On hierarchically closed fractional intersecting families

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Abstract

For a set L of positive proper fractions and a positive integer $r \geqslant 2$, a fractional r-closed L-intersecting family is a collection $\mathcal{F} \subset \mathcal{P}([n])$ with the property that for any $2 \leqslant t \leqslant r$ and $A_1, \ldots, A_t \in \mathcal{F}$ there exists $\theta \in L$ such that $|A_1 \cap \cdots \cap A_t| \in \{\theta|A_1|, \ldots, \theta|A_t|\}$. In this paper we show that for $r \geqslant 3$ and $L = \{\theta\}$ any fractional r-closed θ-intersecting family has size at most linear in n, and this is best possible up to a constant factor. We also show that in the case $\theta = 1/2$ we have a tight upper bound of $\lfloor \frac{3n}{2} \rfloor - 2$ and that a maximal r-closed $(1/2)$-intersecting family is determined uniquely up to isomorphism.

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1 Introduction

The theory of set systems with restricted intersection sizes is a classical and well-studied problem and the basic template of the problem is as follows. Given a set L of non-negative integers, determine the maximum size of a family $\mathcal{F} \subset \mathcal{P}([n])$ of subsets of $[n] := \{1, \ldots, n\}$ such that for distinct $A, B \in \mathcal{F}$ we have $|A \cap B| \in L$. This problem has its origins in the de Bruijn–Erdős theorem with further extensions including the Ray-Chaudhuri–Wilson inequality, the Frankl–Wilson inequality, and the Alon–Babai–Suzuki inequality among a host of other interesting results [5, 12, 7, 6, 1, 11, 13] and has spawned several variants,

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each with its own set of highlights and difficulties besides ushering in a wide range of combinatorial and algebraic tools that are now an integral component of combinatorial techniques for extremal problems.

A recent variant [4] of this problem, which is the principal focus of this paper, introduces the notion of *fractional intersecting families* which goes as follows. Suppose $L = \{\theta_1, \ldots, \theta_\ell\}$ is a set of proper positive fractions with $0 < \theta_i = \frac{a_i}{b_i} < 1$ and $\gcd(a_i, b_i) = 1$ for each i. We say that $\mathcal{F} \subset \mathcal{P}([n])$ is a *fractional L-intersecting family* (or that $\mathcal{F}$ is *fractionally L-intersecting*) if for any two distinct sets $A, B \in \mathcal{F}$ there exists $\theta \in L$ such that $|A \cap B| \in \{\theta|A|, \theta|B|\}$. The most natural question again is: how large can a fractional L-intersecting family be? This problem still remains unresolved; the best known bounds are a poly-logarithmic factor away from optimal bounds [4]. The notion of fractional intersecting families has produced other related variants, including the notion of fractional L-intersecting families of vector spaces [8] and fractional cross-intersecting families [9, 14]. Attempts to obtain a linear upper bound for $|L| = 1$ have led to conjectures on ranks of certain ensembles of matrices [2, 3], so the problem of fractional intersecting families has generated a considerable amount of interest.

In this paper we propose a more hierarchical extension of this notion of fractional intersecting families. But before we get to the notion in more precise terms, we return to the original problem concerning the size of fractional intersecting families for some motivation. For the rest of the paper, we shall always have $L = \{\theta\}$ where $\theta = \frac{a}{b}$ is a proper positive fraction with $\gcd(a, b) = 1$, and we shall also use the term "θ-intersecting" interchangeably with "L-intersecting".

One of the main results in [4] states that if $\mathcal{F}$ is a fractional L-intersecting family with $L = \{\frac{a}{b}\}$ then $|\mathcal{F}| \leqslant O_b(n \log n)$. On the lower bound side, there are constructions of fractional L-intersecting families of size $\Omega(n)$. For $\theta = \frac{1}{2}$, one can improve upon the constant a little more; there exist bisection closed families[1] of size $\lfloor \frac{3n}{2} \rfloor - 2$. What makes the problem of determining the size of maximal bisection closed families more interesting and intriguing is that there are non-isomorphic families of size $\lfloor \frac{3n}{2} \rfloor - 2$. The simplest example (and an instructive one at that) is the following.

Example 1. For the sake of simplicity, denote the set $\{x_1, \ldots, x_\ell\}$ by $x_1 \cdots x_\ell$. Then, the family

$$\mathcal{F} = \begin{cases} \{12, 13, \ldots, 1n, 1234, 1256, \ldots, 12(n-1)n\}, & n \equiv 0 \pmod 2; \\ \{12, 13, \ldots, 1n, 1234, 1256, \ldots, 12(n-2)(n-1)\}, & n \equiv 1 \pmod 2, \end{cases}$$

is not only bisection closed, but also *hierarchically bisection closed* in the following sense: for any sets $A_1, \ldots, A_r \in \mathcal{F}$ we also have $|A_1 \cap \cdots \cap A_r| \in \{\frac{1}{2}|A_1|, \ldots, \frac{1}{2}|A_r|\}$. The easiest way to see this is to note that for this family, the subfamilies of sizes 2 and 4 are sunflowers, and also that any collection of subsets in $\mathcal{F}$ have non-empty intersection.

The other known bisection closed families of size $\lfloor \frac{3n}{2} \rfloor - 2$ arise from a construction using Hadamard matrices, and do not satisfy this stronger property.

[1] When $\theta = 1/2$ a fractional L-intersecting family is also called a bisection closed family in [4].

Example 2. Let H be an $m \times m$ Hadamard matrix, i.e. a matrix whose entries lie in $\{\pm 1\}$, and with all the rows being mutually orthogonal. Assume that H is normalized so that the first row is the all-ones vector. Let J denote the $m \times m$ all-ones matrix. Consider the matrix

$$\begin{bmatrix} H & H \\ H & -H \\ H & -J \end{bmatrix},$$

and delete the first and $(2m+1)$th rows. Viewing the remaining rows as the ± 1 incidence vectors of subsets of $[2m]$, one can verify that this defines a family $\mathcal{F} \subset \mathcal{P}([n])$ that is 2-bisection closed, where $n = 2m$. Since there are $3m-2$ sets in $\mathcal{F}$, we have $|\mathcal{F}| = \frac{3n}{2} - 2$.

One of the principal reasons why a linear bound, let alone a tight bound, for the size of a bisection closed family is elusive is this diffusive nature of the known families of maximal size. But since this last example seems to be structurally different from the others, it raises the following more natural question: how large could a hierarchically bisection closed family be?

In order to make this precise, we make a formal definition.

Definition 3. Let $r \geqslant 2$ and $L = \{\theta_1, \ldots, \theta_m\}$ be a set of fractions in $(0,1)$. So, $\theta_i = a_i/b_i$ for some positive integers a_i, b_i such that $\gcd(a_i, b_i) = 1$, for each $1 \leqslant i \leqslant m$. A family $\mathcal{F}$ of subsets of $[n]$ is called *hierarchically r-closed L-intersecting* (or simply *r-closed L-intersecting*) if, for each $2 \leqslant t \leqslant r$ and any t distinct sets $A_1, \ldots, A_t$ in $\mathcal{F}$ we have $|\bigcap_{i=1}^{t} A_i| \in \{\theta_j |A_i| : 1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant m\}$.

When $L = \{\theta\}$, an r-closed L-intersecting family is also called an *r-closed θ-intersecting* family. In particular, when $\theta = 1/2$, we call such a family *r-bisection closed.*

Note that if a θ-intersecting family is r-closed, then it is also s-closed for all $2 \leqslant s \leqslant r$, which explains why we refer to such a family as hierarchically closed.

The natural question that arises is the following. Suppose $r \geqslant 3$. If $\mathcal{F} \subset \mathcal{P}([n])$ is r-closed θ-intersecting, then determine the optimal upper bound for $|\mathcal{F}|$. Note that if $r = 2$, then we are back to the case of fractional L-intersecting families, so it behooves us to set $r \geqslant 3$ if we hope to see any different emergent phenomenon arising from the definition. And the main thesis of this paper is that setting $r \geqslant 3$ makes a big difference.

It is imperative to compare this notion with another generalization that appears in [10] which goes as follows. For an integer $r \geqslant 2$, and L as above, a family $\mathcal{F}$ is said to be *r-wise fractionally L-intersecting* if for any distinct $A_1, \ldots, A_r \in \mathcal{F}$ there exists $\theta \in L$ such that $|A_1 \cap \cdots \cap A_r| \in \{\theta|A_1|, \ldots, \theta|A_r|\}$. Again, the problem of determining the size of a maximum r-wise fractional L-intersecting family is optimally determined in [10] up to poly-logarithmic factors, and it appears that to get beyond the poly-logarithmic factor needs newer ideas (see [4] for more details on this). Our notion of r-closed θ-intersecting is somewhat related and yet vastly different as the main results of our paper will attest.

We are now in a position to state the main results of the paper.

Theorem 4. *Let $\mathcal{F}$ be an r-bisection closed family over $[n]$, with $r \geqslant 3$. Then,*

$$|\mathcal{F}| \leqslant \lfloor \tfrac{3n}{2} \rfloor - 2 \qquad (*)$$

for all $n \geqslant 2$. Moreover:

1. *(Tightness) For each $n \geqslant 2$, there exists an r-bisection closed family $\mathcal{F}_{\max}$ over $[n]$ which attains the bound in $(*)$.*

2. *(Uniqueness) For any family $\mathcal{F}$ over $[n]$ that attains the bound in $(*)$, there is a permutation σ of $[n]$ such that $\mathcal{F}_{\max} = \sigma(\mathcal{F}) \coloneqq \{\sigma(A) : A \in \mathcal{F}\}$, where $\sigma(E) \coloneqq \{\sigma(a) : a \in E\}$ for any set $E \in \mathcal{P}([n])$.*

3. *(Stability) There exists an absolute constant $C > 0$ such that the following holds. If $|\mathcal{F}| \geqslant (\frac{3}{2} - \epsilon)n$ for some $0 < \epsilon < 0.1$, then for some permutation σ of $[n]$,*

$$|\sigma(\mathcal{F}) \setminus \mathcal{F}_{\max}| < C\epsilon n.$$

When $\mathcal{F}$ is a general r-closed θ-intersecting family, where θ is not necessarily equal to $1/2$, we do not have a tight upper bound on $|\mathcal{F}|$. But, we are able to establish a linear upper bound on $|\mathcal{F}|$ even in this case.

Theorem 5. *Let $\mathcal{F}$ be an r-closed θ-intersecting family over $[n]$, with $r \geqslant 3$. Let $\theta = a/b \in (0, 1)$ with $\gcd(a, b) = 1$, $a, b > 0$.*

1. *If $a > 1$, then $|\mathcal{F}| \leqslant 2\left(\frac{\ln(b)-\ln(a)+1}{b-a}\right)(n - a) + 1$.*

2. *If $a = 1$, then we have two cases:*

   (a) if $b = 2$ and $\mathcal{F}$ contains a set of size 2, then $|\mathcal{F}| \leqslant (1 + \ln(2))(n - 1) + 1$;

   (b) otherwise, $|\mathcal{F}| \leqslant \left(\frac{2\ln(b)}{b-1}\right)(n - 1) + 1$.

The rest of the paper is organized as follows. We start with some preliminary results along with some terminology and develop some tools and lemmas in the next section. In Section 3, we prove Theorem 5, and then use this to prove Theorem 4. We finally conclude with some remarks and open questions in Section 4.

2 Preliminaries

In what follows, we always assume that $\mathcal{F}$ is an r-closed θ-intersecting family with $r \geqslant 3$. We denote by $\mathcal{F}(i)$ the collection of all i-element sets in $\mathcal{F}$, that is, $\mathcal{F}(i) \coloneqq \mathcal{F} \cap \binom{[n]}{i}$.

Our first observation is that the possible sizes that could appear in any intersection of t sets $(2 \leqslant t \leqslant r)$ in $\mathcal{F}$ is quite limited.

Proposition 6. *Let $2 \leqslant t \leqslant r$ and suppose $A_1, \ldots, A_t \in \mathcal{F}$ are distinct sets with $|A_1| \leqslant \cdots \leqslant |A_t|$. Then, $|A_1 \cap \cdots \cap A_t| \in \{\theta|A_1|, \theta|A_2|\}$.*

Proof. Since $2 \leqslant t \leqslant r$, we have $\theta|A_1| \leqslant |A_1 \cap \cdots \cap A_t| \leqslant |A_1 \cap A_2| \leqslant \theta|A_2|$, and so $|A_1 \cap \cdots \cap A_t| \in \{\theta|A_1|, \theta|A_2|\}$. □

Next, we show that one can often define a core of a set $A \in \mathcal{F}$ with certain nice properties.

Definition 7. For $A \in \mathcal{F}$, define the set $\mathsf{Tor}(A)$ of θ-intersectors of A by

$$\mathsf{Tor}(A) \coloneqq \{B \in \mathcal{F} : |B| \geqslant |A|,\ |A \cap B| = \theta|A|\}.$$

Note the condition $|B| \geqslant |A|$ in the definition of $\mathsf{Tor}(A)$.

Proposition 8. *If $\mathsf{Tor}(A) \neq \emptyset$, then $A \cap B = A \cap B'$ for all $B, B' \in \mathsf{Tor}(A)$.*

Proof. We have $\theta|A| \leqslant |A \cap B \cap B'| \leqslant |A \cap B| = |A \cap B'| = \theta|A|$. Hence, $A \cap B \cap B' = A \cap B$ and $A \cap B \cap B' = A \cap B'$. Thus, $A \cap B = A \cap B'$. $\square$

Definition 9. For $A \in \mathcal{F}$ such that $\mathsf{Tor}(A) \neq \emptyset$, define the *core* of A by

$$\mathsf{Cor}(A) \coloneqq A \cap B$$

for any $B \in \mathsf{Tor}(A)$.

Proposition 8 shows that Definition 9 is well-defined. For a set $A \in \mathcal{F}$, $\mathsf{Cor}(A)$ is not defined iff $\mathsf{Tor}(A) = \emptyset$. The next two results describe when this may happen.

Proposition 10. *Let $|\mathcal{F}(i)| \geqslant 2$. Then $\mathsf{Tor}(A) \neq \emptyset$ for all $A \in \mathcal{F}(i)$.*

Proof. If $A, B \in \mathcal{F}(i)$ are two distinct sets, then $|A \cap B| = \theta|A|$, so $B \in \mathsf{Tor}(A)$. Hence, $\mathsf{Tor}(A) \neq \emptyset$. $\square$

Corollary 11. *If $A \in \mathcal{F}(i)$ such that $\mathsf{Tor}(A) = \emptyset$, then $\mathcal{F}(i) = \{A\}$.*

In fact, Proposition 10 implies that the family $\mathcal{F}$ is a union of uniform sunflowers.

Definition 12. A family $\mathcal{F}$ of subsets of $[n]$ is called a *sunflower* if, for $C \coloneqq \bigcap_{A \in \mathcal{F}} A$, we have $A \cap B = C$ for all distinct $A, B \in \mathcal{F}$.

Lemma 13. *Every nonempty $\mathcal{F}(i)$ is a sunflower.*

Proof. If $|\mathcal{F}(i)| \leqslant 2$, then this is trivial. Let $|\mathcal{F}(i)| \geqslant 3$. To show that $|\mathcal{F}(i)|$ is a sunflower, it suffices to show that $\mathsf{Cor}(A) = \mathsf{Cor}(B)$ for any two sets $A, B \in \mathcal{F}(i)$. The proof of Proposition 10 shows that $A \in \mathsf{Tor}(B)$ and $B \in \mathsf{Tor}(A)$ for any two sets $A, B \in \mathcal{F}(i)$. Hence, $\mathsf{Cor}(A) = A \cap B = B \cap A = \mathsf{Cor}(B)$. $\square$

Remark 14.

1. Note that the set C in Definition 12 is usually called the core of the sunflower. In particular, if the sunflower is a singleton set $\{A\}$, then $C = A$.

   However, our definition of core is Definition 9. This matches with the above notion when $|\mathcal{F}(i)| \geqslant 2$. But, when $\mathcal{F}(i) = \{A\}$, $\mathsf{Cor}(A)$ is either undefined (if $\mathsf{Tor}(A) = \emptyset$), or a subset of A having cardinality θi (if $\mathsf{Tor}(A) \neq \emptyset$).

2. The property of being 3-closed is crucially used in the proof of Proposition 8. Thus, if $\mathcal{F}$ is not 3-closed, then Definition 9 cannot be made, and Lemma 13 need not hold. Indeed, Example 2 shows that there are 2-bisection closed families that do not satisfy this property.

We now establish some notations that we will use throughout the rest of this paper. Let

$$\begin{aligned} S &\coloneqq \{i \in [n] : \mathcal{F}(i) \neq \emptyset\}, & i_{\min} &\coloneqq \min(S), \\ S_{\text{nor}} &\coloneqq \{i \in S : \mathsf{Tor}(A) \neq \emptyset \text{ for all } A \in \mathcal{F}(i)\}, & i_{\max} &\coloneqq \max(S_{\text{nor}}), \\ S_{\text{exc}} &\coloneqq \{i \in S : \mathsf{Tor}(A) = \emptyset \text{ for some } A \in \mathcal{F}(i)\}. \end{aligned}$$

Note that $S = S_{\text{nor}} \sqcup S_{\text{exc}}$. We say that $\mathcal{F}(i)$ is a *normal* sunflower if $i \in S_{\text{nor}}$, and we say that it is an *exceptional* sunflower if $i \in S_{\text{exc}}$. Define $\mathcal{F}_{\text{nor}} \coloneqq \bigcup_{i \in S_{\text{nor}}} \mathcal{F}(i)$ and $\mathcal{F}_{\text{exc}} \coloneqq \bigcup_{i \in S_{\text{exc}}} \mathcal{F}(i)$. Then, $\mathcal{F} = \mathcal{F}_{\text{nor}} \sqcup \mathcal{F}_{\text{exc}}$. Define $\mathsf{Pet}(A) \coloneqq A \setminus \mathsf{Cor}(A)$ for each $A \in \mathcal{F}_{\text{nor}}$. For the sake of brevity, we also define the following:

$$\begin{aligned} \mathsf{Set}(\mathcal{F}(i)) &\coloneqq \bigcup_{A \in \mathcal{F}(i)} A && \text{for any } i \in S, \\ \mathsf{Pet}(\mathcal{F}(i)) &\coloneqq \bigcup_{A \in \mathcal{F}(i)} \mathsf{Pet}(A) && \text{for any } i \in S_{\text{nor}}, \\ \mathsf{Cor}(\mathcal{F}(i)) &\coloneqq \mathsf{Cor}(A) && \text{for any } A \in \mathcal{F}(i),\ i \in S_{\text{nor}}. \end{aligned}$$

Furthermore, let

$$\mathcal{F}(\geqslant i) \coloneqq \bigcup_{j \geqslant i} \mathcal{F}(j) \qquad \text{and} \qquad \mathcal{F}(I) \coloneqq \bigcup_{i \in I} \mathcal{F}(i) \quad \text{for any } I \subset [n].$$

Thus, we may also speak of $\mathsf{Pet}(\mathcal{F}(\geqslant i))$ and $\mathsf{Set}(\mathcal{F}(\geqslant i))$, as well as $\mathsf{Pet}(\mathcal{F}(I))$ and $\mathsf{Set}(\mathcal{F}(I))$ for any $I \subset [n]$.

Observation 15. *Proposition 10 and Corollary 11 show that if* $\mathsf{Tor}(A) \neq \emptyset$ *for some* $A \in \mathcal{F}(i)$*, then* $i \in S_{nor}$*, and if* $i \in S_{exc}$*, then* $|\mathcal{F}(i)| = 1$*.*

2.1 The structure of $\mathcal{F}_{\text{nor}}$

The next few results describe the structure of the normal sunflowers in $\mathcal{F}$ in relation to the cores.

Observation 16. *The proof of Lemma 13 shows that if* $A, B \in \mathcal{F}_{nor}$ *with* $|A| = |B|$*, then* $\mathsf{Cor}(A) = \mathsf{Cor}(B)$*.*

Lemma 17. *If* $A, B \in \mathcal{F}_{nor}$ *with* $|A| < |B|$*, then* $\mathsf{Cor}(A) \subsetneq \mathsf{Cor}(B)$*.*

Proof. Let $A' \in \mathsf{Tor}(A)$, $B' \in \mathsf{Tor}(B)$. Consider $A \cap A' \cap B = \mathsf{Cor}(A) \cap B \subseteq \mathsf{Cor}(A)$. Since $\theta|A| \leqslant |A \cap A' \cap B| \leqslant |\mathsf{Cor}(A)| = \theta|A|$, we have $A \cap A' \cap B = \mathsf{Cor}(A)$ and $\mathsf{Cor}(A) \subseteq B$. Since $|B| \leqslant |B'|$, we can run the above argument with B' in place of B to show that $\mathsf{Cor}(A) \subseteq B'$. Hence, $\mathsf{Cor}(A) \subseteq B \cap B' = \mathsf{Cor}(B)$. Lastly, $\mathsf{Cor}(A) \neq \mathsf{Cor}(B)$ because $|\mathsf{Cor}(A)| = \theta|A| \neq \theta|B| = |\mathsf{Cor}(B)|$. $\square$

Lemma 18. *Suppose that $i, j \in S$ such that $i < \theta j$. If $A \in \mathcal{F}(i)$ and $B \in \mathcal{F}(j)$, then $B \in \mathsf{Tor}(A)$. In particular, $i \in S_{nor}$.*

Proof. Since $|A \cap B| \leqslant |A| < \theta j$, we must have $|A \cap B| = \theta i$. Hence, $B \in \mathsf{Tor}(A)$. Thus, $i \in S_{\text{nor}}$ by Observation 15. $\square$

Lemma 19. *Let $A \in \mathcal{F}_{nor}$. If there exists $B \in \mathcal{F}(i_{\max})$ such that $\mathsf{Pet}(A) \cap \mathsf{Cor}(B) \neq \emptyset$, then $\mathsf{Cor}(B) \subseteq A$. Moreover, there is at most one set $A \in \mathcal{F}_{nor}$ for which this happens.*

Proof. Note that $|A| < i_{\max}$ by Observation 16. Let $C \in \mathsf{Tor}(B)$, and consider $A \cap B \cap C = A \cap \mathsf{Cor}(B) \subseteq \mathsf{Cor}(B)$. By Lemma 17, $\mathsf{Cor}(A) \subseteq \mathsf{Cor}(B)$, and $\mathsf{Pet}(A) \cap \mathsf{Cor}(B) \neq \emptyset$ by assumption. Hence, $\theta|A| < |A \cap \mathsf{Cor}(B)|$, which implies that $\theta i_{\max} \leqslant |A \cap B \cap C| = |A \cap \mathsf{Cor}(B)| \leqslant |\mathsf{Cor}(B)| = \theta i_{\max}$. Thus, $\mathsf{Cor}(B) \subseteq A$.

Now, suppose that there exists $A' \in \mathcal{F}_{\text{nor}}$ distinct from A for which there exists $B' \in \mathcal{F}(i_{\max})$ such that $\mathsf{Pet}(A') \cap \mathsf{Cor}(B') \neq \emptyset$. By Lemma 13, $\mathsf{Cor}(B) = \mathsf{Cor}(B')$. So, $\mathsf{Cor}(B) \subseteq A \cap A'$, which implies that $|A \cap A'| \geqslant \theta i_{\max}$, a contradiction. $\square$

Denote by E_{nor} the unique set $A \in \mathcal{F}_{\text{nor}}$ for which there exists $B \in \mathcal{F}(i_{\max})$ such that $\mathsf{Pet}(A) \cap \mathsf{Cor}(B) \neq \emptyset$, whenever it exists. Define $\mathcal{F}^*_{\text{nor}} := \mathcal{F}_{\text{nor}} \setminus \{A \in \mathcal{F}_{\text{nor}} : A = E_{\text{nor}}\}$.

Corollary 20. *For all $A, B \in \mathcal{F}^*_{nor}$, $\mathsf{Pet}(A) \cap \mathsf{Cor}(B) = \emptyset$.*

Proof. If $|A| > |B|$, then $\mathsf{Cor}(A) \supsetneq \mathsf{Cor}(B)$ by Lemma 17, so $\mathsf{Pet}(A) \cap \mathsf{Cor}(B) = \emptyset$. If $|A| = |B|$, then this follows from Observation 16. Let $|A| < |B|$, and suppose $z \in \mathsf{Pet}(A) \cap \mathsf{Cor}(B)$. Then, by Lemma 17, $z \in \mathsf{Cor}(B')$ for any $B' \in \mathcal{F}(i_{\max})$. Hence, $\mathsf{Pet}(A) \cap \mathsf{Cor}(B') \neq \emptyset$, which implies by Lemma 19 that $A = E_{\text{nor}}$, a contradiction. $\square$

Lemma 17 and Corollary 20 say that $\mathcal{F}^*_{\text{nor}}$ has the following structure: the cores of $\mathcal{F}^*_{\text{nor}}$ form an increasing chain, and any petal is disjoint from every core. In fact, these two results can be used to show that, for $\mathcal{F}^*_{\text{nor}}$, "$r$-closed" is equivalent to "s-closed" for any $r, s \geqslant 3$.

Proposition 21. *$\mathcal{F}^*_{nor}$ is s-closed θ-intersecting for all $s \geqslant 2$.*

Proof. It suffices to show this for all $s > r \geqslant 3$, and by induction it is enough to show this for $s = r + 1$. Let $A_1, \ldots, A_{r+1} \in \mathcal{F}^*_{\text{nor}}$ be any $r + 1$ distinct sets. Without loss of generality, suppose that $|A_1| \leqslant \cdots \leqslant |A_{r+1}|$.

First, suppose that $|A_i| = |A_j|$ for some $i < j$. Then, $\mathsf{Cor}(A_1) \subseteq \bigcap_{k=1}^{r+1} A_k \subseteq \mathsf{Cor}(A_i)$ by Lemma 17 and Observation 16. But, by Corollary 20, $\mathsf{Pet}(A_1) \cap \mathsf{Cor}(A_i) = \emptyset$. Hence, $\bigcap_{k=1}^{r+1} A_k = \mathsf{Cor}(A_1)$. Thus, $|\bigcap_{k=1}^{r+1} A_k| = \theta|A_1|$. So, we are done in this case.

Next, suppose that $|A_i| < |A_j|$ for all $i < j$. Consider $U = A_1 \cap \cdots \cap A_r$ and $V = A_1 \cap \cdots \cap A_{r-1} \cap A_{r+1}$. By Proposition 6, we know that $|U|, |V| \in \{\theta|A_1|, \theta|A_2|\}$. Also, $|U \cap V| \leqslant \min\{|U|, |V|\}$. Note that $U \cap V = A_1 \cap \cdots \cap A_{r+1}$.

By Lemma 17, $\mathsf{Cor}(A_1) \subseteq U \cap V$. So, if $|U| = \theta|A_1|$ or $|V| = \theta|A_1|$, then $\theta|A_1| \leqslant |U \cap V| \leqslant \theta|A_1|$, and we are done in this case. So, assume that $|U| = \theta|A_2| = |V|$. Consider $U \subseteq A_1 \cap A_2$. Since $\theta|A_2| = |U| \leqslant |A_1 \cap A_2| \leqslant \theta|A_2|$, we have $U = A_1 \cap A_2$. Similarly, $V \subseteq A_1 \cap A_2$ and $\theta|A_2| = |V| \leqslant |A_1 \cap A_2| \leqslant \theta|A_2|$, so $V = A_1 \cap A_2$. Hence, $U \cap V = A_1 \cap A_2$, and $|U \cap V| = |A_1 \cap A_2| = \theta|A_2|$, so we are done. $\square$

The final result of this section provides a linear upper bound on the size of $\mathcal{F}$ when $\mathcal{F} = \mathcal{F}^*_{\text{nor}}$ and $\mathsf{Tor}(A) = \{B \in \mathcal{F} : |B| \geqslant |A|\}$ for every $A \in \mathcal{F}$. Also, the proof technique will be used later on in the proof of Theorem 5 in Section 3.

Lemma 22. *Suppose that for all $A, B \in \mathcal{F}^*_{nor}$ such that $|A| < |B|$, we have $B \in \mathsf{Tor}(A)$. Then, $|\mathcal{F}^*_{nor}| \leqslant \lfloor \frac{n-a}{b-a} \rfloor$.*

Proof. For simplicity of notation, assume that $\mathcal{F} = \mathcal{F}^*_{\text{nor}}$. Suppose that $S = \{i_1, \ldots, i_k\}$ with $i_1 < \cdots < i_k$. Let $C := \mathsf{Cor}(\mathcal{F}(i_k))$. Define $Y_j := \mathsf{Set}(\mathcal{F}(i_j)) \setminus C$ for each $1 \leqslant j \leqslant k$. By Lemma 17 and Corollary 20, $Y_j = \mathsf{Pet}(\mathcal{F}(i_j))$ for each $1 \leqslant j \leqslant k$. Since $B \in \mathsf{Tor}(A)$ whenever $A \in \mathcal{F}(i_j)$ and $B \in \mathcal{F}(i_{j'})$ for $j < j'$, we must have $\mathsf{Pet}(A) \cap \mathsf{Pet}(B) = \emptyset$. Thus, $Y_j \cap Y_{j'} = \emptyset$ for all $j \neq j'$. Now, notice that

$$|\mathcal{F}(i_j)| = \frac{|Y_j|}{(1-\theta)i_j},$$

since the petals in $\mathcal{F}(i_j)$ are pairwise disjoint sets with each having size $(1-\theta)i_j$. Thus,

$$|\mathcal{F}| = \sum_{j=1}^{k} |\mathcal{F}(i_j)| = \sum_{j=1}^{k} \frac{|Y_j|}{(1-\theta)i_j}.$$

We also have $\sum_{j=1}^{k} |Y_j| \leqslant n - |C| = n - \theta i_k$. It is now easy to see that $|\mathcal{F}|$ is maximized when $|Y_j| = (1-\theta)i_j$ for $2 \leqslant j \leqslant k$, and $|Y_1|$ is the largest integer $\leqslant n - \theta i_k - \sum_{j=2}^{k}(1-\theta)i_j$ which is divisible by $(1-\theta)i_1$. Thus, the maximum of $|\mathcal{F}|$ taken as S varies over all subsets of $[n]$ of size k, with k varying from 1 to n, occurs when $k = 1$ and $i_1 = b$, where $\theta = a/b$ in least form, $a, b > 0$. This maximum is easily seen to be $\lfloor \frac{n-a}{b-a} \rfloor$. $\square$

2.2 The structure of $\mathcal{F}_{\text{exc}}$

The next few results describe the structure of the exceptional sunflowers in $\mathcal{F}$ in relation to the cores.

Lemma 23. *Suppose that $S_{nor} \neq \emptyset$. Let $i \in S_{exc}$ such that $i > i_{\max}$. If $\mathcal{F}(i) = \{A\}$, then $\mathsf{Cor}(\mathcal{F}(i_{\max})) \subseteq A$.*

Proof. Let $B \in \mathcal{F}(i_{\max})$ and $C \in \mathsf{Tor}(B)$. Consider $A \cap B \cap C = A \cap \mathsf{Cor}(B) \subseteq \mathsf{Cor}(B)$. Since, $\theta|B| \leqslant |A \cap B \cap C| \leqslant |\mathsf{Cor}(B)| = \theta|B|$, we have $\mathsf{Cor}(B) \subseteq A$, as required. $\square$

Lemma 24. *Suppose that $S_{nor} \neq \emptyset$. Let $i \in S_{exc}$ such that $i < i_{\max}$. If $\mathcal{F}(i) = \{A\}$, then, either $|A \cap \mathsf{Cor}(\mathcal{F}(i_{\max}))| = \theta i$, or $\mathsf{Cor}(\mathcal{F}(i_{\max})) \subseteq A$. Moreover, there is at most one $i < i_{\max}$ such that the latter case holds.*

Proof. Let $B \in \mathcal{F}(i_{\max})$ and $C \in \mathsf{Tor}(B)$. Consider $A \cap B \cap C = A \cap \mathsf{Cor}(B) \subseteq \mathsf{Cor}(B)$. If $|A \cap B \cap C| < \theta i_{\max}$, then we must have $|A \cap \mathsf{Cor}(B)| = \theta i$, which is the former case. If $|A \cap B \cap C| = \theta i_{\max}$, then $A \cap B \cap C = \mathsf{Cor}(B)$, since $|\mathsf{Cor}(B)| = \theta i_{\max}$. Hence, $\mathsf{Cor}(B) \subseteq A$, which is the latter case. Lastly, suppose for the sake of contradiction that there exists $i' \in S_{\text{exc}}$, $i' \neq i$, such that $i' < i_{\max}$, $\mathcal{F}(i') = \{A'\}$, and $\mathsf{Cor}(B) \subseteq A'$. Then, $A \cap A' \supseteq \mathsf{Cor}(B)$, so $|A \cap A'| \geqslant \theta i_{\max}$, which is a contradiction. □

Denote by E_{exc} the unique set in $\mathcal{F}_{\text{exc}}$ such that $|E_{\text{exc}}| < i_{\max}$ and $\mathsf{Cor}(\mathcal{F}(i_{\max})) \subseteq E_{\text{exc}}$, whenever it exists.

Lemma 25. *Let $\theta = a/b$, $\gcd(a, b) = 1$. Let $A \in \mathcal{F}$ such that $b \nmid |A|$. Then, $A \in \mathcal{F}_{exc}$, and there is at most one such set A in $\mathcal{F}$. Moreover, if $S_{nor} \neq \emptyset$, then $\mathsf{Cor}(\mathcal{F}(i_{\max})) \subseteq A$.*

Proof. For any $A_1 \in \mathcal{F}$ distinct from A, we must have $|A \cap A_1| = \theta|A_1|$, since $\theta|A|$ is not an integer. So, $\mathsf{Tor}(A) = \emptyset$, which implies that $A \in \mathcal{F}_{\text{exc}}$. If there were another such set A', then $|A \cap A'|$ can be neither $\theta|A|$ nor $\theta|A'|$, which is a contradiction.

Let $S_{\text{nor}} \neq \emptyset$, $B \in \mathcal{F}(i_{\max})$, and $C \in \mathsf{Tor}(B)$. Consider $A \cap B \cap C = A \cap \mathsf{Cor}(B)$. Since, $|A \cap B \cap C| \neq \theta|A|$, we have $\theta i_{\max} \leqslant |A \cap B \cap C| \leqslant |\mathsf{Cor}(B)| = \theta i_{\max}$. Hence, $A \cap B \cap C = \mathsf{Cor}(B)$, which implies that $\mathsf{Cor}(B) \subseteq A$. □

Denote by E_θ the unique set in $\mathcal{F}$ such that $b \nmid |E_\theta|$ (where $\theta = a/b$, $\gcd(a, b) = 1$), whenever it exists. Define $\mathcal{F}^*_{\text{exc}} \coloneqq \mathcal{F}_{\text{exc}} \setminus \{A \in \mathcal{F}_{\text{exc}} : A = E_{\text{exc}} \text{ or } E_\theta\}$. Define $\mathcal{F}^* \coloneqq \mathcal{F}^*_{\text{nor}} \cup \mathcal{F}^*_{\text{exc}}$.

2.3 The structure of $\mathcal{F}^*$

Observation 26. *If $\theta = a/b$, $\gcd(a, b) = 1$, then $|A| \equiv 0 \pmod{b}$ for all $A \in \mathcal{F}^*$.*

Proposition 27. $|\mathcal{F}^*| \leqslant |\mathcal{F}| \leqslant |\mathcal{F}^*| + 1$.

Proof. It suffices to show that at most one of E_{nor}, E_{exc}, and E_θ can belong to the family $\mathcal{F}$. If $S_{\text{nor}} = \emptyset$, then neither E_{nor} nor E_{exc} can exist by definition. So, suppose that $S_{\text{nor}} \neq \emptyset$. Then, $\mathsf{Cor}(\mathcal{F}(i_{\max})) \subseteq E_{\text{nor}}$, E_{exc}, and E_θ by Lemmas 19, 24, and 25, respectively. Hence, the size of the intersection of any two of these sets must be at least $|\mathsf{Cor}(\mathcal{F}(i_{\max}))| = \theta i_{\max}$, which is neither $\theta|E_{\text{nor}}|$, nor $\theta|E_{\text{exc}}|$, nor $\theta|E_\theta|$, which is a contradiction. □

3 Proofs of Theorems 4 and 5

Assume that $\mathcal{F} = \mathcal{F}^*$. Lemma 18 motivates us to partition the family $\mathcal{F}$ as $\mathcal{F} = \bigsqcup_{k \geqslant 0} \mathcal{F}(I_k)$, where $I_k \coloneqq (i_{\min}/\theta^{k-1}, i_{\min}/\theta^k]$ for $k \geqslant 1$, and $I_0 \coloneqq \{i_{\min}\}$. Suppose that $S_{\text{nor}} \neq \emptyset$. Let $C \coloneqq \mathsf{Cor}(\mathcal{F}(i_{\max}))$. Define $Y_k \coloneqq \mathsf{Set}(\mathcal{F}(I_k)) \setminus C$.

Observation 28. *If $v > u+1$, then $Y_u \cap Y_v = \emptyset$.*

Proof. It suffices to show that if $A \in \mathcal{F}(i)$ ($i \in I_u \cap S$) and $B \in \mathcal{F}(j)$ ($j \in I_v \cap S$), then $A \cap B \subseteq C$. It follows from the definitions of I_k, $k \geqslant 0$, that $i < \theta j$ for any such i and j. Thus, by Lemma 18, $B \in \mathsf{Tor}(A)$, so $A \cap B = \mathsf{Cor}(A)$. Hence, by Lemma 17, $A \cap B \subseteq C$. ☐

Observation 29.

$$\sum_{k\ odd} |Y_k| \leqslant n - |C| \leqslant n - \theta i_{\min},$$
$$\sum_{k\ even} |Y_k| \leqslant n - |C| \leqslant n - \theta i_{\min}.$$

Proof. This is immediate from the previous observation. ☐

Observation 30. *Let $i \in I_k$. Then,*

$$|\mathcal{F}(i)| \leqslant \frac{|Y_k|}{(1-\theta)i}.$$

Proof. Let $i \in S_{\text{nor}}$. By Lemma 17 and Corollary 20, $A \setminus C = \mathsf{Pet}(A)$ for all $A \in \mathcal{F}(i)$, so $Y_k \supseteq \mathsf{Pet}(\mathcal{F}(i))$. By Lemma 13, $\mathsf{Pet}(A) \cap \mathsf{Pet}(A') = \emptyset$ for all distinct $A, A' \in \mathcal{F}(i)$. Hence, $|Y_k| \geqslant |\mathsf{Pet}(\mathcal{F}(i))| = \sum_{A \in \mathcal{F}(i)} |\mathsf{Pet}(A)|$. Since $|\mathsf{Pet}(A)| = (1-\theta)i$ for all $A \in \mathcal{F}(i)$, we are done.

Let $i \in S_{\text{exc}}$ and $\mathcal{F}(i) = \{A\}$. First, consider the case when $i > i_{\max}$. Since $Y_k \supseteq A \setminus C$, and $C \subseteq A$ by Lemma 23, we have $|Y_k| \geqslant |A| - |C| = i - \theta i_{\max} > i - \theta i$. So, we are done. Next, consider the case when $i < i_{\max}$. Since we assume that $\mathcal{F} = \mathcal{F}^*$, we have $|A \cap C| = \theta i$ by Lemma 24. Hence, $|A \setminus C| = i - \theta i$. Since $Y_k \supseteq A \setminus C$, we are done. ☐

We also need the following result.

Lemma 31. *Let $\eta > 1$, and let $m \geqslant 1$ be an integer. Consider the sequence $(s_k)_{k \geqslant 1}$ given by*

$$s_k := \frac{1}{\lfloor m\eta^{k-1} \rfloor + 1} + \frac{1}{\lfloor m\eta^{k-1} \rfloor + 2} + \cdots + \frac{1}{\lfloor m\eta^k \rfloor}.$$

Then, $\lim_{k\to\infty} s_k = \ln(\eta)$.

When η is an integer, the sequence $(s_k)_{k \geqslant 1}$ is monotonically increasing to $\ln(\eta)$. *In general, $s_k < \ln(\eta) + \frac{1}{m}$ for all $k \geqslant 1$.*

Proof. Let H_n denote the nth harmonic number, $H_n = \sum_{i=1}^n 1/i$. It is well-known that $\lim_{n\to\infty}(H_n - \ln(n)) = \gamma$, the Euler–Mascheroni constant. Hence,

$$s_k = H_{\lfloor m\eta^k \rfloor} - H_{\lfloor m\eta^{k-1} \rfloor} = \ln\left(\frac{\lfloor m\eta^k \rfloor}{\lfloor m\eta^{k-1} \rfloor}\right) + \epsilon(\lfloor m\eta^k \rfloor) - \epsilon(\lfloor m\eta^{k-1} \rfloor),$$

where $\lim_{n\to\infty} \epsilon(n) = 0$. Since

$$\eta - \frac{1}{m\eta^{k-1}} < \frac{\lfloor m\eta^k \rfloor}{\lfloor m\eta^{k-1} \rfloor} < \frac{\eta}{1 - \frac{1}{m\eta^{k-1}}},$$

we have $\lim_{k\to\infty} s_k = \ln(\eta)$.

When η is an integer, the monotonicity of the sequence $(s_k)_{k\geqslant 1}$ is a corollary of the following more general observation, where $n \geqslant 1$ is any integer:

$$\sum_{i=n+1}^{\eta n} \frac{1}{i} < \sum_{i=n+1}^{\eta n} \frac{1}{i} + \left(\frac{1}{\eta n+1} - \frac{1}{\eta n+\eta}\right) + \cdots + \left(\frac{1}{\eta n+\eta-1} - \frac{1}{\eta n+\eta}\right) = \sum_{i=(n+1)+1}^{\eta(n+1)} \frac{1}{i}.$$

To show that $s_k < \ln(\eta) + \frac{1}{m}$ for all $k \geqslant 1$, observe that

$$\begin{aligned} s_k &< \int_{\lfloor m\eta^{k-1} \rfloor}^{\lfloor m\eta^k \rfloor} \frac{1}{t} dt \\ &\leqslant \ln(m\eta^k) - \ln(\lfloor m\eta^{k-1} \rfloor) \\ &= \ln(\eta) + \ln\left(\frac{m\eta^{k-1}}{\lfloor m\eta^{k-1} \rfloor}\right) \\ &< \ln(\eta) + \frac{1}{\lfloor m\eta^{k-1} \rfloor} \\ &\leqslant \ln(\eta) + \frac{1}{m}. \end{aligned}$$

□

3.1 Proof of Theorem 5

Now, we are ready to prove Theorem 5.

Proof. We assume throughout that $\mathcal{F} = \mathcal{F}^*$, since it suffices to compute $|\mathcal{F}^*|$ by Proposition 27.

First, observe that if $\mathcal{F}_{\text{nor}} = \emptyset$, then only $\mathcal{F}(I_0)$ and $\mathcal{F}(I_1)$ can be nonempty by Lemma 18. Furthermore, each nonempty $\mathcal{F}(i)$ is a singleton set. Therefore, $|\mathcal{F}| \leqslant \frac{1}{b}\left(\frac{i_{\min}}{\theta} - i_{\min}\right) + 1$, which is maximized when $n = \frac{i_{\min}}{\theta}$. Hence, this gives the bound $|\mathcal{F}| \leqslant \lfloor\frac{(1-\theta)n}{b}\rfloor + 1$, which is stronger than those in the statement of Theorem 5.

For the rest of the proof, suppose that $\mathcal{F}_{\text{nor}} \neq \emptyset$. Let $i_{\min} = mb$ for some $m \geqslant 1$ by Observation 26. For $k \geqslant 1$, we have

$$|\mathcal{F}(I_k)| = \sum_{i\in I_k\cap S} |\mathcal{F}(i)| \leqslant \sum_{i\in I_k\cap S} \frac{|Y_k|}{(1-\theta)i} \leqslant \begin{cases} \dfrac{|Y_k|}{b-a}\left(\ln(\theta^{-1}) + \dfrac{1}{m}\right), & a > 1; \\ \dfrac{|Y_k|}{b-1}(\ln(b)), & a = 1, \end{cases}$$

from Observations 26 and 30, as well as Lemma 31. For $k = 0$, we have

$$|\mathcal{F}(I_0)| = |\mathcal{F}(i_{\min})| \leqslant \frac{|Y_0|}{(1-\theta)mb} \leqslant \begin{cases} \dfrac{|Y_0|}{b-a}\left(\ln(\theta^{-1}) + \dfrac{1}{m}\right), & a > 1; \\ \dfrac{|Y_0|}{b-1}\left(\dfrac{1}{m}\right), & a = 1, \end{cases}$$

from Observation 30 and Lemma 31. Since

$$|\mathcal{F}| = \sum_{k \geqslant 0} |\mathcal{F}(I_k)| = \sum_{k \text{ odd}} |\mathcal{F}(I_k)| + \sum_{k \text{ even}} |\mathcal{F}(I_k)|,$$

we get the bound

$$|\mathcal{F}| \leqslant 2\left(\frac{\ln(b) - \ln(a) + 1}{b-a}\right)(n - |C|) \tag{1}$$

when $a > 1$ by applying Observation 29.

When $a = 1$, we need to compare the term $1/m$ appearing in the bound for $\mathcal{F}(I_0)$ with the term $\ln(b)$ appearing in the bound for $\mathcal{F}(I_k)$ for k even: since $1/m > \ln(b)$ if and only if $m = 1$ and $b = 2$, and this happens if and only if $\theta = 1/2$ and $i_{\min} = 2$, we get

$$\sum_{k \text{ odd}} |\mathcal{F}(I_k)| \leqslant \frac{\ln(b)}{b-1} \sum_{k \text{ odd}} |Y_k|, \qquad \sum_{k \text{ even}} |\mathcal{F}(I_k)| \leqslant \begin{cases} \dfrac{1}{2-1} \displaystyle\sum_{k \text{ even}} |Y_k|, & \theta = 1/2,\ i_{\min} = 2; \\ \dfrac{\ln(b)}{b-1} \displaystyle\sum_{k \text{ even}} |Y_k|, & \text{otherwise.} \end{cases}$$

Thus, by Observation 29,

$$|\mathcal{F}| \leqslant \begin{cases} (1 + \ln(2))(n - |C|), & \theta = 1/2 \text{ and } i_{\min} = 2; \\ \left(\dfrac{2\ln(b)}{b-1}\right)(n - |C|), & \text{otherwise.} \end{cases} \tag{2}$$

The result now follows immediately from (1) and (2). $\square$

3.2 Proof of Theorem 4

We begin with an outline of the proof of Theorem 4 before presenting the details. Since the theorem is easily verified for $n = 2, 3$, we may assume that $n \geqslant 4$. It also suffices to assume that $\mathcal{F} = \mathcal{F}^*$ by Proposition 27. First, we show that the upper bound on $|\mathcal{F}|$ holds when $S = S_{\text{nor}} = \{2, 4\}$. Second, we show that if $S_{\text{nor}} \not\supseteq \{2, 4\}$, then $\mathcal{F}$ cannot be an extremal family. Finally, we show that if $S_{\text{nor}} \supsetneq \{2, 4\}$, then we can get a family that is strictly larger than $\mathcal{F}$ by removing all the sets of sizes greater than 4 and adding new sets of sizes 2 and 4. The uniqueness and stability are then easily verified, thus completing the proof.

Proof. Example 1 shows that there exists an r-bisection closed family $\mathcal{F}$ such that $|\mathcal{F}| = \lfloor \frac{3n}{2} \rfloor - 2$ for any $n \geqslant 2$, so the bound $(*)$, which we shall establish below, is in fact tight. For the rest of the proof, we assume that $n \geqslant 4$ and that $\mathcal{F} = \mathcal{F}^*$.

Claim 32. *If $S = S_{nor} = \{2,4\}$, then $|\mathcal{F}| \leqslant \lfloor \frac{3n}{2} \rfloor - 3$.*

Let $\mathsf{Cor}(\mathcal{F}(2)) = \{a_1\}$ and $\mathsf{Cor}(\mathcal{F}(4)) = \{a_1, a_2\}$. By Corollary 20 it follows that $|\mathcal{F}(4)| \leqslant \lfloor \frac{n-2}{2} \rfloor$ and $|\mathcal{F}(2)| \leqslant n-2$ so $|\mathcal{F}| = |\mathcal{F}(2)| + |\mathcal{F}(4)| \leqslant \lfloor \frac{3n}{2} \rfloor - 3$.

Claim 33. *If $S_{nor} \not\supseteq \{2,4\}$, then $\mathcal{F}$ is not an extremal family.*

Suppose for the sake of contradiction that $\mathcal{F}$ is extremal. Let $C \coloneqq \mathsf{Cor}(\mathcal{F}(i_{\max}))$.

If $S = \{2,4\}$ but $S_{\text{exc}} \neq \emptyset$, then clearly there cannot be more than n sets in the family $\mathcal{F}$, contradicting its extremality. So, assume that $S \neq \{2,4\}$.

Theorem 5 already shows that $|\mathcal{F}| < \lfloor \frac{3n}{2} \rfloor - 3$ for a bisection closed family unless $i_{\min} = 2$. So, suppose that $2 \in S$.

If $2 \in S_{\text{exc}}$, then there cannot be any $i \in S$ such that $i > 4$ by Lemma 18. So, $S = \{2\} = S_{\text{exc}}$, but this implies that $|\mathcal{F}| = 1$, contradicting the extremality of $\mathcal{F}$. Hence, $2 \in S_{\text{nor}}$.

Next, if $4 \notin S$, then by Lemma 18, $A \cap B = \mathsf{Cor}(A)$ for all $A \in \mathcal{F}(2)$, $B \in \mathcal{F}(\geqslant 6)$. If $n = 4$, then $\mathcal{F}(\geqslant 6) = \emptyset$, so we must have $S = \{2\}$. However, this contradicts the extremality of $\mathcal{F}$, as we have seen earlier, so assume that $n \geqslant 6$. Let $m_1 = |\mathsf{Pet}(\mathcal{F}(2))|$ and $m_2 = |\mathsf{Set}(\mathcal{F}(\geqslant 6))|$. Then, $m_1 + m_2 \leqslant n$, and $|\mathcal{F}| \leqslant m_1 + \lfloor 2\ln(2)(m_2 - |C|) \rfloor \leqslant 1 + \lfloor 2\ln(2)(n-4) \rfloor$ by (2). This is less than $\lfloor \frac{3n}{2} \rfloor - 3$, which contradicts the extremality of $\mathcal{F}$. So, $4 \in S$.

Lastly, if $4 \in S_{\text{exc}}$, then $S \subset \{2,4,6,8\}$ by Lemma 18. Suppose that $\mathcal{F}(8) \neq \emptyset$. Then, if $\mathcal{F}(4) = \{A\}$, we must have $|A \cap B| = \frac{1}{2}|B| = |A|$ for any $B \in \mathcal{F}(8)$. Hence, $A \subset B$ for all $B \in \mathcal{F}(8)$. So, if $8 \in S_{\text{nor}}$, then $A = \mathsf{Cor}(B)$, implying that $A = E_{\text{exc}}$. This contradicts that $\mathcal{F} = \mathcal{F}^*$, so $8 \notin S_{\text{nor}}$. But then $\mathcal{F}(4)$ and $\mathcal{F}(8)$ together contain at most two sets, and it is easy to see by a similar argument as in the previous case that $|\mathcal{F}|$ is strictly less than $\lfloor \frac{3n}{2} \rfloor - 3$, which contradicts the extremality of $\mathcal{F}$. So, $4 \in S_{\text{nor}}$.

To quickly summarize the above observations, $S_{\text{nor}} \supseteq \{2,4\}$ for any extremal family $\mathcal{F}$. We will now show that if $S_{\text{nor}} \supsetneq \{2,4\}$, then $\mathcal{F}$ is not extremal. Assume that $\mathcal{F}$ is an extremal family having the maximum number of sets of size 2.

Claim 34. *If there exists $a \in \mathsf{Pet}(\mathcal{F}(2)) \cap B$ for some $B \in \mathcal{F}(\geqslant 4)$, then $a \in \mathsf{Pet}(\mathcal{F}(4))$.*

This follows from Observation 28 and Corollary 20.

Claim 35. $|\mathsf{Pet}(\mathcal{F}(2)) \setminus \mathsf{Pet}(\mathcal{F}(4))| \leqslant 1$.

Suppose for the sake of contradiction that $a_1, a_2 \in \mathsf{Pet}(\mathcal{F}(2)) \setminus \mathsf{Pet}(\mathcal{F}(4))$ such that $a_1 \neq a_2$. Define $B' \coloneqq \mathsf{Cor}(\mathcal{F}(4)) \cup \{a_1, a_2\}$ and $\mathcal{F}' \coloneqq \mathcal{F} \cup \{B'\}$. By Observation 28, $a_1, a_2 \notin \mathsf{Set}(\mathcal{F}'(\geqslant 6))$, so $\mathcal{F}'$ is r-bisection closed. But, $|\mathcal{F}'| > |\mathcal{F}|$, which contradicts the maximality of $\mathcal{F}$.

Claim 36. *For each $B \in \mathcal{F}(4)$, $\mathsf{Pet}(B) \cap \mathsf{Pet}(\mathcal{F}(2)) = \emptyset$ or $\mathsf{Pet}(B)$.*

Let $a \in \mathsf{Pet}(B) \cap \mathsf{Pet}(\mathcal{F}(2))$, and let $b \in \mathsf{Pet}(B)$ such that $b \neq a$. Suppose for the sake of contradiction that $b \notin \mathsf{Pet}(\mathcal{F}(2))$. If $b \notin \mathsf{Pet}(A)$ for any $A \in \mathcal{F}$ distinct from B, then we contradict the maximality of $\mathcal{F}$ as before by considering the family $\mathcal{F}' := \mathcal{F} \cup \{A'\}$, where $A' = \mathsf{Cor}(\mathcal{F}(2)) \cup \{b\}$. So, $b \in \mathsf{Pet}(\mathcal{F}(\geqslant 6))$ by Corollary 20. Note that $b \notin \mathsf{Pet}(\mathcal{F}(\geqslant 10))$ by Observation 28. Also, if $b \in \mathsf{Pet}(\mathcal{F}(A))$ for some $A \in \mathcal{F}(8)$, then we must have $B \subset A$; in particular, $a \in A$, which is not possible by Observation 28. Hence, $b \in \mathsf{Pet}(A)$ for some $A \in \mathcal{F}(6)$, which is also unique by Lemma 13. Now, consider the family $\mathcal{F}'' := (\mathcal{F} \setminus \{A\}) \cup \{A''\}$, where $A'' := \mathsf{Cor}(\mathcal{F}(2)) \cup \{b\}$. Again, the property of being r-bisection closed is preserved, and $|\mathcal{F}''| = |\mathcal{F}|$, but $|\mathcal{F}''(2)| > |\mathcal{F}(2)|$, which is a contradiction.

We now partition the family $\mathcal{F}$ into two disjoint nonempty subfamilies as follows: let $\mathcal{G}_1$ be the subfamily consisting of the sets in $\mathcal{F}(2)$ as well as those sets B in $\mathcal{F}(4)$ such that $\mathsf{Pet}(B) \cap \mathsf{Pet}(\mathcal{F}(2)) \neq \emptyset$, and let $\mathcal{G}_2$ be the subfamily of $\mathcal{F}$ containing the remaining sets. Let $m_1 = |\mathsf{Pet}(\mathcal{G}_1)|$ and $m_2 = |\mathsf{Set}(\mathcal{G}_2)|$. By Claim 36, $\mathsf{Pet}(A) \cap B = \emptyset$ for all $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$. So, $m_1 + m_2 \leqslant n$. Also, $i_{\min}(\mathcal{G}_2) \geqslant 4$, and $|C| \geqslant 3$ since $S \supsetneq \{2,4\}$. Observe that $|\mathcal{G}_1| = \lfloor \frac{3m_1}{2} \rfloor$ and $|\mathcal{G}_2| \leqslant \lfloor 2\ln(2)(m_2-3) \rfloor$ by (2). But then $|\mathcal{F}| = |\mathcal{G}_1| + |\mathcal{G}_2| \leqslant \lfloor \frac{3m_1}{2} \rfloor + \lfloor 2\ln(2)(m_2-3) \rfloor < \lfloor \frac{3n}{2} \rfloor - 3$ since $m_2 \geqslant 6$, which contradicts the extremality of $\mathcal{F}$. This completes the proof of the bound $(*)$. The tightness, uniqueness, and stability are now easily verified:

1. As noted before, the family constructed in Example 1 is tight for the upper bound $(*)$. Call that family $\mathcal{F}_{\max}$.

   Note that $\mathcal{F}_{\max} = \mathcal{F}_{\max}(2) \sqcup \mathcal{F}_{\max}(4)$, and that $\mathcal{F}_{\max}$ is r-bisection closed for any $r \geqslant 2$ because, for any family of subsets of $[n]$ consisting only of sets of sizes 2 and 4, "r-bisection closed" and "intersecting" are equivalent properties. Also note that $E_{\mathrm{nor}} = \{1,2\}$ belongs to the family $\mathcal{F}_{\max}$.

2. The proof of the upper bound $(*)$ shows that if $\mathcal{F}$ is an extremal r-bisection closed family, then $S_{\mathrm{nor}} = \{2,4\}$. Furthermore, Claim 32 shows that for any extremal $\mathcal{F}$ we must have $|\mathcal{F}^*| = \lfloor \frac{3n}{2} \rfloor - 3$, and in particular $|\mathcal{F}^*(2)| = n-2$ and $|\mathcal{F}^*(4)| = \lfloor \frac{n-2}{2} \rfloor$. That is, assuming $\mathsf{Cor}(\mathcal{F}^*(2)) = \{a_1\}$ and $\mathsf{Cor}(\mathcal{F}^*(4)) = \{a_1, a_2\}$, the sets in $\mathcal{F}^*(2)$ are precisely all those obtained by taking the union of $\{a_1\}$ with singleton sets $\{b\}$ such that $b \neq a_1, a_2$, and the sets in $\mathcal{F}^*(4)$ are precisely all those obtained by taking the union of $\{a_1, a_2\}$ with two-element sets $\{b_1, b_2\}$ that are pairwise disjoint from each other as well as from $\{a_1, a_2\}$. Since $\mathcal{F}^*$ is an intersecting family, it is r-bisection closed, too. A moment's reflection shows that this family $\mathcal{F}^*$ can be obtained simply by applying an appropriate permutation of $[n]$ to $\mathcal{F}^*_{\max}$.

   To complete the analysis, observe that $\mathcal{G} := \mathcal{F}^* \cup \{\mathsf{Cor}(\mathcal{F}(4))\}$ is also r-bisection closed, and the permutation of $[n]$ that mapped $\mathcal{F}^*$ to $\mathcal{F}^*_{\max}$ also maps $\mathcal{G}$ to $\mathcal{F}_{\max}$. Clearly, $E_{\mathrm{nor}}(\mathcal{G}) = \mathsf{Cor}(\mathcal{F}(4))$. To show that $\mathcal{G} = \mathcal{F}$, we verify that neither E_{exc} nor E_θ can belong to $\mathcal{F}$. Suppose $E_{\mathrm{exc}} \in \mathcal{F}$. Then $\mathsf{Cor}(\mathcal{F}(4)) \subsetneq E_{\mathrm{exc}}$. But, if $\{a\} \neq \mathsf{Cor}(\mathcal{F}(2))$, then $a \in \mathsf{Pet}(A)$ for some $A \in \mathcal{F}(2)$. In particular, we must have $A \cap E_{\mathrm{exc}} = A$ which forces $E_{\mathrm{exc}} \in \mathcal{F}(4)$, but this is a contradiction. The same argument also shows that $E_\theta \notin \mathcal{F}$, and this completes the proof of uniqueness of the extremal family.

3. Theorem 5 and the proof of the upper bound $(*)$ show that $|\mathcal{F}| < 2\ln(2)(n-1)+1$ for any r-bisection closed family $\mathcal{F}$ that is not extremal. Since $\frac{3}{2} - 2\ln(2) \approx 0.11$, the claim follows.

□

4 Concluding remarks

We ignore all floors and ceilings here for simplicity.

- While Theorem 4 considers the maximum size among all possible r-bisection closed families, it is possible to consider a more constrained problem:

  Problem 37. For an integer $k \geqslant 2$, determine the maximum size of an r-bisection closed family $\mathcal{F}$ with $i_{\min}(\mathcal{F}^*) \geqslant k$.

  Theorem 5 establishes a linear upper bound, and it is not hard to construct a heirarchically bisection closed family of size at least $(2n-k-4)\left(\frac{1}{k} + \frac{1}{k+2} + \frac{1}{k+4}\right)$ when $k \geqslant 4$. Our methods in this paper suggest that all the possible set sizes must lie in the range $[k, 2k]$ for an optimal family. There could be more than three distinct set sizes in an optimal family, though it seems rather unlikely that sets of all possible sizes in this range can be attained. Settling this question fully may require other new ideas.

- While Theorem 4 gives a tight result for $\theta = 1/2$, the bound in Theorem 5 in the general case is far from best possible. Again, one can mimic the construction for $\mathcal{F}_{\max}$ to get r-closed θ-intersecting families of size $(n-2a)\left(\frac{1}{b-a} + \frac{1}{2(b-a)}\right)$ if $\theta = \frac{a}{b}$, but this is not best possible in general. If $\theta = 1/b$ for b odd, then one can get a heirarchically closed θ-intersecting family $\mathcal{F}$ of size $(n-3)\left(\frac{1}{b-1} + \frac{1}{2(b-1)} + \frac{1}{3(b-1)}\right)$. If $\theta = 1/b$ for b even, then in general one can get a heirarchically closed θ-intersecting family $\mathcal{F}$ of size $(n-4)\left(\frac{1}{b-1} + \frac{1}{2(b-1)} + \frac{1}{4(b-1)}\right)$. Similar constructions can be made in general when $a \neq 1$. The methods in this paper suggest that the best bound ought to be attained when $i_{\min}(\mathcal{F}^*)$ is as small as possible, i.e. $i_{\min}(\mathcal{F}^*) = b$ when $\theta = a/b$ in least form, but a complete answer seems beyond the scope of the methods in this paper.

  Problem 38. For a fraction $\theta = a/b \in (0,1)$, determine the maximum size of an r-closed θ-intersecting family $\mathcal{F}$.

- The following general question naturally arises from the above two problems, and we make the explicit statement for the sake of completeness:

Problem 39. For a fraction $\theta = a/b \in (0, 1)$ and an integer $k \geqslant b$, determine the maximum size of an r-closed θ-intersecting family $\mathcal{F}$ with $i_{\min}(\mathcal{F}^*) = k$.

- Another interesting question arises as an artifact of our proof ideas. If $\mathcal{F} = \mathcal{F}_{\mathrm{exc}}$ then the proof of Theorem 5 also shows that $|\mathcal{F}| \leqslant \left(\frac{1-\theta}{b}\right) n+2$. But, it appears that this bound is far from best possible, and we believe that in this case $|\mathcal{F}| = O(\sqrt{n})$. Since the notion of an exceptional family seems a bit contrived, a more natural question is the following:

  Question 40. Suppose $\mathcal{F} = \{A_1, \ldots, A_m\}$ is an r-closed θ-intersecting family with $|A_i| < |A_j|$ whenever $i < j$. Is $|\mathcal{F}| \leqslant O(\sqrt{n})$?

  One indication that this bound is the correct order comes from the situation when $|A_i \cap A_j| = \theta|A_i|$ whenever $i < j$. This setup is similar to that in Lemma 22, but under the additional constraint that there is at most one set of any fixed size. Indeed, in this case, a straightforward inductive argument shows that $|\bigcup_{i=1}^{k} A_i| \geqslant k^2$, and that gives the bound stated. But in the general case, the methods developed in this paper seem to fall short of being able to settle this conjecture in the affirmative. The following weaker version of the above question could prove to be more amenable to investigation:

  Question 41. Suppose $\mathcal{F} = \{A_1, \ldots, A_m\}$ is an r-closed θ-intersecting family with $|A_i| < |A_j|$ whenever $i < j$. Is $|\mathcal{F}| \leqslant o(n)$?

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