]. While the first have their original role in the representation theory of symmetric groups [25, 36], the latter have their origin in projective representations [21, 22], due to I. Schur [32], and their connection to geometry was shown by Hiller and Boe [20] and Pragacz [28]. One important tool for the study of Young tableaux are Kashiwara crystals [5, 24]. A *Kashiwara crystal* of type A (for GL_n) is a non-empty set \mathcal{B} together with partial maps $e_i, f_i : \mathcal{B} \rightarrow \mathcal{B} \sqcup \{\emptyset\}$, length functions $\varepsilon_i, \varphi_i : \mathcal{B} \rightarrow \mathbb{Z}$, for $i \in I = [n - 1]$, and weight function $\text{wt} : \mathcal{B} \rightarrow \mathbb{Z}^n$ satisfying certain axioms (see, for example, [5, Definition 2.13]). This crystal may be regarded as a coloured and weighted directed acyclic graph, with vertices in \mathcal{B} and i -coloured edges $y \xrightarrow{i} x$ if and only if $f_i(y) = x$, for $i \in I$. The set of semistandard Young tableaux of a given shape, in the alphabet $[n]$, is known to provide

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a model for Kashiwara type A crystals [5, Chapter 3], with coplactic operators e_i and f_i defined in terms of reading words. This crystal is isomorphic to the crystal basis of an irreducible $U_q(\mathfrak{gl}_n)$ -module. The Schützenberger involution [33], also known as Lusztig involution [27], is defined on the type A Young tableau crystals, as a set map on \mathcal{B} , and acts on its graph structure by “flipping” it upside down, while reverting the orientation of arrows and its colours. This involution is realized by the evacuation, for straight shapes [33], or its coplactic extension, often called reversal, for skew shapes [4, 15].

Recently, Gillespie, Levinson and Purbhoo [12] and Gillespie and Levinson [11] introduced a crystal-like structure on shifted tableaux. This structure has vertices the skew shifted tableaux, for a given shape λ/μ , on the primed alphabet $[n]'$, and double edges, corresponding to the action of the primed and unprimed lowering and raising operators which commute with the shifted *jeu de taquin*. Each connected component has a unique highest weight element, a shifted skew tableau where each primed and unprimed raising operator is equal to \emptyset , which is a Littlewood–Richardson–Stembridge (LRS) tableau of shape λ/μ [34]. Similarly, it has a unique lowest weight element, a shifted skew tableau such that each primed and unprimed lowering operator is equal to \emptyset , which is the reversal of the highest weight element. The primed and unprimed operators considered separately yield a type A Kashiwara crystal. We remark that this structure is not a queer crystal¹ and differs from the one in [1, 10], which is indeed a crystal for the quantum queer Lie superalgebra.

Crystal reflection operators were originally defined by Lascoux and Schützenberger [26] in type A tableau crystals and they have been shown to define an action of the symmetric group on those crystals. Kashiwara [23, Theorem 7.2.2], [24, Theorem 11.1] defined the Weyl group action on arbitrary normal crystals. Halacheva [16, 17] has shown that there is an internal action of the cactus group in any normal crystal via partial Schützenberger involutions. When considering subintervals of adjacent letters, the action of the cactus group agrees with the action of the corresponding Weyl group generators. Indeed, the internal action factors through the quotient of this group by the braid relations of the corresponding Weyl group [16, 17, 18].

We introduce a shifted version of the crystal reflection operators in type A , Definition 48. In Theorem 51, we show that, similarly to type A , they coincide with the restrictions of the shifted Schützenberger involution to the primed interval of adjacent letters $[i, i+1]' = \{i', i, (i+1)', i+1\} \subseteq [n]'$, for any $i \in I$. They act on the $\{i', i\}$ -coloured components of the shifted tableau crystal by a double reflection through vertical and horizontal axes, rather than a simple reflection as in the Young tableau crystal.

Unlike type A crystals, they do not define a natural action of the symmetric group \mathfrak{S}_n on the shifted tableau crystal, since the braid relations do not need to hold, as shown in Example 61. Following a similar approach as Halacheva [16, 17], we then show in Theorem 69 that the restrictions of the shifted Schützenberger involution on the primed subintervals of $[n]$ yield an internal action of the cactus group J_n on that crystal. We note that this internal action on the shifted tableau crystal, unlike the one on type A crystals,

¹Hence the terminology “crystal-like structure”. However, we will henceforth use the term *crystal* to refer to this structure, whenever there is no risk of ambiguity.

does not factor through the braid relations of the symmetric group. When the shifted Schützenberger involution is restricted to primed subintervals of two adjacent letters, the cactus group action agrees with the action of the shifted crystal reflection operators on the shifted crystal. This means that both actions agree as permutations of the vertices within each $\{i', i\}$ -coloured component of the shifted crystal.

The cactus group, denoted J_n , first appeared in the works of Devadoss [8] and Davis, Januszkiewicz and Scott [7], as the fundamental group of the quotient orbifold of $\overline{M}_0^{n+1}(\mathbb{R})$, the Deligne–Mumford moduli space of stable curves of genus 0 with $n + 1$ marked points, by the action of \mathfrak{S}_n that permutes the first n of those points. It is expected, although we have not attempted to explore it, that this combinatorial internal action of the cactus group on the shifted crystal carries some geometrical meaning, as this crystal has its origin in the orthogonal Grassmannian [13]. Indeed, this is the case for \mathfrak{g} -crystals, for \mathfrak{g} a semi-simple Lie algebra [16, 17, 18]. Moreover, the tensor product of shifted tableau crystals is not known, and consequently, nor an external action of the cactus group.

This paper has the following structure: Section 2 provides the basic notions on shifted tableaux. We recall the definition of words and tableaux, following [12], as well as the notion of shifted *jeu de taquin* and shifted evacuation. Section 3 recalls the main concepts on the shifted tableau crystal of [12]. We provide more details on the i -string decomposition of such crystals (which corresponds to the axiom (B1) in [11] and (A2) in [12]). We then introduce, in Section 4, the shifted crystal reflection operators and prove their coincidence with the restriction of the shifted Schützenberger involution to a marked alphabet of two adjacent letters, Theorem 51. In Section 5, we recall the definition of the cactus group [19] and then we prove Theorem 69, the main result, which presents an action of the cactus group on the shifted tableau crystal. Some additional examples are discussed in Appendix A.

An extended abstract [29] of this paper was accepted in the Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics.

2 Background

This section is intended to provide the basic definitions and results on shifted tableaux, words, and involutions and algorithms among them. We mainly follow the notations in [11, 12].

A *strict partition* is a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ of positive integers such that $\lambda_1 > \dots > \lambda_k$. The *size* of λ is $|\lambda| = \sum \lambda_i$. The entries λ_i are called the *parts* of λ and the *length* of λ , denoted $\ell(\lambda)$, is the number of non-zero parts of λ . A strict partition λ is identified with its *shifted shape* $S(\lambda)$ which consists of $|\lambda|$ boxes placed in $\ell(\lambda)$ rows, with the i -th row having λ_i boxes and being shifted $i - 1$ units to the right. We use the English (or matrix) notation. The boxes in $\{(1, j), (2, j + 1), (3, j + 2), \dots\}$ form a *diagonal*, for $j \geq 1$. If $j = 1$, it is called the *main diagonal*. Given strict partitions λ and μ such that $S(\mu) \subseteq S(\lambda)$, we write $\mu \subseteq \lambda$ and define the *skew shifted shape* of λ/μ as $S(\lambda/\mu) = S(\lambda) \setminus S(\mu)$ (see Figure 1). Shapes of the form λ/\emptyset are called *straight* (or *normal*). Note that the shifted shape λ lies naturally in the ambient triangle of the

shifted staircase shape $\delta = (\lambda_1, \lambda_1 - 1, \dots, 1)$. We define the *complement* of λ to be the strict partition λ^\vee whose set of parts is the complement of the set of parts of λ in $\{\lambda_1, \lambda_1 - 1, \dots, 1\}$. In particular, $\emptyset^\vee = \delta$ (see Figure 1).

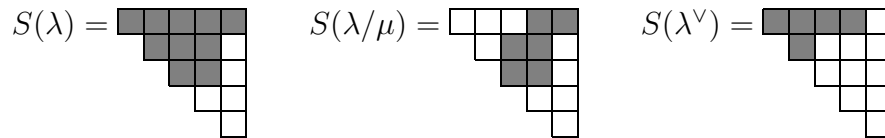


Figure 1: The shapes of λ , λ/μ and λ^\vee , shaded in gray, for $\lambda = (5, 3, 2)$ and $\mu = (3, 1)$.

We set $[n] := \{1 < \dots < n\}$ and define the *primed* (or *marked*) alphabet $[n]' := \{1' < 1 < \dots < n' < n\}$. When referring to the letters i and i' without specifying whether they are primed, we write \mathbf{i} . Given a string $w = w_1 \dots w_m$ in the alphabet $[n]'$, the *first* i or i' will be the leftmost entry of w equal to i or i' , for $1 \leq i \leq n$. The *canonical form* of w is the string obtained from w by replacing the first i or i' (if it exists) with i , for all $1 \leq i \leq n$. Two strings w and v are said to be *equivalent*, denoted by $w \simeq v$, if they have the same canonical form (this is an equivalence relation).

Definition 1 ([12, Definition 2.2]). A *word* \widehat{w} is an equivalence class of strings. The representative in canonical form is called the *canonical representative*. The *weight* of a word \widehat{w} is $\mathbf{wt}(\widehat{w}) = (wt_1, \dots, wt_n)$, where wt_i is equal to the total number of i and i' in \widehat{w} .

Example 2. The string $w = 12'2'1123'2'2$ is equivalent to $\widehat{w} = 122'11232'2$, the latter being the canonical form of the former. The weight of \widehat{w} is $(3, 5, 1)$.

A *partial operator* on a set S is a map $A : T \rightarrow S$, where $T \subseteq S$, such that $A(s) = \emptyset$ for $s \notin T$. In this case, it is said that A is *undefined* on s . Otherwise, it is said to be *defined*. Given a partial operator A on the set of finite strings in the alphabet $[n]'$, we say that A is *defined on words* [12, Definition 2.4] if $A(v)$ is defined on some representative v of \widehat{w} and if $A(u) \simeq A(v)$ for any u and v representatives of \widehat{w} . The *induced operator* \widehat{A} on words is then defined as $\widehat{A}(w) = \widehat{A(v)}$ if such a representative v exists, and $\widehat{A}(w) = \emptyset$ otherwise. For simplicity, from now on we refer to a word by w instead of \widehat{w} and to an operator defined on a word by A instead of \widehat{A} .

Definition 3. Let λ and μ be strict partitions such that $\mu \subseteq \lambda$. A *shifted semistandard (Young) tableau* T of shape λ/μ is a filling of $S(\lambda/\mu)$ with letters in $\{1' < 1 < \dots\}$ such that:

1. The entries are weakly increasing in each row and in each column.
2. There is at most one i per column, for any $i \geq 1$.
3. There is at most one i' per row, for any $i \geq 1$.

The (row) reading word $w(T)$ of such a tableau is formed by reading the entries of T from left to right, going bottom to top. The *weight* of T is defined as $\mathbf{wt}(T) = \mathbf{wt}(w(T))$. A shifted tableau is said to be *standard* if its weight is $(1, \dots, 1)$. We say that a tableau T is in *canonical form* if so is $w(T)$. A tableau T in canonical form is identified with its set of *representatives*, that are obtained by possibly priming the entry corresponding to the first i in $w(T)$, for all i . We denote by $\mathbf{ShST}(\lambda/\mu, n)$ the set of shifted semistandard tableaux of shape λ/μ , on the alphabet $[n]'$, in canonical form.

Example 4. The following is a shifted semistandard tableau, with $\lambda = (6, 4, 2)$, $\mu = (3, 1, 0)$ and $n = 3$, in canonical form, with its word and weight.

$$T = \begin{array}{cccccc} & & & 1 & 1 & 2' \\ & & & 2 & 3' & 3 \\ & & & 3 & 3 & \end{array} \quad w(T) = 3323'3112' \quad \mathbf{wt}(T) = (2, 2, 4)$$

The following shifted semistandard tableaux are the representatives of T that are not in canonical form:

$$\begin{array}{cccc} \begin{array}{cccccc} & & & 1' & 1 & 2' \\ & & & 2 & 3' & 3 \\ & & & 3 & 3 & \end{array} & \begin{array}{cccccc} & & & 1 & 1 & 2' \\ & & & 2' & 3' & 3 \\ & & & 3 & 3 & \end{array} & \begin{array}{cccccc} & & & 1 & 1 & 2' \\ & & & 2 & 3' & 3 \\ & & & 3' & 3 & \end{array} & \begin{array}{cccccc} & & & 1' & 1 & 2' \\ & & & 2' & 3' & 3 \\ & & & 3 & 3 & \end{array} \\ \begin{array}{cccccc} & & & 1' & 1 & 2' \\ & & & 2 & 3' & 3 \\ & & & 3' & 3 & \end{array} & \begin{array}{cccccc} & & & 1 & 1 & 2' \\ & & & 2' & 3' & 3 \\ & & & 3' & 3 & \end{array} & \begin{array}{cccccc} & & & 1' & 1 & 2' \\ & & & 2' & 3' & 3 \\ & & & 3' & 3 & \end{array} & \end{array}$$

A *diagonally-shaped tableau* is a skew shifted tableau of shape

$$(2n-1, 2n-3, \dots, 1)/(2n-2, 2n-4, \dots, 2).$$

Every word $w = w_1 \cdots w_n$ may be regarded as a shifted tableau D_w of such shape.

Example 5. The word $w = 2311'$ is the reading word of

$$D_w = \begin{array}{cccccc} & & & & & 1' \\ & & & & & 1 \\ & & & & & 3 \\ & & & & & 2 \end{array}$$

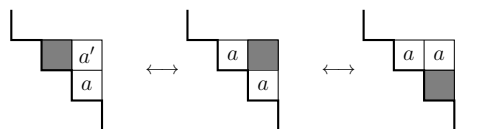
2.1 The shifted *jeu de taquin*, Knuth equivalence and dual equivalence

A skew shape $S(\lambda/\mu)$ is said to be a *border strip* if it contains no subset of the form $\{(i, j), (i+1, j+1)\}$ and a *double border strip* if it contains no subset of the form $\{(i, j), (i+1, j+1), (i+2, j+2)\}$. A shifted semistandard tableau T is a disjoint union of border strips $\bigsqcup_i T^i$, where T^i is the tableau obtained from T considering only the entries filled with \mathbf{i} . Given strict partitions $\nu \subseteq \mu \subseteq \lambda$, we say that λ/μ *extends* μ/ν , and, in this case, we define

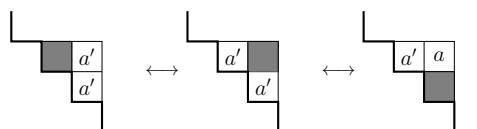
$$(\lambda/\mu) \sqcup (\mu/\nu) := \lambda/\nu.$$

If λ/μ extends a shape B consisting of a single box, then B is said to be an *inner corner* of λ/μ . Similarly, if B , consisting of a single box, extends λ/μ , we say that B is an *outer corner*.

Definition 6 ([35, Section 6.4]). Let $T \in \text{ShST}(\lambda/\mu, n)$. An *inner jeu de taquin slide* is the process in which an empty inner corner of the skew shape of T is chosen and then either the entry to its right or the one below it is chosen to slide into the empty square, in such way that the tableau is still semistandard, and then repeating the process with the obtained empty square until it is an outer corner. An *outer jeu de taquin slide* is the reverse process, starting with an outer corner. This process has an exception to the sliding rules when the empty box of an inner or outer slide enters in the main diagonal. If an inner slide moves a box with a' to the left into the diagonal and then moves a box with a up from the diagonal, to the right of it, the former becomes unprimed (and vice versa for the corresponding outer slide), as illustrated by the following slide:



If T is not in canonical form, there is another exception to consider (observe that this illustration is in the same canonical class of the former):



The *rectification* $\text{rect}(T)$ of T is the tableau obtained by applying a sequence of inner slides until a straight shape is obtained. This is well defined as the rectification process does not depend on the chosen sequence of slides [35], [30, Theorem 11.1]. The *rectification* of a word w is the rectification of any tableau with reading word w (in particular, the diagonally-shaped tableau D_w with word w). Two tableaux are said to be *shifted jeu de taquin equivalent* (or *plactic equivalent*) if they have the same rectification. In Example 5, the rectification of $w = 2311'$ is 1123. The *standardization* of a word w , denoted $\text{std}(w)$, is obtained by replacing in order the letters in any representative of w with $1, \dots, n$, from least to greatest, reading right to left for primed entries, and left to right for unprimed entries. This does not depend on the choice of representative. The *standardization* of a shifted tableau T is the tableau of the same shape as T with reading word $\text{std}(w(T))$. In both cases, the standardization is independent of the choice of representative. For example, the standardization of the word $w = 3323'3112'$ is $\text{std}(w) = 67458123$.

Lemma 7 ([12, Lemma 3.5]). If s is a word in $[n]$, with $n = a_1 + \dots + a_k$, and such that $\text{wt}(s) = (1, \dots, 1)$, then there is at most one word w of weight (a_1, \dots, a_k) with standardization $\text{std}(w) = s$.

Given ν a strict partition, the *Yamanouchi tableau* of shape ν , denoted Y_ν , is the shifted semistandard tableau whose i -th row is filled only with unprimed i 's, for all $i = 1, \dots, \ell(\nu)$.

Proposition 8. If ν is a strict partition, there is a unique shifted tableau of shape and weight ν , up to canonical form, which is Y_ν .

Proof. We construct such a tableau of shape and weight ν , on the primed alphabet $\{1, \dots, \ell(\nu)\}'$, starting in the first row. Since $\text{wt}_1 = \nu_1$ and there are ν_1 boxes in the first row, to ensure semistandardness each box must be filled with 1's, except for the first one which may be filled with $1'$ or 1. If it is $1'$, it is identified with 1 in canonical form, as this is the first occurrence. The process for the remaining rows is the same. \square

Example 9. If $\nu = (4, 3, 1)$, then,

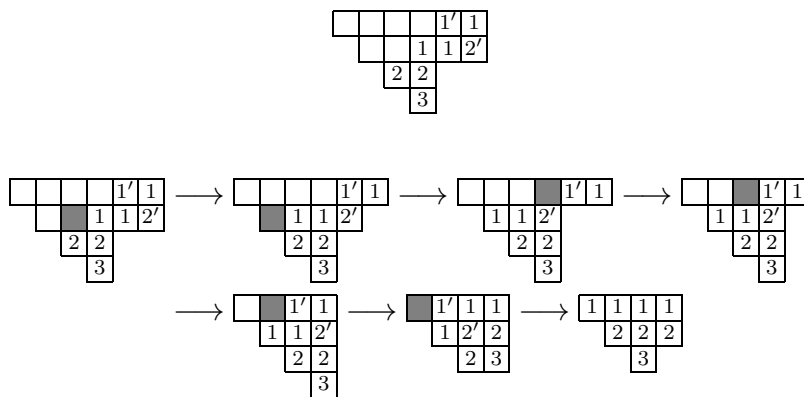
$$Y_\nu = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline & 2 & 2 & 2 \\ \hline & & 3 & \\ \hline \end{array}.$$

Definition 10. A word w on the alphabet $[n]'$ with weight ν , a strict partition, is said to be *ballot* (or *lattice*, or *Yamanouchi*) if its rectification is $w(Y_\nu)$.

Definition 11 ([34, 35]). A shifted semistandard tableau T of weight ν , a strict partition, is said to be *Littlewood–Richardson–Stembridge* (LRS) if $\text{rect}(T) = Y_\nu$. Equivalently, the reading word of such a tableau is a *ballot* word of weight ν .

There is another formulation for LRS tableaux due to Stembridge, using some statistics on its word (for details, see [34, Theorem 8.3]). Given strict partitions λ , μ and ν , such that $|\lambda| = |\mu| + |\nu|$, the *shifted Littlewood–Richardson coefficient* $f_{\mu, \nu}^\lambda$ is defined to be the number of LRS tableaux of shape λ/μ and weight ν .

Example 12. The following tableau of shape $(6, 5, 2, 1)/(4, 2)$ and weight $(4, 3, 1)$ rectifies to $Y_{(4,3,1)}$, thus it is a LRS tableau. Note that its word $322112'1'1$ is a ballot word with weight $(4, 3, 1)$.



Definition 13 ([30]). Two words w and v on an alphabet $[n]'$ are said to be *shifted Knuth equivalent*, denoted $w \equiv_k v$, if one can be obtained from the other by applying a sequence of the following Knuth moves on adjacent letters

- (K1) $bac \longleftrightarrow bca$ if, under the standardization ordering, $a < b < c$.
- (K2) $acb \longleftrightarrow cab$ if, under the standardization ordering, $a < b < c$.
- (S1) $ab \longleftrightarrow ba$ if these are the first two letters.

(S2) $aa \longleftrightarrow aa'$ if these are the first two letters.

Example 14. Let $w = 212'21$. We have $\text{std}(w) = 41352$, and then the last 1 is less than $2'$, which is less than the last 2, in standardization ordering. Thus, $w \equiv_k 212'12$.

Observe that the shifted Knuth moves above may be performed via (inner or outer) *jeu de taquin* slides. If $a < b < c$ in standardization order, then the Knuth moves (K1) and (K2) are illustrated by:

$$\begin{array}{|c|c|c|} \hline & & a \\ \hline & b & c \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline & a & c \\ \hline & b & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline & & b \\ \hline & a & c \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline & a & b \\ \hline & c & \\ \hline \end{array}$$

For the Knuth move (S1), assume, without loss of generality, that $a < b$ in standardization ordering. Then,

$$\begin{array}{|c|c|c|} \hline & & a \\ \hline & b & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline a & b \\ \hline \end{array}$$

Finally the Knuth move (S2) is illustrated by the exception slide

$$\begin{array}{|c|c|} \hline a' & \\ \hline a & \\ \hline \end{array} \longleftrightarrow \begin{array}{|c|c|} \hline a & a \\ \hline \end{array}$$

If w and v are shifted Knuth equivalent words, the diagonally-shaped tableaux D_w and D_v have the same rectification. Thus D_w can be transformed into D_v via some sequence of *jeu de taquin* slides.

Theorem 15 ([30, Theorem 12.2], [35, Theorem 4.4.4]). *Two shifted tableaux are jeu de taquin equivalent if and only if their reading words are shifted Knuth equivalent.*

Therefore, two tableaux in $\text{ShST}(\lambda/\mu, n)$ are said to be shifted Knuth equivalent if so are their reading words. Shifted Knuth equivalence classes and *jeu de taquin* classes on words coincide and are in one-to-one correspondence with shifted semistandard tableaux of straight shape, via rectification (or shifted Schensted insertion [30]). Unlike the classic Knuth relations for unprimed alphabets, the shifted Knuth equivalence is not a congruence, due to rules (S1) and (S2), since $w \equiv_k v$ does not necessarily imply that $tw \equiv_k tv$ for any letter $t \in [n]'$. For instance, $22'1 \equiv_k 221$ but $322'1 \not\equiv_k 3221$. However, under certain conditions we have the following results.

Lemma 16. *Let w and v be two words in $[n]'$ such that $w \equiv_k v$. Let $t \in [n]'$. Then,*

$$wt \equiv_k vt$$

Proof. Let D_w and D_v be diagonally-shaped shifted tableaux with words w and v , respectively. By Theorem 15, $\text{rect}(D_w) = \text{rect}(D_v)$. Let T be this straight-shaped tableau, of shape λ . Hence, we may consider the tableau T^0 of shape $(\lambda_1 + 2, \lambda_1, \dots, \lambda_k)/(\lambda_1 + 1)$ consisting of t on the entry $(1, \lambda_1 + 2)$ and T on the remaining part. Clearly, $\text{rect}(T^0) = \text{rect}(D_w^0) = \text{rect}(D_v^0)$ where D_w^0 and D_v^0 are the diagonally-shaped shifted tableaux with words wt and vt respectively. \square

Lemma 17. *Let w and v be two words in $[n]'$ such that $w \equiv_k v$ and such that there exists a sequence of Knuth relations turning w into v using only (K1) and (K2). Let $t \in [n]'$. Then,*

$$tw \equiv_k tv$$

Proof. If the rules (S1) and (S2) are not used, then w and v are Knuth equivalent as Young tableau words, considering the standardization to avoid primed entries. \square

We define now shifted dual equivalence on words and tableaux. Lemma 7 ensures that it is compatible with standardization, i.e., that shifted *jeu de taquin* commutes with standardization.

Definition 18 ([15]). Two standard shifted tableaux are *shifted dual equivalent* (or *coplactic equivalent*) if they have the same shape after applying any sequence (including the empty sequence) of inner or outer *jeu de taquin* slides to both. Two shifted semistandard tableaux are shifted dual equivalent if so are their standardizations.

Example 19. The following shifted tableaux have the same shape and standardization, and thus are shifted dual equivalent, as any sequence of *jeu de taquin* slides performed on both will yield the same shape:

$$T_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2' \\ \hline & 2 & 3' & 3 \\ \hline & & 3 & \\ \hline \end{array}, \quad T_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline & 2 & 2 & 3 \\ \hline & & 3 & \\ \hline \end{array}.$$

In particular, considering the empty sequence of *jeu de taquin* slides, we have that shifted tableaux that are dual equivalent must have the same shape. In terms of mixed insertion [14], two shifted semistandard tableaux of the same shape are dual equivalent if and only if they have the same mixed insertion recording tableau [15, 30]. This notion is extended to words, with two words being *shifted dual equivalent* if their corresponding diagonally-shaped tableaux are shifted dual equivalent.

The following characterizes dual equivalence on straight-shaped shifted tableaux, in which the dual equivalence classes are determined by the (straight) shapes. Considering mixed-insertion, the recording tableau of a dual class is the recording tableau of the unique shifted Yamanouchi tableau in that class.

Proposition 20 ([15, Corollary 2.5]). *Two tableaux of the same straight shape are dual equivalent.*

An operator on shifted tableaux (in canonical form) of the same shape is said to be *coplactic* if it commutes with all sequences of shifted *jeu de taquin* slides.

2.2 The shifted evacuation and reversal

In this section we recall the *shifted evacuation*, an involution on shifted semistandard tableaux of straight shape that preserves the shape and reverses the weight. This involution was first introduced by Worley [35], as an analogue of the Schützenberger involution

[33] on ordinary Young tableaux. Choi, Nam, and Oh [6] recently gave a new formulation using the shifted switching process and showed that the two definitions coincide.

We define a complementation within the alphabet $[n]'$ as:

$$\begin{aligned} c_n : [n]' &\longrightarrow [n]' \\ k &\longmapsto (n - k + 1)' \\ k' &\longmapsto n - k + 1 \end{aligned} \tag{1}$$

This is extended to shifted semistandard tableaux as follows. Given $T \in \text{ShST}(\lambda/\mu, n)$, we define $c_n(T)$ as the tableau of shape μ^\vee/λ^\vee obtained by taking each box $(i, j) \in S(\lambda/\mu)$, filled with \mathbf{k} , to a box $(\lambda_1 - j + 1, \lambda_1 - i + 1) \in S(\mu^\vee/\lambda^\vee)$, filled with $c_n(\mathbf{k})$. In other words, the operator c_n flips the shape of T across the anti-diagonal of its staircase ambient shape, while complementing the entries using (1). Then, if $\text{wt}(T) = (wt_1, \dots, wt_n)$, we have $\text{wt}(c_n(T)) = \text{wt}(T)^{\text{rev}} := (wt_n, \dots, wt_1)$. Thus, the operator c_n is a weight-reversing and shape-“flipping” bijection between $\text{ShST}(\lambda/\mu, n)$ and $\text{ShST}(\mu^\vee/\lambda^\vee, n)$, and consequently an involution in $\text{ShST}(\lambda/\mu, n) \sqcup \text{ShST}(\mu^\vee/\lambda^\vee, n)$.

If $w = w_1 \cdots w_l$ is a word on $[n]'$, then $c_n(w)$ is defined as the word, in canonical form, of $c_n(D_w)$. Then, we have $c_n(w_1) \cdots c_n(w_l)$, after canonicalizing. The operator c_n on words and diagonally-shaped shifted tableaux is a weight-reversing involution.

Example 21. Consider the following shifted semistandard tableau, with (row) reading word $w = 33223'3112'$ and $n = 3$,

$$T = \begin{array}{ccccc} & & 1 & 1 & 2' \\ & & 2 & 2' & 3 \\ & & 3 & 3 & \end{array}$$

We then have $c_n(w) = 11'22'11'33'2$ and

$$c_n(T) = \begin{array}{ccccc} & & 1' & 2 \\ & & 1' & 1 & 3' \\ & & 1 & 2' & 3 \\ & & 2 & & \end{array}$$

Additionally, we have

$$w_{\text{col}}(c_n(T)) = c_n(w(T)) \tag{2}$$

where $w_{\text{col}}(T)$ denotes the *column reading word* of T , which is read along columns from bottom to top, going left to right. For diagonally-shaped tableaux, it is clear that the row and column reading words coincide. More generally, for T a shifted semistandard tableau of any shape, we have that $w(T) \equiv_k w_{\text{col}}(T)$ ([35, Lemma 6.4.12]), and thus

$$w(c_n(T)) \equiv_k c_n(w(T)). \tag{3}$$

By construction, the operator c_n is coplactic. In particular, it preserves shifted Knuth and dual equivalences. Moreover, it commutes with standardization.

Definition 22 ([35, Definition 7.1.5]). Let T be a shifted semistandard tableau of straight shape. The (*shifted*) *evacuation* evac is defined as $\text{evac}(T) := \text{rect}(c_n(T))$.

The coplacity of c_n ensures that $\text{evac}(T)$ may also be obtained by first “rectifying” T south-eastward, until obtaining a tableau of shape δ/λ^\vee , also known as *anti-straight shape*, and then applying c_n .

Example 23. Consider the following tableau.

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2' & 2 \\ \hline & 2 & 2 & \\ \hline & & 3 & \\ \hline \end{array}$$

Assuming the underlying alphabet to be $\{1, 2, 3\}'$, to obtain $\text{evac}(T)$, we first compute $c_3(T)$ and then rectify it:

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2' & 2 \\ \hline & 2 & 2 & \\ \hline & & 3 & \\ \hline \end{array} \xrightarrow{c_3} c_3(T) = \begin{array}{|c|c|c|c|} \hline & & & 2' \\ \hline & 1 & 2' & 2 \\ \hline & & 2 & 3' \\ \hline & & 3 & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline & & 2' & 2 \\ \hline & 1 & 2' & 3' \\ \hline & & 2 & 3 \\ \hline & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline & 1 & 2' & 2 \\ \hline & 2 & 2 & 3' \\ \hline & & 3 & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 1 & 2' & 2 & 2 \\ \hline & 2 & 3' & \\ \hline & & 3 & \\ \hline \end{array} = \text{evac}(T).$$

If the underlying alphabet was $\{1, 2, 3, 4\}'$, then $\text{evac}(T) = \begin{array}{|c|c|c|c|} \hline 2 & 3' & 3 & 3 \\ \hline & 3 & 4' & \\ \hline & & 4 & \\ \hline \end{array}$.

Proposition 24 ([35, Lemma 7.1.6]). *Let T be a shifted semistandard tableau of straight shape. Then, $\text{evac}(T)$ and T have the same shape and $\text{evac}^2(T) = T$.*

Moreover, since rectification leaves the weight unchanged, we have $\text{wt}(\text{evac}(T)) = \text{wt}(T)^{\text{rev}}$. We remark that $\text{evac}(T)$ is shifted Knuth equivalent to $c_n(T)$ by construction. And since $\text{evac}(T)$ has the same straight shape as T , Proposition 20 ensures that they are shifted dual equivalent. Thus, $\text{evac}(T)$ is the unique tableau that is shifted dual equivalent to T and shifted Knuth equivalent to $c_n(T)$.

Proposition 25. *Let ν be a strict partition. Then, $\text{evac}(Y_\nu)$ is the unique shifted tableau of shape ν and weight ν^{rev} .*

Proof. The tableau $\text{evac}(Y_\nu)$ has shape ν and weight ν^{rev} . Let Q be another shifted tableau in the same conditions. Then, $\text{evac}(Q)$ has shape and weight equal to ν , thus by Proposition 8, $\text{evac}(Q) = Y_\nu$. Since evac is an involution, we have $\text{evac}^2(Q) = Q = \text{evac}(Y_\nu)$. \square

The following result provides a direct way to compute the evacuation of Y_ν .

Proposition 26. *Let $\nu = (\nu_1, \dots, \nu_n)$ be a strict partition, with $n > 1$. Considering $[n]'$ to be the underlying alphabet, $\text{evac}(Y_\nu)$ is the tableau of shape ν such that its n -th row is filled with n^{ν_n} , and its i -th row is filled with $i^{\nu_n}(i+1)^{\nu_{n-1}-\nu_n-1} \dots n'n^{\nu_i-\nu_{i+1}-1}$, reading from left to right, for $i < n$.*

Proof. This filling clearly defines a shifted semistandard tableau. Let T_0 be the tableau with those conditions. By construction, T_0 has shape ν and it is clear that its weight is given by $(\nu_n, \dots, \nu_1) = \nu^{\text{rev}}$. Hence, by Proposition 25, $T_0 = \text{evac}(Y_\nu)$. \square

Example 27. Let $\mu = (4, 3, 1)$ and $n = 3$. Then,

$$Y_\mu = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \\ \hline & & 3 & \\ \hline \end{array} \longrightarrow \text{evac}(Y_\mu) = \begin{array}{|c|c|c|c|} \hline 1 & 2' & 2 & 3' \\ \hline & 2 & 3' & 3 \\ \hline & & 3 & \\ \hline \end{array}.$$

Definition 28. A word v is said to be *anti-ballot* if there exists a shifted semistandard tableau $T \in \text{ShST}(\lambda/\mu, n)$ such that $\text{rect}(T) = \text{evac}(Y_\nu)$, for ν a strict partition.

It is due to Haiman [15, Theorem 2.13] that, given a shifted semistandard tableau T , there is a unique tableau T^e , the *reversal* of T , that is shifted Knuth equivalent to $c_n(T)$ and dual equivalent to T . Since the operator c_n preserves shifted Knuth equivalence [35, Lemma 7.1.4], the reversal operator is the coplactic extension of evacuation, in the sense that, we may first rectify T , then apply the evacuation operator, and then perform outer *jeu de taquin* slides, in the reverse order defined by the previous rectification, to get a tableau T^e with the same shape as T . From [15, Corollary 2.5, 2.8 and 2.9], this tableau T^e is shifted dual equivalent to T , besides being shifted Knuth equivalent to $c_n(T)$. In particular, $\text{evac}(T) = T^e$ for tableaux of straight shape.

Example 29. Let $\lambda = (6, 5, 3, 1)$ and $\mu = (4, 3)$ and consider the following tableau in the alphabet $\{1, 2, 3\}'$

$$T = \begin{array}{|c|c|c|c|c|} \hline & & & 1' & 1 \\ \hline & & & 1 & 2' \\ \hline & & 1 & 2 & 2 \\ \hline & & & 3 & \\ \hline \end{array}.$$

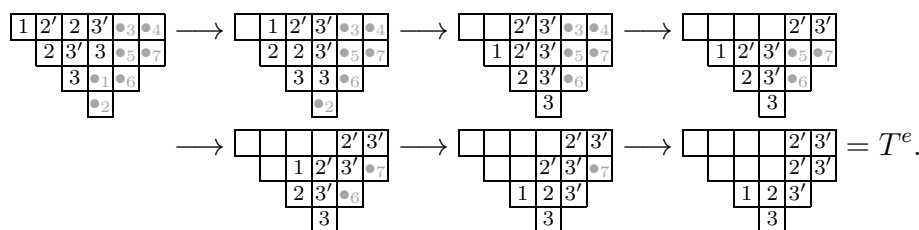
To compute the reversal T^e , we first rectify T , recording the outer corners resulting from the sequence of inner *jeu de taquin* slides:

$$\begin{array}{ccccccc} \begin{array}{|c|c|c|c|c|} \hline & & & 1' & 1 \\ \hline & & & 1 & 2' \\ \hline & & 1 & 2 & 2 \\ \hline & & & 3 & \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|c|c|c|c|} \hline & & & 1' & 1 \\ \hline & & & 1 & 2' \bullet_7 \\ \hline & & 1 & 2 & 2 \\ \hline & & & 3 & \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|c|c|c|c|} \hline & & & 1' & 1 \\ \hline & & & 1 & 2' \bullet_7 \\ \hline & & 1 & 2 & 2 \bullet_6 \\ \hline & & & 3 & \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|c|c|c|c|} \hline & & & 1' & 1 \\ \hline & & & 1 & 2' \bullet_5 \bullet_7 \\ \hline & & 1 & 2 & 2 \bullet_6 \\ \hline & & & 3 & \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|c|} \hline & & & 1' & 1 \bullet_4 \\ \hline & & & 1 & 2' \bullet_5 \bullet_7 \\ \hline & & 1 & 2 & 2 \bullet_6 \\ \hline & & & 3 & \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|c|c|c|c|} \hline & & & 1' & 1 \bullet_3 \bullet_4 \\ \hline & & & 1 & 2' \bullet_5 \bullet_7 \\ \hline & & 1 & 2 & 2 \bullet_6 \\ \hline & & & 3 & \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|c|c|c|c|} \hline & & & 1' & 1 \bullet_3 \bullet_4 \\ \hline & & & 1 & 2' \bullet_5 \bullet_7 \\ \hline & & 1 & 2 & 3 \bullet_6 \\ \hline & & & 3 & \bullet_2 \\ \hline \end{array} & \longrightarrow & \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 1 \bullet_3 \bullet_4 \\ \hline & & & 2 & 2 \bullet_5 \bullet_7 \\ \hline & & 1 & 2 & 2 \bullet_6 \\ \hline & & & 3 & \bullet_1 \bullet_6 \\ \hline & & & & \bullet_2 \\ \hline \end{array} \end{array}$$

Then, we compute the evacuation of the obtained straight-shaped tableau (see Example 27):

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & \bullet_3 & \bullet_4 \\ \hline 2 & 2 & 2 & \bullet_5 & \bullet_7 & \\ \hline 3 & \bullet_1 & \bullet_6 & & & \\ \hline & & \bullet_2 & & & \\ \hline \end{array} \xrightarrow{\text{evac}} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2' & 2 & 3' & \bullet_3 & \bullet_4 \\ \hline 2 & 3' & 3 & \bullet_5 & \bullet_7 & \\ \hline 3 & \bullet_1 & \bullet_6 & & & \\ \hline & & \bullet_2 & & & \\ \hline \end{array}.$$

Finally, we perform outer *jeu de taquin* slides defined by the outer corners of the previous sequence:



By construction, the reversal is an involution on $\text{ShST}(\lambda/\mu, n)$ which permutes the tableaux within each shifted dual class while reversing the weight. It coincides with evacuation on straight-shaped tableaux. In particular, since each dual class has a unique LRS tableau, it yields a bijection $T \mapsto c_n(T^e)$ between the set of LRS tableaux of shape λ/μ and weight ν and the set of LRS tableaux of shape μ^\vee/λ^\vee and weight ν . Hence, we have the symmetry $f_{\mu\nu}^\lambda = f_{\lambda^\vee\nu}^{\mu^\vee}$. The symmetry $f_{\mu\nu}^\lambda = f_{\nu\mu}^\lambda$, as well as the others resulting from these two, may be obtained using the *shifted tableau switching* (see [6]).

Henceforth, we will denote by η either the involution *evac* or its coplactic extension e , and we will refer to it by the *Schützenberger involution*². Later, in Section 4, the involution η will be formulated in the language of a shifted tableau crystal, and it will become clear how the tableaux are permuted within each shifted dual class.

3 A crystal-like structure on shifted tableaux

We recall the main results on the shifted tableau crystal $\mathcal{B}(\lambda/\mu, n)$, the crystal-like structure on $\text{ShST}(\lambda/\mu, n)$ introduced in [11, 12]. Let $\{e_1, \dots, e_n\}$ be the canonical basis of \mathbb{R}^n , and let $\alpha_i = e_i - e_{i+1}$ be the weight vectors, for $i \in I := [n-1]$. We first recall the primed raising and lowering operators on words.

Definition 30 ([12, Definition 3.3]). Given a word w on $[n]'$ and $i \in I$, the *primed raising operator* $E'_i(w)$ is defined as the unique word such that

1. $\text{std}(E'_i(w)) = \text{std}(w)$,
2. $\text{wt}(E'_i(w)) = \text{wt}(w) + \alpha_i$,

if such word exists. Otherwise, $E'_i(w) = \emptyset$, and we say that E'_i is undefined on w . The *primed lowering operator* $F'_i(w)$ is defined in analogous way using $-\alpha_i$.

This notion is well defined due to Lemma 7, and as a direct consequence we have that $E'_i(w) = v$ if and only if $w = F'_i(v)$, for any words w and v [12, Proposition 3.4]. This definition is extended to a shifted semistandard tableau T , putting $E'_i(T)$ as the shifted semistandard tableau with the same shape as T and with (row) reading word

²We remark that our formulation of η , which coincides with the one in [11, Remark 5.7], consists of c_n followed by some *jeu de taquin* slides, to preserve shape. This differs from the η presented in [12], which corresponds to c_n here.

$E'_i(w(T))$. The primed operators preserve semistandardness [12, Proposition 3.6] and they are coplactic [12, Proposition 3.7]. Moreover, the tableaux T , $E'_i(T)$ and $F'_i(T)$ are dual equivalent, since their standardization is unchanged (Definition 18), whenever E'_i and F'_i are defined on T .

In order to simplify the notation, from now on we consider the alphabet $\{1, 2\}'$, but the results hold for any primed alphabet $\{i, i + 1\}'$ of two adjacent letters. The following propositions provide a simple way to compute the primed operators both on words and on shifted tableaux of straight shape.

Proposition 31 ([12, Proposition 3.9]). *To compute $F'_1(w)$ consider all representatives of w . If all representatives have the property that the last 1 is left of the last $2'$ then $F'_1(w) = \emptyset$. If there exists a representative such that the last 1 is right of the last $2'$ then $F'_1(w)$ is obtained by changing the last 1 to $2'$ in that representative. The word $E'_1(w)$ is defined similarly reversing the roles of 1 and $2'$.*

Proposition 32 ([12, Proposition 3.11]). *Let $T \in \text{ShST}(\lambda, n)$ a shifted semistandard tableau of straight shape. If T has one row, then $E'_1(T)$ (respectively $F'_1(T)$) is obtained by changing the leftmost 2 to 1 (respectively, 1 to 2), if possible, and it is \emptyset otherwise. If T has two rows and the first row contains a $2'$, then $E'_1(T)$ is obtained by changing that $2'$ to 1 and $F'_1(T) = \emptyset$. If the first row does not contain a $2'$, then $E'_1(T) = \emptyset$ and $F'_1(T)$ is obtained by changing the rightmost 1 to $2'$.*

Example 33. Let $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2' \\ \hline 2 & 2 & 2 & \\ \hline & & 3 & \\ \hline \end{array}$. Then, $F'_2(T) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2' \\ \hline 2 & 2 & 2 & \\ \hline & & 3 & \\ \hline \end{array}$.

Observe that both tableaux have the same standardization and that $\text{wt}(F'_2(T)) = (3, 4, 2) = \text{wt}(T) - (0, 1, -1)$.

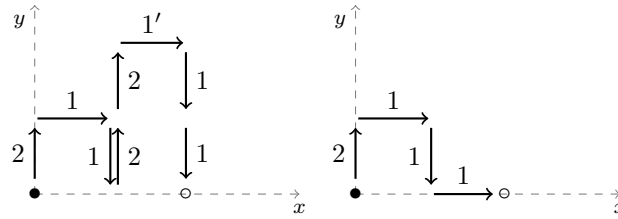
Given a word w on the alphabet $[n]'$ and $i \in I$, the i -th *lattice walk* of w is obtained by considering the subword w_i , consisting of the letters $\{i, i + 1\}'$, and replacing each letter according to the following table, starting at the origin $(0, 0)$ and at step k being in position (x_k, y_k) .

| | | | | |
|------------------|--------------------|-------------------|-------------------|--------------|
| $x_k y_k = 0$ | $\xrightarrow{1'}$ | $\xrightarrow{1}$ | $\uparrow_{2'}$ | \uparrow_2 |
| $x_k y_k \neq 0$ | $\xrightarrow{1'}$ | \downarrow_1 | $\xleftarrow{2'}$ | \uparrow_2 |

The lattice walks of a word may be used to provide another criterion for ballotness.

Proposition 34 ([12, Corollary 4.5]). *A word w is ballot if and only if the i -th lattice walk of the subword w_i consisting of \mathbf{i} and $\mathbf{i} + \mathbf{1}$ ends on the x axis, for all $i \in I$.*

Example 35. Let $w = 3211221'11$. To obtain the 1st and 2nd lattice walks of w , consider the subwords $w_1 = 211221'11$ and $w_2 = 3222$ (which corresponds to 2111, using the alphabet $\{1, 2\}'$). Replacing each letter accordingly, we obtain



Since both lattice walks end on the x axis, the word w is ballot.

If w is a word on the alphabet $\{1', 1, 2', 2\}$ and $u = w_k w_{k+1} \cdots w_l$ is a substring of some representative of w , then u is called a *substring* of w . The coordinates (x, y) of the point of the 1-lattice walk before the start of u is called the *location* of u .

Definition 36 ([12, Definition 5.3]). A substring u is said to be a F_1 -critical substring if any of these conditions on u and its location are satisfied (as well as the adequate transformations to apply), where ab^*c means any string of the form $ab \cdots bc$, including ac :

| Type | Substring | Condition steps | Location | Transformation |
|------|---------------------|---|------------------------------|---------------------------|
| 1F | $u = 1(1')^*2'$ | $\xrightarrow{1} \xrightarrow{1'} \uparrow 2$ $\downarrow 1 \xrightarrow{1'} \uparrow 2$ | $y = 0$ $y = 1, x \geq 1$ | $u \rightarrow 2'(1')^*2$ |
| 2F | $u = 1(2)^*1'$ | $\xrightarrow{1} \uparrow 2 \xrightarrow{1'}$ $\downarrow 1 \uparrow 2 \xrightarrow{1'}$ | $x = 0$ $x = 1, y \geq 1$ | $u \rightarrow 2'(1')^*1$ |
| 3F | $u = 1$ | $\xrightarrow{1}$ | $y = 0$ | $u \rightarrow 2$ |
| 4F | $u = 1'$ | $\xrightarrow{1'}$ | $x = 0$ | $u \rightarrow 2'$ |
| 5F | $u = 1$ $u = 2'$ | $\downarrow 1$ $\xleftarrow{2'}$ | $x = 1, y \geq 1$ | undefined |

The *final* F_1 -critical substring u of the word w is the F_1 -critical substring u with the highest starting index, taking the longest in the case of a tie. If there is still a tie (due to different representatives), take any such u . The unprimed raising and lowering operators may now be recalled.

Definition 37 ([12, Definition 5.4]). Let w be a word. The word $F_1(w)$ is obtained by taking a representative of w containing a final F_1 -critical substring and transforming it according to the previous table. If there is no F_1 -critical substring or it is of type 5F, then put $F_1(w) = \emptyset$, and in this case, F_1 is said to be undefined on w .

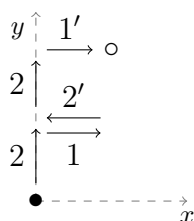
Lemma 38 ([12, Proposition 5.14 (i)]). Let w be a word on the alphabet $\{i', i, (i+1)', (i+1)\}$ and let (x, y) be the endpoint of the i -lattice walk of w . If $x = 0$, then $F_i(w) = \emptyset$.

The operators F_i are called the *unprimed lowering operators*. The *unprimed raising operators* E_i are defined on words by $E_i(w) := c_n F_{n-i} c_n(w)$, for $i \in I$. In particular, for $n = 2$, we have $E_1(w) = c_2 F_1(c_2(w))$, thus E_1 may be obtained in similar way using the following table of E_1 -critical substrings:

| Type | Substring | Condition steps | Location | Transformation |
|------|---------------------|---|------------------------------|--------------------------|
| 1E | $u = 2'(2)^*1$ | $\begin{array}{c} \uparrow_{2'} \uparrow_2 \xrightarrow{1} \\ \xleftarrow{2'} \uparrow_2 \xrightarrow{1} \end{array}$ | $x = 0$ $x = 1, y \geq 1$ | $u \rightarrow 1(2)^*1'$ |
| 2E | $u = 2'(1')^*2$ | $\begin{array}{c} \uparrow_{2'} \xrightarrow{1'} \uparrow_2 \\ \xleftarrow{2'} \xrightarrow{1'} \uparrow_2 \end{array}$ | $y = 0$ $y = 1, x \geq 1$ | $u \rightarrow 1(2)^*2'$ |
| 3E | $u = 2'$ | $\uparrow_{2'}$ | $x = 0$ | $u \rightarrow 1'$ |
| 4E | $u = 2$ | \uparrow_2 | $y = 0$ | $u \rightarrow 1$ |
| 5E | $u = 1$ $u = 2'$ | $\begin{array}{c} \downarrow_1 \\ \xleftarrow{2'} \end{array}$ | $y = 1, x \geq 1$ | undefined |

These definitions can be extended to tableaux: given $T \in \text{ShST}(\lambda/\mu, n)$, $F_i(T)$ is the shifted tableau with the same shape as T with (row) reading word $F_i(w(T))$, for $i \in I$. The definition of $E_i(T)$ is analogous. In both definitions, the row reading word of T may be replaced by the column reading word [12, Proposition 5.21]. This notions are well defined, since $E_i(T)$ and $F_i(T)$ are shifted semistandard tableaux, for all $i \in I$, whenever these operators are not undefined on T [12, Theorem 5.18]. Moreover, the primed and unprimed operators E'_i , E_i , F'_i and F_i commute whenever the compositions among them are defined, for each $i \in I$ [12, Proposition 5.36].

Example 39. Let $T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2' \\ \hline 2 & 3' & 3 & & \\ \hline & 3 & & & \\ \hline \end{array}$. To compute E_2 and F_2 , we consider the subword of $w(T)$ in the alphabet $\{2, 3\}'$, which is $323'32'$, with the following lattice walk (in the alphabet $\{1, 2\}'$):



Since T has a final E_2 -critical substring of type 2E, and a final F_2 -critical substring of type 4F. Thus, applying the correct substitution we obtain

$$E_2(T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2' \\ \hline 2 & 2 & 3' & & \\ \hline & 3 & & & \\ \hline \end{array} \quad F_2(T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 3' \\ \hline 2 & 3' & 3 & & \\ \hline & 3 & & & \\ \hline \end{array}.$$

The primed and unprimed operators may be used to give an alternative formulation for ballot (and anti-ballot) words. Indeed, a word w on the alphabet $\{i, i+1\}'$ is ballot (respectively anti-ballot) if and only if $E_i(w) = E'_i(w) = \emptyset$ (respectively $F_i(w) = F'_i(w) = \emptyset$) [12, Proposition 5.17]. Hence, a word on the alphabet $[n]'$ is ballot (respectively anti-ballot) if and only if $E_i(w) = E'_i(w) = \emptyset$ (respectively $E_i(w) = E'_i(w) = \emptyset$) for all $i \in I$.

Whenever they are defined, $E_i(T)$ and $F_i(T)$ are dual equivalent to T [12, Corollary 5.3]. Since this is also true for the primed operators, as the standardization is unchanged, then any two tableaux that differ by a sequence of any lowering or raising operators are dual equivalent. Moreover, the unprimed operators are coplactic, whenever defined [12, Theorem 5.35].

An *highest weight element* (respectively *lowest weight element*) of $\text{ShST}(\lambda/\mu, n)$ is a tableau T such that $E_i(T) = E'_i(T) = \emptyset$ (respectively $F_i(T) = F'_i(T) = \emptyset$), for any $i \in I$. This means that the reading word of T is ballot (respectively anti-ballot).

Proposition 40 ([12, Proposition 6.4]). *Let ν be a strict partition. We have that Y_ν is the unique $T \in \text{ShST}(\nu, n)$ for which $E_i(T) = E'_i(T) = \emptyset$, for all $i \in I$. Then, every $T \in \text{ShST}(\nu, n)$ may be obtained from every other by a sequence of primed and unprimed lowering and raising operators.*

The set $\text{ShST}(\lambda/\mu, n)$ is closed under the operators E_i, E'_i, F_i, F'_i , for $i \in I$. Moreover, we also have the *partial length functions* [11] given by:

$$\begin{aligned} \varepsilon'_i(T) &:= \max\{k : E_i^{k'}(T) \neq \emptyset\} & \widehat{\varepsilon}_i(T) &:= \max\{k : E_i^k(T) \neq \emptyset\} \\ \varphi'_i(T) &:= \max\{k : F_i^{k'}(T) \neq \emptyset\} & \widehat{\varphi}_i(T) &:= \max\{k : F_i^k(T) \neq \emptyset\}, \end{aligned}$$

and *total length functions* $\varepsilon_i(T)$ and $\varphi_i(T)$, defined as the y -coordinate and x -coordinate, respectively, of the endpoints of the i -th lattice walk of T , for $i \in I$ [12, Section 5.1].

The set $\text{ShST}(\lambda/\mu, n)$, together with primed and unprimed operators, partial and total length functions, and weight function, is called a *shifted tableau crystal* and denoted by $\mathcal{B}(\lambda/\mu, n)$. It may be regarded as a directed acyclic graph with weighted vertices, and i -coloured labelled double edges, the solid ones being labelled with i ($x \xrightarrow{i} y$ if $F_i(x) = y$) and the dashed ones with i' ($x \xrightarrow{i'} y$ if $F'_i(x) = y$), for $i \in I$. Examples are shown in Figures 2, and in Figures 8 and 9 of Appendix A. The connected components of $\mathcal{B}(\lambda/\mu, n)$ are the connected components of the underlying undirected and non-labelled graph. We also remark that the set $\text{ShST}(\lambda/\mu, n)$ together with only the primed (or the unprimed) operators and with the same auxiliary functions, wt , and total length functions φ_i, ε_i , satisfies the axioms of a Kashiwara crystal in type A [12, Proposition 6.9].

Proposition 41 ([12, Corollary 6.5]). *Each connected component of $\mathcal{B}(\lambda/\mu, n)$ has a unique highest weight element T^{high} , which is a LRS tableau, and is isomorphic, as a weighted edge-labelled graph, to the shifted tableau crystal $\mathcal{B}(\nu, n)$, where $\nu = \text{wt}(T^{\text{high}})$. Then, every T in a connected component of $\text{ShST}(\lambda/\mu, n)$ may be obtained from every other by a sequence of primed and unprimed lowering and raising operators.*

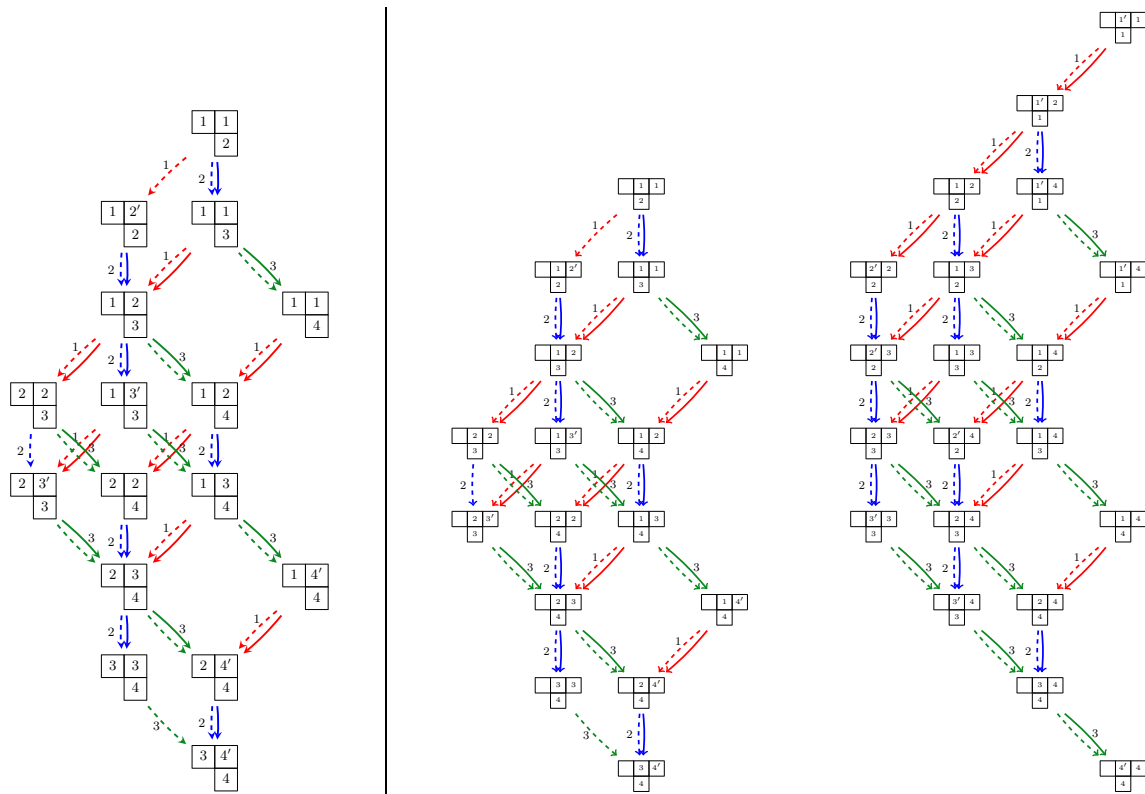


Figure 2: On the left, the shifted tableau crystal graph $\mathcal{B}(\nu, 4)$, for $\nu = (2, 1)$. On the right, the shifted tableau crystal graph $\mathcal{B}(\lambda/\mu, 4)$, for $\lambda = (3, 1)$ and $\mu = (1)$. The operators F_1, F'_1 are in red, the F_2, F'_2 in blue, and F_3, F'_3 in green. Note that $\mathcal{B}(\lambda/\mu, 4)$ has two connected components, one of them being isomorphic to $\mathcal{B}(\nu, 4)$.

Proposition 42 ([12, Corollary 6.6]). *Each connected component of $\mathcal{B}(\lambda/\mu, n)$ forms a shifted dual equivalence class.*

Therefore, by decomposing $\mathcal{B}(\lambda/\mu, n)$ into connected components, we have the crystal isomorphism

$$\mathcal{B}(\lambda/\mu, n) \simeq \bigsqcup_{\nu} \mathcal{B}(\nu, n)^{f_{\mu, \nu}^{\lambda}}$$

where $f_{\mu, \nu}^{\lambda}$ is the shifted Littlewood-Richardson coefficient, which yields the well known decomposition of skew Schur Q -function $Q_{\lambda/\mu} = \sum_{\nu} f_{\mu, \nu}^{\lambda} Q_{\nu}$ (for details, see [12, Section 7]).

3.1 Decomposition into strings

Erasing all arrows of colours $j, j' \notin \{i, i'\}$ in $\mathcal{B}(\lambda/\mu, n)$, for each $i \in I$, one obtains the $\{i', i\}$ -connected components of $\mathcal{B}(\lambda/\mu, n)$. This may also be done by defining an equivalence relation on $\mathcal{B}(\lambda/\mu, n)$ as a set, in which $x \sim y$ if x and y are related by a

sequence of F_i , F'_i , E_i or E'_i . The equivalence classes are called the i -strings, which are the underlying subsets of the $\{i', i\}$ -connected components.

Hence, $\mathcal{B}(\lambda/\mu, n)$ may be partitioned, as a set, into i -strings (see Figure 4). The i -strings, as crystal graphs, are the $\{i', i\}$ -connected components of $\mathcal{B}(\lambda/\mu, n)$, and have two possible arrangements [11, Section 3.1] [12, Section 8], as shown in Figure 3. A string consisting of two i -labelled chains of equal length, connected by i' -labelled edges is called a *separated i -string*. The smallest separated string is formed by two vertices connected by an i' -labelled edge. A string formed by a double chain of both i - and i' -labelled strings is called a *collapsed i -string*. A single vertex (without edges) is the smallest collapsed string.

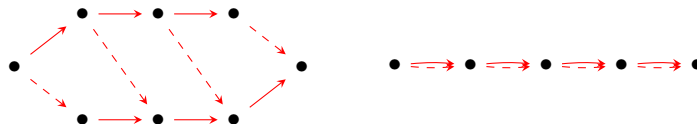


Figure 3: A separated i -string (left) and a collapsed i -string (right).

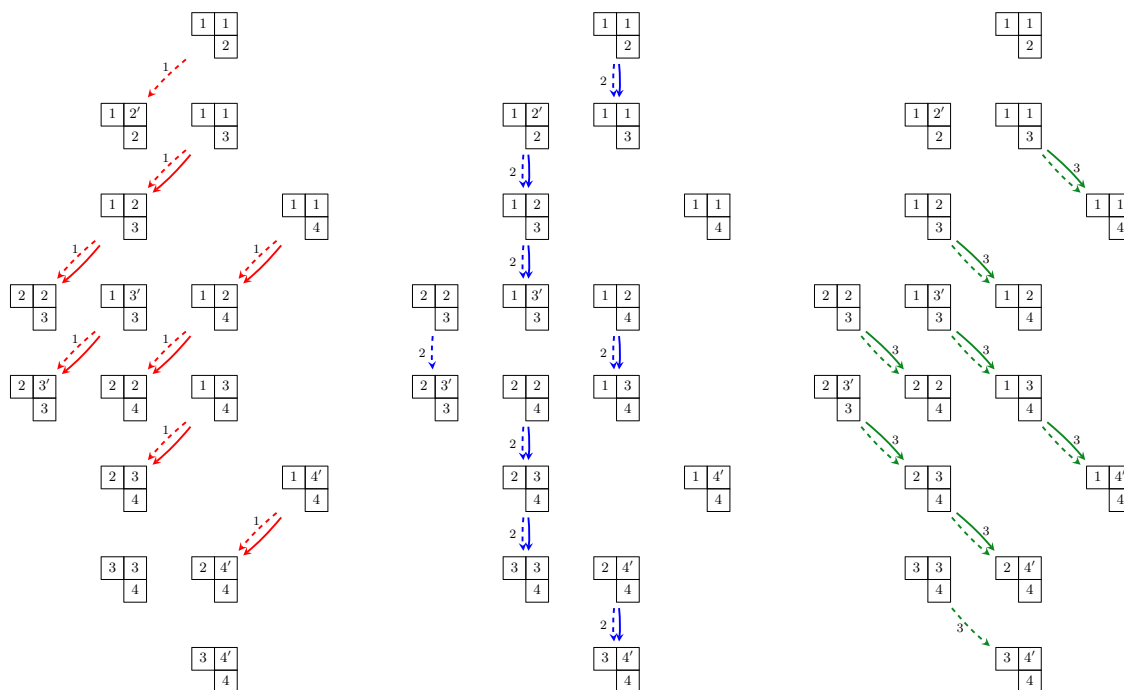


Figure 4: The crystal graph $\mathcal{B}(\nu, 4)$, for $\nu = (2, 1)$, partitioned into 1-strings (left), 2-strings (middle) and 3-strings (right).

The following propositions are intended to detail the possible arrangements for an i -string. This corresponds to the details of the axiom (B1) in [11]. The next result provides

a condition for an i -string to be separated or collapsed in terms of its highest weight element. The highest weight element of an i -string is the unique tableau T in that string such that $E_i(T) = E'_i(T) = \emptyset$, and the lowest weight element is defined similarly.

Proposition 43. *Let $T \in \mathcal{B}(\lambda/\mu, n)$ and $i \in I$. Suppose that T is the highest weight element of its i -string. Then, $F_i(T) = F'_i(T)$ if and only if $\text{wt}(T)_{i+1} = 0$. In this case, the i -string containing T is collapsed.*

Proof. To simplify the notation, we consider the alphabet $\{1, 2\}'$. Suppose that $\text{wt}(T)_2 = 0$. If $\text{wt}(T)_1 = 0$, then $F_1(T) = F'_1(T) = \emptyset$ and the 1-string is a trivial collapsed string. Thus, we may assume that $\text{wt}(T)_1 > 0$. Since $\text{wt}(T)_2 = 0$, there are no occurrences of $2'$ to the right of the last 1. Therefore, $F'_1(T)$ is defined and obtained by changing the last 1 into $2'$. Moreover, since $\text{wt}(F'_1(T))_2 = \text{wt}(T) + 1 = 1$, this $2'$ is the only occurrence in $F'_1(T)$ and is identified with 2 in canonical form. We claim that $w(T)$ has a final F_1 -critical substring that is not of type 5F and thus $F_1(T)$ is defined. Since $\text{wt}(T)_2 = 0$, the location of a possible substring is at $y = 0$, which excludes the type 5F. And since $\text{wt}(T)_1 > 1$, there is necessarily a substring of type 2F (with location at $x = 0$) or 3F (a 4F substring would be the case where either $w(T) = 1'$, which is equal to 1 in canonical form, or $w(T) = 1'(\mathbf{1})^*$, and again the first $1'$ would be identified with 1, and the substring would then be of type 2F or 3F). If $w(T)$ has a final F_1 -critical substring of type 2F, then F_1 changes the substring $11'$ into $2'1$, which is identified with 21 as this is the only occurrence of 2. If it is of type 3F, then F_1 changes the substring 1 into 2. In both cases, F_1 changes the last 1 into 2, coinciding with F'_1 .

We now suppose that $F_1(T) = F'_1(T)$. If they are both undefined, then $\text{wt}(T)_1 = 0$, which is a trivial case. Thus, we may assume that $F_1(T) = F'_1(T) \neq \emptyset$. Since $F'_1(T) \neq \emptyset$, we have $\text{wt}(T)_1 > 0$, and by Proposition 31, there are no occurrences of $2'$ to the right of the last 1 in T . We claim that there are no occurrences of 2 to the right of the last 1 as well. Suppose that there was at least one occurrence of 2 to the right of the last 1 of T . Then, this last 1 would be the first letter of a substring of the form $1(1')^*2$ (that is possibly followed by other occurrences of $1'$ and 2 to the right). By hypothesis, $E_1(T) = \emptyset$, which means that $w(T)$ has no final E_1 -critical substring, or has one of type 5E. The substring $1(1')^*2$ can have the following locations:

- $x \geq 0, y = 0$. In this case, we would have a final E_1 -critical substring of type 4E, which contradicts the hypothesis.
- $x = 0, y > 0$. As $y > 0$, there must be some occurrences of $\mathbf{2}$ to the left of the last 1 of T . The leftmost of these occurrences of $\mathbf{2}$ must be at location $y = 0$ and is identified with 2 in canonical form. Thus, we have a final E_1 -critical substring of type 4E, which is again a contradiction.
- $y = 1, x \geq 1$. For the location of 1 to be $y = 1$, there must be a $\mathbf{2}$ at location $y = 0$, which yields again a final E_1 -critical substring of type 4E.

- $y > 1, x \geq 1$. As there are no occurrences of 1 to the right (as this was the last 1), it is not possible to get any other substring at location $x = 0$. If $x = 1$, this substring is a final F_1 -critical substring of type 5F, and if $x > 1$, then there is no final F_1 -critical substring. Both cases contradict $F_1(T) \neq \emptyset$.

As all the cases lead to a contradiction, we conclude that there are no occurrences of 2 to the right of the last 1 of T . Since $F_1(T) = F'_1(T)$, we either have:

Case 1. F_1 changes the last 1 of T to 2. Then, as F'_1 changes the same 1 to $2'$, it must be the case where 2 and $2'$ are identified in canonical form, and thus, there are no occurrences of **2** to the left of the last 1 of T . Since there were no occurrences of $2'$ to the right of this 1 as well, we conclude that $\text{wt}(T)_2 = 0$.

Case 2. F_1 changes the last 1 of T to $2'$. For this to happen, $w(T)$ must have a final F_1 -critical substring of type 1F or 2F. If it was of type 1F, then there would be a $2'$ to the right of the last 1, and thus F'_1 would be undefined. Thus, the substring must be of type 2F. As there are no occurrences of **2** to the right of the last 1 of T , this must be the case where we have a substring $11'$ that is changed by F_1 to $2'1$. As F'_1 changes the substring $11'$ to $2'1'$, then the last 1 of $F_1(T)$ and the last $1'$ of $F'_1(T)$ must be identified in canonical form. For this to happen, there must be no occurrences of **1** to the left of the last 1 of T . Thus, if there are any letters to the left of the substring $11'$ of T , they must be occurrences of **2**. But as there are no occurrences of **1** to the left of 1, then $w(T)$ would have a final E_1 -critical substring, contradicting $E_1(T) = \emptyset$. Thus, there are no occurrences of **2** to the left of the last 1 of T , and consequently $\text{wt}(T)_2 = 0$. \square

The next lemmas concern the total length functions, which are the total distances from a vertex to the highest and lowest weight vertices of its i -string.

Lemma 44. *Let $T \in \mathcal{B}(\lambda/\mu, n)$ and let $i \in I$. Then,*

$$\varepsilon_i(T) = \begin{cases} \widehat{\varepsilon}_i(T) = \varepsilon'_i(T) & \text{if } T \text{ is in a collapsed } i\text{-string} \\ \widehat{\varepsilon}_i(T) + \varepsilon'_i(T) & \text{if } T \text{ is in a separated } i\text{-string.} \end{cases}$$

The result is also valid for φ_i with the adequate changes.

Proof. Suppose that T is in a collapsed i -string. Then, by Proposition 41, that collapsed string has a highest weight element T_0 , and so $T_0 = E_i^k(T)$, for some $k \geq 0$. Since T_0 is a highest weight element, then $E_i(T_0) = \emptyset$, hence $E_i^{k+1}(T) = \emptyset$. Consequently, $\widehat{\varepsilon}(T) = k$. On the other hand, T_0 is a LRS tableau (for the alphabet $\{i', i, (i+1)', i+1\}$), thus, by Proposition 34, the endpoint of the i -th lattice walk of its word has the y -coordinate equal to 0. The operator F_i shifts the endpoint of the i -th lattice walk by $(-1, 1)$ [12, Corollary 5.12]. Since $T_0 = E_i^k(T)$, then $T = F_i^k(T_0)$, and so the i -th lattice walk of T has the y -coordinate equal to k . Then, $\varepsilon_i(T) = k = \widehat{\varepsilon}_i(T)$.

Now suppose that T is in a separated i -string. This i -string has a highest weight element T_0 , which is a LRS tableau for the mentioned alphabet. Thus, the endpoint of

the i -th lattice walk of its word has the y -coordinate equal to zero. Then, we have two cases:

Case 1. Suppose that T is such that $E'_i(T) \neq \emptyset$. Let $T_1 := E'_i(T)$. By definition of E'_i , T_1 is obtained from T by replacing the last $(i+1)'$ that was right to the last i with i . Hence, $E'_i(T_1) = E_i'^2(T) = \emptyset$ and so

$$\varepsilon'_i(T) = 1. \quad (4)$$

Since T_0 is the highest weight element, we have that $E_i^k E'_i(T) = T_0$, for some $k \geq 0$. This is equivalent to $F'_i F_i^k(T_0) = T$, and since both F_i and F'_i shift the endpoint of the i -th lattice walk by $(-1, 1)$ [12, Proposition 4.9], the y -coordinate of the i -th lattice walk of the word of T must be equal to $k+1$. Hence,

$$\varepsilon_i(T) = k+1. \quad (5)$$

Since the operators E_i and E'_i commute when defined, we have $T_0 = E'_i E_i^k(T)$, and so $E_i^k(T) = F'_i(T_0) \neq \emptyset$ (recall that the shortest i -string is one with an i' -labelled edge). Thus, $E_i^{k+1}(T) = E_i F'_i(T_0) = \emptyset$, and we have

$$\widehat{\varepsilon}_i(T) = k. \quad (6)$$

By (4), (5) and (6), we have

$$\varepsilon_i(T) = \varepsilon'_i(T) + \widehat{\varepsilon}_i(T).$$

Case 2. Suppose now that T is such that $E'_i(T) = \emptyset$. Then, $\varepsilon'_i(T) = 0$. As in the previous case, there exists a highest weight element T_0 in this i -string, and the endpoint of the i -th lattice walk of its words has y -coordinate equal to zero. If $T = T_0$, then $\widehat{\varepsilon}_i(T) = 0$ and the proof is done. Otherwise, there exists $k > 0$ such that $T_0 = E_i^k(T)$, and so, $F_i^k(T_0) = T$. Consequently, the endpoint of the i -th lattice walk of the word of T has its y -coordinate equal to k . Hence,

$$\varepsilon_i(T) = k. \quad (7)$$

Moreover, $E_i^{k+1}(T) = E_i(T_0) = \emptyset$, as T_0 is a highest weight element. So,

$$\widehat{\varepsilon}_i(T) = k. \quad (8)$$

Hence, by (7) and (8), and since $\varepsilon'_i(T) = 0$, we have $\varepsilon_i(T) = \varepsilon'_i(T) + \widehat{\varepsilon}_i(T)$. \square

Lemma 45. Let $T \in \mathcal{B}(\lambda/\mu, n)$ and suppose its i -string has highest weight element T_0^{high} and lowest weight element T_0^{low} , and that $T \neq T_0^{\text{high}}, T_0^{\text{low}}$. The following holds:

1. If the i -string is separated, then

$$\begin{aligned} T_0^{\text{low}} &= F_i'^a F_i^b(T) = F_i^b F_i'^a(T) = F_i^{b-k} F_i'^a F_i^k(T), \text{ for some } k \geq 0 \\ T_0^{\text{high}} &= E_i'^c E_i^d(T) = E_i^d E_i'^c(T) = E_i^{d-k} E_i'^c E_i^k(T), \text{ for some } k \geq 0 \end{aligned}$$

with $a = \varphi'_i(T) \in \{0, 1\}$, $b = \hat{\varphi}_i(T) \geq 0$, $c = \varepsilon'_i(T) \in \{0, 1\}$, and $d = \hat{\varepsilon}_i(T) \geq 0$.

2. If the i -string is collapsed, then

$$\begin{aligned} T_0^{\text{low}} &= F_i^a(T) = F_i'^a(T) \\ T_0^{\text{high}} &= E_i^b(T) = E_i'^b(T) \end{aligned}$$

with $a = \varphi_i(T)$ and $b = \varepsilon_i(T)$.

Proof. We prove the case for the separated i -string and for the raising operators. For collapsed i -string, the proof is similar, noting that by Lemma 44 we have $\varepsilon_i(T) = \hat{\varepsilon}_i(T) = \varepsilon'_i(T)$. Let T be in a separated i -string. We have

$$E_i'(E_i'^c E_i^d(T)) = E_i'^{c+1} E_i^d(T) = E_i^d E_i'^{c+1}(T),$$

since these operators commute. By definition of $c = \varepsilon'_i(T)$, we have $E_i'^{c+1}(T) = \emptyset$. Hence $E_i'(E_i'^c E_i^d(T)) = E_i^d(\emptyset) = \emptyset$. On the other hand, we have $E_i(E_i'^c E_i^d(T)) = E_i'^c E_i^{d+1}(T)$. By definition of $d = \hat{\varepsilon}_i(T)$, $E_i^{d+1} = \emptyset$. Consequently,

$$E_i(E_i'^c E_i^d(T)) = E_i'^c(\emptyset) = \emptyset.$$

Thus, $E_i'^c E_i^d(T)$ must be the highest weight element of this i -string. \square

4 The Schützenberger involution and shifted crystal reflection operators

The Schützenberger involution, or Lusztig involution, is defined on the shifted tableau crystal [11, Section 2.3.1] in the same fashion as for type A Young tableau crystal. Similarly, we also realize it through shifted evacuation, for tableaux of straight shape, and through shifted reversal otherwise. For each $i \in I = [n-1]$, we define the shifted crystal reflection operator σ_i , using the primed and unprimed crystal operators E_i, E'_i, F_i and F'_i . We also show in Example 61 that they do not need to satisfy the braid relations and, therefore, do not yield a natural action of \mathfrak{S}_n on this crystal. Throughout this section ν will denote a strict partition.

Proposition 46. *Let $\mathcal{B}(\nu, n)$ denote the shifted tableau crystal with $T^{\text{high}} = Y_\nu$ as highest weight and $T_{\text{low}} = \text{evac}(Y_\nu)$ as lowest weight. Then, there exists a unique map of sets $\eta : \mathcal{B}(\nu, n) \rightarrow \mathcal{B}(\nu, n)$ that satisfies the following, for all $T \in \mathcal{B}(\nu, n)$ and for all $i \in I$:*

1. $E'_i \eta(T) = \eta F'_{n-i}(T)$.

$$2. E_i \eta(T) = \eta F_{n-i}(T).$$

$$3. F'_i \eta(T) = \eta E'_{n-i}(T).$$

$$4. F_i \eta(T) = \eta E_{n-i}(T).$$

$$5. \text{wt}(\eta(T)) = \text{wt}(T)^{\text{rev}}.$$

This map may be defined on $\mathcal{B}(\lambda/\mu, n)$ by extending it to its connected components. Moreover, it coincides with the evacuation in $\mathcal{B}(\nu, n)$, and with the reversal on the connected components of $\mathcal{B}(\lambda/\mu, n)$.

The map η is called the *Schützenberger or Lusztig involution*. We use the notation η for both straight-shaped and skew tableaux. The map η is indeed an involution on the set of vertices of $\mathcal{B}(\nu, n)$, that reverses all arrows and indices, thus sending the highest weight element to the lowest, i.e. $\eta(T^{\text{high}}) = T^{\text{low}}$, and vice versa. It is coplactic and a weight-reversing, shape-preserving involution. Note that the operator c_n also acts on $\mathcal{B}(\nu, n)$ by reversing arrows and indices, however it does not preserve the shape, although the resulting crystal $c_n(\mathcal{B}(\nu, n))$ is isomorphic as sets to $\mathcal{B}(\nu, n)$ and to $\text{evac}(\mathcal{B}(\nu, n))$, being isomorphic as crystal graph to the latter.

Proof of Proposition 46. It suffices to do the proof for $\mathcal{B}(\nu, n)$. We prove that the evacuation evac satisfies the assertions (1) and (2). Let $T \in \mathcal{B}(\nu, n)$ and let $i \in I$. Since evac is an involution then (3) and (4) are satisfied, and we have seen already that evac satisfies (5). By the definition of c_n , if $a < b$ in standardization ordering, then $c_n(b) < c_n(a)$ in the same ordering. Thus, $c_n(\text{std}(T)) = \text{std}(c_n(T))$, and consequently,

$$\text{std}(\text{evac}(T)) = \text{std}(\text{rect}(c_n(T))) \equiv_k \text{std}(c_n(T)) = c_n(\text{std}(T)).$$

Moreover, by Proposition 20, $\text{std}(\text{evac}(T))$ is shifted dual equivalent to $\text{std}(T)$, as they have the same straight shape. Then, as $\text{std}(\text{evac}(T))$ is shifted dual equivalent to $\text{std}(T)$ and shifted Knuth equivalent to $c_n(\text{std}(T))$, we have

$$\text{evac}(\text{std}(T)) = \text{std}(\text{evac}(T)).$$

By definition of the primed operators, $\text{std}(E'_i \text{evac}(T)) = \text{std}(\text{evac}(T))$, and thus we have

$$\begin{aligned} \text{std}(\text{evac} E'_i \text{evac}(T)) &= \text{evac}(\text{std}(E'_i \text{evac}(T))) \\ &= \text{evac}(\text{std}(\text{evac}(T))) \\ &= \text{std}(\text{evac}^2(T)) \\ &= \text{std}(T). \end{aligned}$$

Moreover, since $\alpha_i^{\text{rev}} = -\alpha_{n-i}$, we have

$$\begin{aligned} \text{wt}(\text{evac} E'_i \text{evac}(T)) &= \text{wt}(E'_i \text{evac}(T))^{\text{rev}} \\ &= (\text{wt}(\text{evac}(T)) + \alpha_i)^{\text{rev}} \\ &= (\text{wt}(T)^{\text{rev}} + \alpha_i)^{\text{rev}} \\ &= \text{wt}(T) + \alpha_i^{\text{rev}} \\ &= \text{wt}(T) - \alpha_{n-i} \end{aligned}$$

We have $\text{std}(\text{evac}E'_i\text{evac}(T)) = \text{std}(T)$ and $\text{wt}(\text{evac}E'_i\text{evac}(T)) = \text{wt}(T) - \alpha_{n-i}$, hence, by the definition of F'_{n-i} , we get $\text{evac}E'_i\text{evac}(T) = F'_{n-i}(T)$ and consequently,

$$E'_i\text{evac}(T) = \text{evac}F'_{n-i}(T).$$

To prove that $E_i\text{evac}(T) = \text{evac}F_{n-i}(T)$ we note that $E_i\text{evac}(T)$ and $F_{n-i}(T)$ are in the same connected component of $\mathcal{B}(\nu, n)$, hence they are dual equivalent due to Proposition 42. Thus, it remains to show that $E_i\text{evac}(T)$ and $\mathbf{c}_n(F_{n-i}(T))$ are shifted Knuth equivalent. We have that $w(\text{evac}(T)) \equiv_k w(\mathbf{c}_n(T))$ and since E_i is coplactic, we have $E_i(w(\text{evac}(T))) \equiv_k E_i(w(\mathbf{c}_n(T)))$. Then, we have

$$\begin{aligned} w(E_i(\text{evac}(T))) &= E_i(w(\text{evac}(T))) \\ &\equiv_k E_i(w(\mathbf{c}_n(T))) \\ &= \mathbf{c}_n F_{n-i} \mathbf{c}_n(w(\mathbf{c}_n(T))). \end{aligned} \tag{9}$$

By (2), we have $\mathbf{c}_n w(\mathbf{c}_n(T)) = w_{\text{col}}(\mathbf{c}_n^2(T)) = w_{\text{col}}(T)$. Moreover, the row and column words of a shifted semistandard tableau are shifted Knuth equivalent (see, for instance, [35, Lemma 6.4.12]). Thus, since \mathbf{c}_n and F_{n-i} are coplactic,

$$\begin{aligned} \mathbf{c}_n F_{n-i} \mathbf{c}_n(w(\mathbf{c}_n(T))) &= \mathbf{c}_n F_{n-i}(w_{\text{col}}(T)) \\ &\equiv_k \mathbf{c}_n F_{n-i}(w(T)) \\ &= \mathbf{c}_n w(F_{n-i}(T)). \end{aligned} \tag{10}$$

Finally, by (2), we have $\mathbf{c}_n w(F_{n-i}(T)) = w_{\text{col}}(\mathbf{c}_n F_{n-i}(T)) \equiv_k w(\mathbf{c}_n F_{n-i}(T))$. Thus, from (9) and (10) we have

$$w(E_i\text{evac}(T)) \equiv_k w(\mathbf{c}_n F_{n-i}(T))$$

and, consequently, $\text{evac}(F_{n-i}(T)) = E_i(\text{evac}(T))$. Finally, we note that the definition of the shifted evacuation ensures that $\text{wt}(\text{evac}(T)) = \text{wt}(\text{evac}(T))^{\text{rev}}$.

For the uniqueness part, suppose that there is another involution ξ on $\mathcal{B}(\nu, n)$ satisfying the previous properties and let Y_ν be the highest weight element of $\mathcal{B}(\nu, n)$. By Proposition 41, we have $T = H_{i_1} \cdots H_{i_k}(Y_\nu)$, where $H_i \in \{F'_i, F_i, E'_i, E_i\}$, with $i_j \in I$. Moreover, let \tilde{H}_i be E'_i (respectively, E_i , F'_i and F_i) if H_i is F'_i (respectively F_i , E'_i and E_i). Then,

$$\begin{aligned} \xi(T) &= \xi H_{i_1} \cdots H_{i_k}(Y_\nu) \\ &= \tilde{H}_{n-i_1} \cdots \tilde{H}_{n-i_k} \xi(Y_\nu) \\ &= \tilde{H}_{n-i_1} \cdots \tilde{H}_{n-i_k} \text{evac}(Y_\nu) \\ &= \text{evac} H_{i_1} \cdots H_{i_k}(Y_\nu) \\ &= \text{evac}(T). \end{aligned} \quad \square$$

Lemma 47. *Let $T \in \text{ShST}(\lambda/\mu, n)$ and let $i \in I$. Then,*

$$1. \varphi_i(T) = \varepsilon_{n-i}\eta(T).$$

$$2. \varepsilon_i(T) = \varphi_{n-i}\eta(T).$$

Proof. We prove the first assertion, the second one is analogous. Using Lemma 45, there are two cases:

1. Suppose that the i -component in which T lies is a collapsed i -string \mathcal{S}_i^0 . Then, by Lemma 45, $F_i^{\varphi_i(T)}(T)$ is the lowest weight element of \mathcal{S}_i^0 . We have that $\eta F_i^{\varphi_i(T)}(T)$ is in a $(n-i)$ -string \mathcal{S}_{n-i}^0 (which is also collapsed) and by Proposition 46,

$$\eta F_i^{\varphi_i(T)}(T) = E_{n-i}^{\varphi_i(T)}(\eta(T)).$$

Hence, by the definition of ε_{n-i} , we have that $\varepsilon_{n-i}(\eta(T)) \geq \varphi_i(T)$. On the other hand, since $E_{n-i}^{\varepsilon_{n-i}(\eta(T))}(\eta(T))$ is in \mathcal{S}_{n-i}^0 , then

$$\eta E_{n-i}^{\varepsilon_{n-i}(\eta(T))}(\eta(T))$$

must be in \mathcal{S}_i^0 . By Proposition 46, and since η is an involution, we have

$$\eta E_{n-i}^{\varepsilon_{n-i}(\eta(T))}(\eta(T)) = F_i^{\varepsilon_{n-i}(\eta(T))}(T).$$

Consequently, by the definition of φ_i , we have $\varphi_i(T) \geq \varepsilon_{n-i}(\eta(T))$.

2. Now suppose that T is in \mathcal{S}_i^0 , a separated i -string. Then, by Lemma 45, the shifted tableau $F_i^{\varphi_i'(T)} F_i^{\hat{\varphi}_i(T)}(T)$ is the lowest weight element of \mathcal{S}_i^0 . Consequently, $\eta F_i^{\varphi_i'(T)} F_i^{\hat{\varphi}_i(T)}(T)$ is in a $(n-i)$ -string \mathcal{S}_{n-i}^0 (which is also separated). As before, we have

$$\eta F_i^{\varphi_i'(T)} F_i^{\hat{\varphi}_i(T)}(T) = E_{n-i}^{\varphi_i'(T)} E_{n-i}^{\hat{\varphi}_i(T)}(\eta(T)),$$

and by definition of ε_{n-i} , we have $\varepsilon_{n-i}(\eta(T)) \geq \varphi_i'(T) + \hat{\varphi}_i(T) = \varphi_i(T)$.

Since $E_{n-i}^{\varepsilon_{n-i}'(\eta(T))} E_{n-i}^{\hat{\varepsilon}_{n-i}(\eta(T))}(\eta(T))$ is in \mathcal{S}_{n-i}^0 , we have that

$$\eta E_{n-i}^{\varepsilon_{n-i}'(\eta(T))} E_{n-i}^{\hat{\varepsilon}_{n-i}(\eta(T))}(\eta(T))$$

is in \mathcal{S}_{n-i}^0 . By Proposition 46, and since η is an involution, we have

$$\eta E_{n-i}^{\varepsilon_{n-i}'(\eta(T))} E_{n-i}^{\hat{\varepsilon}_{n-i}(\eta(T))}(\eta(T)) = F_i^{\varepsilon_{n-i}'(\eta(T))} F_i^{\hat{\varepsilon}_{n-i}(\eta(T))}(T),$$

and then, $\varphi_i(T) \geq \varepsilon_{n-i}'(\eta(T)) + \hat{\varepsilon}_{n-i}(\eta(T)) = \varepsilon_{n-i}(\eta(T))$. □

4.1 The shifted reflection crystal operators

We now introduce a shifted version of the *crystal reflection operators* σ_i (see [5, Definition 2.35]) on $\mathcal{B}(\nu, n)$, for each $i \in I$. Crystal reflection operators were originally defined by Lascoux and Schützenberger [26] in the Young tableau crystal of type A . They are involutions, on type A crystals, so that each i -string is sent to itself by reflection over its middle axis, for all $i \in I$. It coincides with the restriction of the Schützenberger involution to the tableaux consisting of the letters $i, i + 1$, ignoring the remaining ones. On $\mathcal{B}(\nu, n)$, collapsed strings are similar to the i -strings of type A crystals, hence the shifted reflection operator σ_i is expected to resemble the one for Young tableaux. However, for separated strings, a sole reflection of the i -string would not coincide with the restriction of the Schützenberger involution to $\{i, i + 1\}'$, hence we have the next definition.

Definition 48. Let $i \in I$ and $T \in \mathcal{B}(\nu, n)$. Let $k = \langle \text{wt}(T), \alpha_i \rangle$ (usual inner product in \mathbb{R}^n). The *shifted crystal reflection operator* σ_i is defined as follows

$$\sigma_i(T) = \begin{cases} F'_i F_i^{k-1}(T) & \text{if } k > 0 \text{ and } F'_i(T) \neq \emptyset \\ E'_i F_i^{k+1}(T) & \text{if } k > 0 \text{ and } F'_i(T) = \emptyset \\ E_i F'_i(T) & \text{if } k = 0 \text{ and } F'_i(T) \neq \emptyset \\ E'_i F_i(T) & \text{if } k = 0 \text{ and } F'_i(T) = \emptyset \text{ and } F_i(T) \neq \emptyset \\ T & \text{if } k = 0 \text{ and } F'_i(T) = F_i(T) = \emptyset \\ E_i^{-k+1} F'_i(T) & \text{if } k < 0 \text{ and } F'_i(T) \neq \emptyset \\ E_i^{-k-1} E'_i(T) & \text{if } k < 0 \text{ and } F'_i(T) = \emptyset \end{cases}$$

As the definition suggests, the shifted reflection operator σ_i must do a double reflection, by vertical and horizontal middle axes (see Figure 5). As we will see in Theorem 51, a simple reflection in the same fashion as the type A crystal fails to ensure the coincidence of the shifted crystal reflection operators with the adequate restriction of the Schützenberger involution, on separated strings. By coplacity, the operator σ_i is extended to $\mathcal{B}(\lambda/\nu, n)$, for $i \in I$.

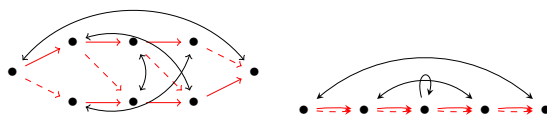


Figure 5: The action of a crystal reflection operator in separated and collapsed strings, which corresponds to the Schützenberger involution.

We remark that this definition is the same for both separated or collapsed strings. However, for the latter there is simpler formulation, as stated in the following lemma, since in this case the primed and unprimed operators coincide.

Lemma 49. Let $i \in I$ and let $T \in \text{ShST}(\lambda/\mu, n)$ be such that $F_i(T) = F'_i(T)$. Let $k = \langle \text{wt}(T), \alpha_i \rangle$. Then,

$$\sigma_i(T) = \begin{cases} F_i^k(T) & \text{if } k > 0 \\ T & \text{if } k = 0 \\ E_i^{-k}(T) & \text{if } k < 0 \end{cases}$$

Proposition 50. For $i \in I$, the operator σ_i satisfies the following:

1. σ_i sends each connected component of $\mathcal{B}(\lambda/\mu, n)$ to itself.
2. σ_i takes each i -string to itself.
3. $\sigma_i^2 = \text{id}$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$, if $|i - j| > 1$.
4. $\text{wt}(\sigma_i(T)) = \theta_i \cdot \text{wt}(T)$, where $\theta_i = (i, i + 1) \in \mathfrak{S}_n$.

Proof. The first two assertions result directly from the definition of the raising and lowering operators. For the third assertion, it is clear that $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i - j| > 1$, since each σ_k acts only on the primed subinterval of adjacent letters $\{k, k + 1\}'$, leaving the remaining ones unchanged. To prove the last statement concerning the weight of $\sigma_i(T)$, we note that in each case we obtain $\text{wt}(\sigma_i(T)) = \text{wt}(T) - k\alpha_i$, and then

$$\begin{aligned} \text{wt}(\sigma_i(T)) &= \text{wt}(T) - k\alpha_i \\ &= \text{wt}(T) - (\text{wt}(T)_i - \text{wt}(T)_{i+1})\alpha_i \\ &= (\text{wt}(T)_1, \dots, \text{wt}(T)_i - \text{wt}(T)_i + \text{wt}(T)_{i+1}, \\ &\quad \text{wt}(T)_{i+1} + \text{wt}(T)_i - \text{wt}(T)_{i+1}, \dots, \text{wt}(T)_n) \\ &= (\text{wt}(T)_1, \dots, \text{wt}(T)_{i+1}, \text{wt}(T)_i, \dots, \text{wt}(T)_n) \\ &= \theta_i \cdot (\text{wt}(T)). \end{aligned}$$

To prove that σ_i is an involution, we must analyse the cases according to Definition 48. Let $T \in \text{ShST}(\lambda/\mu, n)$. If T is the only element of its i -string, then $F'_i(T) = F_i(T) = \emptyset$ and $k = 0$, as $\text{wt}(T)_i = \text{wt}(T)_{i+1} = 0$, and so $\sigma_i^2(T) = \sigma_i(T) = T$. Thus, without loss of generality, we may assume that the i -string of T has at least two elements. Then, we have the following cases:

Case 1. Suppose that $k > 0$ and that $F'_i(T) \neq \emptyset$. Let $S = \sigma_i(T) = F'_i F_i^{k-1}(T)$. We have $\text{wt}(S) = \text{wt}(T) - k\alpha_i$, hence, putting $\tilde{k} := \text{wt}(S)_i - \text{wt}(S)_{i+1}$ we have

$$\begin{aligned} \tilde{k} &= (\text{wt}(T)_i - (k\alpha_i)_i) - (\text{wt}(T)_{i+1} - (k\alpha_i)_{i+1}) \\ &= \text{wt}(T)_i - k - \text{wt}(T)_{i+1} - k \\ &= -k < 0. \end{aligned}$$

Then, if $F'_i(S) = \emptyset$, we get

$$\begin{aligned}\sigma_i(S) &= E_i^{-(k)-1} E'_i(S) \\ &= E_i^{k-1} E'_i(S) \\ &= E_i^{k-1} E'_i F'_i F_i^{k-1}(T) \\ &= E_i^{k-1} F_i^{k-1}(T) = T.\end{aligned}$$

Otherwise, we have $F'_i(S) = F'_i F'_i F_i^{k-1}(T) = F_i^{k-1} \neq \emptyset$, and then T must be in a collapsed i -string. In this case, we have $S = F'_i F_i^{k-1}(T) = F_i^k(T)$ and thus

$$\begin{aligned}\sigma_i(S) &= E_i^{-(k)+1} F'_i(S) \\ &= E_i^{k+1} F'_i(S) \\ &= E_i^{k+1} F'_i F_i^k(T) \\ &= E_i^{k+1} F_i F_i^k(T) = T.\end{aligned}$$

Case 2. Now suppose T is such that $k > 0$ and $F'_i(T) = \emptyset$. As $k = \text{wt}(T)_i - \text{wt}(T)_{i+1} > 0$, T is not the lowest weight of its i -string. Since the string has at least two elements, we have $F_i(T) \neq \emptyset$, and T must be on a separated string and it is not the highest weight of it. Considering $S = \sigma_i(T) = E'_i F_i^{k+1}(T)$, we have $F'_i(S) = F'_i E'_i F_i^{k+1}(T) = F_i^{k+1}(T) \neq \emptyset$, and using the same notation as before,

$$\begin{aligned}\tilde{k} &= (\text{wt}(T)_i - (k\alpha_i)_i) - (\text{wt}(T)_{i+1} - (k\alpha_i)_{i+1}) \\ &= \text{wt}(T)_i - \text{wt}(T)_{i+1} - 2k \\ &= -k < 0.\end{aligned}$$

As $\sigma_i(T) \neq \emptyset$, then $F'_i(S) = F'_i E'_i F_i^{k+1}(T) = F_i^{k+1}(T) \neq \emptyset$, and thus

$$\begin{aligned}\sigma_i(S) &= E_i^{-(k)+1} F'_i(S) \\ &= E_i^{k+1} F_i^{k+1}(T) = T.\end{aligned}$$

Case 3. Suppose that T is such that $k = 0$ and $F'_i(T) \neq \emptyset$ and let $S = \sigma_i(T)$. We have $\text{wt}(S) = \text{wt}(T)$, hence $\tilde{k} = k$. If $F'_i(S) = \emptyset$, we have

$$\begin{aligned}\sigma_i(S) &= E_i F'_i(S) \\ &= E_i F'_i E'_i F_i(T) \\ &= E_i F_i(T) = T.\end{aligned}$$

Otherwise, if $F'_i(S) \neq \emptyset$, then T is in a collapsed string and $E'_i F_i(T) = E_i F'_i(T)$, and thus

$$\begin{aligned}\sigma_i(S) &= E'_i F_i(S) \\ &= E'_i F_i E'_i F_i(T) \\ &= E'_i F_i E_i F'_i(T) = T.\end{aligned}$$

Case 4. Suppose that T is such that $k = 0$ and $F'_i(T) = \emptyset$, and as its i -string has at least two elements, we have $F_i(T) \neq \emptyset$. Then, T must be in a separated string, and since $k = 0$, T is not a highest or lowest weight element of that string. Considering $S = \sigma_i(T) = E'_i F_i(T)$, we have $F'_i(S) = F'_i E'_i F_i(T) = F_i \neq \emptyset$, and thus

$$\begin{aligned}\sigma_i(S) &= E_i F'_i(S) \\ &= E_i F'_i E'_i F_i(T) = T.\end{aligned}$$

Case 5. Suppose that T is such that $k < 0$ and $F'_i(T) \neq \emptyset$. Considering $S = \sigma_i(T) = E_i^{-k+1} F'_i(T)$, we have $\text{wt}(S) = \text{wt}(T) - k\alpha_i$, and thus, using the same notation as before, we obtain $\tilde{k} = -k > 0$. Then, if $F'_i(S) = \emptyset$, we have

$$\begin{aligned}\sigma_i(S) &= E'_i F_i^{-k+1}(S) \\ &= E'_i F_i^{-k+1} E_i^{-k+1} F'_i(T) = T.\end{aligned}$$

Otherwise, if $F'_i(S) \neq \emptyset$, then T is in a collapsed string, in which case the operators F_i and F'_i coincide, and thus

$$\begin{aligned}\sigma_i(S) &= F'_i F_i^{-k-1}(S) \\ &= F'_i F_i^{-k-1} E_i^{-k+1} F'_i(T) \\ &= F_i^{-k+1} E_i^{-k+1}(T) = T.\end{aligned}$$

Case 6. Suppose that T is such that $k < 0$ and $F'_i(T) = \emptyset$. As the i -string of T has at least two elements, we have $F_i(T) \neq \emptyset$, which implies that T is in a separated string and is not its lowest weight element. Moreover, as $k < 0$, T is not the highest weight as well. Considering $S = \sigma_i(T) = E_i^{-k-1} E'_i(T)$, we have, with the same notation as before, $\tilde{k} = -k > 0$. We then have $F'_i(S) = F'_i E_i^{-k-1} E'_i(T) = F'_i E'_i E_i^{-k-1}(T) = E_i^{-k-1} \neq \emptyset$, and thus

$$\begin{aligned}\sigma_i(S) &= F'_i F_i^{-k-1}(S) \\ &= F'_i F_i^{-k-1} E_i^{-k-1} E'_i(T) = T.\end{aligned}$$

□

For each $i \in I$, let $\mathcal{B}_{i,i+1}$ be the subgraph of $\mathcal{B}(\lambda/\mu)$ consisting of the $\{i, i'\}$ -connected components. Given $T \in \text{ShST}(\lambda/\mu, n)$ and $i \in I$, let $T^{i,i+1} := T^i \sqcup T^{i+1}$ be the tableau obtained from T considering only the letters in $\{i, i+1\}'$, and let λ^i/μ^i denote the skew-shape of $T^{i,i+1}$. Let \mathcal{C} be the (unique) connected component of $\mathcal{B}_{i,i+1}$ containing T . We may ignore the entries of the tableaux in \mathcal{C} that are not in $\{i, i+1\}'$ and subtract $i-1$ from each entry and from each edge label, thus obtaining a connected component of the shifted tableau crystal $\mathcal{B}(\lambda^i/\mu^i, 2)$. Then, when we write $\eta(T^{i,i+1})$, we mean that we first apply the Schützenberger involution η to $T^{i,i+1}$ as an element of $\mathcal{B}(\lambda^i/\mu^i, 2)$, and then we add $i-1$ to the entries of the resulting tableau, to obtain one in the alphabet $\{i, i+1\}'$, in \mathcal{C} . Thus, we define

$$\eta_{i,i+1}(T) := T^1 \sqcup \cdots \sqcup T^{i-1} \sqcup \eta(T^{i,i+1}) \sqcup T^{i+2} \sqcup \cdots \sqcup T^n \in \mathcal{C}, \quad (11)$$

where η is the Schützenberger involution on $\mathcal{B}(\lambda^i/\mu^i, 2)$. This makes rigorous the restriction of the Schützenberger involution on $\mathcal{B}(\lambda/\mu)$ to the alphabet $\{i, i', i+1, i+1'\}$.

Theorem 51. *Let T be a shifted semistandard tableau on the alphabet $[n]'$. Then, for any $i \in I$,*

$$\sigma_i(T) = \eta_{i,i+1}(T).$$

It suffices to prove this result for tableaux on the primed alphabet of two adjacent letters and we consider it to be $\{1, 2\}'$, to simplify the notation. Moreover, the raising and lowering operators are coplactic, thus σ_1 is also coplactic. Hence, it suffices to prove the result for tableaux of straight shape. We remark that such tableaux have at most two rows. Furthermore, T and $\sigma_1(T)$ are in the same 1-string (which, in particular, is a connected component), hence by Proposition 42, T and $\sigma_1(T)$ are shifted dual equivalent. It remains to show that $c_2(T)$ and $\sigma_1(T)$ are shifted Knuth equivalent. We remark that another proof may be done by directly verifying the conditions on Proposition 46. The one we choose to present highlights some of the properties of straight-shaped tableaux with at most two rows. First, we introduce some technical results on shifted Knuth equivalence.

Lemma 52. *Let $a_1, \dots, a_n, b_1, \dots, b_m, c \in [n]'$, with $m, n \geq 1$.*

1. *If $b_m < \dots < b_1 < c < a_1 < \dots < a_n$ in standardization ordering, then*

$$ca_1 \dots a_n b_1 \dots b_m \equiv_k cb_1 \dots b_m a_1 \dots a_n.$$

2. *If $a_1 < \dots < a_n < c < b_m < \dots < b_1$ in standardization ordering, then*

$$a_1 \dots a_n b_1 \dots b_m c \equiv_k b_1 \dots b_m a_1 \dots a_n c.$$

Proof. We prove the first part by induction on n , the second part is proved similarly. If $n = 1$, we have

$$\begin{aligned} \underline{ca_1 b_1} b_2 \dots b_m &\equiv_k \underline{cb_1 a_1} b_2 \dots b_m && \text{by (K1)} \\ &\equiv_k \underline{cb_1 b_2 a_1} b_3 \dots b_m && \text{by (K1)} \\ &\dots \\ &\equiv_k \underline{cb_1 \dots b_{m-1} a_1} b_m && \text{by (K1)} \\ &\equiv_k \underline{cb_1 \dots b_{m-1} b_m} a_1 && \text{by (K1)} \end{aligned}$$

Now suppose the result is true for some $n \geq 1$ and let $a_{n+1} > a_n$ in standardization ordering. Then,

$$\begin{aligned} ca_1 \dots \underline{a_n a_{n+1} b_1} b_2 \dots b_m &\equiv_k ca_1 \dots a_n \underline{b_1 a_{n+1} b_2} b_3 \dots b_m && \text{by (K1)} \\ &\dots \\ &\equiv_k ca_1 \dots a_n b_1 \dots \underline{b_{m-1} a_{n+1} b_m} && \text{by (K1)} \\ &\equiv_k ca_1 \dots a_n b_1 \dots b_m a_{n+1} && \text{by (K1)} \\ &\equiv_k \underline{cb_1 \dots b_m} a_1 \dots a_n a_{n+1} && \text{by induction hypothesis + Lemma 17} \end{aligned}$$

□

Lemma 53. Let $a \in [n]'$. Then, for any $m \geq 1$, $a(a')^m \equiv_k a^{m+1}$.

Proof. For $m = 1$ the result corresponds to the (S2) relation. Suppose the result holds for some $m \geq 1$. Then,

$$\begin{aligned} a(a')^{m+1} &= a(a')^m a' \\ &\equiv_k (a)^{m+1} a' && \text{by induction hypothesis + Lemma 16} \\ &= a(a)^m a' \\ &\equiv_k \underline{aa'}(a)^m && \text{by Lemma 52} \\ &\equiv_k aa(a)^m && \text{by (S2)} \\ &= (a)^{m+2}. \end{aligned}$$

□

In order to prove that $c_2(T)$ and $\sigma_1(T)$ are shifted Knuth equivalent, we will present sequences of Knuth moves between their words. We have remarked that T has at most two rows. The case where it has one row is in the following result.

Proposition 54. Let T be a shifted semistandard tableau with one row, filled with the alphabet $\{1, 2\}'$. Then, $\sigma_1(T) = \text{evac}(T)$.

Proof. As stated before, it suffices to show that $\sigma_1(T)$ is shifted Knuth equivalent to $c_2(T)$. Suppose that $w(T) = 1^a$, for $a \geq 1$. If $a = 1$, then $w(c_2(T)) = 2 = w(F_1(T)) = w(\sigma_1(T))$. If $a > 1$, then $w(c_2(T)) = 2(2')^{a-1}$ and $w(\sigma_1(T)) = w(F_1^a(T)) = 2^a$. Hence, by Lemma 53, $w(c_2(T)) \equiv_k w(\sigma_1(T))$. Now suppose that $w(T) = 1^a 2^b$, with $a, b \geq 1$. Then, $w(\sigma_1(T)) = 1^b 2^a$ and $w(c_2(T)) = 2(2')^{a-1} 1(1')^{b-1}$. There are two cases:

Case 1. If $a = 1$, we have

$$\begin{aligned} \underline{21}(1')^{b-1} &\equiv_k 12(1')^{b-1} && \text{by (S1)} \\ &\equiv_k 1(1')^{b-1} 2 && \text{by Lemma 17} \\ &\equiv_k 1^b 2. && \text{by Lemmas 17 and 53} \end{aligned}$$

Case 2. If $a > 1$, then we have

$$\begin{aligned} 2(2')^{a-1} 1(1')^{b-1} &\equiv_k 2^a 1(1')^{b-1} && \text{by Lemmas 53 and 17} \\ &= 22^{a-1} 1(1')^{b-1} \\ &\equiv_k \underline{21}(1')^{b-1} 2^{a-1} && \text{by Lemma 52} \\ &\equiv_k 12(1')^{b-1} 2^{a-1} && \text{by (S1)} \\ &\equiv_k 1(1')^{b-1} 22^{a-1} && \text{by Lemmas 52 and 17} \\ &\equiv_k 1^b 2^a. && \text{by Lemmas 53 and 16} \end{aligned}$$

□

If T has two rows, we remark that it suffices to verify the case where the second row has only one box. To make this statement rigorous, we need to introduce some notation. A shifted semistandard tableau T is called *detached* if its main diagonal has exactly one box. Then, we may define the following operator on shifted semistandard tableaux:

$$r(T) = \begin{cases} T & \text{if } T \text{ is detached} \\ \widehat{T} & \text{otherwise} \end{cases}$$

where \widehat{T} is obtained from T by removing its main diagonal and shifting every box one unit to the left (so that its second diagonal becomes the main diagonal).

Example 55. If $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2' \\ \hline 2 & 2 & 2 & \\ \hline \end{array}$, then $r(T) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2' \\ \hline 2 & 2 & \\ \hline \end{array}$, $r^2(T) = \begin{array}{|c|c|} \hline 1 & 2' \\ \hline 2 & \\ \hline \end{array}$, and $r^m(T) = \begin{array}{|c|} \hline 2 \\ \hline \end{array}$, for $m \geq 3$.

The following lemma states that, if T is not detached and its $(l+1)$ -st diagonal is the first with one box, then $\sigma_1(T)$ is determined by $r^{l-1}(T)$, i.e., one may temporarily remove the first $l-1$ diagonals with two elements, compute σ_1 on the remaining tableau, and then place the diagonals back.

Lemma 56. *Let T be a shifted semistandard tableau of straight shape, with two rows, filled with the alphabet $\{1, 2\}'$. Let l be such that $\{(1, l), (2, l+1)\}$ and $\{(1, l+1)\}$ are adjacent diagonals of T with two and one box, respectively. Then,*

$$r^{l-1}\sigma_1(T) = \sigma_1 r^{l-1}(T).$$

Proof. If $T = Y_\nu$, for $\nu = (\nu_1, \nu_2)$, then $wt(\sigma_1(Y_\nu)) = (\nu_2, \nu_1)$. Consequently, $\sigma_1(Y_\nu) = \text{evac}(Y_\nu)$. Then, $r^{\nu_2-1}\text{evac}(Y_\nu) = \text{evac}(Y_{\nu^0})$, where $\nu^0 = (\nu_1 - \nu_2 + 1, 1)$. Similarly, $r^{\nu_2-1}(Y_\nu) = Y_{\nu^0}$, and using the same argument with the weight, $\sigma_1 r^{\nu_2-1}(Y_\nu) = \text{evac}(Y_{\nu^0})$. The proof for $\text{evac}(Y_\nu)$ is similar.

Suppose now that T is neither Y_ν nor $\text{evac}(Y_\nu)$. Suppose that the word of T is given by $w(T) = 2^a 1^{a+1} 1^b 2^c$, with $a \geq 1$ and $b, c \geq 0$. Then, $\text{wt}(T) = (a+b+1, a+c+1)$ and considering Definition 48, we have $k = (a+b+1) - (a+c+1) = b-c$ (note that it does not depend on a). We show the case when $k > 0$ and $\mathbf{2} = 2$. The proof for the other cases is analogous. If $\mathbf{2} = 2$, then $F'_1(T) \neq \emptyset$ and we have

$$\begin{aligned} \sigma_i(T) &= F'_1 F_1^{b-c-1}(T) \\ &= F'_1 F_1^{b-c-1}(2^a 1^{a+1} 1^b 2^{c+1}) \\ &= F'_1(2^a 1^{a+1} 1^{b-(b-c-1)} 2^{(c+1)+(b-c-1)}) \\ &= F'_1(2^a 1^{a+1} 1^{c+1} 2^b) \\ &= 2^a 1^{a+1} 1^c 2' 2^b \end{aligned}$$

and so, $r^{a-1}\sigma_1(T) = 21^2 1^c 2' 2^b$.

On the other hand, we have $\mathbf{r}^{a-1}(T) = 21^2 1^b 22^c$ and so

$$\begin{aligned}\sigma_1 r^{a-1}(T) &= F_1' F_1^{b-c-1} (21^2 1^b 2^{c+1}) \\ &= F_1' (21^2 1^{c+1} 2^b) \\ &= 21^2 1^c 2' 2^b.\end{aligned}$$

In what follows, we consider T to be of shape $\nu = (m, 1)$, i.e., such that its second row has only one box. To show that $\mathbf{c}_2(T)$ is shifted Knuth equivalent to $\sigma_1(T)$ is equivalent to show that $\mathbf{rect}(\mathbf{c}_2(T)) = \sigma_1(T)$, since T is of straight shape. Moreover, we ask for T to not be Y_ν or $\mathbf{evac}(Y_\nu)$, since the result for those cases is already proved. We have the following lemma, which is easy to prove.

Lemma 57. *Let $\nu = (m, 1)$, for $m \geq 3$. Let $T \in \text{ShST}(\nu, 2)$ such that $T \neq Y_\nu, \text{evac}(Y_\nu)$ and let $k = \varepsilon_1(T)$. Then,*

1. If $T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \dots & 1 & 2' & 2 & \dots & 2 \\ \hline & 2 & & & & & & & \\ \hline \end{array}$, with $\text{wt}(T) = (m - k, k + 1)$, then

$$\sigma_1(T) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \dots & 1 & 2 & 2 & \dots & 2 \\ \hline & 2 & & & & & & & \\ \hline \end{array}$$

with $\text{wt}(\sigma_1(T)) = (k+1, m-k)$.

2. If $T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \dots & 1 & 2 & 2 & \dots & 2 \\ \hline & 2 & & & & & & & \\ \hline \end{array}$, with $\text{wt}(T) = (m - k, k + 1)$, then

$$\sigma_1(T) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \dots & 1 & 2' & 2 & \dots & 2 \\ \hline & 2 & & & & & & & \\ \hline \end{array}$$

with $\text{wt}(\sigma_1(T)) = (k+1, m-k)$.

The rectification process does not depend on the sequence of inner corners, so, for simplicity, we may fix that we always choose the rightmost inner corner in the highest-index row. Thus, we apply *jeu de taquin* slides on $c_2(T)$, following this sequence, to the point where an occurrence of 2 or $2'$ on T (which correspond to $1'$ or 1 in $c_2(T)$) will determine different slides on the next move. For instance, consider the following tableaux:

$T_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline & & 2 & \\ \hline \end{array} \longrightarrow c_2(T_1) = \begin{array}{|c|c|c|c|} \hline & & & 1' \\ \hline & & & 1' \\ \hline & & & 2' \\ \hline & & 1 & 2' \\ \hline & & & 2 \\ \hline \end{array} \equiv_k \begin{array}{|c|c|c|c|} \hline & & & 1' \\ \hline & & & 1' \\ \hline & & 1 & 2' \\ \hline & & & 2 \\ \hline \end{array}$

the box (2, 4) with $\color{red}{1}'$ will go **left** on the next slide.

$T_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2' \\ \hline & & 2 & \\ \hline \end{array} \longrightarrow c_2(T_2) = \begin{array}{|c|c|c|c|} \hline & & & 1' \\ \hline & & & 1 \\ \hline & & & 2' \\ \hline & & 1 & 2' \\ \hline & & & 2 \\ \hline \end{array} \equiv_k \begin{array}{|c|c|c|c|} \hline & & & 1' \\ \hline & & & 1 \\ \hline & & 1 & 2' \\ \hline & & & 2 \\ \hline \end{array}$

the box (3,3) with 1 will go **up** on the next slide.

Continuing the rectification process we obtain, respectively:

$$\text{rect}(\mathbf{c}_2(T_1)) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2' & 2 \\ \hline & 2 & & & \\ \hline \end{array} \quad \text{rect}(\mathbf{c}_2(T_2)) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline & 2 & & & \\ \hline \end{array}.$$

We now state some auxiliary results.

Lemma 58. *Let $T = Y_\nu$ be the highest weight of its 1-string, with $m \geq 3$, for $\nu = (m, 1)$. Then,*

1. $w(\mathbf{c}_2 F_1(T)) = 21(2')^{m-2}1' \equiv_k 212'1'(2)^{m-3}.$
2. $w(\mathbf{c}_2 F'_1(T)) = 21(2')^{m-2}1 \equiv_k 212'1(2)^{m-3}.$

Proof. We prove a more general claim that, if $a \geq 1$, then,

$$\begin{aligned} 21(2')^a 1 &\equiv_k 212'1(2)^{a-1} \\ 21(2')^a 1' &\equiv_k 212'1'(2)^{a-1}. \end{aligned}$$

Then, the result follows, observing that $m \geq 3$ ensures that $a := m - 2 \geq 1$. If $a = 1$, the claim is trivial. Suppose this is true for some $a \geq 1$. We have,

$$21(2')^{a+1}1 = 21(2')^a 2'1.$$

The word $21(2')^a 2'$ has the same standardization as $21(2')^a 1$. Therefore, by the induction hypothesis $21(2')^a 2' \equiv_k 212'2'(2)^{a-1}$. Then, we have

$$\begin{aligned} 21(2')^a 2'1 &\equiv_k \underline{212'2'}(2)^{a-1}1 && \text{Lemma 16} \\ &\equiv_k \underline{22'12'}(2)^{a-1}1 && \text{(K2)} \\ &\equiv_k \underline{2212'}(2)^{a-1}1 && \text{(S2)} \\ &\equiv_k \underline{2122'}(2)^{a-1}1 && \text{(K1)} \\ &\equiv_k \underline{1222'}(2)^{a-1}1 && \text{(S1)} \\ &\equiv_k \underline{122'2}(2)^{a-1}1 && \text{(K1)} \\ &\equiv_k 212'2(2)^{a-1}1 && \text{(S1)} \\ &= 212'(2)^a 1. \end{aligned}$$

Moreover, we have $2'(2)^a 1 \equiv_k 2'1(2)^a$, by Lemma 52, using only (K1) Knuth moves. Hence, by Lemma 17, we have

$$212'(2)^a 1 \equiv_k 212'1(2)^a.$$

Consequently, $21(2')^a 1 \equiv_k 212'1(2)^{a-1}$. The proof that $21(2')^a 1' \equiv_k 212'1'(2)^{a-1}$ is done similarly, since Lemma 52 also ensures that $2'(2)^a 1' \equiv_k 2'1'(2)^a$, using only (K1) Knuth moves. \square

Corollary 59. *Let T be a shifted semistandard tableaux in the 1-string of Y_ν , with $\nu = (m, 1)$ and $m \geq 3$, such that T is not Y_ν or $\text{evac}(Y_\nu)$. Let $a = \varepsilon_1(T)$.*

1. If $T = F_1^a(Y_\nu)$, then $w(c_2(T)) = 21(2')^{m-a-1}1'(1')^{a-1} \equiv_k 212'1'(2)^{m-a-2}(1')^{a-1}$.
2. If $T = F_1'F_1^{a-1}(Y_\nu)$, then $w(c_2(T)) = 21(2')^{m-a-1}1(1')^{a-1} \equiv_k 212'1(2)^{m-a-2}(1')^{a-1}$.

Proof. Since $T \neq \text{evac}(Y_\nu)$ and $\text{evac}(Y_\nu)$ is a lowest weight, then

$$a = \varepsilon_1(T) < \varepsilon_1(\text{evac}(Y_\nu)) = m - 1.$$

Then, $a \leq m - 2$ and so we have that $m - a - 1 \geq 1$. Therefore, using Lemma 58, we have

$$21(2')^{m-a-1}1 \equiv_k 212'1(2)^{m-a-2}.$$

Consequently, by Lemma 16, we have $21(2')^{m-a-1}1(1')^{a-1} \equiv_k 212'1(2)^{m-a-2}(1')^{a-1}$. The proof for the second case is similar. \square

Proposition 60. *Let T be a shifted semistandard tableau in the 1-string of Y_ν , with $\nu = (m, 1)$ and $m \geq 3$, such that T is not Y_ν or $\text{evac}(Y_\nu)$. Let $a = \varepsilon_1(T)$.*

1. *If $T = F_1^a(Y_\nu)$, then $212'1'(2)^{m-a-2}(1')^{a-1} \equiv_k 2(1)^{a+1}2'(2)^{m-a-2} = \sigma_1(T)$.*
2. *If $T = F_1'F_1^a(Y_\nu)$, then $212'1(2)^{m-a-2}(1')^{a-1} \equiv_k 2(1)^{a+1}2(2)^{m-a-2} = \sigma_1(T)$.*

Proof. We first prove the first assertion. We have

$$\begin{aligned}
212'1(2)^{m-a-2}(1')^{a-1} &\equiv_k \underline{212'}1(1')^{a-1}(2)^{m-a-2} && \text{Lemmas 52 and 16} \\
&\equiv_k \underline{22'}11(1')^{a-1}(2)^{m-a-2} && \text{(K2)} \\
&\equiv_k \underline{2211}(1')^{a-1}(2)^{m-a-2} && \text{(S2)} \\
&\equiv_k \underline{2121}(1')^{a-1}(2)^{m-a-2} && \text{(K2)} \\
&\equiv_k \underline{1221}(1')^{a-1}(2)^{m-a-2} && \text{(S1)} \\
&\equiv_k \underline{1212}(1')^{a-1}(2)^{m-a-2} && \text{(K1)} \\
&\equiv_k \underline{121}(1')^{a-1}2(2)^{m-a-2} && \text{Lemma 17} \\
&\equiv_k \underline{211}(1')^{a-1}2(2)^{m-a-2} && \text{(S1)} \\
&\equiv_k \underline{21}(1')^{a-1}12(2)^{m-a-2} && \text{Lemmas 52, 16, and 17} \\
&\equiv_k \underline{12}(1')^{a-1}12(2)^{m-a-2} && \text{(S1)} \\
&\equiv_k 1(1')^{a-1}212(2)^{m-a-2} && \text{Lemmas 52 and 16} \\
&\equiv_k (1)^a212(2)^{m-a-2} && \text{Lemmas 53 and 17} \\
&\equiv_k 2(1)^a12(2)^{m-a-2} && \text{Lemmas 52 and 17} \\
&= 2(1)^{a+1}2(2)^{m-a-2}.
\end{aligned}$$

For the second assertion, we have

$$\begin{aligned}
212'1'(2)^{m-a-2}(1')^{a-1} &\equiv_k 212'1'(1')^{a-1}(2)^{m-a-2} && \text{Lemma 52} \\
&= 212'(1')^a(2)^{m-a-2} \\
&\equiv_k \underline{21}(1')^a2'(2)^{m-a-2} && \text{Lemma 52} \\
&\equiv_k 12(1')^a2'(2)^{m-a-2} && \text{(S1)} \\
&\equiv_k 1(1')^a22'(2)^{m-a-2} && \text{Lemma 52} \\
&\equiv_k (1)^{a+1}22'(2)^{m-a-2} && \text{Lemma 53} \\
&\equiv_k 2(1)^{a+1}2'(2)^{m-a-2}. && \text{Lemma 52}
\end{aligned}$$

□

We are now able to prove Theorem 51.

Proof of Theorem 51. It suffices to show the result for T of straight shape with two rows. Corollary 59 and Proposition 60 ensure that the words $w(\mathbf{c}_2(T))$ and $w(\sigma_1(T))$ are shifted Knuth equivalent, thus $\mathbf{c}_2(T) \equiv_k \sigma_1(T)$. Since T and $\sigma_1(T)$ are dual equivalent, this concludes the proof that $\sigma_1(T) = \mathbf{evac}(T)$. □

As a direct consequence of Proposition 40 and Proposition 46, we have that the evacuation of a Yamanouchi tableau is the lowest weight of its crystal, i.e., $F'_i(\mathbf{evac}(T)) = F_i(\mathbf{evac}(T)) = \emptyset$, for all $i \leq I$.

Unlike the type A crystals, the reflection operators σ_i , for $i \in I$, do not define an action of the symmetric group \mathfrak{S}_n on $\mathcal{B}(\lambda/\mu, n)$. In particular, the braid relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ do not need to hold, as shown in the next example.

Example 61. Let $\mathcal{B}(\nu, 3)$ where $\nu = (5, 3, 1)$, and consider the shifted semistandard tableau

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 3' \\ \hline 2 & 2 & 3' & & \\ \hline & & 3 & & \\ \hline \end{array}$$

The weight of T is given by $\mathbf{wt}(T) = (4, 2, 3)$. Then, since $\langle \mathbf{wt}(T), (1, -1, 0) \rangle = 4 - 2 = 2 > 0$ and $F'_1(T) \neq \emptyset$, we have

$$\sigma_1(T) = F'_1 F_1(T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2' & 2 & 3' \\ \hline 2 & 2 & 3' & & \\ \hline & & 3 & & \\ \hline \end{array}$$

Putting $T_1 := \sigma_1(T)$, we have that $\langle \mathbf{wt}(T_1), (0, 1, -1) \rangle = 4 - 3 = 1 > 0$. As $F'_2(T_1) = \emptyset$, we have

$$\sigma_2\sigma_1(T) = E'_2 F_2^2(T_1) = E'_2(\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2' & 3' & 3 \\ \hline 2 & 3' & 3 & & \\ \hline & & 3 & & \\ \hline \end{array}) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2' & 2 & 3 \\ \hline 2 & 3' & 3 & & \\ \hline & & 3 & & \\ \hline \end{array}$$

And putting $T_2 := \sigma_2\sigma_1(T)$, we have $\langle \mathbf{wt}(T_2), (1, -1, 0) \rangle = 2 - 3 = -1 < 0$ and $F'_1(T_2) = \emptyset$. Thus,

$$\sigma_1\sigma_2\sigma_1(T) = \sigma_1(T_2) = E'_1(T_2) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline & 2 & 3' & 3 & \\ \hline & & 3 & & \\ \hline \end{array} \quad (12)$$

On the other hand, we have that $\langle \text{wt}(T), (0, 1, -1) \rangle = 2 - 3 = -1 < 0$ and $F'_2(T) = \emptyset$, hence

$$\sigma_2(T) = E'_2(T) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 \\ \hline & 2 & 2 & 3' & \\ \hline & & 3 & & \\ \hline \end{array}$$

We put $T_3 := \sigma_2(T)$, thus we have that $\langle \text{wt}(T_3), (1, -1, 0) \rangle = 4 - 3 = 1 > 0$ and $F'_1(T_3) \neq \emptyset$ and consequently

$$\sigma_1\sigma_2(T) = \sigma_1(T_3) = F'_1(T_3) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2' & 2 \\ \hline & 2 & 2 & 3' & \\ \hline & & 3 & & \\ \hline \end{array}$$

Finally, putting $T_4 := \sigma_1\sigma_2(T)$, we have that $\langle \text{wt}(T_4), (0, 1, -1) \rangle = 4 - 2 = 2 > 0$ and $F'_2(T_4) \neq \emptyset$, thus

$$\sigma_2\sigma_1\sigma_2(T) = \sigma_2(T_4) = F'_2F_2(T_4) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2' & 3' & \\ \hline & 2 & 3' & 3 & & \\ \hline & & 3 & & & \\ \hline \end{array} \quad (13)$$

Then, by (12) and (13), we have that $\sigma_1\sigma_2\sigma_1(T) \neq \sigma_2\sigma_1\sigma_2(T)$.

However, we have the following result, as in [3, 2, Section 3.2] for ordinary LR tableaux, ensuring that the longest permutation of \mathfrak{S}_n acts on a connected component of $\mathcal{B}(\lambda/\mu, n)$ by sending the highest weight element to the lowest weight element.

Theorem 62. *Let T be a LRS tableau in $\mathcal{B}(\lambda/\mu, n)$. Let $\omega_0 = \theta_{i_1} \cdots \theta_{i_k}$ be the longest permutation in \mathfrak{S}_n . Then, ω_0 acts on a connected component of $\mathcal{B}(\lambda/\mu, n)$ by sending the highest weight element T to the lowest, $\sigma_{i_1} \cdots \sigma_{i_k}(T) = T^e$.*

Proof. Since the operators σ_i are coplactic, we may consider $Y_\nu = \text{rect}(T)$, $\nu = \text{wt}(T)$. By Proposition 50, σ_j permutes the entries j and $j + 1$ on the weight and keeps the shape ν , and as ω_0 is the longest permutation, $\sigma_{i_1} \cdots \sigma_{i_k}$ reverses the weight of T . Then, from Proposition 25, we have $\sigma_{i_1} \cdots \sigma_{i_k} Y_\nu = \text{evac}(Y_\nu)$. \square

Remark 63. Let $G_n := \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ be the free group generated by the shifted crystal reflection operators σ_i , for $i \in I$, modulo the relations satisfied by them on shifted semistandard tableaux. We know from Proposition 50 that the relations $\sigma_i^2 = id$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i - j| > 1$, hold on G_n , but not the braid relations of \mathfrak{S}_n , $(\sigma_i\sigma_{i+1})^3 = id$, for $i \in [n - 2]$. However, since $\mathcal{B}(\nu, n)$ is finite, we have that, given $T \in \mathcal{B}(\nu, n)$, there must exist some $m > 3$ such that $(\sigma_i\sigma_{i+1})^m(T) = T$, for $i \in [n - 2]$. We computed some examples on the alphabet $\{1, 2, 3\}'$ (see Appendix A), which show that $(\sigma_1\sigma_2)^m = id$, for m a multiple of at least 90, but we do not know if an upper bound valid for any shape ν exists.

5 A cactus group action on the shifted tableau crystal

In this section we show that the adequate restrictions of the shifted Schützenberger involution, which include the crystal reflection operators, define an action of the cactus group J_n on the shifted tableau crystal $\mathcal{B}(\lambda/\mu, n)$. This follows a similar approach to Halacheva [16, 17], where this action is shown for any \mathfrak{g} -crystal, for \mathfrak{g} a complex reductive finite-dimensional Lie algebra, which includes the classical Lie algebras.

Let $1 \leq p < q \leq n$ and define $[p, q] := \{p < \dots < q\}$. Let $\theta_{p,q-1}$ denote the longest permutation in $\mathfrak{S}_{[p,q-1]}$ embedded in \mathfrak{S}_{n-1} , that is, $\theta_{p,q-1}$ sends i to $p + q - i - 1$ if $i \in [p, q - 1]$, leaving i unchanged otherwise, and put $\theta := \theta_{1,n-1}$.

In what follows, and without loss of generality, we will consider the shifted tableau crystal $\mathcal{B}(\nu, n)$ of straight shape. Given $T \in \mathcal{B}(\nu, n)$ and $1 \leq p < q \leq n$, let $T^{p,q} := T^p \sqcup T^{p+1} \sqcup \dots \sqcup T^q$, the tableau obtained from T considering only the entries in $[p, q]'$. In particular, we have $T^{1,n} = T$. By convention, we set $T^{p,p} := T^p$ and $T^{1,0} = T^{n+1,n} := \emptyset$.

For each $1 \leq p < q \leq n$, let $\mathcal{B}_{p,q}$ be the subgraph of $\mathcal{B}(\nu, n)$ where the edges coloured in $I \setminus [p, q - 1]$ were removed, and the vertices are the same as in $\mathcal{B}(\nu, n)$ but the letters in $[n]' \setminus [p, q]'$ are ignored. In particular, $\mathcal{B}_{p,p+1}$ is the collection of all the p -strings. For each $1 \leq p < q \leq n$, the set $\mathcal{B}(\nu, n)$ is partitioned into classes consisting of the underlying sets of the connected components $\mathcal{B}_{p,q}$, by imposing an equivalence relation on the set $\mathcal{B}(\nu, n)$, defined by $x \sim y$ if x and y are related by a sequence of any lowering or raising operators coloured in $[p, q - 1]$. The equivalence classes are the underlying subsets of the $[p, q]'$ -connected components. A highest weight element of $\mathcal{B}_{p,q}$ is a tableau T such that $E_i(T) = E'_i(T) = \emptyset$, for any $i \in [p, q - 1]$ and a lowest weight element is defined similarly.

Lemma 64. *Let $1 \leq p < q \leq n$. Each connected component of $\mathcal{B}_{p,q}$ has unique highest and lowest weight elements.*

Proof. If $\{p, \dots, q\} = \{i, i + 1\}$, then $\mathcal{B}_{i,i+1}$ is a union of i -strings, each one having unique highest and lowest weight elements. Suppose now that the cardinality of $\{p, \dots, q\}$ is larger than 2. Each connected component \mathcal{B}^0 of $\mathcal{B}_{p,q}$ has vertices the shifted tableaux of shape ν filled with letters $[p, q]'$, ignoring the letters in $[n]' \setminus [p, q]'$. Thus, given $T \in \mathcal{B}^0$, we may regard it as shifted tableaux of skew shape λ_0^1/λ_0^2 , for strict partitions $\lambda_0^2 \subseteq \lambda_0^1 \subseteq \lambda$, where λ_0^2 corresponds to the part of tableaux corresponding to the letters $[1, p - 1]'$ and λ/λ_0^1 corresponding to $[q + 1, n]'$. These parts indeed define the same shapes within each connected components, since the operators corresponding to the edges coloured in $\{p, \dots, q - 1\}$ leave them unchanged. Then, similarly to the construction of $\eta_{i,i+1}$ in (11), we may relabel the filling of these tableaux by

$$\begin{aligned} \rho_{p,q}: [p, q]' &\longrightarrow [1, q - p + 1]' \\ i &\longmapsto i - p + 1 \\ i' &\longmapsto (i - p + 1)' \end{aligned} \tag{14}$$

obtaining tableaux of shape λ_0^1/λ_0^2 with the alphabet $[1, q - p + 1]'$. Hence, after rectification, each connected component \mathcal{B}^0 may be identified with a connected component

of $\mathcal{B}(\lambda_0^1/\lambda_0^2, q - p + 1)$, which is isomorphic to $\mathcal{B}(\nu_0, q - p + 1)$, and consequently, by Proposition 41 it has unique highest and lowest weight elements. \square

Recall that, given a shifted semistandard tableau $T \in \mathcal{B}(\nu, n)$, $\eta(T)$ is the unique tableau shifted dual equivalent to T and shifted Knuth equivalent to $c_n(T)$ [15], and coincides with the evacuation if T is of straight shape. The following definition formalizes the restriction of the Schützenberger involution to an interval $[p, q]'$.

Given $T \in \mathcal{B}(\nu, n)$, let \mathcal{B}^0 be the connected component of $\mathcal{B}_{p,q}$ containing T . Similarly to (11), we write $\eta(T^{p,q})$ to mean that we first apply the Schützenberger involution η to $T^{p,q}$ as an element of $\mathcal{B}(\lambda_0^1/\lambda_0^2, q - p + 1)$ and then add $q - p + 1$ to the entries of the obtained tableau, obtaining one in the alphabet $[p, q]'$, in \mathcal{B}^0 . Thus, we have the following definition.

Definition 65. For $1 \leq p < q \leq n$, let $\eta_{p,q} : \mathcal{B}(\nu, n) \rightarrow \mathcal{B}(\nu, n)$ be such that

$$\eta_{p,q}(T) := T^{1,p-1} \sqcup \eta(T^{p,q}) \sqcup T^{q+1,n}.$$

In particular, we have $\eta_{1,n}(T) = \eta(T)$, and, from Theorem 51, $\eta_{i,i+1} = \sigma_i$.

Lemma 66. Let $1 \leq p < q \leq n$ and let $i \in [p, q - 1]$. Given $T \in \mathcal{B}(\nu, n)$, we have, whenever the operators are defined:

1. $E'_i \eta_{p,q}(T) = \eta_{p,q} F'_{\theta_{p,q-1}(i)}(T)$.
2. $E_i \eta_{p,q}(T) = \eta_{p,q} F_{\theta_{p,q-1}(i)}(T)$.
3. $F'_i \eta_{p,q}(T) = \eta_{p,q} E'_{\theta_{p,q-1}(i)}(T)$.
4. $F_i \eta_{p,q}(T) = \eta_{p,q} E_{\theta_{p,q-1}(i)}(T)$.
5. $\text{wt}(\eta_{p,q}(T)) = \theta_{p,q-1} \cdot \text{wt}(T)$.

Proof. If $[p, q - 1] = [1, n - 1]$, this corresponds to Proposition 46. Otherwise, we remark that since the crystal operators are coplactic, we may consider the subinterval $[p, q - 1]$ of $[1, n - 1]$. \square

Definition 67 ([19]). The n -fruit cactus group J_n is the free group with generators $s_{p,q}$, for $1 \leq p < q \leq n$, subject to the relations:

1. $s_{p,q}^2 = id$.
2. $s_{p,q} s_{k,l} = s_{k,l} s_{p,q}$ for $[p, q] \cap [k, l] = \emptyset$.
3. $s_{p,q} s_{k,l} = s_{p+q-l, p+q-k} s_{p,q}$ for $[k, l] \subseteq [p, q]$.

Example 68. For $n = 2$, we have $J_2 = \langle s_{1,2} | s_{1,2}^2 = 1 \rangle = C_2$, the cyclic group of order 2. For $n = 3$, we have $J_3 = \langle s_{1,2}, s_{1,3}, s_{2,3} | s_{1,2}^2 = s_{1,3}^2 = s_{2,3}^2 = 1, s_{1,3} s_{1,2} = s_{2,3} s_{1,3} \rangle$, which is infinite.

Note that there exists an epimorphism $J_n \longrightarrow \mathfrak{S}_n$, that sends $s_{p,q}$ to $\theta_{p,q}$. The kernel of this surjection is known as the *pure cactus group* and denoted by PJ_n (see [19, Section 3.4]). In the next result, we show that the cactus group J_n acts on the shifted tableau crystal $\mathcal{B}(\lambda/\mu, n)$ via the shifted Schützenberger involutions $\eta_{p,q}$ of Definition 65. An example is shown in Figure 6.

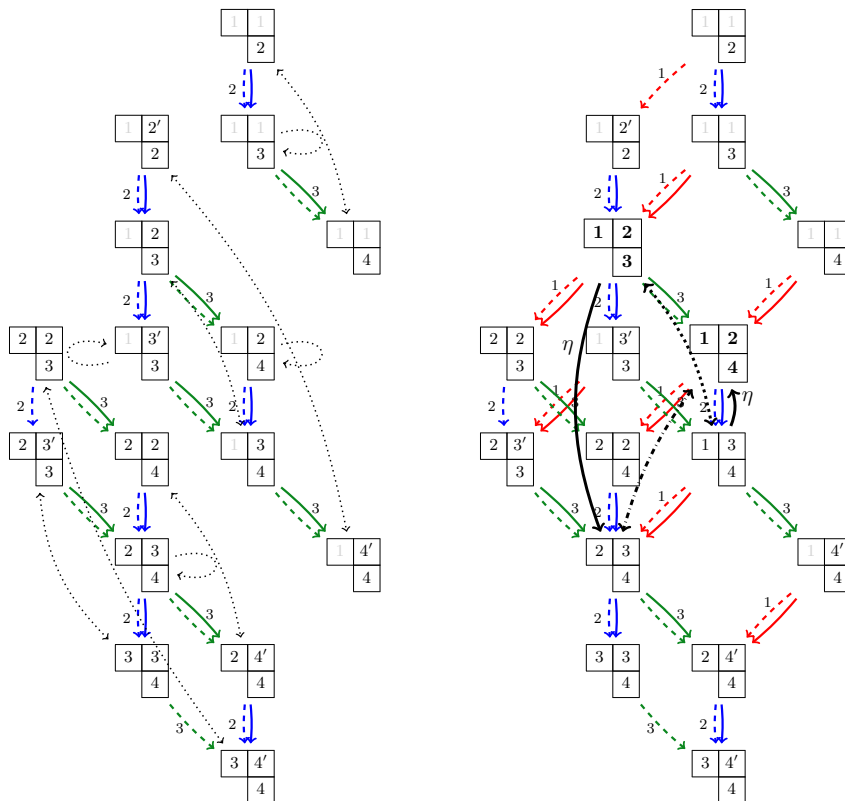


Figure 6: On the left, the action of $s_{2,4}$ on $\mathcal{B}(\nu, 4)$, with $\nu = (2, 1)$. On the right, an illustration of $s_{1,3}s_{1,4} = s_{1,4}s_{2,4}$.

Theorem 69. *There is a natural action of the n -fruit cactus group J_n on the shifted tableau crystal $\mathcal{B}(\lambda/\mu, n)$ given by the group homomorphism:*

$$\begin{aligned} \phi : J_n &\longrightarrow \mathfrak{S}_{\mathcal{B}(\lambda/\mu, n)} \\ s_{p,q} &\longmapsto \eta_{p,q} \end{aligned}$$

for $1 \leq p < q \leq n$.

Proof. To show that ϕ is a group homomorphism, we show that the operators $\eta_{p,q}$ satisfy the cactus group relations of Definition 67, for any $1 \leq p < q \leq n$. Due to Proposition 41, it suffices to do this for shifted crystals of the form $\mathcal{B}(\nu, n)$, as every connected component

of $\mathcal{B}(\lambda/\mu, n)$ is isomorphic to these, via rectification. Since the operator η is an involution on each connected component of $\mathcal{B}_{p,q}$, it follows that $s_{p,q}^2 = id$ for all $1 \leq p < q \leq n$. The second relation is a direct consequence of the definition. Suppose that $[p, q] \cap [k, l] = \emptyset$, and without loss of generality assume that $1 \leq p < q < k < l \leq n$. We remark that each $\eta_{p,q}$ acts on the entries of $[p, q]'$, leaving the entries in $[n]' \setminus [p, q]'$ unchanged. Thus,

$$\begin{aligned}\eta_{p,q}\eta_{k,l}(T) &= T^{1,p-1} \sqcup \eta(T^{p,q}) \sqcup T^{q+1,k-1} \sqcup \eta(T^{k,l}) \sqcup T^{l+1,n} \\ &= \eta_{k,l}\eta_{p,q}(T^{1,p-1} \sqcup T^{p,q} \sqcup T^{q+1,k-1} \sqcup T^{k,l} \sqcup T^{l+1,n}).\end{aligned}$$

For the third relation, we claim that it suffices to show that, for any $[k, l] \subseteq [1, q]$,

$$\eta_{1,q}\eta_{k,l} = \eta_{q+1-l,q+1-k}\eta_{1,q}. \quad (15)$$

We will show that (15) implies the third relation $\eta_{p,q}\eta_{k,l} = \eta_{p+q-l,p+q-k}\eta_{p,q}$ of Definition 67. Given $[k, l] \subseteq [p, q]$, we have, in particular, $[p, q] \subseteq [1, q]$, and thus, by (15),

$$\eta_{1,q}\eta_{p,q} = \eta_{1,q+1-p}\eta_{1,q}. \quad (16)$$

Moreover, $[k, l] \subseteq [p, q]$ implies that $[k-p+1, l-p+1] \subseteq [1, q-p+1]$ and thus, by (15),

$$\eta_{1,q+1-p}\eta_{k+1-p,l+1-p} = \eta_{q+1-l,q+1-k}\eta_{1,1+q-p}. \quad (17)$$

Likewise, $[k, l] \subseteq [p, q]$ implies that $[p+q-l, p+q-k] \subseteq [p, q] \subseteq [1, q]$ and again by (15), we have

$$\eta_{1,q}\eta_{p+q-l,p+q-k} = \eta_{k+1-p,l+1-p}\eta_{1,q}. \quad (18)$$

Then, for $[k, l] \subseteq [p, q]$, and using the fact that $\eta_{p,q}$ is an involution, for any $[p, q]$, we have

$$\begin{aligned}\eta_{p,q}\eta_{k,l} &= \eta_{p,q}\eta_{1,q}\eta_{q+1-l,q+1-k}\eta_{1,q} && \text{by (15)} \\ &= \underline{\eta_{1,q}\eta_{1,q+1-p}\eta_{1,q}}\eta_{1,q}\eta_{q+1-l,q+1-k}\eta_{1,q} && \text{by (16)} \\ &= \eta_{1,q}\eta_{1,q+1-p}\eta_{q+1-l,q+1-k}\eta_{1,q} \\ &= \eta_{1,q}\eta_{1,q+1-p}\underline{\eta_{1,q+1-p}\eta_{k+1-p,l+1-p}\eta_{1,q+1-p}}\eta_{1,q} && \text{by (17)} \\ &= \eta_{1,q}\eta_{k+1-p,l+1-p}\eta_{1,q+1-p}\eta_{1,q} \\ &= \eta_{1,q}\underline{\eta_{1,q}\eta_{p+q-l,p+q-k}\eta_{1,q}}\eta_{1,q+1-p}\eta_{1,q} && \text{by (18)} \\ &= \eta_{p+q-l,p+q-k}\eta_{1,q}\eta_{1,q+1-p}\eta_{1,q} \\ &= \eta_{p+q-l,p+q-k}\eta_{p,q} && \text{by (15)}\end{aligned}$$

We will now prove (15). Given $T \in \mathcal{B}(\nu, n)$, if T is an isolated vertex of $\mathcal{B}_{k,l}$, then the result is trivially true. Hence, we assume that T is in a connected component \mathcal{B}_0 of $\mathcal{B}_{k,l}$ with at least two vertices. By Lemma 64, this connected component \mathcal{B}_0 has a unique highest weight element, T_0^{high} , as well as a lowest weight element,

$$T_0^{\text{low}} = \eta_{k,l}(T_0^{\text{high}}), \quad (19)$$

which is also unique by Lemma 66. Since we assume that \mathcal{B}_0 has at least two elements, we have that $T_0^{\text{high}} \neq T_0^{\text{low}}$. Moreover, \mathcal{B}_0 is contained in $\mathcal{B}_{1,q}$, and the latter has a unique highest weight element T_1^{high} and a lowest weight element $\eta_{1,q}(T_1^{\text{high}})$. Then, we have:

$$\begin{aligned} T &= F_{i_1}^{m_1} F_{i_1}^{n_1} \cdots F_{i_r}^{m_r} F_{i_r}^{n_r} (T_0^{\text{high}}) \\ T_0^{\text{low}} &= E_{j_1}^{a_1} E_{j_1}^{b_1} \cdots E_{j_s}^{a_s} E_{j_s}^{b_s} \eta_{1,q}(T_1^{\text{high}}) \end{aligned} \quad (20)$$

for some $i_1, \dots, i_r \in [k, l-1]$, $j_1, \dots, j_s \in [1, q-1]$, $m_i, a_j \in \{0, 1\}$, $n_i, b_j \geq 0$. Consequently,

$$\begin{aligned} \eta_{1,q} \eta_{k,l}(T) &= \eta_{1,q} \eta_{k,l} F_{i_1}^{m_1} F_{i_1}^{n_1} \cdots F_{i_r}^{m_r} F_{i_r}^{n_r} (T_0^{\text{high}}) && \text{by (20)} \\ &= \eta_{1,q} E_{\theta_{k,l-1}(i_1)}^{m_1} E_{\theta_{k,l-1}(i_1)}^{n_1} \cdots E_{\theta_{k,l-1}(i_r)}^{m_r} E_{\theta_{k,l-1}(i_r)}^{n_r} \eta_{k,l}(T_0^{\text{high}}) && \text{by Lemma 66} \\ &= \eta_{1,q} E_{\theta_{k,l-1}(i_1)}^{m_1} E_{\theta_{k,l-1}(i_1)}^{n_1} \cdots E_{\theta_{k,l-1}(i_r)}^{m_r} E_{\theta_{k,l-1}(i_r)}^{n_r} (T_0^{\text{low}}) && \text{by (19)} \\ &= \eta_{1,q} E_{\theta_{k,l-1}(i_1)}^{m_1} E_{\theta_{k,l-1}(i_1)}^{n_1} \cdots E_{\theta_{k,l-1}(i_r)}^{m_r} E_{\theta_{k,l-1}(i_r)}^{n_r} \\ &\quad E_{j_1}^{a_1} E_{j_1}^{b_1} \cdots E_{j_s}^{a_s} E_{j_s}^{b_s} \eta_{1,q}(T_1^{\text{high}}) && \text{by (20)} \\ &= F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{m_1} F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{n_1} \cdots F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{m_r} F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{n_r} \\ &\quad F_{\theta_{1,q-1}(j_1)}^{a_1} F_{\theta_{1,q-1}(j_1)}^{b_1} \cdots F_{\theta_{1,q-1}(j_s)}^{a_s} F_{\theta_{1,q-1}(j_s)}^{b_s} (\eta_{1,q}^2 T_1^{\text{high}}) && \text{by Lemma 66} \\ &= F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{m_1} F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{n_1} \cdots F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{m_r} F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{n_r} \\ &\quad F_{\theta_{1,q-1}(j_1)}^{a_1} F_{\theta_{1,q-1}(j_1)}^{b_1} \cdots F_{\theta_{1,q-1}(j_s)}^{a_s} F_{\theta_{1,q-1}(j_s)}^{b_s} (T_1^{\text{high}}). \end{aligned}$$

Thus, we have

$$\begin{aligned} \eta_{1,q} \eta_{k,l}(T) &= F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{m_1} F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{n_1} \cdots F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{m_r} F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{n_r} \\ &\quad F_{\theta_{1,q-1}(j_1)}^{a_1} F_{\theta_{1,q-1}(j_1)}^{b_1} \cdots F_{\theta_{1,q-1}(j_s)}^{a_s} F_{\theta_{1,q-1}(j_s)}^{b_s} (T_1^{\text{high}}). \end{aligned} \quad (21)$$

If X and Y are shifted tableaux in the connected component \mathcal{B}_0 of $\mathcal{B}_{k,l}$, then $X \sim Y$ and thus $X = H_{i_1} \cdots H_{i_s}(Y)$, where $H_i \in \{F'_i, F_i, E'_i, E_i\}$, for some $i_1, \dots, i_s \in [k, l-1]$. Then, denoting the partial inverse of H_i by \tilde{H}_i , Lemma 66 ensures that

$$\eta_{1,q}(X) = \eta_{1,q} H_{i_1} \cdots H_{i_s}(Y) = \tilde{H}_{\theta_{1,q-1}(i_1)} \cdots \tilde{H}_{\theta_{1,q-1}(i_s)} \eta_{1,q}(Y),$$

which implies that $\eta_{1,q}(X) \sim \eta_{1,q}(Y)$ in $\mathcal{B}_{\theta_{1,q-1}[k,l]} = \mathcal{B}_{q+1-l, q+1-k}$. Thus, $\eta_{1,q}$ takes the connected component \mathcal{B}_0 to a connected component $\eta_{1,q}(\mathcal{B}_0)$ of $\mathcal{B}_{q+1-l, q+1-k}$. Then, Lemma 47 ensures that, for any Q in \mathcal{B}_0 and for any $i \in [k, l-1] \subseteq [1, q-1]$, $\varepsilon_i(T) = \varphi_{\theta_{1,q-1}(i)} \eta_{1,q}(T)$ and $\varphi_i(T) = \varepsilon_{\theta_{1,q-1}(i)} \eta_{1,q}(T)$. Consequently, $\eta_{1,q}(T_0^{\text{low}})$ and $\eta_{1,q}(T_0^{\text{high}})$ are, respectively, the highest and lowest weight elements of the connected component $\eta_{1,q}(\mathcal{B}_0)$. On the other hand, as $\eta_{1,q}(\mathcal{B}_0)$ is a connected component of $\mathcal{B}_{q+1-l, q+1-k}$, we have that $\eta_{q+1-l, q+1-k}$ interchanges its highest and lowest weight elements. Thus,

$$\eta_{q+1-l, q+1-k} \eta_{1,q}(T_0^{\text{high}}) = \eta_{1,q}(T_0^{\text{low}}). \quad (22)$$

Then, noting that $\theta_{1,q-1}\theta_{k,l-1} = \theta_{q+1-l,q-k}\theta_{1,q-1}$, we have

$$\begin{aligned}
\eta_{q+1-l,q+1-k}\eta_{1,q}(T) &= \eta_{q+1-l,q+1-k}\eta_{1,q}F_{i_1}^{m_1}F_{i_1}^{n_1}\cdots F_{i_r}^{m_r}F_{i_r}^{n_r}(T_0^{\text{high}}) && \text{by (20)} \\
&= \eta_{q+1-l,q+1-k}E_{\theta_{1,q-1}(i_1)}^{m_1}E_{\theta_{1,q-1}(i_1)}^{n_1}\cdots \\
&\quad \cdots E_{\theta_{1,q-1}(i_r)}^{m_r}E_{\theta_{1,q-1}(i_r)}^{n_r}(\eta_{1,q}(T_0^{\text{high}})) && \text{by Proposition 46} \\
&= F_{\theta_{q+1-l,q-k}\theta_{1,q-1}(i_1)}^{m_1}F_{\theta_{q+1-l,q-k}\theta_{1,q-1}(i_1)}^{n_1}\cdots \\
&\quad \cdots F_{\theta_{q+1-l,q-k}\theta_{1,q-1}(i_r)}^{m_r}F_{\theta_{q+1-l,q-k}\theta_{1,q-1}(i_r)}^{n_r} \\
&\quad \eta_{q+1-l,q+1-k}\eta_{1,q}(T_0^{\text{high}}) && \text{by Lemma 66} \\
&= F_{\theta_{q+1-l,q-k}\theta_{1,q-1}(i_1)}^{m_1}F_{\theta_{q+1-l,q-k}\theta_{1,q-1}(i_1)}^{n_1}\cdots \\
&\quad \cdots F_{\theta_{q+1-l,q-k}\theta_{1,q-1}(i_r)}^{m_r}F_{\theta_{q+1-l,q-k}\theta_{1,q-1}(i_r)}^{n_r}\eta_{1,q}(T_0^{\text{low}}) && \text{by (22)} \\
&= F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{m_1}F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{n_1}\cdots \\
&\quad \cdots F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{m_r}F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{n_r} \\
&\quad \eta_{1,q}(E_{j_1}^{a_1}E_{j_1}^{b_1}\cdots E_{j_s}^{a_s}E_{j_s}^{b_s}\eta_{1,q}(T_0^{\text{high}})) && \text{by (20)} \\
&= F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{m_1}F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{n_1}\cdots \\
&\quad \cdots F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{m_r}F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{n_r} \\
&\quad F_{\theta_{1,q-1}(j_1)}^{a_1}F_{\theta_{1,q-1}(j_1)}^{b_1}\cdots F_{\theta_{1,q-1}(j_s)}^{a_s}F_{\theta_{1,q-1}(j_s)}^{b_s}(T_0^{\text{high}}), && \text{by Proposition 46}
\end{aligned}$$

and thus

$$\begin{aligned}
\eta_{q+1-l,q+1-k}\eta_{1,q}(T) &= F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{m_1}F_{\theta_{1,q-1}\theta_{k,l-1}(i_1)}^{n_1}\cdots F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{m_r}F_{\theta_{1,q-1}\theta_{k,l-1}(i_r)}^{n_r} \\
&\quad F_{\theta_{1,q-1}(j_1)}^{a_1}F_{\theta_{1,q-1}(j_1)}^{b_1}\cdots F_{\theta_{1,q-1}(j_s)}^{a_s}F_{\theta_{1,q-1}(j_s)}^{b_s}(T_0^{\text{high}})
\end{aligned} \tag{23}$$

Then, comparing (21) and (23), we get $\eta_{1,q}\eta_{k,k}(T) = \eta_{q+1-l,q+1-k}\eta_{1,q}(T)$, which concludes the proof. \square

Corollary 70. *Let $T \in \text{ShST}(\lambda/\mu, n)$ and $1 \leq p < q \leq n$. Then,*

$$\eta_{p,q}(T) = \eta_{1,q}\eta_{1,q-p+1}\eta_{1,q}(T).$$

In particular, for $T \in \text{ShST}(\nu, n)$, we have

$$\eta_{p,q}(T) = \text{evac}_q \text{evac}_{q-p+1} \text{evac}_q(T),$$

where $\text{evac}_q(T) := \text{evac}(T^{1,q}) \sqcup T^{q+1,n}$.

Proof. Since $[p, q] \subseteq [1, q]$, the third relation of the cactus group ensures that $s_{p,q} = s_{1,q}s_{1,q-p+q}s_{1,q}$ in J_n . Then, since ϕ from Theorem 69 is a group homomorphism, we have

$$\eta_{p,q} = \eta_{1,q}\eta_{1,q-p+1}\eta_{1,q}.$$

The second identity follows from Proposition 46 which ensures that $\eta_{1,q}$ coincides with evac_q in $\mathcal{B}(\nu, n)$. \square

As shown by Halacheva [16, 17], the cactus group J_n acts on the crystal of Young tableaux of a given shape filled with $[n]$, via the type A Schützenberger involutions $\eta_{p,q}$, for $1 \leq p < q \leq n$. The symmetric group \mathfrak{S}_n also acts on this crystal via the type A crystal reflection operators σ_i , for $1 \leq i < n$. Hence, we have that the action of J_n on type A crystals factorizes in the quotient of this group by the normal subgroup $\langle (s_{i,i+1}s_{i+1,i+2})^3, i \in [n-2] \rangle$ generated by the braid relations, which is contained in the kernel of this action. This is not the case for the action ϕ in Theorem 69 on the shifted tableau crystal $\mathcal{B}(\nu, n)$, since the braid relations of the symmetric group do not need to hold (see Example 61), hence $\langle (s_{i,i+1}s_{i+1,i+2})^3, i \in [n-2] \rangle \not\subseteq \ker \phi$.

A Shifted tableau crystals for $n = 3$

In this appendix we present some larger examples of the shifted tableau crystal $\mathcal{B}(\nu, n)$, for $n = 3$. Shifted tableaux will be enumerated according to their list enumeration in SageMath (this enumeration starts on 0 by default, we choose to start on 1). We remark that the definition of shifted tableaux used in SageMath is the one in [1], which is not the same as in [12]. Here, shifted tableaux are not required to be in canonical form and the main diagonal must have no primed entries. However, we remark that both definitions coincide for straight-shaped tableaux that are filled with $[n]'$, where n is equal to the number of rows, as this ensures that the first occurrence of each letter i or i' appears in the main diagonal, thus being unprimed in canonical form. For instance, the tableau $T \in \mathcal{B}(\nu, 3)$, with $\nu_1 = (5, 3, 1)$ (see Figure 9), in Example 61 is T_{61} .

```
sage: SPT=ShiftedPrimedTableaux([5,3,1], max_entry=3)
sage: L=SPT.list()
sage: T=SPT([[1,1,1,1,1,'3p'],[2,2,'3p'],[3]])
sage: L.index(T)+1
61
```

Figure 7: A SageMath code listing the shifted semistandard tableaux of shape $\nu = (5, 3, 1)$, in the alphabet $\{1, 2, 3\}'$.

We have seen in Example 61 that $(\sigma_1\sigma_2)^3(T_{61}) \neq T_{61}$. However, we have $(\sigma_1\sigma_2)^9(T_{61}) = T_{61}$. If we set $m_i := \min\{m : (\sigma_1\sigma_2)^m(T_i) = T_i\}$, for $i \in [|\mathcal{B}(\nu_1, 3)|] = [64]$, then, we have the following in $\mathcal{B}(\nu, n)$:

$$m_i = \begin{cases} 3 & \text{for } i \in \{1, 2, 3, 9, 10, 11, 15, 18, 19, 22, 23, 27, 28, 31, 37, 38, 44, 47\} \\ 5 & \text{for } i \in \{6, 13, 25, 33, 42, 50, 54, 58, 62, 64\} \\ 9 & \text{otherwise} \end{cases}$$

Therefore, taking $m = \text{lcm}(3, 5, 9) = 45$, we have that $(\sigma_1\sigma_2)^m(T) = T$, for all $T \in \mathcal{B}(\nu_1, 3)$.

Similarly, for $\nu_2 = (5, 2, 1)$ (see Figure 8), we have that, for all $i \in [|\mathcal{B}(\nu_2, 3)|] = [48]$,

$$m_i = \begin{cases} 3 & \text{for } i \in \{1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 14, 15, 16, 19, 20, \\ & 21, 23, 24, 25, 26, 30, 31, 32, 33, 35, 36, 39, 40, 42, 45\} \\ 9 & \text{otherwise} \end{cases}$$

Hence, putting $m = \text{lcm}(3, 9) = 9$, we have that $(\sigma_1\sigma_2)^m(T) = T$ for all $T \in \mathcal{B}(\nu_2, 3)$. The following table summarizes these and other computations we did. We remark that Lemma 56 ensures that the effect of σ_i on $\text{rect}(T^{i,i+1})$ (which has, at most, two rows) does not depend on the first diagonals, except for one, with two elements. Thus, it suffices to check strict partitions whose last part is equal to one. This means that the results obtained for $(3, 2, 1)$ are the same for $(3 + k, 2 + k, 1 + k)$, for $k \geq 1$.

| ν | $ \mathcal{B}(\nu, 3) $ | least m such that $(\sigma_1\sigma_2)^m(T) = T$ for all $T \in \mathcal{B}(\nu, 3)$ |
|-------------|-------------------------|--|
| $(3, 2, 1)$ | 8 | 3 |
| $(4, 2, 1)$ | 24 | 3 |
| $(4, 3, 1)$ | 24 | 3 |
| $(5, 2, 1)$ | 48 | 9 |
| $(5, 3, 1)$ | 64 | 45 |
| $(5, 4, 1)$ | 48 | 9 |
| $(6, 2, 1)$ | 80 | 18 |
| $(6, 3, 1)$ | 120 | 18 |
| $(6, 4, 1)$ | 120 | 18 |

Given ν a strict partition, since $\mathcal{B}(\nu, 3)$ is finite, we know that there exists a $m > 3$ such that $(\sigma_1\sigma_2)^m(T) = T$ for all $T \in \mathcal{B}(\nu, 3)$ (see Remark 63). These computations show that, if there exists an m such that $(\sigma_1\sigma_2)^m = \text{id}$, for any ν , then it should be greater or equal to $\text{lcm}(3, 9, 18, 45) = 90$. However, an upper bound for any ν is not known.

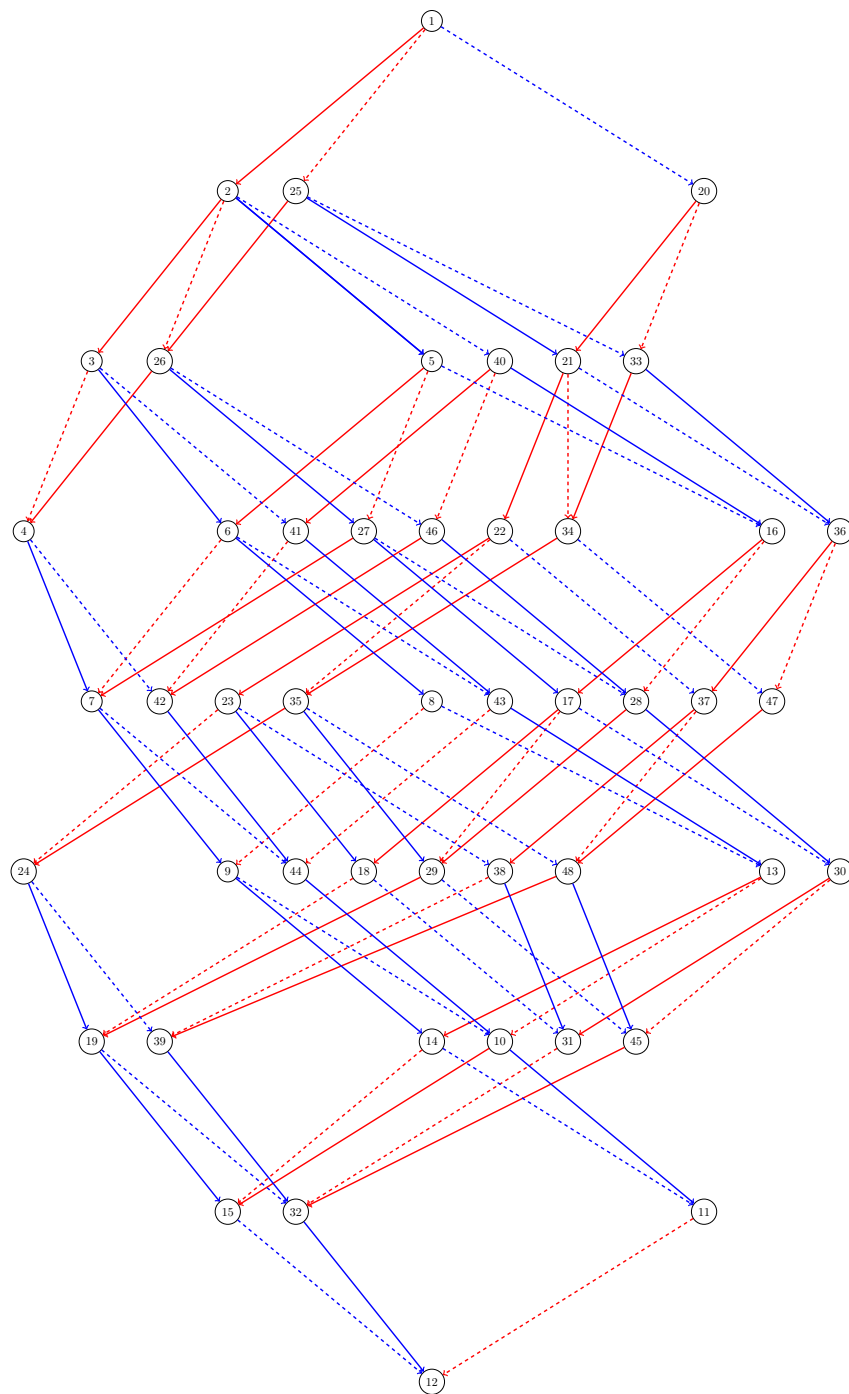


Figure 8: Shifted tableau crystal graph $\mathcal{B}(\nu, 3)$, with $\nu = (5, 2, 1)$. The operators F_1, F'_1 are in red and the F_2, F'_2 are in blue. Vertices with the same weight are grouped together.

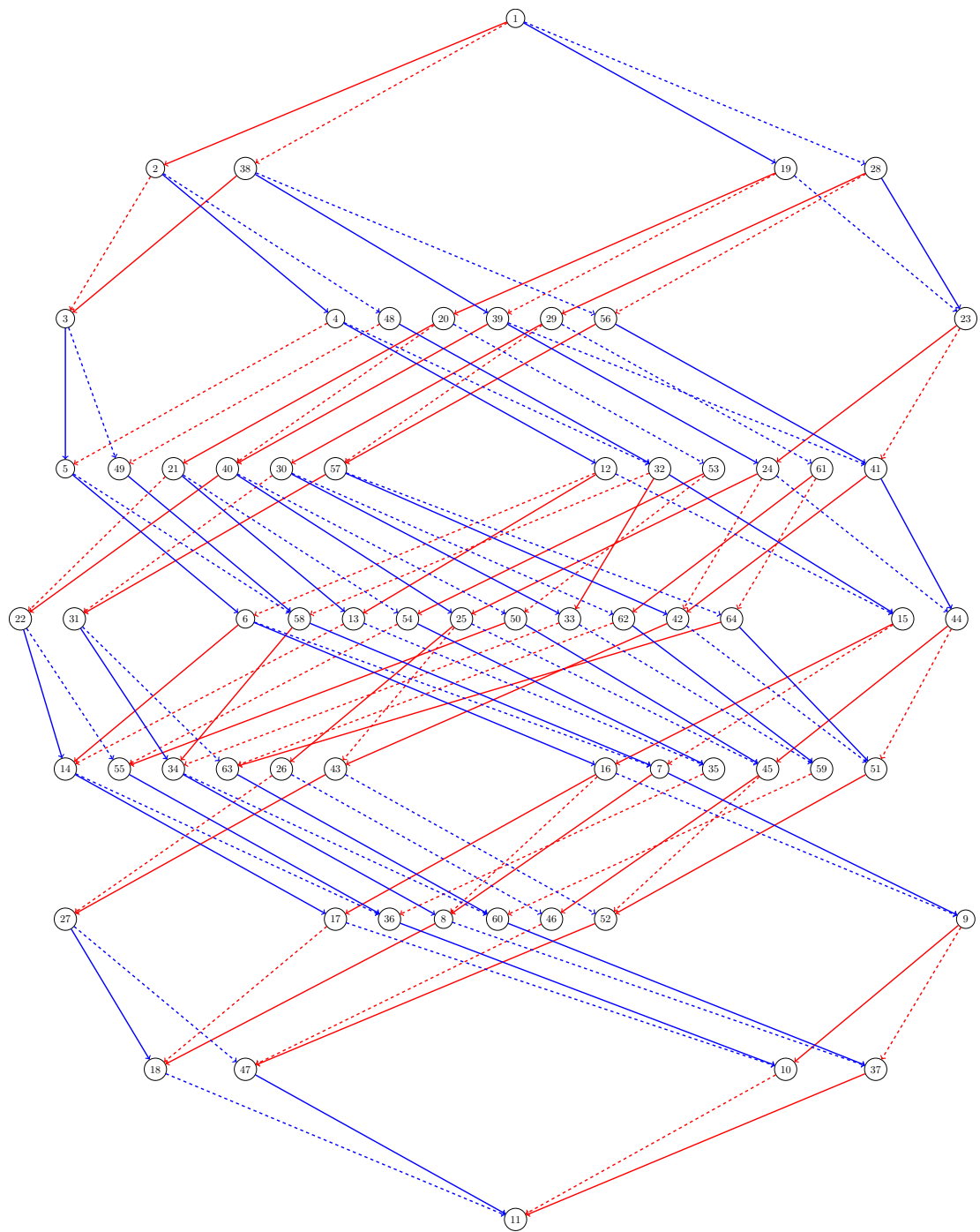


Figure 9: Shifted tableau crystal graph $\mathcal{B}(\nu, 3)$, with $\nu = (5, 3, 1)$. The operators F_1, F'_1 are in red and the F_2, F'_2 are in blue. Vertices with the same weight are grouped together.

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