RSK tableaux and the weak order on fully commutative permutations

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Abstract

For each fully commutative permutation, we construct a "boolean core," which is the maximal boolean permutation in its principal order ideal under the right weak order. We partition the set of fully commutative permutations into the recently defined crowded and uncrowded elements, distinguished by whether or not their RSK insertion tableaux satisfy a sparsity condition. We show that a fully commutative element is uncrowded exactly when it shares the RSK insertion tableau with its boolean core. We present the dynamics of the right weak order on fully commutative permutations, with particular interest in when they change from uncrowded to crowded. In particular, we use consecutive permutation patterns and descents to characterize the minimal crowded elements under the right weak order.
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1 Introduction

First introduced in [Ste96a], the fully commutative elements of a Coxeter group have the property that every pair of reduced words are related by a sequence of commutation relations. This set of objects is combinatorially rich and has been studied extensively (see, for example, [MPPS20, Nad15, Ste98]). A permutation is fully commutative if and only if it avoids the pattern 321 [BJS93], and the fully commutative permutations are exactly

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those with fewer than three rows in their Robinson–Schensted–Knuth (RSK) tableaux [Sch61]. In this paper, following up on recent work in [GPRT22], we examine the interplay between reduced words and RSK tableaux for fully commutative permutations and analyze the set of fully commutative permutations under the weak order.

Our previous work, which is a companion to this paper, proves that the RSK insertion tableaux for boolean permutations satisfy a certain sparsity condition that we call *uncrowded* [GPRT22]. Boolean permutations are an important subset of fully commutative permutations, characterized by the fact that their principal order ideals in the Bruhat order are isomorphic to boolean algebras. Motivated by those results, we call a fully commutative permutation with an uncrowded insertion tableau an *uncrowded permutation*. In other words, an uncrowded fully commutative permutation shares its insertion tableau with some boolean element. A fully commutative permutation that is not uncrowded is called *crowded*. Central to this paper is the partition of the set of fully commutative permutations into crowded and uncrowded elements.

For each fully commutative element w, we identify a particular boolean element $\hat{w}$ that is below w in the weak order and has the same support as w; we call this $\hat{w}$ the *boolean core* of w (Theorem 18). We then view the fully commutative permutation w as an "elongation" of its boolean core, and we investigate the evolution of RSK insertion tableaux along chains of fully commutative elements in the right weak order. We prove that the second rows of insertion tableaux obey a containment property along covering relations in the right weak order (Theorem 20).

Applying this containment property, we show that if two fully commutative elements with the same support satisfy a covering relation in the right weak order and have different insertion tableaux then the larger one is necessarily crowded (Theorem 37). This has two important implications. First, a fully commutative element is uncrowded exactly when it has the same insertion tableau as its boolean core (Corollary 38). Second, within the set of fully commutative permutations under the right weak order, the uncrowded permutations form an order ideal and the crowded permutations form a dual order ideal (Lemma 40). Thus, knowing the minimal crowded elements in the poset is, in fact, enough information to identify each fully commutative element as being either crowded or uncrowded. Our final result, Theorem 54, proves a set of necessary and sufficient conditions for a fully commutative permutation to be minimal in the dual order ideal of crowded permutations.

This paper is organized as follows. Section 2 provides necessary background information and notation including several results from our companion paper on boolean RSK tableaux. Section 3 defines the boolean core of a fully commutative element and proves a containment property for RSK tableaux under the right weak order. Section 4 explores covering relations between fully commutative elements in the right weak order when the two permutations have the same support but different insertion tableaux. Finally, Section 5 characterizes the minimal elements of the dual order ideal of crowded fully commutative permutations in the right weak order, thus providing the key to classifying each fully commutative permutation as being either crowded or uncrowded.

Several relevant open questions are mentioned throughout the paper.

2 Background and notation

Denote the symmetric group on n elements by S_n. For a permutation $w \in S_n$, we use the *one-line notation* $w = w(1)w(2)\cdots w(n)$ to represent w. For each $i \in \{1,\ldots,n-1\}$, we write $s_i \in S_n$ to denote the *simple reflection* (or *adjacent transposition*) that swaps i and $i+1$ and fixes all other letters. Every permutation can be expressed as a product of simple reflections. Given $w \in S_n$, the minimum number of simple reflections among all such expressions for w is called the *(Coxeter) length* of w, and is denoted by $\ell(w)$. An *inversion* in the one-line notation for w is a pair of positions $i < j$ such that $w(i) > w(j)$. It is often convenient to recognize that $\ell(w)$ is the number of inversions in the one-line notation for w. A *reduced decomposition* of w is an expression $w = s_{i_1}\cdots s_{i_{\ell(w)}}$ realizing the Coxeter length of w. To simplify notation, we refer to such a decomposition via its *reduced word* $\left[i_1\cdots i_{\ell(w)}\right]$. Let $R(w)$ denote the set of reduced words for w.

The set of letters appearing in reduced words of a permutation w is the *support* $\mathsf{supp}(w)$ of w. For example, consider $w = 51342 = s_4s_2s_3s_2s_4s_1 \in S_5$. Then $\mathsf{supp}(w) = \{1,2,3,4\}$. Because w has six inversions, we see that $\ell(w) = 6$ and $[423241] \in R(w)$.

The following technical lemma is related to the support of a permutation. It introduces a pair of values M and m which depend on the choice of $v \in S_n$ and $i \in \{1,\ldots,n-1\}$. These values play a central role in the arguments in Section 4.

Lemma 1. *[Ten12, Lemma 2.8] Fix a permutation $v \in S_n$ and $i \in \{1,\ldots,n-1\}$, and let $M := \max\{v(j) : j \leqslant i\}$ and $m := \min\{v(j) : j \geqslant i+1\}$. Then the following statements are equivalent:*

- $i \in \mathsf{supp}(v)$,
- $\{v(1),\ldots,v(i)\} \neq \{1,2,\ldots,i\}$,
- $\{v(i+1),\ldots,v(n)\} \neq \{i+1,i+2,\ldots,n\}$,
- $M > i$,
- $m < i+1$,
- $M > m$.

The *right weak order*, denoted by $\leqslant$, is a partial order on S_n obtained by taking the transitive closure of the cover relation $w < ws_i$ whenever $\ell(w) < \ell(ws_i)$. We use $w < w'$ to denote when $w \leqslant w'$ and $w \neq w'$. The *left weak order* is defined analogously, with left multiplication by s_i instead of right. In each order, the minimum element is the identity permutation and the maximum element is the *long element* $n(n-1)\cdots 21$. More details on the weak order can be found in, for example, [BB05, Section 3.1].

An *order ideal* of a poset is a subset C such that if $y \in C$ and $x \leqslant y$, then $x \in C$. A *dual order ideal* (or *order filter*, or *upper order ideal*) of a poset is a subset C such that if $x \in C$ and $x \leqslant y$, then $y \in C$.

2.1 Fully commutative permutations and boolean permutations

Let $m \leqslant n$. The permutation $w \in S_n$ is said to *contain the pattern* $\sigma \in S_m$ if w has a (not necessarily contiguous) subsequence whose elements are in the same relative order as σ. In the case that w does not contain σ, we say w *avoids* σ. For instance, the permutation $w = 314592687$ contains the pattern 1423 because the subsequence 1927 (among others) has the same relative order as 1423. On the other hand, w avoids 3241 since it has no subsequences that follow the pattern 3241. We note also that the inversions of a permutation are exactly the instances of 21-patterns.

For $|i-j| > 1$, simple reflections satisfy *commutation relations* of the form $s_i s_j = s_j s_i$. An application of a commutation relation to a product of simple reflections is called a *commutation move*. In the context of reduced words, we will say adjacent letters i and j in a reduced word *commute* when $|i-j| > 1$. For a reduced word $[u]$ of a permutation, the equivalence class of all words obtained from $[u]$ by sequences of commutation moves is called the *commutation class* of $[u]$. A permutation is called *fully commutative* if all of its reduced words form a single commutation class. As the following proposition shows, fully commutative permutations can be characterized in terms of pattern avoidance.

Proposition 2 ([BJS93])**.** *Let w be a permutation. The following are equivalent:*

- *w is fully commutative,*
- *w avoids the pattern 321,*
- *no reduced word of w contains $i(i+1)i$ or $(i+1)i(i+1)$ as a factor, for any i.*

Boolean permutations are an important subset of the set of fully commutative permutations. The following result gives a description of boolean permutations analogous to that of Proposition 2.

Proposition 3 ([Ten07])**.** *Let w be a permutation. The following are equivalent:*

- *w is boolean,*
- *w avoids the pattern 321 and 3412,*
- *there exists a reduced word of w consisting of all distinct letters, and*
- *every reduced word of w consists of all distinct letters.*

2.2 Heaps and commutation class

In this section, we review the classical theory of heaps, which was used in [Ste96b] to study fully commutative elements of a Coxeter group. For a detailed list of attributions on the theory of heaps, see [Sta12, Solutions to Exercise 3.123(ab)].

Given a reduced word $[u]$ of a permutation, we can associate to $[u]$ a *heap*, a poset whose elements are labeled by the simple reflections in $[u]$. A *heap diagram* is the Hasse diagram for a heap in which poset elements are replaced by their labels.

Definition 4. Given an arbitrary reduced word $[u] = [u_1 \cdots u_\ell]$ of a permutation, consider the partial order $\preccurlyeq$ on the set $\{1, \ldots, \ell\}$ obtained via the transitive closure of the relations

$$x \prec y$$

for $x < y$ such that $|u_x - u_y| \leqslant 1$. For each $1 \leqslant x \leqslant \ell$, the *label* of the poset element x is u_x. This labeled poset is called the *heap* for $[u]$. The Hasse diagram for this poset with elements $\{1, \ldots, \ell\}$ replaced by their labels is called the *heap diagram* for $[u]$.

The following lemma follows directly from this definition.

Lemma 5. *Let $[u]$ be an arbitrary reduced word for a permutation, and let $x < y$ be elements of the heap for $[u]$. If y covers x, then the labels of x and y differ by exactly one.*

Note that a heap is, in some sense, a partial ordering on the multiset of simple reflections occurring in a reduced word. For a fully commutative permutation, the heap structure on this multiset is, in fact, independent of the choice of reduced word (see Proposition 7). Throughout this paper, for a fully commutative permutation w, we will use H_w to denote both the heap diagram for w and the poset of simple reflections of any reduced word $[u]$ of w. The context should make it clear to which object H_w refers.

From a linear extension of the heap, one can define a *labeled linear extension* essentially by replacing elements of the heap with their labels.

Definition 6. A *labeled linear extension* of the heap of a reduced word $[u] = [u_1 \cdots u_\ell]$ is a word $\left[u_{\pi(1)} \cdots u_{\pi(\ell)}\right]$, where $\pi = \pi(1) \cdots \pi(\ell)$ is a total order on $\{1, \ldots, \ell\}$ that is consistent with the structure of the heap. That is, $x \prec y$ implies $\pi(x) < \pi(y)$.

As the next proposition illustrates, labeled linear extensions are related to reduced words and commutation classes.

Proposition 7 ([Ste96b, Proof of Proposition 2.2] and [Sta12, Solutions to Exercise 3.123(ab)])**.** *Given a reduced word $[u]$, the set of labeled linear extensions of the heap for $[u]$ is the commutation class of $[u]$.*

By definition, a fully commutative permutation has exactly one commutation class. Hence Proposition 7 implies that given any reduced word $[u]$ for a fully commutative permutation w, the set of labeled linear extensions of the heap for $[u]$ is exactly $R(w)$, the set of reduced words of w.

Example 8. The heap diagram H_w of the fully commutative permutation $w = 345619278 \in S_9$ is depicted in Figure 1. Two of the labeled linear extensions correspond to the reduced words [87234561234] and [23451234876].

Proposition 3 states that a boolean permutation is a fully commutative permutation with no repeated letters in any of its reduced words. In the sense of heaps, this means that there are no two elements corresponding to the same simple reflection. For boolean-specific descriptions of heaps, see [GPRT22, Section 2.2].

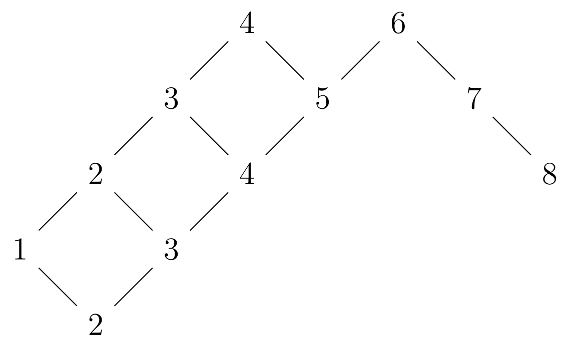

Figure 1: The heap diagram for the fully commutative permutation $345619278 \in S_9$.

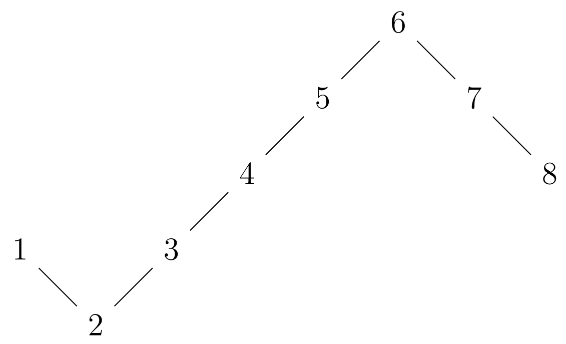

Figure 2: The heap diagram for the boolean permutation $314569278 \in S_9$.

2.3 Robinson–Schensted–Knuth tableaux

The well-known Robinson–Schensted–Knuth (RSK) insertion algorithm, as described in [Sch61], is a bijection

$$w \mapsto (\mathrm{P}(w), \mathrm{Q}(w))$$

from S_n onto pairs of standard tableaux of size n having identical shape. The tableau $\mathrm{P}(w)$ is called the *insertion tableau* of w, and the tableau $\mathrm{Q}(w)$ is the *recording tableau* of w. The shape of these tableaux is the *RSK partition of* w. We will also write $\mathrm{P}_i(w)$ to denote the partial insertion tableau constructed by the first i letters in the one-line notation for w. For more details, see for example [Sta99, Section 7.11].

The following symmetry result is an important feature of the algorithm, and one that will simplify our own work.

Proposition 9 ([Sch63]). *For any permutation w,*

$$\mathrm{P}(w^{-1}) = \mathrm{Q}(w).$$

Schensted's theorem [Sch61, Theorem 1], presented here as Theorem 10, articulates an important relationship between the RSK partition shape and the one-line notation for w.

Theorem 10. *Given a permutation w, the length of the longest increasing (resp., decreasing) subsequence in the one-line notation of w is the size of the first row (resp., column) of $\mathrm{P}(w)$.*

Due to Schensted's theorem, we can see that a permutation is fully commutative if and only if its RSK partition has at most two rows. We denote the set of values in the second row of the RSK insertion tableau of a permutation w by $\mathsf{Row}_2(\mathrm{P}(w))$. More generally, we denote the set of values in the second row (resp., first row) of any tableau T by

$$\mathsf{Row}_2(T) \quad \text{(resp., } \mathsf{Row}_1(T)\text{)}.$$

Next we list some basic features of RSK insertion, which we may use without specific mention in the future. The following lemma is a consequence of the definition of RSK insertion.

Lemma 11. *Let $w \in S_n$, and suppose b bumps z in the RSK insertion process for w. Then $b < z$ and b appears to the right of z in the one-line notation of w.*

For permutation $v \in S_n$ and value $q \in \{1, \ldots, n\}$, let $c_v(q)$ be the column of $\mathrm{P}(v)$ into which q is first inserted. Let $\mathrm{LIS}_v(q)$ be the length of a longest increasing subsequence of v that ends with q. The following is a key result we will reference in our analysis.

Lemma 12 ([Sag01, Lemma 3.3.3])**.** *For $v \in S_n$ and $q \in \{1, \ldots, n\}$, we have $c_v(q) = \mathrm{LIS}_v(q)$.*

One consequence of Lemma 12 is that certain values must be part of every longest increasing subsequence of a permutation.

Corollary 13. *For a permutation v, if q is the only value in v first inserted into column $c_v(q)$ of $\mathrm{P}(v)$, then q is in every longest increasing subsequence in v.*

The last result in this subsection highlights basic properties of RSK tableaux for fully commutative permutations.

Lemma 14. *Let w be a fully commutative permutation with $\mathsf{Row}_2(\mathrm{P}(w)) = \{z_1 < z_2 < \cdots < z_t\}$. For each $i \in \{1, \ldots, t\}$, let b_i be the value that bumps z_i from the first row to the second row during the construction of $\mathrm{P}(w)$. Then we have the following.*

(a) The sequence $z_1 z_2 \cdots z_t$ is an (increasing) subsequence of w. In other words, the values $z_1, z_2, \ldots, z_t$ appear from left to right in the one-line notation of w.

(b) The sets $\{z_1, \ldots, z_t\}$ and $\{b_1, \ldots, b_t\}$ are disjoint. In other words, during RSK insertion, no value can both bump something and be bumped by something.

(c) The sequence $b_1 b_2 \ldots b_t$ is an increasing subsequence of w.

(d) Let $1 \leqslant i < j \leqslant t$. During RSK insertion, the value z_i is bumped before z_j.

Proof.

(a) Suppose, to the contrary, that z_i appears to the right of z_{i+1} for some i. Since b_i bumps z_i during the insertion algorithm, we know $b_i < z_i$, and the value b_i occurs to the right of z_i in the one-line notation of w. This means $z_{i+1}z_ib_i$ is a 321-pattern in w, which is a contradiction.

(b) By (a), we have that $z_1 \cdots z_t$ is an increasing subsequence of w. Hence, there are no i and j such that z_i bumps z_j, and the sets $\{z_1, \ldots, z_t\}$ and $\{b_1, \ldots, b_t\}$ are therefore disjoint.

(c) First, we show that $b_1 < \cdots < b_t$. Suppose, to the contrary, that $b_i > b_{i+1}$ for some i. Since z_i appears to the left of b_i in the one-line notation for w and $z_i > b_i$, the value b_{i+1} must appear to the left of b_i in order to avoid a 321-pattern in w. We also know z_{i+1} appears to the left of b_{i+1} in the one-line notation for w and $z_{i+1} > b_{i+1}$. From (a), we know z_iz_{i+1} is a subsequence of the one-line notation for w. Combining all of these observations, we conclude that

$$z_iz_{i+1}b_{i+1}b_i$$

is a subsequence of the one-line notation of w. So, since z_i is bumped by b_i, immediately before b_{i+1} is inserted, the value z_i is still in the first row. This means that b_{i+1} must bump a number no larger than z_i, which contradicts the assumption that b_{i+1} bumps z_{i+1}. Therefore $b_1 < \cdots < b_t$.

Now say for some i that b_{i+1} occurs to the left of b_i in the one-line notation for w. Since $b_i < b_{i+1}$, we would have the 321-pattern $z_{i+1}b_{i+1}b_i$ in w, which is a contradiction. Hence $b_1, \ldots, b_t$ occur from left to right in w.

(d) This follows from (c). $\square$

2.4 Characterization of boolean RSK tableaux

While Schensted's Theorem (Theorem 10) guarantees the insertion tableau of a boolean permutation has at most two rows, not every 2-row standard tableau is the insertion tableau of some boolean permutation. For example, the tableau T_1 below is the insertion tableau of the boolean permutation $w = 315264 = [21435] \in S_6$, but T_2 cannot be obtained as the insertion tableau of any boolean permutation.

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array}, \quad T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}$$

We review the characterization of these tableaux from [GPRT22]. First we need to define when a set of integers is "uncrowded."

Definition 15. Let L be a set of integers. If, for all integers x and y, with $x > 0$, we have

$$|[y, y + 2x] \cap L| \leqslant x + 1,$$

then we will say that L is *uncrowded*. Otherwise, we say that L is *crowded*.

Let T be a standard tableau with at most two rows. When $\mathsf{Row}_2(T)$ is uncrowded, we also call the tableau T an *uncrowded tableau*, and T is a *crowded tableau* otherwise. In the example above, we can see that T_1 is an uncrowded tableau because its second row $\{3, 5, 6\}$ is an uncrowded set, while T_2 is a crowded tableau because

$$|[4, 4 + 2 \cdot 1] \cap \{4, 5, 6\}| = 3 > 1 + 1\,.$$

The following proposition, which is the combination of several results in [GPRT22], provides a characterization of RSK tableaux coming from boolean permutations.

Proposition 16. *A standard tableau T with at most two rows is the insertion (or recording) tableau of a boolean permutation if and only if T is uncrowded.*

We define an *uncrowded (respectively, crowded) permutation* to be a permutation with an uncrowded (respectively, crowded) insertion tableau. By Proposition 16, a permutation is uncrowded exactly when it shares an insertion tableau with some boolean permutation.

3 Fully commutative elements and the weak order

From Theorem 10, we know that the RSK partition for a permutation has at most two rows if and only if the permutation is 321-avoiding; that is, if and only if it is fully commutative. Boolean permutations, which avoid patterns 321 and 3412, are a special class of fully commutative permutations, and Proposition 16 fully characterized their RSK tableaux. In this section, we build upon Proposition 16 to study the insertion tableaux of fully commutative, but not necessarily boolean, permutations.

3.1 Boolean core

We set the stage using the following lemma, which is little more than a restatement of the definition of fully commutative element.

Lemma 17. *Let w be a fully commutative permutation and $[u] \in R(w)$. If j is a repeated letter in $[u]$, then each pair of copies of j must be separated by both $j + 1$ and $j - 1$ in $[u]$. Put another way, if $x \prec y$ are elements of the heap H_w both with label j (i.e., $u_x = u_y = j$), then H_w contains elements p and $\tilde{p}$ with labels $k + 1$ and $k - 1$ such that $x \prec p \prec y$ and $x \prec \tilde{p} \prec y$.*

A key feature of boolean permutations is that their reduced words contain no repeated letters. This property fails to hold for arbitrary fully commutative permutations, but, as we will show in the next result, every fully commutative permutation can be thought of as having a "boolean core." More precisely, we can write any fully commutative permutation as the product of two permutations, one of which is boolean with the same support as the original permutation. As a result, every fully commutative permutation has a reduced word in which any repetition of letters occurs only after every letter in the support has appeared.

Theorem 18. *Let w be a fully commutative permutation. Then we can uniquely write $w = \hat{w}w'$, where $\ell(w) = \ell(\hat{w}) + \ell(w')$, the permutation $\hat{w}$ is boolean, and $\mathsf{supp}(\hat{w}) = \mathsf{supp}(w)$.*

Proof. Fix a fully commutative permutation w and $[u] = [u_1 \cdots u_\ell] \in R(w)$. Because w is fully commutative, it has a unique heap H_w. Elements with the same label are comparable in H_w. Thus, for each $i \in \mathsf{supp}(w)$, we can take the smallest element x in H_w such that $u_x = i$. Let C denote the set of all such smallest elements, for $i \in \mathsf{supp}(w)$.

We claim that C is an order ideal of H_w, and we will show that this is true using a proof by contradiction. Suppose $x, y \in H_w$ such that $y \in C$ and x is covered by y. Let $u_x = j$, and so by Lemma 5 $u_y = j \pm 1$. Suppose, for the purpose of obtaining a contradiction, that $x \notin C$. Thus there exists $\tilde{x} \prec x$ with $u_{\tilde{x}} = j$. Then, by Lemma 17, there exist $p, \tilde{p} \in H_w$ such that $\tilde{x} \prec p \prec x$, $\tilde{x} \prec \tilde{p} \prec x$, $u_p = j + 1$, and $u_{\tilde{p}} = j - 1$. But then we would have $y \notin C$, which is a contradiction.

Because C is an order ideal of H_w, we can choose a labeled linear extension of H_w whose first $|C|$ letters are precisely $\mathsf{supp}(w)$. This produces a reduced word for w whose leftmost $|C|$ letters are precisely $\mathsf{supp}(w)$.

Finally we show that this $\hat{w}$ is also unique. Recall that any prefix of a reduced word for w corresponds to an order ideal of H_w. The condition $\mathsf{supp}(\hat{w}) = \mathsf{supp}(w)$ requires that we pick an order ideal of H_w having $|\mathsf{supp}(w)|$ elements of distinct labels. Elements with the same label are comparable in H_w, meaning that we are forced to select the smallest one for each label. $\square$

We refer to the boolean permutation $\hat{w}$ in Theorem 18 as the *(right) boolean core* of a fully commutative permutation, where "right" refers to the fact that $\hat{w}$ is the maximal boolean permutation that is less than w in the right weak order.

Example 19. The heap of the permutation $w = 345619278$ in Example 8 is given in Figure 1. The boolean core of w is $\hat{w} = 314569278$, and its heap is given in Figure 2. Note that the reduced word $[21873456] \in R(\hat{w})$ appears as the left prefix of the reduced word $[21873456234] \in R(w)$.

Theorem 18 can also be proved without the language of heaps, by inducting on the length of a permutation.

3.2 Containment under the weak order

Theorem 18 identifies the boolean core of a fully commutative permutation, which gives some sense of how fully commutative permutations can be viewed as "elongations" of boolean permutations. We can similarly consider lengthening a fully commutative permutation. This leads to an important property about insertion tableaux.

Theorem 20. *Let v and w be fully commutative permutations such that $w = vs_i$ with $\ell(w) = \ell(v)+1$. Then $\mathsf{Row}_1(\mathrm{P}(v)) \supseteq \mathsf{Row}_1(\mathrm{P}(w))$; equivalently, $\mathsf{Row}_2(\mathrm{P}(v)) \subseteq \mathsf{Row}_2(\mathrm{P}(w))$.*

Proof. Let v and w be as in the statement of the result. So

$$w = v(1)\cdots v(i-1)\, v(i+1)\, v(i)\, v(i+2)\cdots v(n),$$

with $v(i) < v(i+1)$. The permutation w is fully commutative by assumption, so it is 321-avoiding. Therefore, in fact, we have

$$\begin{aligned} &v(j) < v(i+1) \text{ for all } 1 \leqslant j \leqslant i, \text{ and} \\ &v(j) > v(i) \text{ for all } i+1 \leqslant j \leqslant n. \end{aligned} \tag{3.1}$$

Set $\mathrm{P}_{i-1} := \mathrm{P}_{i-1}(v) = \mathrm{P}_{i-1}(w)$ to be the insertion tableau for the shared prefix $v(1)\cdots v(i-1)$ in the two permutations. To compute $\mathrm{P}_{i+1}(v)$, we insert $v(i)$ first and then $v(i+1)$; to compute $\mathrm{P}_{i+1}(w)$, we insert $v(i+1)$ first and then $v(i)$.

Consider first what happens when we insert $v(i)$ into P_{i-1}. There are two cases to consider: either $v(i)$ bumps something out of the first row of P_{i-1}, or $v(i)$ gets appended to the end of the first row of P_{i-1}.

Suppose first that $v(i)$ bumps some z out of the first row of P_{i-1}. For $v(i)$ to do this, the value z must have been the smallest number in that row larger than $v(i)$. To create $\mathrm{P}_{i+1}(v)$ from $\mathrm{P}_i(v)$, the value $v(i+1)$ must be appended to the first row of $\mathrm{P}_i(v)$, because $v(i+1) > v(j)$ for all $1 \leqslant j \leqslant i$, by (3.1). To construct $\mathrm{P}_i(w)$, we again have that $w(i) = v(i+1)$ gets appended to the end of the first row of P_{i-1}. When $w(i+1) = v(i)$ is inserted into $\mathrm{P}_i(w)$, it must bump the smallest value in $\mathsf{Row}_1(\mathrm{P}_{i-1}) \cup \{v(i+1)\}$ that is larger than $v(i)$; this value must be z, as above, because $z < v(i+1)$. Therefore, $\mathrm{P}_{i+1}(v) = \mathrm{P}_{i+1}(w)$, with $\{v(i), v(i+1)\}$ in the top row and z in the second row.

Because the rest of the entries in the one-line notations of v and w are identical, we can conclude from here that $\mathrm{P}(v) = \mathrm{P}(w)$.

Now suppose, for the remainder of the proof, that when $v(i)$ is inserted into P_{i-1} it is appended to the end of the first row of P_{i-1}. In other words, $v(i)$ is larger than all values in $\mathsf{Row}_1(\mathrm{P}_{i-1})$. Then, when $v(i+1)$ is inserted into $\mathrm{P}_i(v)$, this new value is also appended to the end of the first row because $v(i+1) > v(i)$. In other words, $\mathrm{P}_{i+1}(v)$ is created by appending both $v(i)$ and $v(i+1)$ to the first row of P_{i-1}.

To construct $\mathrm{P}_i(w)$, on the other hand, we first insert $v(i+1)$. This gets appended to the end of the first row of P_{i-1} because $v(i+1)$ is larger than all other values seen so far, by (3.1). In contrast, $v(i) < v(i+1)$, so $v(i)$ will bump something out of the first row of $\mathrm{P}_i(w)$ in order to form $\mathrm{P}_{i+1}(w)$. Everything in $\mathsf{Row}_1(\mathrm{P}_{i-1})$ is greater than $v(i)$, so $v(i)$ must bump $v(i+1)$ itself. Therefore, $\mathsf{Row}_1(\mathrm{P}_{i+1}(v)) = \mathsf{Row}_1(\mathrm{P}_{i+1}(w)) \cup \{v(i+1)\}$. And more to the point, $\mathsf{Row}_1(\mathrm{P}_{i+1}(v)) \supset \mathsf{Row}_1(\mathrm{P}_{i+1}(w))$.

Combining (3.1) with the fact that $v(i)$ is larger than every letter in $\mathsf{Row}_1(\mathrm{P}_{i-1})$, we have that $v(i+1), \ldots, v(n)$ must each be larger than every letter in $\mathsf{Row}_1(\mathrm{P}_{i-1}) \cup \{v(i)\}$. Therefore, all future insertions performed during the computation of both $\mathrm{P}(v)$ and $\mathrm{P}(w)$ will not bump any letter of $\mathsf{Row}_1(\mathrm{P}_{i-1}) \cup \{v(i)\}$ out of the first row. That is, everything in the first row from $v(i)$ leftward will remain unchanged during the remaining steps of the insertion algorithm.

We will prove that $\mathsf{Row}_1(\mathrm{P}(v))$ contains all of $\mathsf{Row}_1(\mathrm{P}(w))$, using an inductive argument with $\mathrm{P}_k(v)$ and $\mathrm{P}_k(w)$, for $i+1 \leqslant k \leqslant n$. We have shown the base case: $\mathsf{Row}_1(\mathrm{P}_{i+1}(v)) \supset \mathsf{Row}_1(\mathrm{P}_{i+1}(w))$. Assume, inductively, that for some $k \geqslant i+1$, we have $\mathsf{Row}_1(\mathrm{P}_k(v)) \supseteq \mathsf{Row}_1(\mathrm{P}_k(w))$. There are two ways for $v(k+1)$ to be inserted into $\mathrm{P}_k(v)$: either it gets appended to the end of the top row of the tableau, or it bumps some value z.

- If $v(k+1)$ gets appended to $\mathsf{Row}_1(\mathrm{P}_k(v))$, then everything in $\mathsf{Row}_1(\mathrm{P}_k(v))$ is less than $v(k+1)$. Because $\mathsf{Row}_1(\mathrm{P}_k(v)) \supseteq \mathsf{Row}_1(\mathrm{P}_k(w))$, all numbers in $\mathsf{Row}_1(\mathrm{P}_k(w))$ must also be less than $v(k+1)$. Therefore, $\mathrm{P}_{k+1}(w)$ is formed from $\mathrm{P}_k(w)$ by appending $v(k+1)$ to the end of the first row as well, and thus $\mathsf{Row}_1(\mathrm{P}_{k+1}(v)) \supseteq \mathsf{Row}_1(\mathrm{P}_{k+1}(w))$.
- If $v(k+1)$ bumps some $z \in \mathsf{Row}_1(\mathrm{P}_k(v))$, then z is the smallest value in $\mathsf{Row}_1(\mathrm{P}_k(v))$ that is larger than $v(k+1)$. We must now consider whether or not z was in $\mathsf{Row}_1(\mathrm{P}_k(w))$. If not, then there is nothing to worry about and we are done. On the other hand, if $z \in \mathsf{Row}_1(\mathrm{P}_k(w))$, then, because $\mathsf{Row}_1(\mathrm{P}_k(w)) \subseteq \mathsf{Row}_1(\mathrm{P}_k(v))$, this z must also be the smallest number in $\mathsf{Row}_1(P_k(w))$ that is larger than $v(k+1)$. Therefore, when we insert $v(k+1)$ into $\mathrm{P}_k(w)$, we will also bump z.

Thus the induction holds at all stages of the insertion algorithm, and hence $\mathsf{Row}_1(\mathrm{P}(v)) \supseteq \mathsf{Row}_1(\mathrm{P}(w))$. The tableaux have height at most 2, and so $\mathsf{Row}_2(\mathrm{P}(v)) \subseteq \mathsf{Row}_2(\mathrm{P}(w))$, as well. □

We highlight several facts relevant to upcoming arguments in Section 4.

Remark 21. For v and w fully commutative permutations with $w = vs_i$, $\ell(w) = \ell(v)+1$, and $\mathrm{P}(v) \neq \mathrm{P}(w)$, the following are established within the proof of Theorem 20:

(a) $v(k) < v(i+1)$ for $k < i$, and $v(k) > v(i)$ for $k > i$;

(b) the value $v(i)$ does not bump anything in $\mathrm{P}(v)$, and $v(i) \in \mathsf{Row}_1(\mathrm{P}(v))$;

(c) $v(i)$ bumps $v(i+1)$ in $\mathrm{P}(w)$, and $v(i) \in \mathsf{Row}_1(\mathrm{P}(w))$;

(d) $\mathsf{Row}_1(\mathrm{P}(v)) \cap [1, v(i)] = \mathsf{Row}_1(\mathrm{P}_i(v)) = \mathsf{Row}_1(\mathrm{P}_{i+1}(w)) = \mathsf{Row}_1(\mathrm{P}(w)) \cap [1, v(i)]$.

Because the length of the first row of a permutation's shape is determined by the length of a longest increasing subsequence in the permutation, we can use Theorem 20 to characterize when the insertion tableaux of v and vs_i are unequal.

Corollary 22. *Let v and w be fully commutative permutations such that $w = vs_i$, with $\ell(w) = \ell(v)+1$. Then $\mathrm{P}(v) \neq \mathrm{P}(w)$ if and only if every longest increasing subsequence in v uses both $v(i)$ and $v(i+1)$. In particular, when $\mathrm{P}(v) \neq \mathrm{P}(w)$, we have $|\mathsf{Row}_2(\mathrm{P}(w))| = |\mathsf{Row}_2(\mathrm{P}(v))| + 1$.*

Proof. Note that $|\mathsf{Row}_2(\mathrm{P}(w)) \setminus \mathsf{Row}_2(\mathrm{P}(v))| \leqslant 1$, because the length of the longest increasing subsequence changes by at most one after swapping adjacent values in a position. Since $\mathsf{Row}_2(\mathrm{P}(v)) \subseteq \mathsf{Row}_2(\mathrm{P}(w))$ by Theorem 20, we have that $\mathsf{Row}_2(\mathrm{P}(v)) \subsetneq \mathsf{Row}_2(\mathrm{P}(w))$ if and only if the size of the first row of $\mathrm{P}(v)$ is one more than the size of the first row of $\mathrm{P}(w)$. By Schensted's theorem (Theorem 10), this holds if and only if the length of a longest increasing subsequence of v is one more than the length of a longest increasing subsequence of w. Swapping $v(i)$ and $v(i+1)$ changes this length if and only if every longest increasing subsequence in v uses both $v(i)$ and $v(i+1)$. It follows that when $\mathsf{Row}_2(\mathrm{P}(v)) \subsetneq \mathsf{Row}_2(\mathrm{P}(w))$, the set $\mathsf{Row}_2(\mathrm{P}(w))$ contains exactly one more element than $\mathsf{Row}_2(\mathrm{P}(v))$. □

Theorem 20 has other implications for the weak order on fully commutative elements.

Corollary 23. *Let v and w be fully commutative permutations.*

(a) If v is less than w in the right weak order, then $\mathsf{Row}_2(\mathrm{P}(v)) \subseteq \mathsf{Row}_2(\mathrm{P}(w))$.

(b) If v is less than w in the left weak order, then $\mathsf{Row}_2(\mathrm{Q}(v)) \subseteq \mathsf{Row}_2(\mathrm{Q}(w))$.

Proof. Statement (a) follows immediately from Theorem 20. Statement (b) follows from (a) and Proposition 9. □

There is another important implication of Theorem 20, in conjunction with Theorem 18. This allows us to show the relationship between the insertion tableaux of a fully commutative element and that of its boolean core.

Corollary 24. *Let w be a fully commutative permutation and $\hat{w}$ its boolean core. Then*

$$\mathsf{Row}_1(\mathrm{P}(\hat{w})) \supseteq \mathsf{Row}_1(\mathrm{P}(w)) \quad \textit{and} \quad \mathsf{Row}_2(\mathrm{P}(\hat{w})) \subseteq \mathsf{Row}_2(\mathrm{P}(w)).$$

The following example illustrates this result.

Example 25. Let $v = 41623785 = [32154673]$, $w = vs_5 = 41627385 = [321546735]$, and let $\hat{v} = 41263785 = [3215467]$ denote their common boolean core. We can see that $\hat{v} < v < w$ in the right weak order. The RSK insertion algorithm produces

$$\mathrm{P}(\hat{v}) = \mathrm{P}(v) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 8 \\ \hline 4 & 6 & 7 \\ \cline{1-3}\end{array} \qquad \text{and} \qquad \mathrm{P}(w) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 7 & 8 \\ \hline\end{array}.$$

We have $\mathsf{Row}_1(\mathrm{P}(\hat{v})) \supseteq \mathsf{Row}_1(\mathrm{P}(v)) \supseteq \mathsf{Row}_1(\mathrm{P}(w))$ and $\mathsf{Row}_2(\mathrm{P}(\hat{v})) \subseteq \mathsf{Row}_2(\mathrm{P}(v)) \subseteq \mathsf{Row}_2(\mathrm{P}(w))$.

These results suggest a natural next step of research.

Question 26. The set of 4321-avoiding permutations are exactly those whose RSK tableaux have fewer than 4 rows. What is an analog of Theorem 20 for 4321-avoiding permutations?

4 Insertion tableaux dynamics

Throughout this section, we will restrict our attention to certain important scenarios, and we will highlight our assumptions for the reader in centered boxed text. To begin, we will assume throughout this section that

$$\boxed{v \text{ and } w \text{ are fully commutative permutations with } w = vs_i \text{ and } \ell(w) = \ell(v) + 1.}$$

In Theorem 20, we learned that

$$\mathsf{Row}_2(\mathrm{P}(v)) \subseteq \mathsf{Row}_2(\mathrm{P}(w)).$$

Corollary 22 gave conditions that determine exactly when $\mathrm{P}(v) \neq \mathrm{P}(w)$ in terms of the longest increasing subsequences of v. We next want to understand the entries of these tableaux when they are unequal. In particular, if $\mathrm{P}(v) \neq \mathrm{P}(w)$, is it possible for $\mathrm{P}(w)$ to be uncrowded? Said another way, if the insertion tableau changes along a covering relation in the right weak order, can the covering permutation be uncrowded? If $i \notin \mathsf{supp}(v)$, then this could certainly be the case. Consider, for example, when v is the identity. On the other hand, if $i \in \mathsf{supp}(v)$, then, as we shall see, the answer to the question is *no*.

Recall our assumptions in this section: v and w are fully commutative permutations (that is, they avoid 321) with $w = vs_i$ and $\ell(w) = \ell(v) + 1$. Let M and m be the values defined in Lemma 1:

$$\boxed{M := \max\{v(j) : j \leqslant i\} \qquad \text{and} \qquad m := \min\{v(j) : j \geqslant i+1\}.}$$

Our first lemma shows these values are part of a 3142-pattern in v whenever $i \in \mathsf{supp}(v)$.

Lemma 27. *Suppose $i \in \mathsf{supp}(v)$. Then v has a 3142-pattern formed by $M\, v(i)\, v(i+1)\, m$.*

Proof. Because $i \in \mathsf{supp}(v)$, it follows from Lemma 1 that $m < M$, and $M \geqslant v(i)$ and $m \leqslant v(i+1)$ by definition. Because $w = vs_i$ and $\ell(w) > \ell(v)$, we must have $v(i) < v(i+1)$.

Next we argue that $M > v(i)$. Suppose $M = v(i)$. Then $m < M = v(i) < v(i+1)$, so $v(i+1)\, v(i)\, m$ will form a 321-pattern in w, which is a contradiction. Therefore $M > v(i)$. Similarly we can show that $m < v(i+1)$.

Since w cannot have a 321-pattern, we also must have $M < v(i+1)$ and $m > v(i)$. Therefore the subsequence $M\, v(i)\, v(i+1)\, m$ is a 3142-pattern in v. □

In fact, 321-avoidance, the maximality of M, and the minimality of m force even more structure upon v.

Corollary 28. *Suppose $i \in \mathsf{supp}(v)$. Then*

$$v = \cdots M \ a_1 \ \cdots \ a_h \ v(i) \ v(i+1) \ e_1 \ \cdots \ e_j \ m \cdots,$$

where

$$a_1 < a_2 < \cdots < a_h < v(i) < m < M < v(i+1) < e_1 < e_2 < \cdots < e_j. \tag{4.1}$$

Let us now further suppose, for the remainder of this section, that

$$\boxed{i \in \mathsf{supp}(v) \text{ and } \mathrm{P}(v) \neq \mathrm{P}(w).}$$

Furthermore, we will

$$\boxed{\text{maintain the notation established in Corollary 28.}}$$

Example 29. Consider $v = 41623785 = [32154673]$, $w = vs_5 = 41627385 = [321546735]$ from Example 25, where we have $i = 5 \in \mathsf{supp}(v)$ and $\mathrm{P}(w) \neq \mathrm{P}(v)$. We now verify that $M = \max\{4, 1, 6, 2, 3\} = 6$ and $m = \min\{7, 8, 5\} = 5$. Furthermore, the value immediately to the right of M in the one-line notation of v is $a_1 = 2$, and the value immediately to the right of $v(5 + 1)$ is $e_1 = 8$, satisfying the inequalities in Corollary 28.

Corollary 22 tells us that every longest increasing subsequence in v must use both $v(i)$ and $v(i + 1)$. In particular, this means that $h \geqslant 1$ and $j \geqslant 1$.

Since $\mathsf{Row}_2(\mathrm{P}(v)) \subsetneq \mathsf{Row}_2(\mathrm{P}(w))$, it also follows from Corollary 22 that there is a unique value

$$e \in \mathsf{Row}_1(\mathrm{P}(v)) \cap \mathsf{Row}_2(\mathrm{P}(w)).$$

We will show that e occurs after $v(i + 1)$ in v, that $e > M$, and, finally, that $\mathsf{Row}_2(\mathrm{P}(w))$ is crowded as it contains too many integers in the interval $\{M, \ldots, e\}$.

The next sequence of lemmas describe certain values in the rows of $\mathrm{P}(v)$ and $\mathrm{P}(w)$. Recall for a permutation $v \in S_n$ and value $q \in \{1, \ldots, n\}$, we define $c_v(q)$ to be the column of $\mathrm{P}(v)$ into which q is first inserted.

Lemma 30. *In the construction of $\mathrm{P}(v)$ and $\mathrm{P}(w)$, the value M is bumped to the second row by one of $a_1, \ldots, a_h$.*

Proof. By Remark 21(b), $v(i)$ does not bump anything in $\mathrm{P}(v)$, so we have that $c_v(M) < c_v(v(i))$. Since $v(i) < M$, this means some value that occurs between M and $v(i)$ in v must bump M in $\mathrm{P}(v)$. Because the values prior to $v(i)$ are unchanged in w, M will be bumped by that same value in $\mathrm{P}(w)$. □

Just as we can track M in the RSK insertion algorithm, we can determine the role of m in the construction of $\mathrm{P}(v)$.

Lemma 31. *The value $v(i + 1)$ is bumped by m in $\mathrm{P}(v)$.*

Proof. Corollary 28 and Remark 21(b) tell us that, just before m is inserted in the process of constructing $\mathrm{P}(v)$, the first row contains $v(i) < v(i+1) < e_1 < \cdots < e_j$ with no element between $v(i)$ and $v(i + 1)$. By Lemma 27, $v(i) < m < v(i + 1)$, so m will bump $v(i + 1)$ in $\mathrm{P}(v)$. □

Next, we apply Remark 21 and Lemma 31 to determine the position of e in the one-line notation for v.

Lemma 32. *The value e occurs after $v(i + 1)$ in v.*

Proof. By Remark 21(d), we have

$$\mathsf{Row}_1(\mathrm{P}(v)) \cap [1, v(i)] = \mathsf{Row}_1(\mathrm{P}(w)) \cap [1, v(i)].$$

Since $e \in \mathsf{Row}_1(\mathrm{P}(v))$ and $e \notin \mathsf{Row}_1(\mathrm{P}(w))$, we know $e > v(i)$. By Remark 21(c), $v(i)$ does not bump anything in $\mathrm{P}(v)$ and $v(i) \in \mathsf{Row}_1(\mathrm{P}(v))$. Thus we have $c_v(e) > c_v(v(i))$, and e occurs after $v(i)$ in v. By Lemma 31, $v(i+1) \in \mathsf{Row}_2(\mathrm{P}(v))$, so $e \neq v(i+1)$. Hence e occurs after $v(i+1)$ in v. □

Using Lemma 32, we can show that e does not bump anything during the construction of $\mathrm{P}(v)$.

Lemma 33. *The value e does not bump anything in* $\mathrm{P}(v)$*.*

Proof. Suppose, for the purpose of obtaining a contradiction, that e bumps something in $\mathrm{P}(v)$. Then there exists a value q such that $e < q$ and q occurs before e in v. By Lemma 32, e occurs after $v(i+1)$ in v, so q occurs before e in w as well. However, e is bumped in w, so there is a value q' with $q' < e$ and q' occurring after e in w. This yields a 321-pattern in both v and w, which is not possible. Hence e does not bump anything in $\mathrm{P}(v)$. □

Define $e_0 := v(i+1)$. By Corollary 28, we see that $c_v(e_k) = c_v(v(i+1)) + k$ for $0 \leqslant k \leqslant j$. For $k > j$, we can then define (if any) e_k to be the first value in the one-line notation for v with $c_v(e_k) = c_v(v(i+1)) + k$. Let r be maximal so that $\{e_0, e_1, \ldots, e_r\} \subseteq \mathsf{Row}_2(\mathrm{P}(v))$. For all $0 \leqslant k \leqslant r$, let $t_0 := m, t_1, \ldots, t_r$ be the values such that t_k bumps e_k in $\mathrm{P}(v)$. By Lemma 14(c) we have $t_0 < t_1 < \cdots < t_r$, and these values appear from left to right in the one-line notation of v.

For a permutation $v \in S_n$ and a value $q \in \{1, \ldots, n\}$, recall that we define $\mathrm{LIS}_v(q)$ to be the length of a longest increasing subsequence of v that ends with q. The next lemma shows that the columns into which the values t_k are first inserted are the same in $\mathrm{P}(v)$ and $\mathrm{P}(w)$, for $0 \leqslant k \leqslant r$.

Lemma 34. *For all $0 \leqslant k \leqslant r$, $c_v(t_k) = c_w(t_k)$.*

Proof. By construction, we have $c_v(t_{k+1}) = c_v(t_k) + 1$ and $c_w(t_{k+1}) > c_w(t_k)$ for $0 \leqslant k < r$. Furthermore, since $\mathrm{LIS}_w(t_k) \leqslant \mathrm{LIS}_v(t_k)$, we know by Lemma 12 that $c_w(t_k) \leqslant c_v(t_k)$ for $0 \leqslant k \leqslant r$. We prove the statement by induction on k.

First we show $c_v(t_0) = c_w(t_0)$. By Corollary 28 and Remark 21(c), the first row of $\mathrm{P}_{i+j+1}(w)$ contains $v(i)$ and e_1, with no element between them. Because $v(i) < m < e_1$, we know that $m = t_0$ bumps e_1 in $\mathrm{P}(w)$. Since $c_w(e_1) = c_v(v(i+1))$ and t_0 bumps $v(i+1)$ in $\mathrm{P}(v)$, we have $c_w(t_0) = c_w(e_1) = c_v(v(i+1)) = c_v(t_0)$.

Next assume for some $0 \leqslant k < r$ that $c_v(t_k) = c_w(t_k)$. Then we have

$$c_w(t_k) + 1 \leqslant c_w(t_{k+1}) \leqslant c_v(t_{k+1}) = c_v(t_k) + 1.$$

Therefore $c_w(t_{k+1}) = c_v(t_{k+1})$, proving the statement. □

Since $e \in \mathsf{Row}_1(\mathrm{P}(v))$ occurs after $v(i+1)$ in v and does not bump anything in $\mathrm{P}(v)$, it follows that $e = e_k$ for some $k > r$. Therefore the value e_{r+1} exists, and by the definition of r, we have $e_{r+1} \in \mathsf{Row}_1(\mathrm{P}(v))$ with $e_{r+1} \leqslant e$. In fact, as a corollary to Lemma 34, we can show that $e_{r+1} = e$.

Corollary 35. *We have $e_{r+1} \in \mathsf{Row}_2(P(w))$, and so $e_{r+1} = e$.*

Proof. Since e_{r+1} is the only value in v inserted into column $c_v(e_{r+1})$ of $\mathrm{P}(v)$, we can apply Corollary 13 to conclude that e_{r+1} is in every longest increasing subsequence in v. By Corollary 22, this implies that every longest increasing subsequence in v ending with e_{r+1} must use both $v(i)$ and $v(i+1)$. As a result, $\mathrm{LIS}_w(e_{r+1}) = \mathrm{LIS}_v(e_{r+1}) - 1$. By Lemma 12, $c_w(e_{r+1}) = c_v(e_{r+1}) - 1$. We know $c_v(e_{r+1}) - 1 = c_v(t_r)$ by definition, and by Lemma 34, $c_v(t_r) = c_w(t_r)$. Hence $c_w(e_{r+1}) = c_w(t_r)$. Since $t_r < e_{r+1}$, we conclude that t_r bumps e_{r+1} in $\mathrm{P}(w)$. Since $e_{r+1} \in \mathsf{Row}_1(\mathrm{P}(v))$, it follows that $e_{r+1} = e$. □

Next we show that e_r and e are consecutive values.

Lemma 36. *With notation as above, $e = e_r + 1$.*

Proof. Since e occurs after e_r and $e_r, e \in \mathsf{Row}_2(\mathrm{P}(w))$, Lemma 14(c) shows that $e_r < e$. Suppose, for the purpose of obtaining a contradiction, that $e \neq e_r + 1$, and so $e > e_r + 1$. We analyze where $e_r + 1$ could occur in the one-line notation of v. First we argue that $e_r + 1$ cannot occur after e. Suppose it occurs after e. Before $e_r + 1$ is inserted into $\mathrm{P}(v)$, e is in the first row and the element to the left of e is either e_r or t_r. Since $t_r < e_r < e_r + 1 < e$, the value $e_r + 1$ will bump e, which contradicts the fact that $e \in \mathsf{Row}_1(\mathrm{P}(v))$.

Next we argue that $e_r + 1$ cannot occur prior to e_r. Suppose $e_r + 1$ is to the left of e_r. Before e_r is inserted into $\mathrm{P}(v)$, if $e_r + 1$ is in the first row, then e_r will bump $e_r + 1$, which contradicts Lemma 14(b). This forces $e_r + 1 \in \mathsf{Row}_2(\mathrm{P}(v))$, which, then, contradicts Lemma 14(d).

Therefore $e_r + 1$ must occur after e_r and before e, which implies $c_v(e_r) < c_v(e_r + 1) < c_v(e)$. However, this is impossible since $c_v(e_r) + 1 = c_v(e)$. Hence $e = e_r + 1$. □

The maximality of M means that $M + 1$ appears to the right of $v(i)$ in the one-line notation of v. Consider the set

$$[M + 1, e_r] \setminus \{v(i + 1) = e_0, e_1, \ldots, e_r\},$$

which has $e_r - (M + 1) + 1 - (r + 1)$ elements. These elements occur after $v(i)$ and are in $\mathsf{Row}_1(\mathrm{P}(v))$, so they must bump (some of) the r elements $\{e_1, \ldots, e_r\}$ and nothing else, by definition of r. Therefore we get

$$e_r - M - (r + 1) \leqslant r,$$

and hence

$$e_r - M \leqslant 2r + 1. \tag{4.2}$$

Now consider the interval

$$I := [M, e].$$

This is a set of size $e - M + 1$, and we can use Lemma 36 and Equation (4.2) to get

$$\big|I\big| = e - M + 1 = e_r + 1 - M + 1 \leqslant 2r + 3.$$

Moreover, the $(r + 2)$-element set

$$\{M, v(i + 1) = e_0, e_1, \ldots, e_r\}$$

is a subset of $\mathsf{Row}_2(\mathrm{P}(v))$.

We are now able to state the main result.

Theorem 37. *Suppose that v and w are fully commutative permutations with $w = vs_i$, $\ell(w) = \ell(v) + 1$, and $i \in \mathsf{supp}(v)$. Suppose, moreover, that $\mathrm{P}(v) \neq \mathrm{P}(w)$. Then w is a crowded permutation.*

Proof. As discussed above, there are $r + 2$ elements of the interval I in $\mathsf{Row}_2(\mathrm{P}(v))$, and the interval I contains at most $(2r + 3)$ elements. By Theorem 20, Corollary 35, and Lemma 36, there are $r + 3$ elements of the interval I in $\mathsf{Row}_2(\mathrm{P}(w))$, which means that w must be crowded. $\square$

A corollary of this result is an alternate characterization of uncrowded permutations

Corollary 38. *Let w be a fully commutative permutation with boolean core $\hat{w}$. Then w is uncrowded if and only if $\mathrm{P}(\hat{w}) = \mathrm{P}(w)$.*

Theorem 37 addresses the case $i \in \mathsf{supp}(v)$, and it is unclear how that result would change when $i \notin \mathsf{supp}(v)$.

Question 39. What can we say about $\mathrm{P}(vs_i)$ and $\mathrm{P}(v)$ for $i \notin \mathsf{supp}(v)$, when v is a fully commutative permutation? For example, consider $v = [3243] = 14523$ and $w = vs_1 = [32431] = 41523$. Now $1 \notin \mathsf{supp}(v)$, and we get that both of their insertion tableaux are the following tableau.

1	2	3
4	5	

.

5 Minimal crowded permutations under the weak order

Consider the poset of fully commutative (that is, 321-avoiding) permutations in S_n under the right weak order. The RSK partitions of such permutations have at most two rows, and we saw in Theorem 20 that the content of their second rows obeys a subset relation along covering relations in the weak order. We also saw, in Proposition 16, that a 2-row tableau is an insertion tableau for a boolean permutation if and only if it is an uncrowded tableau.

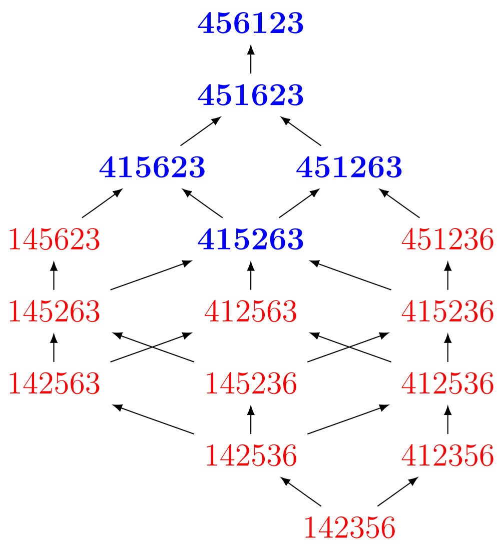

Figure 3: The dual order ideal of all crowded permutations in S_6 (in bold blue).

Recall that a fully commutative permutation w is called "uncrowded" if its insertion tableau is an uncrowded tableau. Otherwise a permutation is "crowded." Set

$$\mathsf{uncrowded}_n := \{w \in S_n \mid w \text{ is fully commutative and uncrowded}\}, \text{ and}$$
$$\mathsf{crowded}_n := \{w \in S_n \mid w \text{ is fully commutative and crowded}\}.$$

The subset relation in Theorem 20 allows us to conveniently partition the fully commutative elements into two sets: uncrowded and crowded permutations.

Lemma 40. *Consider the fully commutative elements of S_n, partially ordered according to the right weak order. The uncrowded permutations form an order ideal of this poset, and the crowded permutations form a dual order ideal of this poset.*

See Figure 3 for the dual order ideal of all crowded permutations in S_6.

Proof of Lemma 40. This follows from Theorem 20 and Proposition 16. □

Thus we can identify this partition of the fully commutative permutations in S_n by characterizing the maximal elements of the set $\mathsf{uncrowded}_n$ or, equivalently, the minimal elements of the set $\mathsf{crowded}_n$. The minimal elements of this latter set satisfy a pattern containment condition. Before we state and prove that property, consider what it means for w to be a minimal element of $\mathsf{crowded}_n$: the fully commutative permutation w is crowded, while every fully commutative permutation ws_i that it covers is uncrowded.

For the remainder of this section, we will assume that

$$\boxed{w \text{ is fully commutative; i.e., } w \text{ is 321-avoiding.}}$$

We begin by recalling a standard definition: an integer $d \in \{1, \ldots, n-1\}$ is a *descent* of $w \in S_n$ if $w(d) > w(d+1)$.

Lemma 41. *Suppose that d is a descent of w, and that $w(d+1)$ does not bump $w(d)$ during RSK insertion. Then $\mathrm{P}(w) = \mathrm{P}(ws_d)$. In other words, if $\mathrm{P}(w) \neq \mathrm{P}(ws_d)$, then either d is not a descent of w, or $w(d+1)$ bumps $w(d)$ during RSK insertion.*

Proof. Set $v := ws_d$, and $\mathrm{P}' := \mathrm{P}_{d-1}(w) = \mathrm{P}_{d-1}(v)$. Because w is 321-avoiding and d is a descent of w, the value $w(d)$ must be larger than everything to its left in the one-line notation of w. Thus, in forming $\mathrm{P}_d(w)$, this $w(d)$ gets appended to the end of the first row of P', without bumping anything. In forming $\mathrm{P}_{d+1}(w)$, the value $w(d+1)$, which is less than $w(d)$ because d is a descent, will bump something. Let z be the value that it bumps; i.e., z is the smallest value in $\mathsf{Row}_1(\mathrm{P}_d(w))$ that is larger than $w(d+1)$. We know by assumption that $z \neq w(d)$. In particular, $z < w(d)$ and z appears to the left of $w(d)$ in w. This last fact means that $z \in \mathsf{Row}_1(\mathrm{P}')$.

In forming $\mathrm{P}_d(v)$, the value $v(d) = w(d+1)$ bumps z from the first row of P' to the second row. In forming $\mathrm{P}_{d+1}(v)$, the value $v(d+1) = w(d)$ is the largest value we have seen so far, so it gets appended to the end of the first row of $\mathrm{P}_d(v)$, without bumping anything. Therefore $\mathrm{P}_{d+1}(w) = \mathrm{P}_{d+1}(v)$, and hence $\mathrm{P}(w) = \mathrm{P}(v)$. □

Somewhat akin to Lemma 41, we can make the following additional statement, which we phrase in terms of Knuth relations.

Definition 42. Two permutations w and v *differ by one Knuth relation* if w is the result of replacing a consecutive 312-pattern in v by a consecutive 132-pattern (or vice versa), or replacing a consecutive 231-pattern in v by a consecutive 213-pattern (or vice versa).

Lemma 43. *Let d be a descent of w. If $w(d+2) < w(d)$, then $\mathrm{P}(w) = \mathrm{P}(ws_d)$.*

Proof. The permutation w is 321-avoiding, so $w(d)w(d+1)w(d+2)$ must be a 312-pattern. Knuth's theorem [Knu70] says that the insertion tableau is preserved under a Knuth relation, so $\mathrm{P}(w) = \mathrm{P}(ws_d)$. □

We now return to the characterization motivated by Lemma 40: identification of the minimal elements of $\mathsf{crowded}_n$ in the poset of fully commutative permutations of S_n.

5.1 Consequences of minimality in $\mathsf{crowded}_n$

As it turns out, knowing that a permutation is minimal in the dual order ideal $\mathsf{crowded}_n$ imposes substantial structure on the permutation. In this subsection, we will collect many of these consequences of minimality, with the ultimate goal of proving a characterization of minimality in Section 5.2.

Throughout this subsection we will consider permutations that are minimal elements of the dual order ideal $\mathsf{crowded}_n$, and we will identify features of the permutations that follow from that property.

We begin with an immediate corollary of Lemma 41.

Corollary 44. *Let w be a minimal crowded permutation.*

(a) *Then d is a descent of w if and only if $w(d+1)$ bumps $w(d)$ to the second row during RSK insertion.*

(b) *Furthermore, every $w(d) \in \mathsf{Row}_2(\mathrm{P}(w))$ is bumped by $w(d+1)$.*

Proof. It remains to prove Part (b). Suppose $w(d)$ is an element of $\mathsf{Row}_2(\mathrm{P}(w))$, bumped by $w(j)$ with $j > d+1$. Part (a) tells us that d is not a descent of w. Because $w(d) > w(j)$, there exists a descent $d' \in [d+1, j-1]$. Lemma 41 implies that $w(d'+1)$ bumps $w(d')$ during RSK insertion. The values $w(j)$ and $w(d'+1)$ cannot be equal, so d' is in fact in $[d+1, j-2]$. However, the fact that $w(j)w(d'+1)$ is not a subsequence of w violates Lemma 14(c), and so in fact we must have $j = d+1$. □

Lemma 43 and Corollary 44 impose rules on the values that are unaffected by bumping during RSK insertion.

Corollary 45. *Let w be a minimal crowded permutation, with first descent d and last descent d'. Then the permutation w fixes all $i \in [1, d-1] \cup [d'+2, n]$.*

Proof. Suppose, first, that some $i < d$ is not fixed by w. Let i be minimal with this property, and let j be such that $w(j) = i$. Minimality of i means that $j > i$, and that $j-1$ is a descent of w. By Corollary 44, the value i must bump $w(j-1)$ to the second row during RSK insertion. Moreover, this minimality means that $w(d+1) \geqslant i$. To avoid $w(d)w(d+1)i$ forming a 321-pattern in w, we must have that $w(d+1) = i$. Minimality of i and the fact that d is the first descent mean that $w(d+1) = i < w(i) < w(i+1) < \cdots < w(d)$, and so i will actually bump $w(i)$ during RSK insertion, contradicting the assumption that $i < d$ and Corollary 44.

Now suppose that some $i > d'+1$ is not fixed by w. Let i be maximal with this property, and let j be such that $w(j) = i$. Maximality of i means that $j < i$, and that j is a descent of w. And, by Corollary 44, this i must be bumped by $w(j+1)$ during RSK insertion. To avoid $w(j)w(d')w(d'+1)$ forming a 321-pattern in w, we must have that $j = d'$. Moreover, maximality of $i > d'$ means that $w(j+2) < i$, and so Lemma 43 contradicts the minimality of w. □

At this point, we have established several properties about the one-line representation of minimal elements of $\mathsf{crowded}_n$. In fact, we can go even further, showing that values in the interval $[d, d'+1]$, in the language of Corollary 45 must be, in a sense, interwoven.

For the remainder of this subsection, define:

$$\boxed{\begin{array}{l}\mathsf{Row}_2(\mathrm{P}(w)) = \{z_1 < \cdots < z_t\}, \text{ and } b_i \text{ is the value that bumps } z_i \text{ to} \\ \mathsf{Row}_2(\mathrm{P}(w)) \text{ during the construction of } \mathrm{P}(w), \text{ for each } i = 1, \ldots, t.\end{array}}$$

We will also want to be able to refer to "minimally crowded sets," and so for positive integers x and y, we will write

$$\boxed{S_{x,y} := \{y, y+1, y+3, y+5, \ldots, y+2x-1, y+2x\}}$$

Lemma 46. *Let w be minimal in $\mathsf{crowded}_n$. Then*

$$z_1b_1z_2b_2\cdots z_tb_t$$

is a consecutive subsequence of the one-line notation for w.

Proof. By Corollary 44, each z_ib_i is a consecutive subsequence. From Lemmas 11 and 14, we know that $z_1z_2z_3\cdots z_t$ and $b_1b_2b_3\cdots b_t$, are subsequences of the one-line notation for w. We next prove that b_jz_{j+1} is a subsequence of w for all $1 \leqslant j < t$.

Suppose, for the sake of contradiction, there is some j such that b_j appears to the right of z_{j+1}, as in

$$w = \cdots z_j \cdots z_{j+1} \cdots b_j \cdots b_{j+1} \cdots .$$

But $z_{j+1} > z_j > b_j$ by Lemma 11, hence there exists a descent d such that $w(d)$ occurs at or after z_{j+1}, and before b_j. By Corollary 44, this is impossible. Therefore, for all $1 \leqslant j < t$, we must have b_j appearing to the left of z_{j+1} in the one-line notation for w.

We now prove that this subsequence is consecutive. Suppose that some value $q \neq z_{i+1}$ follows b_i. To avoid 321-patterns, we must have that $q < z_{i+1}$. If $q < z_i$, then Lemma 43 would produce a contradiction with the fact that w is minimal in $\mathsf{crowded}_n$. Thus it remains only to consider when $q > z_i$.

Because q is necessarily in $\mathsf{Row}_1(\mathrm{P}(w))$ and b_{i+1} bumps z_{i+1}, we have $q < b_{i+1} < z_{i+1}$. We have assumed $q > z_i$, so, in fact, $z_{i+1} \geqslant z_i + 3$. The set $\mathsf{Row}_2(\mathrm{P}(w)) = \{z_1 < z_2 < \cdots < z_t\}$ is crowded, so there exist positive integers x and y such that $S_{x,y} \subseteq \mathsf{Row}_2(\mathrm{P}(w))$. Since $z_{i+1} \geqslant z_i + 3$, the values z_i and z_{i+1} cannot both be in $S_{x,y}$. Define j and j' so that $w(j) = z_i$ and $w(j') = z_{i+1}$, and there are two options.

- If $y + 2x < z_{i+1}$, set $\tilde{w} := ws_{j'}$:

  $$\tilde{w} = \cdots z_ib_iq \cdots b_{i+1}z_{i+1} \cdots .$$

  Thus $S_{x,y} \subseteq \{z_1 < \cdots < z_i\} \subseteq \mathsf{Row}_2(\mathrm{P}(\tilde{w}))$. Therefore $\tilde{w} < w$, and $\tilde{w}$ is crowded, contradicting the minimality of w.

- If $z_i < y$, then construct $\tilde{w} := ws_j$:

  $$\tilde{w} = \cdots b_iz_iq \cdots z_{i+1}b_{i+1} \cdots .$$

  Then $S_{x,y} \subseteq \{z_{i+1} < \cdots < z_t\} \subseteq \mathsf{Row}_2(\mathrm{P}(\tilde{w}))$. Therefore $\tilde{w} < w$ and $\tilde{w}$ is crowded, again contradicting the minimality of w.

Thus there can be no such q, and the subsequence $z_1b_1z_2b_2\cdots z_tb_t$ is consecutive in w. □

Recall from Section 2.4 that a fully commutative permutation w is crowded if and only if $\mathsf{Row}_2(\mathrm{P}(w))$ is a crowded set. This means that there exist positive integers x and y such that

$$|[y, y + 2x] \cap \mathsf{Row}_2(\mathrm{P}(w))| > x + 1.$$

In fact, we can choose x and y so that

$$S_{x,y} = [y, y + 2x] \cap \mathsf{Row}_2(\mathrm{P}(w)).$$

Note that the set $S_{x,y}$ contains at least three elements.

Lemma 47. *Let w be a crowded permutation, and let $S_{x,y} \subseteq \mathsf{Row}_2(\mathrm{P}(w))$ be as described above. The value that bumps the third smallest element of $S_{x,y}$ during RSK insertion is less than y.*

Proof. Let c be the value that bumps the third smallest element of $S_{x,y}$. By Lemma 14(b), we know $c \notin \mathsf{Row}_2(\mathrm{P}(w))$. We will prove our result in two cases: $x = 1$ and $x > 1$. First consider $x = 1$. Since $S_{1,y} \subseteq \mathsf{Row}_2(\mathrm{P}(w))$ and $c < y + 2$, we see that $c < y$.

Now consider the case $x > 1$, so $c = y + 2$. By Lemma 14, this implies $y + 2k$ will bump $y + 2k + 1$ for all $1 \leqslant k \leqslant x - 1$. However, if $y + 2x - 2$ bumps $y + 2x - 1$, there is no value in $\mathsf{Row}_1(\mathrm{P}(w))$ both smaller than $y + 2x$ and larger than $y + 2x - 2$ that could have bumped $y + 2x$, and yet $S_{x,y} \subseteq \mathsf{Row}_2(\mathrm{P}(w))$. Thus $c < y$. □

Corollary 48. *Let w be a minimal element of $\mathsf{crowded}_n$. Then w contains a consecutive occurrence of the pattern 415263. Moreover, w has an occurrence $w(i) \cdots w(i + 5)$ of the pattern 415263 in which*

$$\{w(i), \dots, w(i + 5)\} \cap \mathsf{Row}_2(\mathrm{P}(w)) = \{w(i), w(i + 2), w(i + 4)\}.$$

Proof. The permutation w is crowded, so there exist integers x and y such that

$$\{y, y + 1, y + 3, \dots, y + 2x - 1, y + 2x\} = [y, y + 2x] \cap \mathsf{Row}_2(\mathrm{P}(w)).$$

Since $\mathsf{Row}_2(\mathrm{P}(w)) = \{z_1 < \cdots < z_t\}$, there is some $1 \leqslant r \leqslant t - 2$ such that $y = z_r$.

Lemma 46 tells us that

$$z_r b_r z_{r+1} b_{r+1} z_{r+2} b_{r+2}$$

is a consecutive subsequence of w. By Lemma 47, we know that $b_{r+2} < z_r$. By Lemma 14, we have $z_r < z_{r+1} < z_{r+2}$ and $b_r < b_{r+1} < b_{r+2}$. Combining these with the fact that $b_i < z_i$ for each i, the consecutive subsequence that $z_r b_r z_{r+1} b_{r+1} z_{r+2} b_{r+2}$ is a 415263-pattern. By Lemma 14, the values b_r, b_{r+1}, and b_{r+2} are not in $\mathsf{Row}_2(\mathrm{P}(w))$. □

In fact, we can say more about this set $S_{x,y}$.

Lemma 49. *Let w be minimal in $\mathsf{crowded}_n$. Any crowded subset of $\mathsf{Row}_2(\mathrm{P}(w))$ must include the largest element of $\mathsf{Row}_2(\mathrm{P}(w))$.*

Proof. Recall that $z_t = \max\{\mathsf{Row}_2(\mathrm{P}(w))\}$, and let j be such that $w(j) = z_t$. The set $\mathsf{Row}_2(\mathrm{P}(w))$ is crowded, so there are positive integers x and y such that $S_{x,y} \subseteq \mathsf{Row}_2(\mathrm{P}(w))$. If $z_t \notin S_{x,y}$, then $S_{x,y} \subseteq \mathsf{Row}_2(\mathrm{P}(w)) \setminus \{z_t\}$ and the permutation $\tilde{w} := ws_j < w$ would be crowded because $\mathsf{Row}_2(\mathrm{P}(\tilde{w})) = \mathsf{Row}_2(\mathrm{P}(w)) \setminus \{z_t\}$, contradicting the minimality of w. Thus every crowded subset $S_{x,y} \subseteq \mathsf{Row}_2(\mathrm{P}(w))$ must have $y + 2x = z_t$. □

This property about crowded subsets implies that when w is minimal in $\mathsf{crowded}_n$, there is, in fact, a unique crowded subset of the second row of $\mathrm{P}(w)$ that is inclusion-wise minimal.

Corollary 50. *Let w be minimal in $\mathsf{crowded}_n$. Then $\mathsf{Row}_2(\mathrm{P}(w))$ contains exactly one inclusion-wise minimal crowded subset.*

Proof. By Lemma 49, every crowded subset of $\mathsf{Row}_2(\mathrm{P}(w))$ includes the maximal element $z_t \in \mathsf{Row}_2(\mathrm{P}(w))$. Let $S_{x,y}$ be the crowded subset for which y is maximal. Then $\{y, y+1\} \in \mathsf{Row}_2(\mathrm{P}(w))$. To avoid $\{y-1, y, y+1\}$ contradicting Lemma 49, we must have $y-1 \notin \mathsf{Row}_2(\mathrm{P}(w))$. This means that there are no crowded sets $S_{x',y'}$ for $y' < y$. $\square$

In Lemma 46, we proved the consecutivity of the subsequence $z_1 b_1 \cdots z_t b_t$ in w. We already know several inequalities among these letters, and there is now one more that we can establish.

Lemma 51. *Let w be minimal in $\mathsf{crowded}_n$. For all $i \in [1, t-3]$, we have $z_i < b_{i+3}$.*

Proof. By Corollary 45, we can assume, without loss of generality, that $b_1 = 1$ and $z_t = n$. Furthermore, Corollary 50 forces $z_{t-1} = n-1$. Now suppose, for the purpose of obtaining a contradiction, that there exists $i \in [1, t-3]$ such that $z_i > b_{i+3}$.

The set $\{z_i, \ldots, z_{t-1}\}$ contains $t-i$ elements. Because $z_i < z_{i+1} < \cdots$, we have

$$\{z_i, \ldots, z_{t-1}\} \subseteq [z_i, n-1].$$

The interval $[z_i, n-1]$ can be partitioned into

$$\{z_i, \ldots, z_{t-1}\} \sqcup \{b_j : b_j > z_i\},$$

meaning that the cardinality of $[z_i, n-1]$ is at most $(t-i) + (t-(i+3)) = 2(t-i) - 3$. Therefore $\{z_i, \ldots, z_{t-1}\}$ is crowded, contradicting Corollary 49. $\square$

From this property, we learn how 415263-patterns can appear in a minimal element of $\mathsf{crowded}_n$.

Corollary 52. *Let w be minimal in $\mathsf{crowded}_n$. Every 415263-pattern in w is consecutive.*

Proof. Lemma 14, Corollary 45, and Lemma 51 mean that $w(i) < w(j)$ for all $j \geqslant i+6$, and so it is impossible to find a non-consecutive 415263-pattern in w. $\square$

In fact, any consecutive subsequence of a minimal element of $\mathsf{crowded}_n$ that both begins and ends with a descent must have one of two forms.

Lemma 53. *Let w be a minimal element in $\mathsf{crowded}_n$, with descent set $\{d, d+2, \ldots, d+2k\}$. For every $i \in [0, k-2]$, the consecutive subsequence*

$$w(d+2i) \cdots w(d+2i+5)$$

is either a 415263- or a 315264-pattern.

Proof. From Lemmas 14 and 46, it remains to show that in any such sequence

$$z_i b_i z_{i+1} b_{i+1} z_{i+2} b_{i+2},$$

we have that z_i is greater than b_{i+1}. Suppose, for the sake of contradiction, that $z_i < b_{i+1}$, with j defined so that $w(j) = z_i$. In other words, $\{w(1), \ldots, w(j+1)\} = [1, j+1]$. The permutation w is crowded, so let $S_{x,y} \subseteq \mathsf{Row}_2(\mathrm{P}(w))$ be the unique containment-wise minimal crowded set guaranteed by Corollary 50. If $z_{i+1} \notin S_{x,y}$, then $z_i \notin S_{x,y}$, by Lemma 49. Moreover, $\mathsf{Row}_2(ws_j) = \mathsf{Row}_2(w) \setminus \{z_i\}$, so $S_{x,y} \subseteq \mathsf{Row}_2(ws_j)$, and $ws_j < w$ is an element of $\mathsf{crowded}_n$, contradicting the assumption of minimality.

Now suppose, on the other hand, that $z_{i+1} \in S_{x,y}$. Then $z_{i+1} + 1 \notin S_{x,y}$, so it must be that $z_{i+1} + 2 \in S_{x,y}$, and this can only happen if $z_{i+1} + 1$ bumps $z_{i+1} + 2$ to $\mathsf{Row}_2(w)$. In fact, a similar argument shows that $z_{i+1} + 2i$ might be bumped by $z_{i+1} + 2i - 1$, but $z_{i+1} + 2i + 1$ cannot then also be bumped, contradicting the fact that $S_{x,y}$ is crowded. ☐

5.2 Characterization of minimality in $\mathsf{crowded}_n$.

Having established a variety of properties of minimal elements of $\mathsf{crowded}_n$ in Section 5.1, we are now able to completely characterize those elements.

Theorem 54. *A permutation w is a minimal element of $\mathsf{crowded}_n$ if and only if it satisfies the conditions below.*

(a) The set of descents of w has the form $\{d, d+2, d+4, \ldots, d+2k\}$ for some $k \geqslant 2$.

(b) The set $\{w(d), w(d+2), \ldots, w(d+2k)\}$ is crowded.

(c) The permutation fixes all $i \in [1, n] \setminus [d, d+2k+1]$.

(d) The pattern 415263 occurs in w, and every occurrence of 415263 is consecutive.

(e) For each $i \in [0, k-2]$, the consecutive subsequence

$$w(d+2i) \cdots w(d+2i+5)$$

is either a 415263- or a 315264-pattern.

Proof. First suppose that w is a minimal element of $\mathsf{crowded}_n$. Then Corollary 44 and Lemma 46 establish Properties (a) and (b). Property (c) is proved in Corollary 45, and Property (d) is a result of Corollaries 48 and 52. Finally, Property (e) follows from Lemma 53. Finally, we know from Property (a) and Corollary 44 that $\mathsf{Row}_2(\mathrm{P}(w)) = \{w(d), w(d+2), \ldots, w(d+2k)\}$.

Now suppose that a permutation w has Properties (a)–(e) in the statement of the theorem. It follows from (a) and (c) and the fact that w is 321-avoiding that $\mathsf{Row}_2(\mathrm{P}(w)) = \{w(d), w(d+2), \ldots, w(d+2k)\}$.

This and Property (b) mean that $w \in \mathsf{crowded}_n$. It remains, now, to prove that w is minimal in that set.

Suppose, for the purpose of obtaining a contradiction, that w is not minimal in $\mathsf{crowded}_n$. In fact, suppose that w is minimal with this property, meaning that anything covered by w is either not crowded, or minimal in $\mathsf{crowded}_n$. In particular, there must be at least one $v = ws_i$ in the latter category, by our assumption about w. Given Property (c), let us assume, without loss of generality, that $d = 1$ and $d + 2k = n - 1$. Because $v < w$, we have that $w(i) > w(i+1)$. Moreover, our assumptions about w mean that $w(2j-1) > w(2j)$ for all j, and each $w(2j)$ bumps $w(2j-1)$ to $\mathsf{Row}_2(\mathrm{P}(w))$. In particular, $w(n-1) = n$ and $w(n-3) = n-1$. On the other hand, Properties (a), (d), and (e) mean that in v, those bumping rules are no longer the case when $2j-1 > i$. Indeed, in v, it is $w(2j+2)$ that bumps $w(2j-1)$ to $\mathsf{Row}_2(\mathrm{P}(v))$ when $2j-1 \geqslant i$. Thus

$$\mathsf{Row}_2(\mathrm{P}(v)) = \mathsf{Row}_2(\mathrm{P}(w)) \setminus \{w(n-1)\}.$$

Since we have assumed that $v \in \mathsf{crowded}_n$ is minimal, we know from previous results that $\mathsf{Row}_2(\mathrm{P}(v))$ contains $n-1$, $n-2$ (which would have been $w(n-5)$, and either $n-3$ or $n-4$ (which would have been $w(n-7)$.

- If $n-3 \in \mathsf{Row}_2(\mathrm{P}(v))$, then $w(n) < n-3$, and so $w(n-7)w(n-6)w(n-5)w(n-4)w(n-1)w(n)$ would be a non-consecutive 415263-pattern, violating Property (d).
- If, instead, $n-4 \in \mathsf{Row}_2(\mathrm{P}(v))$, then $n-4 = w(n-7)$ and hence $n-3 = w(n)$. If we try to understand the rest of w while satisfying Property (d), we find that $n-5 = w(n-2)$, $n-6 = w(n-9)$, $n-7 = w(n-4)$, $n-8 = w(n-11)$, and so on, meaning that the set $\mathsf{Row}_2(\mathrm{P}(v))$ will never actually be crowded.

Thus there can be no such $v < w$, and so $w \in \mathsf{crowded}_n$ is minimal. □

Remark 55. Continuing the notation of Theorem 54, it followed immediately that if w is a minimal element of $\mathsf{crowded}_n$ then $\mathsf{Row}_2(\mathrm{P}(w)) = \{w(d), w(d+2), \ldots, w(d+2k)\}$.

Example 56. The permutation $w = 41627385$ is a minimal element of $\mathsf{crowded}_8$. We check each of the conditions to confirm this.

(a) The descents of w are $\{1,3,5,7\}$, so $d = 1$ and $k = 3$.

(b) The set $\{w(1), w(3), w(5), w(7)\}$ is $\{4,6,7,8\}$, which is crowded due to $\{6,7,8\}$.

(c) The third condition holds vacuously.

(d) The permutation w contains two occurrences of the 415263 pattern: 416273 and 627385. Both are consecutive subsequences of w.

(e) We check $i = 0, 1$: the subsequence 416273 is a 415263-pattern and the subsequence 627385 is a 415263-pattern.

The insertion tableau in this case is

$$\mathrm{P}(w) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 7 & 8 \\ \hline \end{array},$$

and indeed $\mathsf{Row}_2(\mathrm{P}(w)) = \{w(1), w(3), w(5), w(7)\}$.

The analyses of uncrowded and crowded permutations throughout this section suggest several possible areas of future research.

Question 57. How can the maximal elements of $\mathsf{uncrowded}_n$ be characterized?

Question 58. How many uncrowded permutations are there? How many crowded permutations?

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