Polynomial extension of the Stronger Central Sets Theorem

Sayan Goswami[a] Lorenzo Luperi Baglini[b] Sourav Kanti Patra[c]

Submitted: Sep 21, 2022; Accepted: Oct 26, 2023; Published: Dec 1, 2023

Abstract

In [F81], Furstenberg introduced the notion of central set and established his famous Central Sets Theorem. Since then, several improved versions of Furstenberg's result have been found. The strongest generalization has been published by De, Hindman and Strauss in [DHS08], whilst a polynomial extension by Bergelson, Johnson and Moreira appeared in [BJM17].

In this article, we will establish a polynomial extension of the stronger version of the central sets theorem, and we will discuss properties of the families of sets that this result leads to consider.

Mathematics Subject Classifications: 05D10, 22A15

1 Introduction

A core problem in Ramsey Theory over the naturals is the characterization of which families $\mathcal{F}$ of subsets of $\mathbb{N}$ are partition regular, *i.e.* which families have the property that whenever $\mathbb{N} = \bigcup_{i=1}^{r} A_i$ is a finite partition of $\mathbb{N}$ (throughout our article we will assume $\mathbb{N} = \{1, 2, \ldots\}$), at least one of the A_i's belongs to $\mathcal{F}$. If the family $\mathcal{F}$ has the stronger property that whenever any $A \in \mathcal{F}$ is finitely partitioned one of the pieces in the partition belongs to $\mathcal{F}$, the family is said to be strongly partition regular.

Two fundamental and classical results in Ramsey theory state, respectively, that the family of sets that contain arbitrarily long arithmetic progressions (called AP-rich sets) and the family of sets that contain an infinite subset X and all the finite sums of distinct elements of X, called IP-sets, are partition regular[1]. The first result is called van der Waerden's theorem [vdW27], the latter is called Hindman's theorem [H74].

[a]The Institute of Mathematical Sciences, A CI of Homi Bhabha National Institute, CIT Campus, Taramani, Chennai 600113, India (sayangoswami@imsc.res.in.

[b]Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italy (lorenzo.luperi@unimi.it).

[b]Department of Mathematics, Centre for Distance and Online Education, University of Burdwan, Burdwan-713104, West Bengal (souravkantipatra@gmail.com).

[1]It has been proven that they are indeed strongly partition regular.

In his seminal work [F81], Furstenberg used methods and notions from topological dynamics to define the notion of the central set and provided a joint extension of both van der Waerden's and Hindman's theorems, known as the Central Sets Theorem. In its statement, and in what follows, we will use the following notation: given any set X, we let $\mathcal{P}_f(X)$ be the set of all nonempty finite subsets of X.

Central Sets Theorem. Let $l \in \mathbb{N}$, and $A \subseteq \mathbb{N}$ be a central set. For each $i \in \{1, 2, \ldots, l\}$ let $\langle x_{i,m} \rangle_{m=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Then there exist sequences $\langle b_m \rangle_{m=1}^{\infty}$ in $\mathbb{N}$ and $\langle K_m \rangle_{m=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that

1. For each m, $\max K_m < \min K_{m+1}$ and
2. For each $i \in \{1, 2, \ldots, l\}$ and $H \in \mathcal{P}_f(\mathbb{N})$, $\sum_{m \in H}(b_m + \sum_{t \in K_m} x_{i,t}) \in A$.

The original definition of the notion of central set was dynamical; however, the following equivalent simpler ultrafilters[2] characterization was found in [BH90].

Definition 1. $A \subseteq \mathbb{N}$ is central if A belongs to a minimal idempotent $\mathcal{U} \in \beta\mathbb{N}$.

Several generalizations of the Central Sets Theorem to semigroups have been found in the literature; for details, we refer to [H20]. As we are interested to provide a new general version of the Central Sets Theorem for[3] $\mathbb{N}$, we will recall the specification to $\mathbb{N}$ of some of these generalizations.

We will polynomialize the following result of De, Hindman and Strauss, published in [DHS08], which is a general commutative version of the Central Sets theorem.

For any two sets A, B let ${}^{B}A$ be the set of all functions from B to A.

Stonger Central Sets Theorem. Let $C \subseteq S$ be central. There exists $\alpha : \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N}) \to \mathbb{N}$ and $H : \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N}) \to \mathcal{P}_f(\mathbb{N})$ such that

1. if $F, G \in \mathcal{P}_f(\tau)$ and $F \subsetneq G$ then $\max H(F) < \min H(G)$, and
2. whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\tau)$, $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_m$ and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, one has

$$\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in C.$$

We are interested in arriving to a nonlinear version of the Stronger Central Sets Theorem. Nonlinear versions of linear statements in combinatorics are usually much harder to obtain. Before we proceed, let us define a shorthand notation to denote polynomials.

For any $m_1, m_2, m_3 \in \mathbb{N}$, and $A \subseteq \mathbb{R}^{m_1}$, $B \subseteq \mathbb{R}^{m_2}$, and $C \subseteq \mathbb{R}^{m_3}$, let $\mathbb{P}_A(B, C)$ be the set of all polynomials from B to C with zero constant term and coefficients are in

[2] In this paper we assume the reader to know the basics of the algebra of $\beta\mathbb{N}$.

[3] Anyhow, most of our proofs could easily be generalized to the case of countable commutative semigroups.

A. Whenever $A = \mathbb{Z}$, we will simply write $\mathbb{P}(B, C)$ instead of $\mathbb{P}_{\mathbb{Z}}(B, C)$. We will use the notation $\mathbb{P}$ to denote $\mathbb{P}_{\mathbb{N}\cup\{0\}}(\mathbb{N}, \mathbb{N})$.

A nonlinear version of van der Waerden's theorem, known as the Polynomial van der Waerden's Theorem[4] was established by Bergelson and Liebman in [BL96], using the methods of topological dynamics and PET induction.

Polynomial van der Waerden Theorem. Let $r \in \mathbb{N}$, and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ be a r-coloring of $\mathbb{N}$. Then for any $F \in \mathcal{P}_f(\mathbb{P})$, there exist $a, d \in \mathbb{N}$ and $1 \leqslant j \leqslant r$ such that $\{a + p(d) : p \in F\} \subset C_j$.

The polynomial van der Waerden's theorem and the central sets theorem have a joint extension, proven by Hindman and McCutcheon in [HM99].

Polynomial Central Sets Theorem. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$, let $T \in \mathcal{P}_f(\mathbb{P})$ and let A be a central subset of $\mathbb{N}$. Then there exist sequences $\langle b_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that

1. for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and
2. for each $f \in F$, each $P \in T$ and each $K \in \mathcal{P}_f(\mathbb{N})$

$$\sum_{n \in K} b_n + P\left(\sum_{n \in K} \sum_{t \in H_n} f(t)\right) \in A.$$

Inspired by the above result, our goal in this paper is to provide the polynomial extension, in the flavor of the Polynomial Central Sets Theorem, of the Stronger Central Sets Theorem. This will be done in Section 2, where new special classes of sets related to our result, called J_p- and C_p-sets, will be introduced. In Section 3, we will discuss many open problems that arise as consequences of our main result.

2 Polynomial extension of the Stronger Central Sets Theorem

As we will need this definition specialized to some different settings, we start this section by recalling that, given an arbitrary $(S, +)$, an IP-set in S is a subset of S of the form

$$FS(\langle x_n \rangle_{n \in \mathbb{N}}) = \left\{\sum_{n \in H} x_n : H \in \mathcal{P}_f(\mathbb{N})\right\}$$

for some injective sequence $\langle x_n \rangle_{n \in \mathbb{N}}$.

Moreover, for any $j \in \mathbb{N}$, $\mathbb{N}^j$ is piecewise syndetic in $\mathbb{Z}^j$, so it follows from [GJ21, Corollary 2.3] that any set $A \subseteq \mathbb{N}$ which is piecewise syndetic in $\mathbb{N}$, is also piecewise syndetic in $\mathbb{Z}$. Hence, the following version of the IP-Polynomial van der Waerden's theorem (that we specialize here to $\mathbb{N}$) is a special case of [BJM17, Corollary 2.12].

[4]Actually, the authors proved a generalized version of the much stronger Szémeredi's Theorem, but we are not going to discuss it in this paper.

Abstract IP-Polynomial van der Waerden theorem. Let $j \in \mathbb{N}$ and $A \subseteq \mathbb{N}$ be a piecewise syndetic set. Let $(x_\alpha)_{\alpha\in\mathcal{P}_f(\mathbb{N})}$ be an IP-set in $\mathbb{N}^j$. Then for any $F \in \mathbb{P}(\mathbb{N}^j, \mathbb{N})$, there exists $a \in \mathbb{N}$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ such that for all $P \in F$ $a + P(x_\beta) \in A$.

The following simple consequence of the above theorem will motivate us to introduce the notion of J_p-set.

Theorem 2. *Let $l, m \in \mathbb{N}$ and $A \subset \mathbb{N}$ be a piecewise syndetic set. For each $i = 1, 2, \ldots, l$, let $(x^i_\alpha)_{\alpha\in\mathcal{P}_f(\mathbb{N})}$ be an IP-set in $\mathbb{N}$. Then for every $F \in \mathcal{P}_f(\mathbb{P})$, there exist $a \in \mathbb{N}$, $\beta \in \mathcal{P}_f(\mathbb{N})$ such that* $\min\beta > m$ *and for all $i \in \{1, 2, \ldots, l\}$ and for all $P \in F$ $a + P(x^i_\beta) \in A$.*

Proof. Consider the IP set $\left(x^1_\alpha, x^2_\alpha, \ldots, x^l_\alpha\right)_{\alpha\in\mathcal{P}_f(\mathbb{N}\setminus\{1,2,\ldots,m\})}$ in $\mathbb{N}^l$. For $i \in \{1, 2, \ldots, l\}$ and $P \in F$, let $f^i_P : \mathbb{N}^l \to \mathbb{N}$ be defined by $f^i_P(x_1, \ldots, x_l) := P(x_i)$. Finally, let $G = \{f^i_P : P \in F \text{ and } i \in \{1, 2, \ldots, l\}\}$. Then G is a finite set of polynomials from $\mathbb{N}^l$ to $\mathbb{N}$ without constant term. Applying the abstract IP-polynomial van der Waerden theorem we find $a \in \mathbb{N}$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ such that $\min\beta > m$ and, for all $i \in \{1, 2, \ldots, l\}$ and $P \in F$,

$$a + f^i_P\left(x^1_\beta, x^2_\beta, \ldots, x^l_\beta\right) = a + P\left(x^i_\beta\right) \in A,$$

as desired. $\square$

Theorem 2 leads to strengthen polynomially the notion of J-set (that we recall), introducing that of J_p-set.

Definition 3. Let $A \subseteq \mathbb{N}$. Then:

1. A is called a J-set if and only if for every $H \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$, there exists $a \in \mathbb{N}$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ such that for all $f \in H$, $a + \sum_{t\in\beta} f(t) \in A$.
2. A is called a J_p-set if and only if for every $F \in \mathcal{P}_f(\mathbb{P})$, and every $H \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$, there exists $a \in \mathbb{N}$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ such that for all $P \in F$ and all $f \in H$, $a + P\left(\sum_{t\in\beta} f(t)\right) \in A$.

By the definition, it trivially holds that a J_p-set is a J-set. We discuss the converse in Section 4.

The family of J_p set is actually quite rich. To prove this, let us recall the notion of (upper) Banach density:

Definition 4. For any $n \in \mathbb{N}$, the upper Banach density of a set $A \subseteq \mathbb{N}^n$ is defined as

$$d^*(A) = \sup_{[a_1,b_1]\times\cdots\times[a_n,b_n]} \limsup_{n\to+\infty} \frac{|A \cap [a_1, b_1] \times \cdots \times [a_n, b_n]|}{|(b_1 - a_1) \cdots (b_n - a_n)|}.$$

Now we will prove that any subset of $\mathbb{N}$ that has a positive upper Banach density is a J_p-set. In our proof we will use the following restricted version of the multidimensional IP polynomial Szemerédi theorem. To derive the following theorem one has to consider the set of polynomials $F \cup \{\mathbf{0}\}$ instead of F, where $\mathbf{0}$ is a polynomial map from $\mathbb{Z}^l$ to $\mathbb{Z}$ defined as $\mathbf{0}(x) = 0$ for all $x \in \mathbb{Z}^l$.

Theorem 5. ([BJM17, Theorem 2.9]) *Let $l \in \mathbb{N}$, and $B \subseteq \mathbb{N}$ has positive upper Banach density. Let $\langle y_\alpha \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})}$ be an IP-set in $\mathbb{N}^l$. Then for any $F \in \mathcal{P}_f\left(\mathbb{P}(\mathbb{Z}^l, \mathbb{Z})\right)$, there exists $x \in \mathbb{Z}$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ such that $x + P(y_\beta) \in B$ for all $P \in F$.*

The proof of the following corollary is similar to the proof of Theorem 2.

Corollary 6. *Let $l, m \in \mathbb{N}$ and $A \subset \mathbb{N}$ have positive upper Banach density. For each $i = 1, 2, \ldots, l$, let $(x^i_\alpha)_{\alpha \in \mathcal{P}_f(\mathbb{N})}$ be an IP-set. Then for all finite $F \subset \mathbb{P}(\mathbb{Z}, \mathbb{Z})$, there exist $a \in \mathbb{N}$, $\beta \in \mathcal{P}_f(\mathbb{N})$ such that $\min \beta > m$, and for all $i \in \{1, 2, \ldots, l\}$ and $P \in F$, $a + P(x^i_\beta) \in A$. In particular, any set with positive upper Banach density is a J_p-set.*

Proof. Let G be a finite set of polynomials from $\mathbb{N}^l$ to $\mathbb{N}$ constructed as in the proof of Theorem 2. Let $H = G \cup \mathbf{0}$. Then from Theorem 5, there exist $a \in \mathbb{N}$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ with $\min \beta > m$ such that for all $i \in \{1, 2, \ldots, l\}$ and $P \in F$, $a + P(x^i_\beta) \in A$. $\square$

In [H09, Theorem 2.1], Hindman produced an example to show that not all J sets have positive upper Banach density. We show here that the same example shows that not all J_p sets have positive upper Banach density. First, let us recall the construction of the set.

For $n \in \mathbb{N}$, let

$$a_n = \min\left\{t \in \mathbb{N} \mid \left(\frac{2^n - 1}{2^n}\right)^t \leqslant \frac{1}{2}\right\},$$

and let $S_n = \sum_{i=1}^n a_i$. Let $b_0 = 0$, let $b_1 = 1$, and for $n \in \mathbb{N}$, $t \in \{S_n, S_{n+1}, \ldots, S_{n+1} - 1\}$, let $b_{t+1} = b_t + n + 1$.

For $k \in \mathbb{N}$, let $B_k = \{b_k, b_k + 1, b_k + 2, \ldots, b_{k+1} - 1\}$. Finally, let

$$A = \{x \in \mathbb{N} \mid (\forall k \in \mathbb{N})\,(B_k \setminus Supp(x)) \neq \emptyset)\}.$$

It follows from [H09, Theorem 2.1] that the upper Banach density of A is 0. Let $A' = A \cup \{0\}$ and let $S \in \mathcal{P}_f(\mathbb{P})$, let F be a finite collection of IP-sets, and $r = |F|$. Pick $k \in \omega$, and $H \in \mathcal{P}_f(\mathbb{N})$ such that

- $b_{k+1} - b_k > r$,
- $b_k > n$,
- $\min\{x : x \in H\} > m$, and
- for all $f \in F$ and $P \in S$, $P(\sum_{t \in H} f(t)) \in 2^{b_k}\mathbb{Z}$.

Pick $c \in 2^{b_k}\mathbb{N}$ such that $\forall f \in F$ and $\forall P \in S$ $c + P(\sum_{t \in H} f(t)) > 0$.

Let

$$l = \max(\bigcup\{supp(c + P(\sum_{t \in H} f(t)) : f \in F, P \in S\})$$

and pick j such that $l < b_j$. Pick $r_0 \in B_k$ such that for all $P \in S, f \in F$ $B_k \setminus supp\left(2^{r_0} + c + P(\sum_{t\in H} f(t))\right) \neq \emptyset$. Inductively, for $i \in \{1, 2, \dots j-k\}$, pick $r_i \in B_{k+i}$ such that for all $f \in F, P \in S$

$$B_{k+i} \setminus supp\left(2^{r_i} + \sum_{t=0}^{i-1} 2^{r_t} + c + P(\sum_{t\in H} f(t))\right) \neq \emptyset.$$

Let $d = c + \sum_{i=0}^{j-k} 2^{r_i}$. Then we have that for all $\forall f \in F$, and $\forall P \in S$, $d + P(\sum_{t\in H} f(t)) \in A \cap 2^n\mathbb{N}$. This shows A is a J_p-set, as desired.

As every set with positive upper Banach density is a J_p-set, and the family of sets with positive upper Banach density is partition regular, by [HS12, Theorem 5.7] there exist ultrafilters each of whose members are J_p sets.

Definition 7. We set $\mathcal{J}_p = \{p \in \beta\mathbb{N} \mid \forall A \in p\ A \in J_p\}$.

Piecewise syndetic sets have a positive Banach density, hence they are J_p-sets; it immediately follows that $\overline{K(\beta\mathbb{N}, +)} \subseteq \mathcal{J}_p$. More in general, the following result, whose trivial routine proof we omit, holds.

Theorem 8. *$\mathcal{J}_p$ is a two sided ideal of $(\beta\mathbb{N}, +)$.*

Less routine are some multiplicative properties of $\mathcal{J}_p$ that we will discuss at the end of this Section.

By Ellis' theorem [HS12, Corollary 2.39], a straightforward consequence of Theorem 8 is that there are idempotent ultrafilters, and even minimal idempotent ultrafilters, in $\mathcal{J}_p$. We denote the set of all idempotents in $\mathcal{J}_p$ by $E(\mathcal{J}_p)$.

A long studied family of sets in the literature are C-sets, namely J-sets that belong to some idempotent made of J-sets. In complete analogy, in our setting it makes sense to introduce the following polynomial version of C-sets.

Definition 9. $A \subseteq \mathbb{N}$ is a C_p-set if $A \in p$ for some idempotent ultrafilter $p \in E(\mathcal{J}_p)$.

By the definition, we see immediately that all central and all C-sets are C_p-sets. C_p-sets will play a fundamental role in our Theorem 11, that establishes the polynomial extension of the stronger Central Sets Theorem of De, Hindman and Strauss. To prove it, we will need the following lemma, which states that the conclusion of the Theorem 2 holds also for J_p-sets.

Lemma 10. *Let $l, m \in \mathbb{N}$ and $A \subset \mathbb{N}$ be a J_p-set. For each $i = 1, 2, \dots, l$, let $(x^i_\alpha)_{\alpha\in\mathcal{P}_f(\mathbb{N})}$ be an IP-set in $\mathbb{N}$. Then for all finite $F \in \mathcal{P}_f(\mathbb{P})$, there exist $a \in \mathbb{N}$, $\beta \in \mathcal{P}_f(\mathbb{N})$ such that $\min\beta > m$ and*

$$a + P(x^i_\beta) \in A$$

for all $i \in \{1, 2, \dots, l\}$ and $P \in F$.

Proof. For each $\alpha \in \mathcal{P}_f(\mathbb{N})$, let $\alpha + m = \{i + m : i \in \alpha\}$. Now for each $i = 1, 2, \ldots, l$, and each $(x^i_\alpha)_{\alpha \in \mathcal{P}_f(\mathbb{N})}$, let $\left((x')^i_\alpha\right)_{\alpha \in \mathcal{P}_f(\mathbb{N})}$ be a new IP-set defined by $\sum_{i\in\alpha} x'_i = \sum_{i\in\alpha+m} x_i$. By applying Definition 3 we conclude. $\square$

We are thankful to the referee for his/her comments that helped us to simplify the proof of the following theorem.

Theorem 11. *Let A be a C_p-set and let $T \in \mathcal{P}_f(\mathbb{P})$. There exist functions $\alpha : \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N}) \to \mathbb{N}$ and $H : \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N}) \to \mathcal{P}_f(\mathbb{N})$ such that*

1. *if $G, K \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ and $G \subsetneqq K$ then $\max H(G) < \min H(K)$ and*
2. *if $n \in \mathbb{N}$, $G_1, G_2, \cdots, G_n \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$, $G_1 \subsetneqq G_2 \subsetneqq \cdots \subsetneqq G_n$ and for all $i \in \{1, 2, \ldots, n\}$, $f_i \in G_i$, then for all $P \in T$,*

$$\sum_{i=1}^{n} \alpha(G_i) + P\left(\sum_{i=1}^{n} \sum_{t\in H(G_i)} f_i(t)\right) \in A.$$

Proof. Choose an idempotent $p \in \mathcal{J}_p$ with $A \in p$. Given any $B \in p$, let $B^* = \{x \in B : -x + B \in p\}$. Then by [HS12, Lemma 4.14] for all $x \in B^*$, $-x + B^* \in p$.

For $K \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ we define $\alpha(K) \in \mathbb{N}$ and $H(K) \in \mathcal{P}_f(\mathbb{N})$ by induction on $|K|$.

For the base case $|K| = 1$ let $K = \{f\}$. Since A is a J_p-set, pick $a \in \mathbb{N}$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ such that for all $P \in T$, $a + P(\sum_{t\in\beta} f(t)) \in A^*$. In this case, we let $\alpha(K) = a$ and $H(K) = \beta$.

Now assume that $|K| > 1$ and that $\alpha(G)$ and $H(G)$ have been defined for all nonempty proper subsets G of K satisfying (1) and (2) with A^* replacing A. Let $F = \bigcup\{H(G) : \emptyset \neq G \subsetneqq K\}$ and let $m = \max F$. Let

$$R = \left\{\sum_{i=1}^{n} \sum_{t\in H(G_i)} f_i(t) \mid n \in \mathbb{N}, \emptyset \neq G_1 \subsetneqq \cdots \subsetneqq G_n \subsetneqq K,\right.$$
$$\left.\text{and for } i \in \{1, 2, \ldots, n\}, f_i \in G_i\right\}$$

and let

$$M = \left\{\sum_{i=1}^{n} \alpha(G_i) + P\left(\sum_{i=1}^{n} \sum_{t\in H(G_i)} f_i(t)\right) \mid n \in \mathbb{N}, \emptyset \neq G_1 \subsetneqq \cdots \subsetneqq G_n \subsetneqq K,\right.$$
$$\left.P \in T, \text{ and for } i \in \{1, 2, \ldots, n\}, f_i \in G_i\right\}.$$

Then $R \subseteq \mathbb{N}$ and, by assumption, $M \subseteq A^*$. Let $B = A^* \cap \bigcap_{x\in M}(-x + A^*)$. As this is an intersection of sets in p, we have that $B \in p$.

For $P \in T$ and $d \in R$, define the polynomial $Q_{P,d}$ by

$$Q_{P,d}(y) = P(y + d) - P(d).$$

Since the coefficients of P come from ω, $Q_{P,d} \in \mathbb{P}$.

Let $S = T \cup \{Q_{P,d} : P \in T \text{ and } d \in R\}$. By Theorem 10, pick $a \in \mathbb{N}$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ such that $\min \beta > m$ and for all $Q \in S$ and all $f \in K$, $a + Q(\sum_{t\in\beta} f(t)) \in B$. Let $\alpha(K) = a$ and $H(K) = \beta$. We are left to verify conditions (1) and (2).

Since $\min \beta > m$, (1) is satisfied.

To verify (2), let $n \in \mathbb{N}$, let $G_1, G_2, \cdots, G_n \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ with $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n = K$, for each $i \in \{1, 2, \ldots, n\}$, let $f_i \in G_i$, and let $P \in T$.

If $n = 1$, then $\alpha(G_n) + P(\sum_{t\in H(G_n)} f_n(t)) = a + P(\sum_{t\in\beta} f_n(t)) \in B \subseteq A^*$.

If $n > 1$, then

$$\begin{gathered}
\textstyle\sum_{i=1}^{n} \alpha(G_i) + P\left(\sum_{i=1}^{n} \sum_{t\in H(G_i)} f_i(t)\right) \\
= a + \textstyle\sum_{i=1}^{n-1} \alpha(G_i) + P\left(\sum_{i=1}^{n-1} \sum_{t\in H(G_i)} f_i(t)\right) + \\
P\left(\textstyle\sum_{t\in\beta} f_n(t) + \sum_{i=1}^{n-1} \sum_{t\in H(G_i)} f_i(t)\right) - P\left(\sum_{i=1}^{n-1} \sum_{t\in H(G_i)} f_i(t)\right) \\
= a + x + Q_{P,d}(\textstyle\sum_{t\in\beta} f_n(t)),
\end{gathered}$$

where $x = \sum_{i=1}^{n-1} \alpha(G_i) + P\left(\sum_{i=1}^{n-1} \sum_{t\in H(G_i)} f_i(t)\right)$, $d = \sum_{i=1}^{n-1} \sum_{t\in H(G_i)} f_i(t)$.

Since $P \in T$ and $d \in R$, $Q_{P,d} \in S$ so, since $x \in M$, we have that $a + Q_{P,d}(\sum_{t\in\beta} f_n(t)) \in B \subseteq -x + A^*$.

Therefore $\sum_{i=1}^{n} \alpha(G_i) + P\left(\sum_{i=1}^{n} \sum_{t\in H(G_i)} f_i(t)\right) \in A^*$ as required. $\square$

Notice that now the Stronger Central Sets Theorem can be seen as a particular case of the above theorem obtained by taking $P(x) = x$.

By observing that any $\beta \in \mathcal{P}_f(\mathbb{N})$ is a subset of $\{1, \ldots, n\}$ for some large enough $n \in \mathbb{N}$, we deduce the following seemingly stronger version of Theorem 11.

Corollary 12. *Let A be a C_p-set and $S \in \mathcal{P}_f(\mathbb{P})$. Then there exist $\alpha : \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N}) \to \mathbb{N}, H : \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N}) \to \mathcal{P}_f(\mathbb{N})$ such that*

1. *if $F, G \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N}), F \subset G$, then $\max H(F) < \min H(G)$;*
2. *if $\langle G_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ such that $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n \subsetneq \cdots$ and $f_i \in G_i, i \in \mathbb{N}$, then*

$$\sum_{i\in\beta} \alpha(G_i) + P\left(\sum_{i\in\beta} \sum_{t\in H(G_i)} f_i(t)\right) \in A.$$

for all $\beta \in \mathcal{P}_f(\mathbb{N})$.

Having settled the desired polynomial version of the Stronger Central Sets Theorem, we now turn back to J_P-sets to show that, in their definition, we could be a little more general and enlarge $\mathbb{P}$ to $\mathbb{P}_{\mathbb{Q}}$, where $\mathbb{P}_{\mathbb{Q}} = \mathbb{P}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q})$.

Theorem 13. *Let $l \in \mathbb{N}$ and $A \subset \mathbb{N}$ be a J_P set. For each $i = 1, 2, \ldots, l$, let $(x^i_\alpha)_{\alpha \in \mathcal{P}_f(\mathbb{N})}$ be an IP-set. Then for any $F \in \mathcal{P}_f(\mathbb{P}_\mathbb{Q})$ there exist $a \in \mathbb{N}$, $\beta \in \mathcal{P}_f(\mathbb{N})$ such that for all $i \in \{1, 2, \ldots, l\}$ and $P \in F$, $a + P(x^i_\beta) \in A$.*

Proof. Let $M \in \mathbb{N}$ be the smallest common multiple of all denominators that appear among the coefficients of the polynomials in F. For each $P \in F$ and $n \in \mathbb{N}$, let b^P_n be the coefficient of x^n in P.

We let P' be the polynomial obtained from P by multiplying each b^P_n by M^n. With this construction, P' is a polynomial with integer coefficients. We let

$$F' = \{P' \mid P \in F\}.$$

Given an IP-set $(x^1_\alpha)_{\alpha \in \mathcal{P}_f(\mathbb{N})}$, one can use the pigeonhole principle to choose a sequence $\langle H^1_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{P}_f(\mathbb{N})$ so that x^1_α is a multiple of M for each $\alpha \in \langle H^1_n \rangle_{n \in \mathbb{N}}$ and $n \in \mathbb{N}$. Now again applying the pigeonhole principle over the IP-set $FU(\langle H^1_n \rangle_{n \in \mathbb{N}})$ in the semigroup $(\mathcal{P}_f(\mathbb{N}), \cup)$, we obtain another sequence $\langle H^2_n \rangle_{n \in \mathbb{N}}$ in $FU(\langle H^1_n \rangle_{n \in \mathbb{N}})$ such that $M | x^2_\alpha$ for each $\alpha \in \langle H^2_n \rangle_{n \in \mathbb{N}}$ and $n \in \mathbb{N}$. As there are l sequences, this process will terminate after l steps. Then we end up with an IP set $I = FU(\langle K_n \rangle_{n \in \mathbb{N}})$ in $\mathcal{P}_f(\mathbb{N})$ such that $M | x^i_\beta$ for all $\beta \in I$ and $i \in \{1, 2, \ldots, l\}$.

Now fix this IP set I and for each $(x^i_\alpha)_{\alpha \in \mathcal{P}_f(\mathbb{N})}$, consider the subsystems $\left(y^i_\beta\right)_{\beta \in \mathcal{P}_f(\mathbb{N})}$, where $y^i_n = \sum_{t \in K_n} x^i_t$. Hence $M | y^i_\beta$ for all $i = 1, 2, \ldots, l$ and $\beta \in \mathcal{P}_f(\mathbb{N})$. Now for each $i \in \{1, 2, \ldots, l\}$, let $\widetilde{x}^i_\alpha = \frac{y^i_\alpha}{M}$. Take the finite set of IP sets $(\widetilde{x}^i_\alpha)_{\alpha \in \mathcal{P}_f(\mathbb{N})}$ for all $i \in \{1, 2, \ldots, l\}$.

For this new finite set of IP sets $(\widetilde{x}^i_\alpha)_{\alpha \in \mathcal{P}_f(\mathbb{N})}$, and finite set of polynomials F', there exists $a \in \mathbb{N}$ and $\beta \in \mathcal{P}_f(\mathbb{N})$ such that $a + P'(\widetilde{x}^i_\beta) \in A$ for all $i \in \{1, 2, \ldots, l\}$ and $P' \in F'$. As for each $n \in \mathbb{N}$, $i \in \{1, 2, \ldots, l\}$, the n^{th} monomial is of the form

$$a_n M^n \left(\widetilde{x}^i_\beta\right)^n = a_n M^n \left(\frac{y^i_\alpha}{M}\right)^n = a_n \left(y^i_\alpha\right)^n = a_n \left(\sum_{t \in \bigcup_{n \in \alpha} K_n} x^i_t\right)^n,$$

so it is the n-th monomial of P. Hence for $\beta = \bigcup_{n \in \alpha} K_n$ and $a \in \mathbb{N}$ we have that

$$a + P\left(x^i_\beta\right) \in A$$

for all $i \in \{1, 2, \ldots, l\}$ and $P \in F$, as desired. □

The aforementioned theorem gives us the following multiplicative property of J_p-sets.

Corollary 14. *If $A \subset \mathbb{N}$ is a J_P-set and $n \in \mathbb{N}$, then $n \cdot A$ is a J_P-set.*

Proof. Let $F \in \mathcal{P}_f(\mathbb{P})$ and $(x^i_\alpha)_{\alpha \in \mathcal{P}_f(\mathbb{N})}$ be IP-sets for all $i = 1, 2, \ldots, l$. Let

$$F' = \left\{\frac{1}{n} P \mid P \in F\right\}.$$

As A is a J_P set, by Theorem 13 we find $a \in \mathbb{N}$, $\beta \in \mathcal{P}_f(\mathbb{N})$ such that $a + \frac{1}{n} P(x^i_\beta) \in A$ for all $i \in \{1, 2, \ldots, l\}$ and $P \in F$, which implies that $na + P\left(x^i_\beta\right) \in n \cdot A$.

$\square$

Corollary 15. *If $A \subset \mathbb{N}$ is a C_p-set, then for any $n \in \mathbb{N}$, $n \cdot A$, and $n^{-1}A$ are C_p-sets.*

Proof. Let $A \subset \mathbb{Z}$ be a C_p-set and let $A \in q$ for some $q \in E(\mathcal{J}_p)$. First of all, observe that $nq, n^{-1}q$ are idempotent.

nA is a C_p-set as it belongs to nq, which is an idempotent made of J_p-sets. In fact, by definition $B \in nq$ if and only if $B \supseteq nB'$ for some $B' \in q$, and since B' is a J_p-set also B is by Corollary 14.

As $n^{-1}A \in n^{-1}q$ and $n^{-1}q$ is an idempotent, if we can show that each element of $n^{-1}q$ is a J_p-set, we will have $n^{-1}A$ is a C_p-set.

Suppose $B \in n^{-1}q$, then we have $n \cdot B \in q$. So, for any finite $F \in \mathcal{P}_f(\mathbb{P})$ and for any l ($l \geqslant 1$) IP-sets, $(x^i_\alpha)_{\alpha \in \mathcal{F}}$, we have from Theorem 13, $a + n \cdot P(x^i_\beta) \in n \cdot B$ for some $a \in \mathbb{N}$, $\beta \in \mathcal{P}_f(\mathbb{N})$ for all $P \in F$. Hence $a \in n \cdot B$. This implies $\frac{a}{n} + P(x^i_\beta) \in B$ and so B is a J_p-set.

This completes the proof. $\square$

As a trivial consequence, we obtain the following multiplicative property of $\mathcal{J}_p$.

Corollary 16. *$\mathcal{J}_p$ is a left ideal of $(\beta\mathbb{N}, \cdot)$.*

3 Discussions and further possibilities

The introduction of J_p- and C_p-sets rises many questions. We want to list some of them here, providing some comments on why we believe these are relevant.

Question 17. Is the family of J_p-sets strongly partition regular?

We have not been able to prove this fact; simple modifications of the proof of the same result for J-sets seems not to work. Anyhow, we do believe that the answer to the above question is positive, a reason being that it is possible to prove that the related family of PP-rich sets is strongly partition regular. Let us first recall its definition.

Definition 18. Let $F \in \mathcal{P}_f(\mathbb{P})$. A polynomial progression of F is a pattern of the form $\{a + p(x) \mid P \in F\}$ for some $a, x \in \mathbb{N}$.

A set $A \subseteq \mathbb{N}$ is called PP-rich if for all $F \in \mathcal{P}_f(\mathbb{P})$ A contains a polynomial progression of F.

For example, if $F = \{x, 2x, 3x, \ldots, kx\}$, a polynomial progression of F is just an arithmetic progression of length k.

Theorem 19. *The family of PP-rich sets is strongly partition regular.*

Proof. Let A be a PP-rich set, and let $A = A_1 \cup A_2$. By contrast, let us assume that A_1 and A_2 are not PP-rich sets. Let A_1 does not contain the polynomial progression of the finite set of polynomials $F_1 \in \mathcal{P}_f(\mathbb{P})$, and A_2 does not contain polynomial progression of the finite set of polynomials $F_2 \in \mathcal{P}_f(\mathbb{P})$. Let $F = F_1 \cup F_2$, and

$$n = \max\{\deg P : P \in F\}, l = \max\{Coef(P) : P \in F\},$$

where $Coef(P)$ is the maximum coefficient of polynomial P.

By [BL96, Theorem B_0] and a compactness argument, there exists a sufficiently large $N \in \mathbb{N}$ such that if $[1, N]^n$ is 2-colored, then one of the color classes contain a monochromatic structure of the form

$$\left\{(z_1 + j_1 w, z_2 + j_2 w^2, \ldots, z_n + j_n w^n) : 0 \leqslant j_k \leqslant l \text{ for } 1 \leqslant k \leqslant n\right\}.$$

For $\vec{a} = (a_1, a_2, \ldots, a_n) \in \omega^n$, let us define the polynomoial $P_{\vec{a}}(y) = a_1 y + a_2 y^2 + \ldots + a_n y^n$, where $\omega = \mathbb{N} \cup \{0\}$. Let us define

$$G = \{P_{\vec{a}} : \vec{a} = (a_1, a_2, \ldots, a_n) \in \omega^n \text{ with } 0 \leqslant a_i \leqslant N \, and \, 1 \leqslant i \leqslant n\}.$$

Now choose $x, y \in \mathbb{N}$ such that $\{x + P(y) : P \in G\} \subseteq A$. Color the set $[1, N]^n = C_1 \cup C_2$ as $\vec{a} \in C_i$ if and only if $x + P_{\vec{a}}(y) \in A_i$. So there exist, $i \in \{1, 2\}$ and $z_1, z_2, \ldots, z_n \in \mathbb{N}$, $w \in \mathbb{N}$ such that

$$\left\{(z_1 + j_1 w, z_2 + j_2 w^2, \ldots, z_n + j_n w^n) : 0 \leqslant j_k \leqslant l\right\} \subset C_i.$$

Hence, $x + P_{\vec{a}_j}(y) \in A_i$, where $\vec{a}_j = (z_1 + j_1 w, z_2 + j_2 w^2, \ldots, z_n + j_n w^n)$, $0 \leqslant j_k \leqslant l$ for $1 \leqslant k \leqslant n$. Hence for every $0 \leqslant j_k \leqslant l, 1 \leqslant k \leqslant n$, we have

$$x + (z_1 + j_1 w)y + (z_2 + j_2 w^2)y^2 + \ldots + (z_n + j_n w^n)y^n \in A_i,$$

and thus

$$(x + z_1 y + z_2 y^2 + \ldots + z_n y^n) + j_1(wy) + j_2(wy)^2 + \ldots + j_n(wy)^n \in A_i,$$

for $0 \leqslant j_k \leqslant l$, $1 \leqslant k \leqslant n$.

In particular, $a + P(wy) \in A_i$, where $a = x + z_1 y + z_2 y^2 + \ldots + z_n y^n$. Therefore, PP-rich sets are strongly partition regular. □

A related important question is the following one:

Question 20. Does there exist a J-set that is not a J_p-set?

Again, we do not have an answer to the above question. The reason is that this question is much harder to answer than it seems. In fact, the following is a similar question that has now been open for some years:

Question 21. Are J-sets PP-rich?

Notice that J_p-sets are PP-rich, so if we would be able to prove that all J-sets are J_p-sets, the above question would be solved affirmatively. On the other end, the precise relationship between PP-rich sets and J_p-sets is still unknown, as the following question remains open.

Question 22. Are PP-rich sets also J_p-sets?

We believe that the answer to the previous question should be no. In fact, the similar linear question relating AP-rich sets and J-sets has been solved in [HJ12, Lemma 4.3], where the authors have demonstrated that there exists an AP-rich set that is not a J-set. However, it seems that the argument can not be easily lifted from the linear to the polynomial case.

Finally, maybe the most relevant open question that has to be mentioned is the following:

Question 23. Is our polynomial extension of the stronger central sets theorem actually more general than the original polynomial central sets theorem?

It has been shown in [DHS08, Theorem 4.4] that the stronger central sets theorem for arbitrary semigroups is indeed stronger than the original central sets Theorem for semigroups by considering a special free semigroup. However, it is still an open question if this is true or not on $\mathbb{N}$, or on any countable Abelian group.

Acknowledgements

We are thankful to the referee for his/her detailed comments on our draft that helped us improve the quality of our paper.

L. Luperi Baglini was supported by the project PRIN 2022 "Logical methods in combinatorics", 2022BXH4R5, MIUR (Italian Ministry of University and Research).

References

[BH90] V. Bergelson and N. Hindman, Nonmetrizable topological dynamics and Ramsey theory, *Trans. Am. Math. Soc.*, 320: 293–320, 1990.

[BJM17] V. Bergelson, J. H. Johnson Jr. and J. Moreira, New polynomial and multidimensional extensions of classical partition results, *J. Comb. Theory Ser. A*, 147: 119–154, 2017.

[BL96] Bergelson and A. Leibman: Polynomial extensions of van der Waerden and Székeredi theorems, *J. Amer. Math. Soc.*, 9: 725–753, 1996.

[DHS08] D. De, N. Hindman and D. Strauss: A new and stronger Central Sets Theorem, *Fund. Math.*, 199: 155–175, 2008.

[F81] H. Furstenberg: Recurrence in Ergodic Theory and Combinatorial Number Theory, Princeton University Press, Princeton, N.J., 1981.

[GJ21] S. Goswami and S. Jana: A combinatorial viewpoint on preserving notion of largeness and an abstract Rado theorem, *Integers* 21: #A72, 2021.

[H74] N. Hindman: Finite sums from sequenceses within cells of partitions of $\mathbb{N}$, *J. Comb. Theory Ser. A*, 17: 1–11, 1974.

[H09] N. Hindman: Small sets satisfying the Central Sets Theorem, *Integers*, 9: #A5, 2007.

[H20] N. Hindman: A history of central sets, *Ergodic Theory Dynam. Systems*, 40: 1–33, 2020.

[HJ12] N. Hindman and J. H. Johnson Jr.: Images of C sets and related large sets under nonhomogeneous spectra, *Integers*, 12B: 1–25, 2012.

[HM99] N. Hindman and R. McCutcheon: Weak VIP systems in commutative semigroups, *Topology Proc.*, 24: 199–221, 1999.

[HS12] N. Hindman and D. Strauss: Algebra in the Stone-Čech Compactification: Theory and Applications, second edition, de Gruyter, Berlin, 2012.

[vdW27] B. van der Waerden: Beweis einer Baudetschen Vermutung, *Nieuw Arch. Wiskunde*, 19: 212–216, 1927.