Some Exact Results for Non-Degenerate Generalized Turán Problems

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Abstract

The generalized Turán number $\mathrm{ex}(n, H, F)$ is the maximum number of copies of H in n-vertex F-free graphs. We consider the case where $\chi(H) < \chi(F)$. There are several exact results on $\mathrm{ex}(n, H, F)$ when the extremal graph is a complete $(\chi(F) - 1)$-partite graph. We obtain multiple exact results with other kinds of extremal graphs.

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1 Introduction

One of the most studied questions in extremal Combinatorics is the following: what is the largest number $\mathrm{ex}(n, F)$ of edges that an n-vertex graph can have, if it does not contain F as a subgraph? Turán [25] showed that in the case $F = K_{r+1}$, the largest number of edges are in the Turán graph $T(n, r)$, which is the complete r-partite graph with each part of order $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Erdős, Stone and Simonovits [8, 10] showed that if $\chi(F) = r + 1 > 2$, then $\mathrm{ex}(n, F) = (1 + o(1))|E(T(n, r))|$.

A straightforward generalization of the above question is when instead of the number of edges, we consider the number of subgraphs isomorphic to a given graph H. Let $\mathcal{N}(H, G)$ denote the number of copies of H in G. Given graphs H and F and a positive integer n, their generalized Turán number is $\mathrm{ex}(n, H, F) = \max\{\mathcal{N}(H, G) : G \text{ is an } n\text{-vertex } F\text{-free graph}\}$. The systematic study of these numbers was initiated by Alon and Shikhelman [1].

One particular topic that has attracted a lot of attention is the study of when the Turán graph contains the most copies of H among F-free graphs. More generally, when does a complete $(\chi(F) - 1)$-partite graph contain the most copies of H? We say that H is *F-Turán-good* if $\chi(H) < \chi(F)$ and $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T(n, r))$ for sufficiently large n. We say that H is *weakly F-Turán-good* if $\chi(H) < \chi(F)$ and for sufficiently large n, $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T)$ for some complete $(\chi(F) - 1)$-partite graph T. Note that for a given graph H, a straightforward but complicated computation determines which n-vertex complete r-partite graph contains the most copies of H. The very first result in the area is due to Zykov [27] and states that if $k < r$, then K_k is K_r-Turán-good. The

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systematic study of F-Turán-good graphs was initiated by Győri, Pach and Simonovits [18] in the case $F = K_r$ and by Gerbner and Palmer [16] for general F.

A phenomenon often seen in this area is the so-called *stability*. It refers to the property that if some F-free graph contains almost $\mathrm{ex}(n, F)$ edges (almost $\mathrm{ex}(n, H, F)$ copies of H), then it is in some sense close to the extremal graph. There are different versions of stability based on what "almost" and "close" means. The most studied version is based on the following notion. Given two graphs H and G on the same vertex set, the *edit distance* of H and G is the least number of edges that need to be added to H and removed from H to obtain G. Slightly imprecisely, when we say that H has edit distance at most k from $T(n, r)$, we mean that we can obtain a graph isomorphic to $T(n, r)$ on the vertex set $V(H)$ by adding and deleting at most k edges.

The classical Erdős-Simonovits stability theorem [5, 6, 24] states that if F is not bipartite and an n-vertex F-free graph G has $\mathrm{ex}(n, F)-o(n^2)$ edges, then the edit distance of G from $T(n, \chi(F) - 1)$ is $o(n^2)$. We say that H is *F-Turán-stable* if for every n-vertex F-free graph G that contains $\mathrm{ex}(n, H, F) - o(n^{|V(H)|})$ copies of H, the edit distance of G from $T(n, \chi(F)-1)$ is $o(n^2)$. We say that H is *weakly F-Turán-stable* if for every n-vertex F-free graph G that contains $\mathrm{ex}(n, H, F) - o(n^{|V(H)|})$ copies of H, the edit distance of G is $o(n^2)$ from some complete $(\chi(F) - 1)$-partite graph T. Using this language, the Erdős-Simonovits stability theorem states that K_2 is F-Turán-stable for every non-bipartite graph F.

The first stability result in generalized Turán problems is due to Ma and Qiu [21]. They showed that if $\chi(F) = r + 1$ and $k \leq r$, then K_k is F-Turán-stable. They used this result to show that if F has a color-critical edge, then K_k is F-Turán-good. More precisely, they proved a bound for every F that happens to give an exact result if F has a color-critical edge. The *decomposition family* $\mathcal{D}(F)$ of F consists of every bipartite graph that can be obtained by deleting $r-1$ classes from an $(r+1)$-coloring of F. Let $\mathrm{biex}(n, F)$ denote the maximum number of edges in an n-vertex graph that does not contain any member of the decomposition family of F. In particular, if F has a color-critical edge, then K_2 is in the decomposition family of F, thus $\mathrm{biex}(n, F) = 0$.

Theorem 1 (Ma, Qiu [21])**.** *For any $k \leq r$, we have $\mathrm{ex}(n, K_k, F) = \mathcal{N}(K_k, T(n, r)) + \mathrm{biex}(n, F)\Theta(n^{k-2})$.*

This approach where we prove that H is F-Turán-stable and use it to prove that H is F-Turán-good can also be found in [20, 23, 19, 12]. Finally, Gerbner [14] provided the following general formulation.

Theorem 2. *Let $\chi(F) > \chi(H)$ and assume that F has a color-critical edge. If H is weakly F-Turán-stable, then H is weakly F-Turán-good.*

Let us remark that the same property is also implied by the weaker assumption that there is an n-vertex F-free graph G with $\mathcal{N}(H, G) = \mathrm{ex}(n, H, F)$ that has edit distance $o(n^2)$ from a complete $(\chi(F)-1)$-partite graph. We emphasize that what makes the above theorem especially useful is the simple observation from [14] that if H is weakly $K_{\chi(F)}$-Turán-stable, then H is weakly F-Turán-stable, thus one stability result gives infinitely many exact results.

In this paper we extend this result by going beyond $(\chi(F) - 1)$-partite graphs: we obtain exact results when the extremal construction is obtained by adding further edges to a complete $(\chi(F)-1)$-partite graph. One such result has been obtained by Gerbner and Patkós [17]. Let $B_{r,1}$ denote the graph consisting of two copies of K_r sharing a vertex. It was shown in [4] that among $B_{r,1}$-free graphs, the most edges are contained in the Turán graph plus an additional edge. It was extended in [17] to any K_r with $r<k$ in place of K_2. There are further results when $B_{3,1}$ is forbidden in [12].

Our first result is a common generalization of Theorems 1 and 2. Let σ denote the smallest number of vertices that a color class of H can have in a proper $\chi(H)$-coloring. We say that a coloring of a graph is *almost proper* if there is exactly one edge whose endpoints have the same color. Note that an r-chromatic graph does not necessarily have an almost proper coloring with at most r colors, for example any almost proper coloring of C_4 uses three colors.

Theorem 3.

(i) *Let $r+1=\chi(F)>\chi(H)$ and assume that H is weakly F-Turán-stable. Then $\mathrm{ex}(n,H,F)\leq \mathcal{N}(H,T)+\mathrm{biex}(n,F)\Theta(n^{|V(H)|-2})$ for some n-vertex complete r-partite graph T. Moreover, for every n-vertex F-free graph G with $\mathcal{N}(H,G)=\mathrm{ex}(n,H,F)$ there is an r-partition of $V(G)$ to $A_1,\dots,A_r$, a constant $K=K(F)$ and a set B of at most $rK(\sigma(F)-1)$ vertices such that each member of $\mathcal{D}(F)$ inside a part shares at least two vertices with B, every vertex of B is adjacent to $\Omega(n)$ vertices in each part, and every vertex of $A_i\setminus B$ is adjacent to $o(n)$ vertices in A_i and all but $o(n)$ vertices in A_j with $j\neq i$.*

(ii) $\mathrm{ex}(n,H,F)=\mathcal{N}(H,T)+\mathrm{biex}(n,F)\Theta(n^{|V(H)|-2})$ *if there is a connected component H' of H such that one of the following hold.*

- **(a)** *H' has an almost proper coloring with at most r colors.*
- **(b)** $\mathrm{biex}(n,H')=o(\mathrm{biex}(n,F))$.

In particular, if F has a color-critical vertex, there are no members of $\mathcal{D}(F)$ inside the parts A_i. An example where **(ii)** does not hold is $\mathrm{ex}(n,C_4,F_2)$. It was shown in [16] that C_4 is F_2-Turán-good, i.e., $\mathrm{ex}(n,C_4,F_2)=\max\{\mathcal{N}(H,T)\}$, while $\mathrm{biex}(n,F_2)=1$. A more general result is in [11], showing that for any F with a color-critical vertex there is a graph H that is F-Turán-good. Again, the error term is 0 instead of $\mathrm{biex}(n,F)\Theta(n^{|V(H)|-2})$.

We prove this theorem in Section 2. Section 3 contains several exact results proved using Theorem 3, and some new constructions of weakly F-Turán-stable graphs.

2 Proof of Theorem 3

We will use the following lemma from [14].

Lemma 4. *Let us assume that $\chi(H)<\chi(F)$ and H is weakly F-Turán-stable, thus $\mathrm{ex}(n,H,F)=\mathcal{N}(H,T)+o(n^{|V(H)|})$ for some complete $(\chi(F)-1)$-partite n-vertex graph T. Then every part of T has order $\Omega(n)$.*

We will use the following simple observation (and variations of it) multiple times.

Lemma 5. *Let G be an n-vertex graph with a partition of $V(G)$ to $A_1, \dots, A_r$ such that $|A_i| = \Theta(n)$ for each i. Assume that each vertex of A_i is connected to all but $o(n)$ vertices outside A_i. Let F' be an induced subgraph of F such that the remaining part of F is s-colorable. Assume that a copy of F' is embedded into G avoiding $A_1, \dots, A_s$. Then we can extend this copy of F' to a copy of F in G.*

Proof. Let $U_1 \dots, U_s$ be the color classes of the graph we obtain by deleting F' from F. We will embed the sets U_i one by one in an arbitrary order into A_i. When we embed U_i, the already embedded at most $|V(F)|$ vertices are adjacent to all the $\Theta(n)$ vertices in A_i with $o(n)$ exceptions, thus we can pick the necessary vertices. □

Now we are ready to prove Theorem 3 that we restate here for convenience.

Theorem 3.

(i) *Let $r + 1 = \chi(F) > \chi(H)$ and assume that H is weakly F-Turán-stable. Then $\mathrm{ex}(n, H, F) \leq \mathcal{N}(H, T) + \mathrm{biex}(n, F)\Theta(n^{|V(H)|-2})$ for some n-vertex complete r-partite graph T. Moreover, for every n-vertex F-free graph G with $\mathcal{N}(H, G) = \mathrm{ex}(n, H, F)$ there is an r-partition of $V(G)$ to $A_1, \dots, A_r$, a constant $K = K(F)$ and a set B of at most $rK(\sigma(F)-1)$ vertices such that each member of $\mathcal{D}(F)$ inside a part shares at least two vertices with B, every vertex of B is adjacent to $\Omega(n)$ vertices in each part, and every vertex of $A_i \setminus B$ is adjacent to $o(n)$ vertices in A_i and all but $o(n)$ vertices in A_j with $j \neq i$.*

(ii) $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T) + \mathrm{biex}(n, F)\Theta(n^{|V(H)|-2})$ *if there is a connected component H' of H such that one of the following hold.*

- **(a)** *H' has an almost proper coloring with at most r colors.*
- **(b)** $\mathrm{biex}(n, H') = o(\mathrm{biex}(n, F))$.

Proof. To prove **(i)**, we will pick the numbers $\alpha, \beta, \gamma, \varepsilon > 0$ in this order, such that each is sufficiently small compared to the previous one, and after that we pick n that is sufficiently large. Let us consider an n-vertex F-free graph G with $\mathrm{ex}(n, H, F)$ copies of H. Because of the weakly F-Turán-stable property, for any $\varepsilon > 0$ there is a complete r-partite graph T on $V(G)$ that can be obtained from G by adding and removing at most εn^2 edges. Let us pick T such that we need to remove the least number of edges and let $A_1, \dots, A_r$ be the parts of T. Note that by the choice of T, each vertex in A_i has at least as many neighbors in every part as in A_i, and we have $|A_i| \geq \alpha n$ for some $\alpha > 0$ using Lemma 4.

Let $\beta > 0$ be sufficiently small and $\gamma > 0$ be sufficiently small compared to β. Let B_i denote the set of vertices in A_i with at least γn neighbors in A_i and $B = \cup_{i=1}^r B_i$.

Claim 6. *There is a K depending on γ and F such that $|B| \leq K(\sigma(F) - 1)$.*

Proof. This is an extension of Claim 4.2 in [21], and the proof also extends to our case, thus we only give a sketch here. Clearly $|B| \leq 2\varepsilon n/\gamma$ by the definition of ε. Therefore, $v \in B_i$ has at least $\gamma n - 2\varepsilon n/\gamma \geq \gamma n/2$ neighbors in every $A_j \setminus B_j$, using that ε is small enough compared to β.

Let A'_j denote an arbitrary set of $\gamma n/2$ neighbors of v in $A_j \setminus B_j$, and let $G(v)$ denote the subgraph of G induced on $\cup_{j=1}^r A'_j$. Then at most εn^2 edges are missing in $G(v)$

between parts A'_j. Therefore, the edge density in $G(v)$ is larger than $(r-2)/(r-1)$ (as that is the edge density of the $(r-1)$-partite Turán graph, but we have almost all the edges of the r-partite Turán graph). Thus, we can apply the Erdős-Simonovits supersaturation theorem [9] to obtain that $G(v)$ contains at least cn^{br} copies for some constant $c>0$ of the complete r-partite graph $K_{b,\dots,b}$ for $b=|V(F)|$.

Consider the following auxiliary bipartite graph. Part A consists of the copies of $K_{b,\dots,b}$ in $\cup_{i=1}^r A_i \setminus B_i$, while the other part is B. A vertex $u\in A$ is adjacent to $v\in B$ if the corresponding complete r-partite graph is in the neighborhood of v. Then clearly $|A|\le n^{br}$. Each vertex of A has at most $\sigma(F)-1$ neighbors in B, since otherwise we can find a complete $(r+1)$-partite graph in G with parts of order $\sigma(F),|V(F)|,\dots,|V(F)|$, which obviously contains F. This implies that the number of edges in this auxiliary bipartite graph is at most $(\sigma(F)-1)n^{br}$ and at least $|B|cn^{br}$, completing the proof with $K=1/c$. □

Let us return to the proof of the theorem. Let U_i denote the set of vertices v in $A_i\setminus B_i$ such that there are at least βn vertices in $V(G)\setminus A_i$ that are not adjacent to v. Clearly, $|U_i|\le \varepsilon n/\beta$. Let $V_i=A_i\setminus (B_i\cup U_i)$.

For each vertex $v\in U_i$ for each i, let us delete the edges from v to vertices in A_i, and connect v to every vertex of V_j with $j\neq i$. Let G' be the resulting graph. Observe that we deleted at most γn edges incident to v, thus at most $\gamma n^{|V(H)|-1}$ copies of H containing v. On the other hand, we added at least $\beta n - r\varepsilon n/\beta - rK(\sigma(H)-1)\ge \beta n/2$ edges incident to v (here we use that ε is small enough compared to β and n is large enough).

We claim that these edges are in at least $\beta n^{|V(H)|-1}/2^{|V(H)|-1}$ copies of H. Indeed, let us fix an r-coloring of H with color classes $W_1,\dots,W_r$. We count only the copies of H where W_j is embedded into V_j for $j\neq i$ and W_i is embedded into $V_i\cup\{v\}$. We apply the same greedy idea that we used in the proof of Lemma 5. First we embed W_i: One vertex to v and the other vertices into V_i arbitrarily. Then we embed the other parts W_j in an arbitrary order. Each time, when we want to embed W_ℓ, the already embedded at most $|V(H)|$ vertices have at least $|V_\ell|-|V(H)|\beta n > n/2+|V(H)|$ common neighbors in V_ℓ (using that β is small enough), as each vertex of V_j has at most βn non-neighbors in other parts. Therefore, embedding the vertices of W_ℓ one by one, we always have at least $n/2$ choices.

We also claim that G' is F-free. Assume not and pick a copy F_0 of F with the smallest number of vertices from $\cup_{i=1}^r U_i$. Clearly F_0 contains a vertex $v\in U_i$, as all the new edges are incident to such a vertex. Let Q be the set of vertices in F_0 that are adjacent to v in G'. They are each from $\cup_{j\neq i}V_j$. Their common neighborhood in V_i is of order at least $\alpha n-|V(F)|\beta n>|V(F)|$ (here we use that β is small enough compared to α). Therefore, at least one of them is not in F_0, thus we can replace v with that vertex to obtain another copy of F with less vertices from $\cup_{i=1}^r U_i$, a contradiction.

We obtained that G' is F-free and contains more copies of H than G (a contradiction) unless U_i is empty for every i. We also claim that there is no member of $\mathcal{D}(F)$ inside V_i. Indeed, by Lemma 5 that would extend to a copy of F. Moreover, if there is a member of $\mathcal{D}(F)$ that contains only one vertex u from B, then we can restrict ourselves to the neighbors of u and apply Lemma 5 to obtain a copy of F. Indeed, u has at least γn neighbors in each part A_i, thus at least $\gamma n - K(\sigma(F)-1)=\Theta(n)$ neighbors in $A_i\setminus B$. We restrict G' to these vertices and the already embedded vertices. Then we let F' be the already embedded member of $\mathcal{D}(F)$ and $s=r-2$, and apply Lemma 5.

Let us count now the copies of H in G. We have $\mathcal{N}(H,T)$ copies inside T. There are at most $r\mathrm{biex}(n,F)$ edges inside V_i, thus there are at most $r\mathrm{biex}(n,F)n^{|V(H)|-2}$ copies of H using some of those edges. It is left to count the copies of H which contain a vertex from B. As $|B|\le rK(\sigma(F)-1)$, clearly there are at most $rK(\sigma(F)-1)n^{|V(H)|-1}$ such copies of H. If $\sigma(F)=1$, then this is 0. If $\sigma(F)>1$, then $\mathrm{biex}(n,F)\ge n-1$, since the star is not in the decomposition family. Therefore, $rK(\sigma(F)-1)n^{|V(H)|-1}=O(\mathrm{biex}(n,F)n^{|V(H)|-2})$, completing the proof of **(i)**.

The lower bound in **(ii)** is given by taking T and adding into one part A_i of T a $\mathcal{D}(F)$-free graph with $\mathrm{biex}(|A_i|,F)$ edges. As $|A_i|=\Omega(n)$ by Lemma 4, $\mathrm{biex}(|A_i|,F)=\Theta(\mathrm{biex}(n,F))$. In the proofs of both **(a)** and **(b)**, we will find $\Theta(\mathrm{biex}(n,F)n^{|V(H')|-2})$ copies of H'. Clearly we can find $\Theta(n^{|V(H'')|})$ copies of every connected component H'' of H in T, and we can pick them one by one such that they are vertex-disjoint. This way we find $\Theta(\mathrm{biex}(n,F)n^{|V(H)|-2})$ copies of H, which completes the proof in both cases.

Observe that in **(a)**, each new edge is in $\Theta(n^{|V(H')|-2})$ copies of H'. To show **(b)**, observe that there are $\Theta(\mathrm{biex}(n,F))$ edges in A_i that are contained in a member of $\mathcal{D}(H')$. Each such copy is clearly extended to a copy of H' using edges of T, and there are $\Theta(n)$ ways to choose each new vertex. □

3 Exact results

We can obtain several exact results. First we generalize a theorem of Moon [22] that determines $\mathrm{ex}(n,F)$ where F consists of s vertex-disjoint copies of K_{r+1}.

Theorem 7. *Let F consist of $s>1$ components with chromatic number $r+1$, each with a color-critical edge, and any number of components with chromatic number at most r. Let H be a weakly F-Turán-stable graph and n sufficiently large. Then $\mathrm{ex}(n,H,F)=\mathcal{N}(H,T)$ for a complete $(s+r-1)$-partite graph T with $s-1$ parts of order 1.*

Proof. One can check that T does not contain F, and therefore $\mathrm{ex}(n,H,F)$ is at least $\mathcal{N}(H,T)$. Let G be an n-vertex F-free graph with $\mathrm{ex}(n,H,F)$ copies of H. We will apply Theorem 3 to obtain a partition to $A_1,\dots,A_r$ and a set B with $|B|=O(1)$ such that each member of $\mathcal{D}(F)$ inside parts A_i contains a vertex from B.

Assume first that there are s independent edges $u_1v_1,\dots,u_sv_s$ inside the parts such that for each i, at least one of u_i and v_i are not in B. Observe that u_i and v_i have $\Omega(n)$ common neighbors in each part besides the one containing them.

Let $F_1,\dots,F_s$ denote the components of F with chromatic number $r+1$. We apply the same greedy idea as in the proof of Lemma 5. We go through the edges u_iv_i one by one and extend them to F_i. Without loss of generality, $v_i\in A_1$. Let $B_1,\dots,B_r$ denote the parts of the graph we obtain by deleting a color-critical edge from F_i, with the endpoints of the deleted edge being in B_1. We will embed the vertices in B_j to A_j. First we map the two endpoints of the deleted edge to u_i and v_i and the other vertices of B_1 arbitrarily. Recall that the $O(1)$ embedded vertices have $\Omega(n)$ common neighbors in A_2. We pick $|B_2|$ of them that avoid the vertices we already picked to be in the copy of F and the avoid each u_j and v_j. Then the vertices we picked to be in the copy of F_i have $\Omega(n)$ common neighbors in A_3, we pick $|B_3|$ of them that we have not picked to be in our copy of F, and so on. We always have to avoid $O(1)$ already picked vertices. This way we obtain an F_i and ultimately $F_1, \dots, F_s$. Clearly we can pick the remaining components in a similar way to obtain F, a contradiction.

If $|B| \geq s$, then clearly we can find s distinct vertices among their neighbors not in B, resulting in the contradiction. Consider now a largest set of independent edges inside parts. By the above, there are at most $s-1$ edges, and each edge inside parts is incident to at least one of their at most $2s-2$ endvertices. Observe that outside B, each vertex is in $o(n)$ edges inside parts. Thus, there are $o(n)$ edges inside parts that are not incident to B. Therefore, deleting all the edges inside parts that are not incident to B, we lose $o(n^{|V(H)|-1})$ copies of H. If $|B| < s-1$, then we can add a vertex to B creating $\Theta(n^{|V(H)|-1})$ copies of H, a contradiction. We obtained that $|B| = s-1$. Clearly, for any edge $u_s v_s$ inside parts but outside B we could add independent edges $u_1v_1, \dots, u_{s-1}v_{s-1}$ with $u_i \in B$, to obtain the forbidden configuration. This implies that G is a subgraph of a complete $(s+r-1)$-partite graph with $s-1$ parts of order 1, completing the proof. □

From now on we will focus on the case where F has a color-critical vertex. An example where we can obtain an exact result is when all the parts can contain any $\mathcal{D}(F)$-free graphs at the same time. Note that this does not necessarily mean that we embed a $\mathcal{D}(F)$-free graph with the maximum number of edges into the parts.

Let us consider the complete $(r+1)$-partite graph $K_{1,a,\dots,a}$. Clearly its decomposition family contains the star $K_{1,a}$, thus graphs with maximum degree at most $a-1$ can be embedded into each part. Let us call a graph *almost a-regular* if it is either a-regular or has one vertex of degree $a-1$ and each other vertex has degree a.

Let $\mathcal{T}_0^{(s)}(n,r)$ denote the following family of graphs. We take a complete r-partite graph T, and for each part A_i, we embed an almost s-regular graph. It is easy to see that these graphs are $K_{1,a,\dots,a}$-free, where $s = a-1$. Let $\mathcal{T}^{(s)}(n,r)$ denote the subfamily of $\mathcal{T}_0^{(s)}(n,r)$ where T is the Turán graph $T(n,r)$. Simonovits [24] showed that the graphs in $\mathcal{T}^{(a-1)}(n,r)$ have the most edges among $K_{1,a,\dots,a}$-free graphs.

Theorem 8. *Let F be the complete $(r+1)$-partite graph $K_{1,a,\dots,a}$, H be a weakly F-Turán-stable graph and n sufficiently large.*

(i) *If $\chi(H) < r$, then $\mathrm{ex}(n,H,F) = \mathcal{N}(H,T)$ for some $T \in \mathcal{T}_0^{(a-1)}(n,r)$.*

(ii) *If H is a forest, then $\mathrm{ex}(n,H,F) = \mathcal{N}(H,T)$ for every $T \in \mathcal{T}_0^{(a-1)}(n,r)$ where the graph embedded into each part has girth at least $|V(H)|$.*

We remark that **(i)** is not immediate from the preceding argument, as it is possible that a graph embedded into a part with less edges contains more copies of some (subgraph of) H. Consider now cliques K_k. It is not hard to show that we need to embed into each part of a complete r-partite graph an almost a-regular graph with the maximum number of triangles, and among those we need a graph with the maximum number of K_4s, and so on. However, to determine these graphs does not seem to be a simple problem. A very special case is solved in [26].

We also remark that linear forests, (thus paths) are F-Turán-stable for every r [15], and forests H containing a matching of size $\lfloor |V(H)|/2 \rfloor$ are F-Turán-stable for $r=3$ by [14]. Thus, **(ii)** gives an exact result for those graphs. Before proving the theorem, we need a lemma.

Lemma 9. *Let G be an n-vertex graph with vertex set partitioned into $A_1, \dots, A_r$ such that for each $i \leq r$, $|A_i| = \Theta(n)$ and each vertex $v \in A_i$ adjacent to at most a vertices in A_i and to all but $o(n)$ vertices in A_j for $j \neq i$. If $\chi(H) < r$ or H is a forest, then for each edge uv inside A_i, there are $\Omega(n^{|V(H)|-2})$ copies of H containing uv. Moreover, there are $\Omega(n^{|V(H)|-2})$ copies of H containing uv and no other edges inside any A_j.*

Proof. We apply a greedy argument similar to Lemma 5. We assume without loss of generality that $u, v \in A_r$ and embed an arbitrary edge of H to uv. If $\chi(H) < r$, then take a proper $r-1$-coloring of the remaining part of H, and we will embed the jth color class of H to A_j. We embed the color classes one by one. The $O(1)$ vertices already embedded have $|A_j| - o(n) = \Theta(n)$ common neighbors in A_j, we pick vertices arbitrarily among those. Therefore, we have $\Theta(n)$ choices for each vertex, and altogether we pick $|V(H)|-2$ vertices, implying the required bound.

If H is a forest and $r = 2$, the embedded edge cuts H to at least two connected components, we embed the components one by one. For each component that has a vertex adjacent to u, we embed that vertex first to a neighbor of u in A_2 that we have not used earlier. Then we embed the neighbors of that vertex to neighbors in A_1, and so on. We proceed similarly for components with a vertex incident to v, and we start with an arbitrary vertex for other components. Each time we have to pick one of the $\Theta(n)$ neighbors and have to avoid the at most $|V(H)|-1$ vertices that we already used. □

Now we are ready to prove Theorem 8.

Proof of Theorem 8. First we prove **(i)** and that the same statement holds for forests H. Observe first that by Theorem 3, an n-vertex F-free graph G with $\mathcal{N}(H, G) = \mathrm{ex}(n, H, F)$ can be obtained the following way: we embed a $K_{1,a}$-free graph into each part of an r-partite graph T. Indeed, we have $\sigma(F) = 1$, thus in the extremal graph we have $|B| = 0$, which in turn implies that there is no member of $\mathcal{D}(F) = \{K_{1,a}, K_{a,a}\}$ inside any of the parts. One can also easily see that G is F-free. We can assume that T is a complete r-partite graph, as adding further edges between parts does not create F.

We need to show that the graphs G_i embedded into the parts A_i are $(a-1)$-almost regular. Those graphs G_i do not affect the number of copies of H that contain only edges between parts. Each other copy of H intersects some part A_i in a subgraph H_i that contains least one edge. Let p_i denote the number of isolated edges in H_i and q_i denote the number of components of H_i of order more than 2. By Lemma 9 we have that for any edge inside A_i, there are at least $cn^{|V(H)|-2}$ copies of H containing that edge (and no other edges inside any A_j) for some c.

Observe that for a fixed H_i, there are $O(n^{|V(H)|-p_i-2q_i})$ copies of H intersecting A_i in a copy of H_i. Indeed, we pick one vertex from each component inside A_i, then there are $O(1)$ ways to pick the other vertices of that component. There are at most n ways to pick each other vertex of H. This means that for any edge uv inside A_i, there are at least $cn^{|V(H)|-2}$ copies of H containing uv and no other edge from A_i, and for some c', there are at most $c'n^{|V(H)|-2}$ copies of H containing another subgraph inside A_i.

Assume now that G_i is not an $(a-1)$-almost regular graph. If $|E(G_i)| \leq (a-1)|A_i|/2 - c'/c$, then we are done, since we lose at least $cn^{|V(H)|-2}c'/c$ copies of H. Observe that if $|E(G_i)| > (a-1)|A_i|/2 - c'/c$, then at least $|A_i| - 2c'/c$ vertices have degree $a-1$. It is easy to see that we can turn G_i to an $(a-1)$-almost regular graph G_i' by adding and removing $O(1)$ edges.

Indeed, let u and v be of degree less than $a-1$ in G_i, then there are at most $2a-4$ vertices in A_i adjacent to u or v. If n is sufficiently large, then we can find an edge xy in G_i such that xu and yv are not in G_i. We delete the edge xy and add the edges xu and yv. This way we find a $K_{1,a}$-free graph with more edges. We can repeat this until we arrive to an $(a-1)$-almost regular graph G_i'.

We claim that $\mathcal{N}(H_i,G_i)-\mathcal{N}(H_i,G_i')=O(n^{p_i+q_i-1})$. Let U be the set of vertices with different neighborhood in G_i and G_i', then $|U|=O(1)$. The number of copies of H_i avoiding U is the same in G_i and G_i'. The other copies in G_i' can be counted by picking a vertex in U and extending it to a component, $O(1)$ many ways, then picking one vertex in each component, $O(n)$ ways each, then adding adjacent vertices $O(1)$ ways.

Since $|V(H)|\geq 2(p_i+q_i)$, we have that $p_i+q_i-1\leq |V(H)|-3$ unless H_i consists of an isolated edge plus some isolated vertices. Clearly the number of such subgraphs H_i increases by $\Omega(n^{|V(H_i)|-2})$, while the number of other subgraphs H_i decreases by $O(n^{|V(H_i)|-3})$. Therefore, replacing G_i with G_i', we lose $O(n^{|V(H)|-3})$ copies of H and we gain $\Omega(n^{|V(H)|-2})$ copies, a contradiction.

To prove **(ii)**, by Theorem 3, we need to maximize the number of copies of H in graphs G which are obtained from a complete r-partite graph by placing $K_{1,a}$-free subgraphs into each part. Each copy of H intersects each part in a forest. As observed by Cambie, de Verclos and Kang [3], the largest number of copies of each forest in m-vertex $K_{1,a}$-free graphs is in almost a-regular graphs with girth at least the diameter of the forest. Such graphs exist if m is sufficiently large [7], completing the proof. □

Assume now that $\mathrm{biex}(n,F)=O(1)$. This will be useful since we add to a complete r-partite graph T and delete from T $O(1)$ edges, thus it is enough to obtain asymptotic results on the number of copies of H containing those edges.

Let H be a graph with chromatic number less than r and uv be an edge of H that cuts H into at least two connected components. Let us fix an edge $u'v'$ of $T(n,r)$ and consider the number $f(u'v')$ of copies of H in $T(n,r)$ where $u'v'$ corresponds to uv. Note that $f(u'v')=\Theta(n^{|V(H)-2})$, $f(u'v')$ depends on u' and v', but clearly $f(u'v')=(1+o(1))f(u''v'')$ for any $u'v',u''v''$. Let $f(n)=\max f(u'v')$ where the maximum is taken on all the edges $u'v'$ of $T(n,r)$.

Lemma 10. *Let G be obtained from $T(n,r)$ by adding and removing $o(n)$ edges at every vertex. Then each edge of G corresponds to uv in $(1+o(1))f(n)$ copies of H in G. If we add any non-edge to G, the resulting edge also corresponds to uv in $(1+o(1))f(n)$ copies of H.*

Proof. Let H' denote the graph we obtain by deleting u and v from H, and let H'' be a component of H' such that some of its vertices are connected to, say, u.

Observe that $\mathcal{N}(H',T(n,r)=\Theta(n^{|V(H')|})$, since we can embed each color class of H' in a proper r-coloring of H to a distinct color class of $T(n,r)$. By embedding the vertices of the color classes one by one, we have at least $\lfloor n/r\rfloor-|V(H')|+1$ choices each time, and we count each copy of H $O(1)$ times this way. As each edge is in $O(n^{|V(H')|-2})$ copies of H', adding and deleting $o(n^2)$ edges from $T(n,r)$ adds and deletes $o(n^{|V(H')|})$ copies of H'. Therefore, a graph with edit distance $o(n^2)$ from the Turán graph has $(1+o(1))\mathcal{N}(H',T(n,r))$ copies of H'.

For each edge and non-edge $u'v'$ of G, the rest of G has edit distance $o(n^2)$ from the Turán graph, thus has $(1+o(1))\mathcal{N}(H',T(n,r))$ copies of H'. Some of those copies are extended to H with u' and v', while others are not. As the parts A_i have roughly the same size, the only difference is whether u' and v' belong to the same part or not.

We can embed H'' into G $(1+o(1))\mathcal{N}(H'', T(n,r))$ ways, even after we already embedded some of the other components of H'. Indeed, the $O(1)$ already embedded vertices cannot be used, but there are $O(n^{|V(H'')|-1}) = o(\mathcal{N}(H'', T(n,r)))$ copies of H'' in G containing at least one of those vertices. More importantly, such an embedding is good for us if the neighbors of u in H' are the neighbors of u' in G. This happens when they belong to parts of $T(n,r)$ distinct from the part of u, with a lower order term of exceptions. Thus we can embed the components one by one, obtaining the same asymptotic each time. By symmetry, the same argument works for components of H' where some of the vertices are connected to v. Finally, for components where no vertex is connected to any of u and v, the situation is even simpler, since we can use any embedding of such a component avoiding the already embedded vertices. □

Theorem 11. *Let $r+1 = \chi(F) > \chi(H) = 2$, assume that H is F-Turán-stable and K_{r+1}-Turán-good and n is sufficiently large. Assume that H is a forest and* $\mathrm{biex}(n,F) = O(1)$. *Then* $\mathrm{ex}(n,H,F) = \mathcal{N}(H,G)$ *for some G with* $\mathrm{ex}(n,F) = |E(G)|$.

Proof. Assume indirectly that there is an n-vertex F-free graph G_0 with less than $\mathrm{ex}(n,F)$ edges such that $\mathcal{N}(H,G_0) = \mathrm{ex}(n,H,F)$. Then we can apply Theorem 3 to obtain that G_0 can be obtained from a complete r-partite graph T with parts $A_1,\dots,A_r$ by adding a edges inside parts and removing b edges between parts. Moreover, each vertex is connected to $o(n)$ vertices in its part and all but $o(n)$ vertices in the other parts. We also have $a,b = O(1)$. Observe that each part of T has order $(1+o(1))n/r$, otherwise T contains $\Omega(n^{|V(H)|})$ less copies of H, and the $O(1)$ extra edges create $O(n^{|V(H)|-2})$ copies of H. Let G_1 be an n-vertex F-free graph with $\mathrm{ex}(n,F)$ edges.

As K_2 is F-Turán-stable, we have that G_1 can be obtained from $T(n,r)$ by adding a' and deleting b' edges. There are no graphs from $\mathcal{D}(F)$ inside the parts of $T(n,r)$ in G_1, thus $a' = O(1)$. Observe that $a-b < a'-b'$ since G_0 has less than $\mathrm{ex}(n,F)$ edges. We will apply Lemma 10 to both G_0 and G_1.

Let us return to G_0. By deleting the a edges inside parts, we delete $(1+o(1))af(n)$ copies of H; the number of copies containing more than one such edge is $O(n^{|V(H)|-3})$, thus negligible. By adding the b edges to obtain a complete r-partite graph, we create $(1+o(1))bf(n)$ copies of H; again, the number of copies containing more than one such edge is negligible. Then we turn the resulting graph to the Turán graph, this does not decrease the number of copies of H because H is K_{r+1}-Turán-good. Finally, we turn this graph to G_1 by adding a' and removing b' edges. This way altogether the number of copies of H increases by $(1+o(1))(a'-b'+b-a)f(n)$. As we have $0 < (a'-b'+b-a) = O(1)$, we obtain that the number of copies of H increases, a contradiction completing the proof. □

Given F with chromatic number $r+1$, let us call an F-free n-vertex graph G *nice* for F if G contains $T(n,r)$ and has $\mathrm{ex}(n,F)$ edges.

Theorem 12. *Let $\chi(F) = 3$, $\chi(H) = 2$, assume that H is F-Turán-stable and K_{r+1}-Turán-good,* $\mathrm{biex}(n,F) = O(1)$ *and n is sufficiently large. If there is a nice graph for F, then there is a nice graph G_0 for F with* $\mathrm{ex}(n,H,F) = \mathcal{N}(H,G_0)$.

As the proof is similar to the proof of Theorem 11, we will present it briefly. One can look at the above proof to find how the arguments can be completed.

Proof. Let G be an n-vertex F-free graph G with $\mathrm{ex}(n,H,F) = \mathcal{N}(H,G)$ and assume indirectly that G is not nice. Let us apply Theorem 3 to obtain T. Let us fix an edge uv of H. If uv cuts H into two or more parts, we will apply Lemma 10 to show that each edge of G corresponds to uv in $(1+o(1))f(n)$ copies of H. Otherwise uv is contained in a cycle, which must be of even length. Observe that if an even cycle contains an edge of G not in T, it must contain another such edge. This implies that an edge of G not in T corresponds to uv in $O(n^{|V(H)|-3})$ copies of H. On the other hand, an edge of T corresponds to uv in $\Theta(n^{|V(H)|-2})$ copies of H.

We obtained that every edge of T corresponds to uv in more or asymptotically the same number of copies of H as the edges of G outside T, which implies the statement. $\square$

An even simpler special case is when $\mathrm{biex}(n,F) = 1$, i.e., when $\mathcal{D}(F)$ contains both the star and the matching with two edges. The main example is $B_{r,1}$. Another example is the following graph Q_r. Let $r \geq 3$. We take $K_{1,2,a_3,\dots,a_{r+1}}$ with parts A_i of order $a_i \geq 2$, and remove all but two independent edges between A_r and A_{r+1}.

Let $\mathcal{G}_m$ denote the family of graphs obtained the following way. We take a complete r-partite graph T, add one edge u_iv_i into each of m parts, and for each $i,j \leq m$, we delete either the edges u_iu_j and v_iv_j or the edges u_iv_j and v_iu_j. Let G_m denote the element of $\mathcal{G}_m$ where $T = T(n,r)$ and the m additional edges are placed into m smallest parts.

Theorem 13.

(i) *Let H be weakly Q_r-Turán-stable. Then $\mathrm{ex}(n,H,Q_r) = \mathcal{N}(H,G)$ for some $G \in \mathcal{G}_m$, $m \leq r$. In particular, if $q := (2r-k)/(r-k+1)$ is not an integer, then $\mathrm{ex}(n,K_k,Q_r) = \mathcal{N}(K_k,G_{\lceil q \rceil})$. If q is an integer, then either $\mathrm{ex}(n,K_k,Q_r) = \mathcal{N}(K_k,G_q)$ or $\mathrm{ex}(n,K_k,Q_r) = \mathcal{N}(K_k,G_{q+1})$.*

(ii) *Let H be $B_{r,1}$-Turán-stable. Then $\mathrm{ex}(n,H,B_{r,1}) = \mathcal{N}(H,G')$, for some G' that is obtained from a complete $(r-1)$-partite graph by adding an edge into one of the parts.*

The first statement of the above theorem gives an example where the extremal graph is not the same for $\mathrm{ex}(n,H,F)$ as for $\mathrm{ex}(n,F)$, even though both contain the Turán graph. The second statement generalizes most of the known exact results when $\chi(H) < \chi(F)$ and F does not have a color-critical edge.

Proof. We start with the upper bound in **(i)**. We apply Theorem 3. We obtain that an extremal graph G is obtained by adding at most one edge into each part B_i of an r-partite graph T. Moreover, every vertex of T is connected to all but $o(n)$ vertices in the other parts. Assume that u_1v_1 and u_2v_2 are added to parts B_1 and B_2 of T and assume indirectly that one of these four vertices, say v_2 is adjacent to both vertices of the other edge, i.e., u_1v_2 and v_1v_2 are in G. We show that in this case we can embed Q_r into G, a contradiction. We embed an edge between A_r and A_{r+1} to u_1v_1. For the other edge xy between A_r and A_{r+1}, we embed x to a common neighbor of u_2 and v_2 in B_1 and embed y to u_2. We embed the single vertex in the first part of Q_r to v_2, and embed all the other vertices in A_r and A_{r+1} to neighbors of v_2 in B_1. Finally, we embed the remaining parts of Q_r by Lemma 5. By symmetry, for each $i,j \leq m$ either the edges u_iu_j and v_iv_j or the edges u_iv_j and v_iu_j are missing, showing that G is a subgraph of an element of $\mathcal{G}_m$.

To show the lower bound in **(i)**, it is enough to deal with the case $a_2 = \dots = a_{r+1} = 2$. Assume that Q_r is embedded into G_m. Let v denote the vertex in the part of order 1 in Q_r. If v is embedded in a part B_i of T together with any other vertex, then they are the endpoints of the extra edge of G inside B_i, and the other extra edges of G are not used, since one of the endpoints is not adjacent to the image of v. But then Q_r is embedded into T plus one edge, which is impossible. Thus we can assume that B_i contains only v.

Observe that if $x \in A_i$ with $i < r$ and x is embedded into B_i of T, then at most one other vertex of Q_r is embedded into B_i. Indeed, x has only one non-neighbor in Q_r, and the same holds for that vertex, thus every set of three vertices in Q_r containing x induces at least two edges. The vertices in parts A_i, $i < r$ are embedded into either at least $r-1$ or exactly $r-2$ parts of T.

In the first case, those $r-1$ parts of T contain at most $2r-2$ embedded vertices and the last part contains one vertex, a contradiction. In the second case, those $r-2$ parts contain the images of the parts of order 2, thus $A_r \cup A_{r+1}$ is embedded into one part, a contradiction.

Consider now $\mathrm{ex}(n, K_k, Q_r)$. If T has a part of order not $(1+o(1))n/r$, then clearly T has $\Theta(n^k)$ less copies of K_k then the Turán graph, and the extra edges create $O(n^{k-2})$ copies of K_k. Therefore, G is close to $T(n,r)$, in particular, every edge of G inside parts is in $(1+o(1))(n/r)^{k-2}\binom{r-1}{k-2}$ copies of K_k. Every non-edge between parts would create $(1+o(1))(n/r)^{k-2}\binom{r-2}{k-2}$ copies of K_k is added to G.

By the first part of the statement in **(i)**, we know that there are m edges added inside parts and $2\binom{m}{2}$ edges removed between parts. Therefore, the number of copies of K_k is $\mathcal{N}(K_k, T) + (1+o(1))m(n/r)^{k-2}\binom{r-1}{k-2} - (1+o(1))m(m-1)(n/r)^{k-2}\binom{r-2}{k-2}$. If $m < q$, then increasing m increases this number, if $m > q$ then increasing m decreases this number. If $m = q$, then it can go either way because of the $o(1)$ term. This shows that if q is not an integer, then $m = \lceil q \rceil$, while if q is an integer, then either $m = q$ or $m = q+1$. If there are two parts of T such that $|B_i| > |B_j| + 1$, then we move a vertex u from B_i to B_j. We pick u that is not incident to an edge of G inside B_i. This creates $\Theta(n^{k-2})$ more copies in the complete r-partite graph T and destroys $O(n^{k-3})$ copies of K_k that contains an edge of G not in T (since those copies each contain u and one of m edges). Finally, it is easy to see that placing extra edges inside the smaller parts creates more copies of K_k.

The lower bound in **(ii)** is obvious. The upper bound in the case $r = 2$ follows from Theorem 12, thus we assume that $r > 2$, consider an n-vertex $B_{r,1}$-free graph G with $\mathrm{ex}(n, H, B_{r,1})$ copies of H and apply Theorem 3. In this case we need to show that if there are two edges inside parts in G, then G' contains no less copies of H than G. Instead of this, we show the stronger statement that there cannot be two edges inside parts in G.

Assume that there are at least two edges u_1v_1 and u_2v_2 inside parts. We pick a vertex v that is a common neighbor of these vertices in another part. Then we embed the intersection of the two cliques of $B_{r,1}$ to v. We embed two vertices of one of the cliques to u_1 and v_1 and two vertices of the other clique to u_2 and v_2. We embed the remaining vertices one by one to the other parts, each time picking a vertex that is in the common neighborhood of the vertices already picked for that clique. $\square$

Finally, we show two simple ways to obtain weakly F-Turán-stable graphs. Let us consider the longest odd cycle C_{2k+1} such that F is subgraph of a p-blow-up of C_{2k+1} for some p, i.e., of the graph obtained the following way: we replace each vertex v of C_{2k+1}

with p vertices $v_1, \dots, v_p$, and each edge uv with p^2 edges u_iv_j, $i, j \leq p$. Let $c(F) = 2k+1$ and $b(F)$ denote the smallest p such that the above property holds. The connection of $c(F)$ to Turán-goodness was studied in [13].

Theorem 14.

(i) *Let $\chi(F) = r+1$ and H be a weakly F-Turán-stable graph. Assume that H has a unique r-coloring and H' is an r-chromatic graph obtained by adding edges but no vertices to H. Then H' is weakly F-Turán-stable.*

(ii) *Let $\chi(F) = 3$ and H be a weakly F-Turán-stable graph. Let us assume that H contains the $b(F)$-blow-up of $P_{c(F)-1}$, where $u_1, \dots, u_{b(F)}$ replace the first vertex and $v_1, \dots, v_{b(F)}$ replace the last vertex of the path. Let H' be the graph obtained by adding vertices $w_1, \dots, w_s, x_1, \dots x_t$ and edges u_iw_j, v_ix_ℓ for $i \leq b(F)$, $j \leq s$ and $\ell \leq t$. Assume that $\binom{t-s}{2} \leq s \leq t$. Then H' is F-Turán-stable.*

We remark that **(i)** is a straightforward extension of a proposition from [12], which states the same result with weakly F-Turán-good instead of weakly F-Turán-stable. Observe that in **(ii)**, if $F = K_3$, then we just add leaves to the endpoints of an edge.

Proof. To prove **(i)**, assume that there are p ways to obtain H' from H by adding edges, and H' contains q copies of H. Then we have $\mathcal{N}(H', G) \leq q\mathcal{N}(H, G)/p$ for every graph G. Observe that if a copy of H is in a complete r-partite graph G_0, then all the p ways to obtain H' create a subgraph of G_0, thus $\mathcal{N}(H', G_0) = q\mathcal{N}(H, G_0)/p$.

We claim that $\mathrm{ex}(n, H, F) = q\mathrm{ex}(n, H', F)/p - o(n^{|V(H)|})$. Let G' be an n-vertex F-free graph with $\mathrm{ex}(n, H, F)$ copies of H.

Let G_0 be the complete r-partite graph obtained by adding and deleting $o(n^2)$ edges from G', then G_0 contains $\mathcal{N}(H, G') - o(n^{|V(H)|})$ copies of H. Since G_0 contains $p\mathcal{N}(H, G_0)/q$ copies of H', we have that

$$\begin{aligned}
\mathrm{ex}(n, H, F) &= \mathcal{N}(H, G') = \mathcal{N}(H, G_0) - o(n^{|V(H)|}) \\
&= q\mathcal{N}(H', G_0)/p - o(n^{|V(H)|}) \\
&= q\mathrm{ex}(n, H', F)/p - o(n^{|V(H)|}).
\end{aligned}$$

Let G be an n-vertex F-free graph with $\mathrm{ex}(n, H', F) - o(n^{|V(H)|})$ copies of H'. Clearly G contains at least $q\mathrm{ex}(n, H', F)/p - o(n^{|V(H)|}) = \mathrm{ex}(n, H, F) - o(n^{|V(H)|})$ copies of H. Therefore, by the weak F-Turán-stability of H, G can be obtained from a complete r-partite graph by adding and removing $o(n^2)$ edges, completing the proof.

To prove **(ii)**, let G be an n-vertex F-free graph. We first embed H and then the additional vertices. Let Q be a copy of H and observe that there are at most $b(F) - 1$ common neighbors of $u_1, \dots, u_{b(F)}, v_1, \dots, v_{b(F)}$ in G. Let A denote the set of vertices that are common neighbors of $u_1, \dots, u_{b(F)}$ and B denote the set of vertices that are common neighbors of $v_1, \dots, v_{b(F)}$ in G, then $|A \cap B| \leq b(F) - 1$. We need to pick s vertices from A and t vertices from B.

Observe that the number of ways to do this while avoiding the vertices in $A \cap B$ is the same as the number of ways to pick $K_{s,t}$ from $K_{|A \setminus B|, |B \setminus A|}$. This is asymptotically the largest if $|A| = |B|$ by a result of Brown and Sidorenko [2], using that $\binom{t-s}{2} \leq s \leq t$. Note that they did this optimization in a slightly different context. Observe that $A \cap B$

is negligible, thus we have at most $(1+o(1))\mathcal{N}(K_{s,t},T(n-|V(H)|,2))$ ways to extend a copy of H to H'. Therefore, the number of copies of H' in G is at most the number of copies of H times $(1+o(1))\mathcal{N}(K_{s,t},T(n-|V(H)|,2))$, divided by some fixed number q of automorphisms of H'.

Observe that removing $|V(H)|$ vertices of $T(n,2)$, the resulting graph has edit distance $O(n)$ from $T(n-|V(H)|,2)$. Thus $\mathcal{N}(H',T(n,2)) = (1+o(1))\mathcal{N}(H,T(n,2))\mathcal{N}(K_{s,t},T(n-|V(H)|,2))/q$. Assume now that G has

$$\mathrm{ex}(n,H',F) - o(n^{|V(H')|}) \geq (1+o(1))\mathcal{N}(H,T(n,2))\mathcal{N}(K_{s,t},T(n-|V(H)|,2))/q$$

copies of H'. Then G has to contain $\mathcal{N}(H,T(n,2)) - o(n^{|V(H)|}) = \mathrm{ex}(n,H,F) - o(n^{|V(H)|})$ copies of H, thus G has edit distance $o(n^2)$ from $T(n,2)$ by the F-Turán-stable property of H, completing the proof. □

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