Improved lower bounds for multiplicative square-free sequences

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Abstract

In this short paper, we improve an almost 30-year-old result of Erdős, Sárközy and Sós on lower bounds for the size of multiplicative square-free sequences. Our construction uses Berge-cycle free hypergraphs that is interesting in its own right.

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1 Introduction

Erdős, Sárközy and Sós [7] defined and started to investigate the following property (in connection with the multiplicative Sidon problem).

Definition 1. For $k \geqslant 2$ we say that a set of positive integers $\mathcal{A}$ has property P_k if the equation

$$a_1 a_2 \dots a_k = x^2,\ a_1, a_2, \dots, a_k \in \mathcal{A},\ a_1 < a_2 < \dots < a_k$$

can not be solved for any $x \in \mathbb{N}$. Let Γ_k denote the set of subsets of $\mathbb{N}$ that satisfy P_k.

For $k, n \geqslant 2$ let

$$F_k(n) := \max\{|\mathcal{A}| : \mathcal{A} \subset \{1, 2, \dots, n\}, \mathcal{A} \in \Gamma_k\}.$$

Let us denote by $\pi(n)$ the number of prime numbers that are at most n. Erdős, Sárközy and Sós [7] proved the following.

Theorem A ([7] Theorem 5, Theorem 6)**.** *There exists $c > 0$ and for every $k \in \mathbb{N}$ there exist $c_k > 0$ and $n_0(k)$ such that for $n > n_0(k)$ we have*

- $c_k(n^{\frac{1}{2}}(\log n)^{-1})^{1+\frac{1}{4k-1}} \leqslant F_{4k}(n) - \pi(n) \leqslant cn^{\frac{3}{4}}(\log n)^{-\frac{3}{2}}$, *and*
- $c_k(n^{\frac{1}{2}}(\log n)^{-1})^{1+\frac{1}{4k+1}} \leqslant F_{4k+2}(n) - (\pi(n) + \pi(\frac{n}{2})) \leqslant cn^{\frac{7}{9}} \log n$.

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Let us note that there is a typo in the exponent of the lower bound of $F_{4k}(n)$ in [7].

They also proved sharper results for small values of k. The following two results were achieved in [7]: there exist positive constants c_1 and c_2 such that

$$c_1 n^{\frac{3}{4}} (\log n)^{-\frac{3}{2}} \leqslant F_4(n) - \pi(n) \leqslant c_2 n^{\frac{3}{4}} (\log n)^{-\frac{3}{2}},$$

so the order of magnitude of $F_4(n) - \pi(n)$ was determined; and there exist positive constants c_3 and c_4 such that

$$c_3 n^{\frac{2}{3}} (\log n)^{-\frac{3}{4}} \leqslant F_6(n) - (\pi(n) + \pi(n/2)) \leqslant c_4 n^{\frac{7}{9}} \log n.$$

Later, in [8] Győri improved the upper bound on $F_6(n)$ and proved that there exists a positive constant c_5 with $F_6(n) - (\pi(n) + \pi(n/2)) \leqslant c_5 n^{\frac{2}{3}} \log n$. The first author could improve further this upper bound in [14, 15] and finally could prove that there exists a constant c_6 with $F_6(n) - (\pi(n) + \pi(n/2)) \leqslant c_6 n^{\frac{2}{3}} (\log n)^{2^{1/3} - 1/3 + o(1)}$.

In this paper, we study $F_k(n)$ when k is even. Note that for odd k the function $F_k(n)$ has a different growth rate, in fact, $F_k(n) = \Theta(n)$ holds. [7] For instance, the set of those positive integers that are the product of an odd number of (not necessarily different) primes satisfy property P_k for every odd k.

Notation. We use standard notation for the order of a function. For two functions $f, g : \mathbb{N} \to \mathbb{N}$ we write $f \ll g$ if there is a (positive) constant c and a natural number n_0 such that we have $f(n) \leqslant cg(n)$ for all $n \geqslant n_0$. If the constant c depends on certain parameters, we indicate this by using a lower index.

Structure of the paper. The structure of the paper is the following: in Subsection 1.1 we provide an improvement of Theorem A just by using a better (known) graph theoretic result; then in Subsection 1.2 we state one of our results that connects the hypergraph girth problem to lower bound constructions of multiplicative square-free sequences; in Subsection 1.3 we provide constructions concerning the hypergraph girth problem; while in Subsection 1.4 we give the concrete improvements. In Section 2 we give the proofs of our results and finally, in Section 3 we provide some analysis.

1.1 Some improvement of Theorem A

In this subsection we provide an improvement of Theorem A by following the proof of Erdős, Sárközy and Sós from [7] and using a better extremal graph-theoretic result.

For $k \geqslant 2$ we denote the cycle of length k by C_k and the set of cycles $\{C_3, \ldots, C_k\}$ by $\mathcal{C}_k$. For two graphs F and G, we say that G is *F-free*, if G does not contain F as a subgraph and for a set of graphs $\mathcal{F}$ we say that G is $\mathcal{F}$-free if it is F-free for all $F \in \mathcal{F}$. For an integer n and a set of graphs $\mathcal{F}$ we denote by $\mathrm{ex}(n, \mathcal{F})$ the maximum number of edges that a simple graph G on n vertices can have if G is $\mathcal{F}$-free. We say that this function is the extremal or Turán function of $\mathcal{F}$.

In [7] to prove the lower bound of Theorem A the authors use the result of Erdős stating that for $k \geqslant 3$ we have $\mathrm{ex}(n, \mathcal{C}_k) \gg_k n^{1+\frac{1}{k-1}}$. We note that exactly the same way they derived the lower bounds of Theorem A, if one has $\mathrm{ex}(n, \mathcal{C}_k) \gg_k n^{1+\alpha(k)}$ for some function $\alpha : \mathbb{N} \to \mathbb{R}$, then it implies the existence of a positive constant c_k with

$$c_k(n^{\frac{1}{2}}(\log n)^{-1})^{1+\alpha(4k)} \leqslant F_{4k}(n) - \pi(n)$$

and

$$c_k(n^{\frac{1}{2}}(\log n)^{-1})^{1+\alpha(4k+2)} \leqslant F_{4k+2}(n) - (\pi(n) + \pi(n/2)).$$

To the best of our knowledge, the next theorem provides the currently known best lower bound on the order of magnitude of $\mathrm{ex}(n, \mathcal{C}_{2k})$, due to Benson [2]; and Lazebnik, Ustimenko, and Woldar [12]:

Theorem B. *For $k \geqslant 2$ we have the following:*

- ([12] Corollary 3.3) $\mathrm{ex}(n, \mathcal{C}_{2k}) \gg_k n^{1+\frac{2}{3k-3+\varepsilon}}$, *where $\varepsilon = 0$, if k is odd and $\varepsilon = 1$, if k is even, and*
- ([2] Theorem 2) $\mathrm{ex}(n, \mathcal{C}_{10}) \gg n^{\frac{6}{5}}$.

Theorem B implies the way described above the following.

Corollary 2. *For $k \geqslant 2$ we have*

- $F_{4k}(n) - \pi(n) \gg_k (n^{\frac{1}{2}}(\log n)^{-1})^{1+\frac{1}{3k-1}}$,
- $F_{4k+2}(n) - (\pi(n) + \pi(\frac{n}{2})) \gg_k (n^{\frac{1}{2}}(\log n)^{-1})^{1+\frac{1}{3k}}$,
- $F_{10}(n) - (\pi(n) + \pi(\frac{n}{2})) \gg (n^{\frac{1}{2}}(\log n)^{-1})^{\frac{6}{5}}$.

Note that the so-called *Graph Girth*[1] *Problem*, i.e. to give better lower (or upper) bounds on $\mathrm{ex}(n, \mathcal{C}_{2k})$ is a notoriously difficult problem. To improve the results in Corollary 2 just by improving the lower bound on $\mathrm{ex}(n, \mathcal{C}_{2k})$ seems hard.

1.2 General lower bound construction: one of our results

So – because of the obstacle mentioned in the previous paragraph – instead of using graphs our new idea is to use hypergraphs in the lower bound constructions to give better lower bounds for $F_{4k}(n)$ and $F_{4k+2}(n)$ for certain values of k using extremal results for Berge hypergraphs.

First, we state a general theorem that connects lower bounds for $F_{4k}(n)$ and $F_{4k+2}(n)$ with extremal numbers of 3-uniform Berge hypergraphs of appropriate girth (similarly as in the graph case). Then we state concrete lower bounds and finally compare them with the previously known lower bounds.

To be able to state our general theorem, we need some definitions. Similarly to the graph case one can introduce the Turán function of (a set of) hypergraphs. For two

[1] Girth is the length of the shortest cycle in a graph.

hypergraphs H and G we say that a hypergraph H is *G-free*, if H does not contain G as a subhypergraph. For an integer n and a family of r-uniform[2] hypergraphs $\mathcal{G}$ the *Turán function* – denoted by $\mathrm{ex}_r(n, \mathcal{G})$ – is the maximum number of hyperedges in an r-uniform hypergraph H on n vertices such that H is G-free for every $G \in \mathcal{G}$.

There are many different ways one can generalize the notion of graph cycles to the case of hypergraphs. The one that will be useful for us is due to Berge [3].

Definition 3. For an integer $t \geqslant 2$ a *Berge-cycle* of length t is an alternating sequence of t distinct vertices and t distinct hyperedges (of a hypergraph), $v_1, e_1, v_2, e_2, v_3, \ldots, v_t, e_t$, such that $v_i, v_{i+1} \in e_i$, for $i \in \{1, 2, \ldots, t\}$, where the indices are taken *modulo t*. The vertices $v_1, v_2, \ldots, v_t$ are called *defining vertices* and the hyperedges $e_1, e_2, \ldots, e_t$ are called *defining hyperedges* of the Berge-cycle. We denote the set of all Berge-cycles of length t by $\mathcal{B}C_t$. Let us denote the set $\{\mathcal{B}C_3, \ldots, \mathcal{B}C_k\}$ by $\mathcal{BC}_k$.

Note that a cycle in $\mathcal{B}C_2$ is just 2 distinct hyperedges whose intersection has cardinality at least 2. Note also that the notion of being a Berge-cycle of length k means rather a family of hypergraphs than just a single one. Now we state our general result.

Theorem 4. *Let $\beta : \mathbb{N} \to \mathbb{R}$ be a function. If we have $\mathrm{ex}_3(n, \mathcal{BC}_{2k+1}) \gg_k n^{\beta(2k+1)}$ for an integer k, then*

$$F_{4k+2}(n) - (\pi(n) + \pi(n/2)) \gg_k (n^{\frac{1}{2}}(\log n)^{-1})^{\frac{2\cdot\beta(2k+1)}{1+\beta(2k+1)}}.$$

1.3 Berge Hypergraph Girth Problem

According to Theorem 4 the Berge Hypergraph Girth Problem plays a crucial role in the lower bound constructions for $F_{4k+2}(n)$. Now we list the results that we will use.

For 3-uniform hypergraphs, we have the following theorem (and we are not aware of any better lower bound). The proof is coming from the (bipartite) graph girth problem.

Theorem 5. *For $k \geqslant 2$ we have*

- $\mathrm{ex}_3(n, \mathcal{BC}_{2k+1}) \gg_k n^{1+\frac{2}{3k-3+\varepsilon}}$, *where $\varepsilon = 0$, if k is odd and $\varepsilon = 1$, if k is even, and*
- $\mathrm{ex}_3(n, \mathcal{BC}_{11}) \gg n^{\frac{6}{5}}$.

1.4 The main result

Theorem 4 and Theorem 5 imply our main result, that is

Theorem 6. *For $k \geqslant 2$ we have*

- $F_{4k+2}(n) - (\pi(n) + \pi(\frac{n}{2})) \gg_k (n^{\frac{1}{2}}(\log n)^{-1})^{1+\frac{1}{3k-2+\varepsilon}}$, *where $\varepsilon = 0$, if k is odd and $\varepsilon = 1$, if k is even, and*
- $F_{22}(n) - (\pi(n) + \pi(\frac{n}{2})) \gg (n^{\frac{1}{2}}(\log n)^{-1})^{\frac{12}{11}}$.

[2]For an integer $r \geqslant 1$ we call a hypergraph r-uniform if the cardinality of each hyperedge is r.

Comparing our main result with previous results

We put the first few values of the exponent of $n^{\frac{1}{2}}(\log n)^{-1}$ in Theorem A, Corollary 2 and Theorem 6 into the following table.

k	2	3	4	5	6
Theorem A	$\frac{10}{9}$	$\frac{14}{13}$	$\frac{18}{17}$	$\frac{22}{21}$	$\frac{26}{25}$
Corollary 2	$\frac{7}{6}$	$\frac{10}{9}$	$\frac{13}{12}$	$\frac{16}{15}$	$\frac{19}{18}$
Theorem 6	$\frac{6}{5}$	$\frac{8}{7}$	$\frac{12}{11}$	$\frac{12}{11}$	$\frac{18}{17}$

One can easily check that Corollary 2 improves the result of Theorem A for all $k \geqslant 2$. It can also be computed that Theorem 6 improves the exponent of $n^{\frac{1}{2}}(\log n)^{-1}$ in Corollary 2 by

$$\frac{1}{3k-2+\varepsilon} - \frac{1}{3k},$$

where $\varepsilon = 0$, if k is odd and $\varepsilon = 1$, if k is even; for all $k \geqslant 2$, with the exception of $k = 5$. For $k = 5$ check the table for the precise result.

2 Proofs

2.1 The proof of Theorem 4, a construction

In this section, we describe a general construction that can be considered as the hypergraph analog of the construction used in [7] to prove the lower bounds of Theorem A and it will serve as a core of the proof of Theorem 6.

Construction. For a set of integers X let us denote by $P(X)$ the set of primes in X. For two real numbers a and b we denote by $(a, b]$ the set of integers x in the interval $a < x \leqslant b$. To describe the construction we give three different sets of integers as follows. Let the first two sets be the following (where we specify $\frac{1}{2} < \alpha < 1$ and $S = n^{\alpha+o(1)}$ later):

- $P_1 := \{p : \ p \in P((S, n])\}$
- $P_2 := \{2p : \ p \in P((S, \frac{n}{2}])\}$

To define the third set of integers (and the parameters α, S) we need some preparation. Let us divide the set $P((0, S])$ into two disjoint sets A and B in such a way that there exists a 3-uniform hypergraph $\mathcal{H}(B)$ on B for which the number of hyperedges in $\mathcal{H}(B)$ is $|E(\mathcal{H}(B))| = |A|$ and let us assign different prime numbers from A to the hyperedges of $\mathcal{H}(B)$. So let the hyperedge set of $\mathcal{H}(B)$ be $\{e_p = \{r_p, s_p, t_p\} : p \in A, \ r_p, s_p, t_p \in B\}$. Finally, let us set

- $P_3 := \{p \cdot q : \ p \in A, q \in e_p\}$

Note that if the product of the largest element of A and the largest element of B is at most n, then $P_1 \cup P_2 \cup P_3 \subseteq \{1, 2, \ldots, n\}$.

Now we prove the following lemma (we also state it in case of $F_{4k}(n)$ that we refer to in the last Section) that connects those subsets of $P_1 \cup P_2 \cup P_3$ whose product is a square with Berge-cycles in $\mathcal{H}(B)$.

Lemma 7. *Suppose that we have distinct elements $a_1, a_2, \ldots, a_k \in P_1 \cup P_2 \cup P_3$ and an integer x such that $a_1 a_2 \ldots a_k = x^2$. Then we have*

1. *$a_1, a_2, \ldots, a_k \in P_1 \cup P_2$, or*
2. *if $k = 4\ell$, then there is a $\mathcal{B}C_j$ in $\mathcal{H}(B)$ for some $j \in \{2, 3, \ldots, 2\ell\}$, or*
3. *if $k = 4\ell + 2$, then there is a $\mathcal{B}C_j$ in $\mathcal{H}(B)$ for some $j \in \{3, \ldots, 2\ell + 1\}$.*

Proof. First note that for any $a_1, a_2, \ldots, a_k \in P_1 \cup P_2 \cup P_3$ and integer x with $a_1 a_2 \ldots a_k = x^2$, we have that $a_1, a_2, \ldots, a_k$ contains exactly $4s$ numbers from the set $P_1 \cup P_2$ (for some integer $s \geqslant 0$).

Observe also that if for some $1 \leqslant i \leqslant k$ and $q \in e_p$ we have $pq = a_i \in P_3$, then p must occur in some other $a_{i'} = pq'$ with $1 \leqslant i' \leqslant k$, $i \neq i'$, $q \neq q' \in e_p$ and similarly q must occur in some $a_{i''} = p'q$ with $1 \leqslant i'' \leqslant k$, $i \neq i''$, $p' \neq p$. This observation means that if we have at least one element a_i in P_3, then we get a Berge-cycle of length j with $2 \leqslant j \leqslant k/2$ and actually the elements in P_3 gives the union of some Berge-cycles in $\mathcal{H}(B)$.

As a $\mathcal{B}C_2$ is assigned to 4 elements, in case $k = 4\ell + 2$ (using the remark above) we get that there is a Berge-cycle of length j with $3 \leqslant j \leqslant 2\ell + 1 (= k/2)$ also.

□

To finish the proof of Theorem 4 we define A and B the following way. Let $c > 0$ and

$$B := P((0, cn^{1-\alpha}(\log n)^{\frac{\beta(2k+1)-1}{\beta(2k+1)+1}}]) \text{ and } A := P((cn^{1-\alpha}(\log n)^{\frac{\beta(2k+1)-1}{\beta(2k+1)+1}}, S]),$$

where S is chosen in such a way that the number of hyperedges of our 3-uniform $\mathcal{BC}_{2k+1}$-free hypergraph on B is exactly $|A|$. If $c > 0$ and α are chosen in such a way that

$$S \leqslant (1/c) n^{\alpha} (\log n)^{-\frac{\beta(2k+1)-1}{\beta(2k+1)+1}}$$

holds, then we can carry out the construction described above. To get this the following inequality should hold:

$$\left[\pi\left(cn^{1-\alpha}(\log n)^{\frac{\beta(2k+1)-1}{\beta(2k+1)+1}}\right)\right]^{\beta(2k+1)} \ll_k \pi\left((1/c)n^{\alpha}(\log n)^{-\frac{\beta(2k+1)-1}{\beta(2k+1)+1}}\right) - \pi\left(cn^{1-\alpha}(\log n)^{\frac{\beta(2k+1)-1}{\beta(2k+1)+1}}\right) \quad (1)$$

Note that we want to choose P_3 as large as possible, and that means we would like to choose $\alpha > \frac{1}{2}$ as small as possible. One can easily check that inequality (1) is satisfied with $\alpha = \frac{\beta(2k+1)}{1+\beta(2k+1)}$ and a sufficiently small constant $c > 0$. So we are able to carry out the construction with this exponent.

The improvement compared to the bound $\pi(n) + \pi(n/2)$ is $(1 - o(1))|E(\mathcal{H}(B))|$, so we are done with the proof of Theorem 4, since for the above choice of the parameters we have

$$|E(\mathcal{H}(B))| \gg_k (n^{\frac{1}{2}}(\log n)^{-1})^{\frac{2\cdot\beta(2k+1)}{1+\beta(2k+1)}}.$$

□

2.2 Proof of Theorem 5, the Turán function of large girth Berge hypergraphs

Now we introduce a construction that helps us in the proof. [3] First we prove that if we have a lower bound $\mathrm{ex}(n, \mathcal{C}_k) \gg_k n^{\gamma(k)}$ for some $\gamma : \mathbb{N} \to \mathbb{R}$, then from that construction we can get another implying $\mathrm{ex}(n, \mathcal{BC}_k) \gg_k n^{\gamma(k)}$.

Construction 8. *Let us suppose that $G = ((A, B), E)$ is a bipartite graph with vertex set $A \cup B$ $(A \cap B = \emptyset)$ and edge set E. Let us define the 3-uniform hypergraph $\mathcal{H}(G, A)$ in the following way:*

- *let the vertex set of $\mathcal{H}(G, A)$ be $A_1 \cup A_2 \cup B$ with $A_1 \cap A_2 = \emptyset$ and $|A| = |A_1| = |A_2|$ (for any $a \in A$ we denote the corresponding vertices in A_1 and A_2 by a_1 and a_2, respectively), and*
- *let the set of hyperedges be $\{(a_1, a_2, b) : (a, b) \in E\}$.*

Lemma 9. *If $G = ((A, B), E)$ is $\mathcal{C}_k$-free for some $k \geqslant 3$, then $\mathcal{H}(G, A)$ is $\mathcal{BC}_k$-free.*

Proof. Suppose by contradiction that there exists a Berge-cycle of length at most k in $\mathcal{H}(G, A)$ and G is $\mathcal{C}_k$-free. Then consider its defining vertices: $v_1, v_2, \ldots, v_j$ for some $j \leqslant k$. Note that for two consecutive vertices, there are two possibilities: either v_ℓ and $v_{\ell+1}$ (where $\ell \leqslant j$ and indices are meant modulo j) are both coming from $A_1 \cup A_2$ or one of them is coming from $A_1 \cup A_2$ and the other one is coming from B. However, also note that

Case 1: if $v_\ell, v_{\ell+1} \in A_1 \cup A_2$, then $\{v_\ell, v_{\ell+1}\} = \{a_1, a_2\}$ for some $a \in A$ (observe that this also implies that we can not have for 3 consecutive vertices $v_\ell, v_{\ell+1}, v_{\ell+2} \in A_1 \cup A_2$), and

Case 2: if $v_\ell \in A_1 \cup A_2$ and $v_{\ell+1} \in B$, then $(a, v_{\ell+1})$ is an edge in G with a for which $v_\ell \in \{a_1, a_2\}$. (Or symmetrically: if $v_{\ell+1} \in A_1 \cup A_2$ and $v_\ell \in B$ and then (v_ℓ, a) is an edge in G with a for which $v_{\ell+1} \in \{a_1, a_2\}$.)

So if we replace in the defining vertex set $v_1, v_2, \ldots, v_k$ each pair of vertices $v_\ell, v_{\ell+1}$ for which $\{v_\ell, v_{\ell+1}\} = \{a_1, a_2\}$ (i.e., we are in Case 1) with the corresponding vertex a, then we get a cycle in G whose length is at most k.

□

[3] We note that this kind of construction is known in the literature (see e.g. [9]).

Note that if we have a series of graphs showing $\mathrm{ex}(n, \mathcal{C}_k) \gg_k n^{\gamma(k)}$, then we also have a series of bipartite graphs as we can make any graph bipartite by deleting at most half of its edges. Then by Lemma 9 and Theorem B we get Theorem 5.

□

3 Analysis of the results and some remarks

There are natural questions that emerge concerning the construction provided. What can be the limit of different methods just by improving bounds in the different girth questions? Could we give a similar construction for $F_{4k}(n)$? We answer these questions in this section.

3.1 Improving the lower bound using lower bound results for the graph girth problem

An old conjecture of Erdős states the following:

Conjecture 10 (Erdős' Girth Conjecture [5] for k)**.** For any positive integer k, there exist a constant $c > 0$ depending only on k, and a family of graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geqslant cn^{1+1/k}$ and the girth of G_n is more than $2k$.

If Erdős' Girth Conjecture holds, then we can increase the exponent of $n^{\frac{1}{2}}(\log n)^{-1}$ in Corollary 2 to the following:

- $F_{4k}(n) - \pi(n) \gg_k (n^{\frac{1}{2}}(\log n)^{-1})^{1+\frac{1}{2k}}$
- $F_{4k+2}(n) - (\pi(n) + \pi(n/2)) \gg_k (n^{\frac{1}{2}}(\log n)^{-1})^{1+\frac{1}{2k+1}}$

However, note that if Erdős' Girth Conjecture holds, then the exponent is tight by a result of Alon, Hoory, and Linial [1] who proved the following upper bound on $\mathrm{ex}(n, \mathcal{C}_{2k})$.

Theorem C ([1] Theorem 1)**.** *For any $k \geqslant 2$ we have*

(i) $\mathrm{ex}(n, \mathcal{C}_{2k}) < \frac{1}{2}n^{1+1/k} + \frac{1}{2}n$,

(ii) $\mathrm{ex}(n, \mathcal{C}_{2k+1}) < \frac{1}{2^{1+1/k}}n^{1+1/k} + \frac{1}{2}n$.

So Theorem C implies that the above mentioned "possible lower bound" is the best we can hope for F using the technique of Erdős, Sárközy and Sós.

3.2 Improving the lower bounds for $F_{4k+2}(n)$ using hypergraphs

In the proof of Theorem 6 we used lower bound results for the Berge Hypergraph Girth Problem. However, note that Győri and Lemons [10] proved the following result:

Theorem D ([10] Theorem 1.5)**.** *For every $\ell \geqslant 3$, $r \geqslant 3$ and $k = \lfloor \frac{\ell}{2} \rfloor$ we have*

$$\mathrm{ex}_r(n, \mathcal{B}C_\ell) \ll_k n^{1+\frac{1}{k}}.$$

This means that the best possible lower bound result we can get is $F_{4k+2}(n) - (\pi(n) + \pi(n/2)) \gg_k (n^{\frac{1}{2}}(\log n)^{-1})^{1+\frac{1}{2k}}$.

3.3 Possible improved lower bounds for $F_{4k}(n)$

It is a natural question to ask whether a similar improvement that worked in Theorem 6 would work in the case of $F_{4k}(n)$ also.

We say that a hypergraph is *linear* if it does not contain two hyperedges whose intersection has cardinality at least 2. So for a hypergraph being linear is equivalent to being $\mathcal{B}C_2$-free. For an integer n and a family of r-uniform linear hypergraphs $\mathcal{G}$ the *linear Turán number* – denoted by $\mathrm{ex}_r^{\mathrm{lin}}(n, \mathcal{G})$ – is the maximum number of hyperedges in an r-uniform linear hypergraph H on n vertices such that H is G-free for every $G \in \mathcal{G}$.

We can prove the following theorem for $F_{4k}(n)$ (that can be considered as the analog of Theorem 4; see e.g. Lemma 7 for the main ingredient of the proof).

Theorem 11. *Let $\beta_{\mathrm{lin}} : \mathbb{N} \to \mathbb{R}$ be a function. If we have $\mathrm{ex}_3^{\mathrm{lin}}(n, \mathcal{BC}_{2k}) \gg_k n^{\beta_{\mathrm{lin}}(2k)}$, then we have*

$$F_{4k}(n) - \pi(n) \gg_k n^{\frac{\beta_{\mathrm{lin}}(2k)}{1+\beta_{\mathrm{lin}}(2k)}}.$$

So the Linear Berge Hypergraph Girth Problem could play a similar role in possible lower bound constructions for $F_{4k}(n)$ as the Berge Hypergraph Girth Problem plays in lower bound constructions for $F_{4k+2}(n)$.

There is a conjecture concerning the Turán number of linear hypergraphs of high girth (see e.g., [17]).

Conjecture 12. For every $\ell \geqslant 3$, $r \geqslant 2$ and $k = \lfloor \frac{\ell}{2} \rfloor$ we have

$$\mathrm{ex}_r^{\mathrm{lin}}(n, \mathcal{BC}_\ell) = \Theta(n^{1+\frac{1}{k}-o(1)}).$$

This conjecture is known to be true for $\ell = 3, 4$ and $r \geqslant 3$, see e.g., [6, 13, 16, 18], and wide open for $\ell \geqslant 5$ and $r \geqslant 3$. Note that by Győri and Lemons [10] we have $\mathrm{ex}_r(n, \mathcal{B}C_\ell) \ll_{r,k} n^{1+\frac{1}{k}}$ and also note that the $o(1)$ term in Conjecture 12 is necessary for $\ell = 3$ by [16] and for $\ell = 5$ by [4].

By a standard probabilistic argument (see e.g., [11]) for $r \geqslant 2$ and $\ell \geqslant 3$ one can prove

$$\mathrm{ex}_r^{\mathrm{lin}}(n, \mathcal{BC}_\ell) = \Omega(n^{1+\frac{1}{\ell-1}}).$$

We are not aware of any better result and the known results do not give better lower bounds than those in Corollary 2.

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