Chromatic Polynomials of 2-Edge-Coloured Graphs

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Abstract

Using the definition of colouring of 2-edge-coloured graphs derived from 2-edge-coloured graph homomorphism, we extend the definition of chromatic polynomial to 2-edge-coloured graphs. We find closed forms for the first three coefficients of this polynomial that generalize known results for the chromatic polynomial of a graph. We classify those graphs that admit a 2-edge-colouring for which the chromatic polynomial of the graph and the chromatic polynomial of the 2-edge-colouring is equal. Finally, we examine the behaviour of the roots of this polynomial, highlighting behaviours not seen in chromatic polynomials of graphs.

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1 Introduction and Preliminary Notions

A 2-*edge-coloured graph* G is a triple (Γ, R_G, B_G) where Γ is a simple graph, $R_G \subseteq E(\Gamma)$, and $B_G \subseteq E(\Gamma)$ such that $R_G \cap B_G = \emptyset$ and $R_G \cup B_G = E(\Gamma)$. We call G a 2-*edge-colouring* of Γ. We call Γ the simple graph *underlying* G. We note $\{R_G, B_G\}$ need not be a partition of $E(\Gamma)$; we permit $R_G = \emptyset$ or $B_G = \emptyset$. When $\{R_G, B_G\}$ is a partition of $E(\Gamma)$, we say that G is *bichromatic*. Otherwise we say G is *monochromatic*. With each 2-edge-coloured graph G we associate an edge-colour indication function $c_G : E(\Gamma) \to \{R, B\}$ defined such that $c_G(e) = R$ (respectively, $c_G(e) = B$) when $e \in R$ (respectively, $e \in B$). The set of such functions is in bijection with the set of 2-edge-coloured graphs that can be obtained from a fixed underlying graph. When there is no chance for confusion we refer to R_G, B_G and c_G as R, B and c, respectively. At various points, we will refer to the graph induced by the set of red edges (respectively blue edges) of G. Such a graph is formed from G

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by removing all blue edges (respectively red) and then removing all isolated vertices. We denote such a graph as $G[R_G]$ (respectively $G[B_G]$).

We assume all graphs are loopless and have no parallel edges. Thus we drop the descriptor of simple when we refer to simple graphs. In various corners of the literature [11, 18] 2-edge-coloured graphs are referred to as *signified graphs* to highlight the absence of a switching operation as in [12] so as to disambiguate from the notion of *signed graph.* Vertex colouring 2-edge-coloured graphs in the sense defined below is not the same as Zaslavksy's notion of signed graph colouring

For other graph-theoretic notions not defined we refer the reader to [4]. Throughout we use Greek majuscules to refer to graphs and Latin majuscules to refer to 2-edge-coloured graphs.

Let $G = (\Gamma_G, R_G, B_G)$ and $H = (\Gamma_H, R_H, B_H)$ be 2-edge-coloured graphs. There is a *homomorphism* of G to H when there exists a homomorphism $\phi : \Gamma_G \to \Gamma_H$ such that $\phi : G[R_G] \to H[R_H]$ and $\phi : G[B_G] \to H[B_H]$ are both graph homomorphisms. Equivalently there is a *homomorphism* of G to H when there exists $\phi : V(\Gamma_G) \to V(\Gamma_H)$ such that $\phi(u)\phi(v) \in E(\Gamma_H)$ and $c_G(uv) = c_H(\phi(u)\phi(v))$ for all $uv \in E(\Gamma_G)$. Informally, a homomorphism of 2-edge-coloured graphs is a vertex mapping that preserves the existence and colour of each edge. When H has k vertices we call ϕ a k-colouring of G.

Equivalently one may define colouring without appealing to homomorphism. A k-colouring of a 2-edge-coloured graph $G = (\Gamma, R, B)$ can be defined as a function $d : V(G) \to \{1, 2, 3, \ldots, k\}$ satisfying the following two conditions

1. for all $yz \in E(\Gamma)$, we have $d(y) \neq d(z)$; and

2. for all $ux \in R$ and $vy \in B$, if $d(u) = d(v)$ then $d(x) \neq d(y)$.

Such a colouring d defines a homomorphism to the 2-edge-coloured graph H with vertex set $\{1, 2, 3, \ldots k\}$ in which $ij \in R_H$ (respectively, $\in B_H$) if and only if there is an edge $wx \in R_G$ (respectively, $\in B_G$) such that $d(w) = i$ and $d(x) = j$.

The *chromatic number of* G, denoted $\chi(G)$, is the least integer k such that G admits a k-colouring. Observe that when G is monochromatic the definitions above are equivalent to the usual definitions for graph homomorphism, k-colouring and chromatic number. Thus the choice of notation, χ, to denote the chromatic number is appropriate. Below we introduce the notion of a mixed 2-edge-coloured graph and a corresponding notion of vertex colouring. For consistency we use the notation $\chi(\cdot)$ throughout the manuscript to refer to chromatic number. In the case of a 2-edge-coloured graph the notation $\chi(G)$ is used in place of the more common notation $\chi_2(G)$.

For a 2-edge-coloured graph $G = (\Gamma, R, B)$, the chromatic number of G may differ vastly from that of Γ. There exist 2-edge-colourings of bipartite graphs that have chromatic number equal to their number of vertices For example $K_{n,n}$ with edges of a perfect matching red and all remaining edges blue has chromatic number $2n$.

Homomorphisms of edge-coloured graphs have received increasing attention in the literature in recent years. Early work by Alon and Marshall [1] gave an upper bound on the chromatic number of a 2-edge-coloured planar graph. More recent work has bounded the chromatic number of 2-edge-coloured graphs from a variety of graph families [11, 15] as well as considered questions of computational complexity [5, 6]. The study of these objects in the context of colourings and homomorphisms is often compared and contrasted with similar questions for oriented graphs. Our work here continues with this comparison and seeks analogues to results for the oriented chromatic polynomial introduced in [7]. Of particular interest to this on-going work in comparing 2-edge-coloured graphs and oriented graphs is our results in Section 3 and their contrasting results in [7]. Together our work and the work in [7] give an example of results for these two types of graphs where the results are strikingly dissimilar. We comment more on this in Section 5.

A k-colouring of a 2-edge-coloured graph G is a proper k-colouring of the underlying graph Γ satisfying extra constraints. Thus the number of k-colourings of G is bounded by the chromatic polynomial of Γ evaluated at k. We show that the number of k-colourings of a 2-edge coloured graph G is itself a polynomial in k (Theorem 1 below), which in the monochromatic case coincides with the chromatic polynomial.

To study the chromatic polynomial for 2-edge-coloured graphs, we introduce a chromatic polynomial of a more generalized graph object. A *mixed 2-edge-coloured graph* is a pair $M = (G, F_M)$ where G is 2-edge-coloured graph with $G = (\Gamma, R_G, B_G)$ and $F_M \subseteq \overline{E(\Gamma)}$. We consider F_M as a set of edges that belong to neither R_G nor B_G. We denote by $S(M)$ the graph with vertex set $V(\Gamma)$ and edge set $R_G \cup B_G \cup F_M$. When there is no chance of confusion we refer to F_M as F.

Let $M = (G, F)$ be a mixed 2-edge-coloured graph with $G = (\Gamma, R, B)$. We define a *k-colouring* of M to be a function $d : V(G) \to \{1, 2, 3, \ldots, k\}$ such that

1. $d(u) \neq d(v)$ for all $uv \in R \cup B \cup F$; and
2. for all $ux \in R$ and for all $vy \in B$, if $d(u) = d(v)$ then $d(x) \neq d(y)$.

Informally, d is a k-colouring of the 2-edge-coloured graph G with the extra condition that vertices at the ends of an element of F receive different colours. Notice when $F = \emptyset$, d is a k-colouring of G. When $R = B = \emptyset$, d is a k-colouring of the graph with vertex set $V(\Gamma)$ and edge set F. The definition of k-colouring for mixed 2-edge-coloured graphs captures both the definition of k-colouring for 2-edge-coloured graphs and the definition of k-colouring for graphs.

At this point it may be tempting to consider a mixed 2-edge-coloured graph as a 3-edge-coloured graph. Note however that the definition of vertex colouring given above for mixed 2-edge-coloured graph does not coincide with the usual definition for vertex colouring for a 3-edge-coloured graph.

Let M be a mixed 2-edge-coloured graph. In Theorem 1 we show there is a polynomial, $P(G, \lambda)$ such that $P(M, k)$ is the number of k-colourings of M. We refer to $P(M, \lambda)$ as the chromatic polynomial of M. Notice when $F_M = \emptyset$, the polynomial $P(M, \lambda)$ enumerates colourings of a 2-edge-coloured graph and when $R = B = \emptyset$, this polynomial is identically the chromatic polynomial of the graph with vertex set $V(\Gamma)$ and edge set F.

Our work proceeds as follows. In Section 2 we study properties of chromatic polynomials of mixed 2-edge-coloured graphs. We use a recurrence reminiscent of the standard recurrence for the chromatic polynomial of a graph to give closed forms for the first three coefficients of this polynomial. These results generalize known results for chromatic polynomials of graphs. As every colouring of a 2-edge-coloured graph is a colouring of the underlying graph, we are led naturally to considering those 2-edge-coloured graphs that have the same chromatic polynomial as their underlying graph. We study this problem in Section 3. We give a full classification of graphs that admit a 2-edge-colouring for which the chromatic polynomial of the graph and the chromatic polynomial of the 2-edge-colouring are equal. In Section 4 we study roots of chromatic polynomials of 2-edge-coloured graphs. We find the closure of the real roots of the 2-edge-coloured chromatic polynomials to be $\mathbb{R}$ and that the non-real roots can have arbitrarily large modulus.

2 The Chromatic Polynomial of a Mixed 2-edge-coloured Graph

Consider the case $x = v$ for condition (2) in the definition of k-colouring of a mixed graph. When $x = v$ the sequence uvy is a path in which one edge is red and the other is blue. And so condition (2) enforces that vertices at the end of induced 2-path whose edges are respectively red and blue receive different colours. Let $M = (G, F)$ be a mixed 2-edge-coloured graph with n vertices. An induced path uvy is called an *induced bichromatic 2-path* when $uv \in R$ and $vy \in B$ or $uv \in B$ and $vy \in R$. Let $\mathcal{P}_M$ be the set of induced bichromatic 2-paths of M. If every pair of vertices of M is either adjacent in M or at the ends of an induced bichromatic 2-path in G, then in any colouring of M each vertex receives a distinct colour. Thus

$$P(M, \lambda) = \Pi_{i=0}^{n-1} (\lambda - i) = P(K_n, \lambda) \tag{1}$$

As every vertex of M receives a distinct colour in every colouring of M, the chromatic polynomial of such a mixed 2-edge-coloured graph M is exactly that of K_n. We return to this observation in the proofs of Theorems 2 and 4.

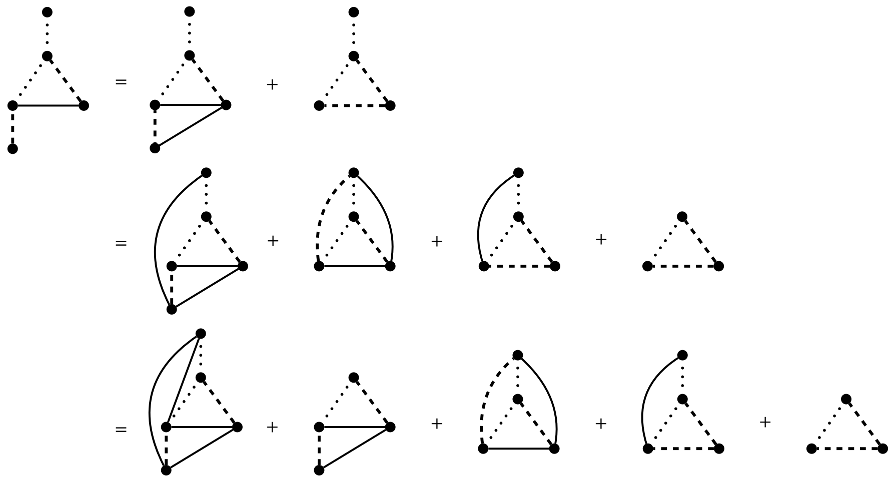
Let x and y be a pair of vertices that are neither adjacent in M nor at the ends of a bichromatic 2-path in M. The k-colourings of M can be partitioned into those in which x and y receive the same colour and those in which x and y receive different colours. Thus

$$P(M, \lambda) = P(M + xy, \lambda) + P(M_{xy}, \lambda) \tag{2}$$

where

- $M + xy$ is the mixed 2-edge-coloured graph formed from M by adding xy to F; and
- M_{xy} is the mixed 2-edge-coloured graph formed from identifying vertices x and y and deleting any edge that is parallel with a coloured edge.

See Figure 1 for a sample computation. Elements of R and B are denoted respectively by dotted and dashed lines. Elements of F are denoted by solid lines. We follow the usual convention for chromatic polynomials of having the picture of the graph stand in for its polynomial. Notice that in the third line each of the mixed 2-edge-coloured graphs has



$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)+3\lambda(\lambda-1)(\lambda-2)(\lambda-3)+\lambda(\lambda-1)(\lambda-2)$$

$$= \lambda^5-7\lambda^4+18\lambda^3-20\lambda^2+8\lambda$$

Figure 1: Computing the chromatic polynomial of a mixed 2-edge-coloured graph

the property that two distinct vertices are either adjacent or end vertices of an induced bichromatic 2-path.

From Equations (1) and (2) we directly obtain the following.

Theorem 1. *Let $M = (G, F)$ be a mixed graph with n vertices.*

1. *$P(M, \lambda)$ is a polynomial of degree n in λ;*
2. *the coefficient of λ^n is 1; and*
3. *$P(M, \lambda)$ has no constant term.*

Since P has bounded degree and it interpolates the points $(k, P(G, k))$ for all $k > 0$, it is necessarily unique.

As with chromatic polynomials, the coefficient of λ^{n-1} can be computed directly by counting a particular set of induced subgraphs.

Theorem 2. *For a mixed graph $M = (G, F)$ the coefficient of λ^{n-1} is given by*

$$-\left(|R_G| + |B_G| + |F_M| + |\mathcal{P}_M|\right).$$

Proof. We proceed considering the existence of a counterexample. Let n be the least integer such that there exists a mixed 2-edge-coloured graph with n vertices such that the statement of the theorem is false. Among all counterexamples on n vertices, let $M = (G, F)$ be a counterexample that maximizes the number of edges in $S(M)$. If $\mathcal{P}_M \neq \emptyset$, then there exists vertices u and v which are the ends of an induced bichromatic path in M. Let M' be the mixed 2-edge-coloured graph formed from M by adding uv to F. We observe that every colouring of M is a colouring of M' and also that every colouring of M' is a colouring of M. Thus M and M' have the same chromatic polynomial. This contradicts our choice of M as a counterexample on n vertices with the maximum number of edges in $S(M)$. Thus we may assume $\mathcal{P}_M = \emptyset$.

We claim there exists a pair of vertices u and v such that u and v are not adjacent in $S(M)$. If u and v do not exist, then $S(M) \cong K_n$ and so $P(M, \lambda) = P(K_n, \lambda)$. The coefficient of λ^{n-1} in $P(K_n, \lambda)$ is given by $-\binom{n}{2}$ [17]. Recall $\mathcal{P}_M = \emptyset$. For M we observe

$$-\left(|R_G| + |B_G| + |F_M| + |\mathcal{P}_M|\right) = -\binom{n}{2}$$

Thus we consider $u, v \in V(M)$ such that u and v are not adjacent in $S(M)$. Let c_{n-1} be the coefficient of λ^{n-1} in $P(M, \lambda)$. By Theorem 1, our choice of M, and Equation (2), we have

$$c_{n-1} = -\left(|R_G| + |B_G| + |F_{M+uv}| + |\mathcal{P}_M|\right) + 1$$

We observe $|F_{M+uv}| = |F_M| + 1$. Simplifying yields

$$c_{n-1} = -\left(|R_G| + |B_G| + |F_M| + |\mathcal{P}_M|\right).$$

Thus M is not a counterexample. And so by choice of M, no counterexample exists. □

Corollary 3. *For a 2-edge-coloured graph G, the coefficient of λ^{n-1} in $P(G, \lambda)$ is given by $-\left(|R_G| + |B_G| + |\mathcal{P}_G|\right)$*

Consider the case in condition (2) in the definition of k-colouring of a mixed 2-edge-coloured graph where u, v, x and y are distinct vertices in M. For d, a k-colouring of M, we must have $|\{d(u), d(x), d(v), d(z)\}| \in \{3, 4\}$. As the subgraph induced by $\{u, v, y, z\}$ has chromatic number 2 in $S(M)$, such pairs of edges, in a sense, obstruct a colouring of $S(M)$ from being a colouring of M.

Let $M = (G, F)$ be a mixed 2-edge-coloured graph. Let Λ be the graph formed from M by adding to F the edge between any pair of vertices that are the ends of an induced bichromatic 2-path in M and considering coloured edges in G as edges in Λ. (See Figure 2 for an example)

For $ux \in R$ and $vy \in B$ we say that ux and vy are pair of *obstructing edges* when $\chi(\Lambda[u, x, v, y]) = 2$. Let $\mathcal{O}_M$ be the set of pairs of obstructing edges in M.

Using the notion of obstructing edges, we find a closed form for the coefficient of λ^{n-2} of a chromatic polynomial of a mixed 2-edge-coloured graph.

Theorem 4. *For $M = (G, F)$ a mixed 2-edge-coloured graph, the coefficient of λ^{n-2} in $P(M, \lambda)$ is given by*

$$\binom{|R_G| + |B_G| + |\mathcal{P}_M| + |F_M|}{2} - |T_{S(M)}| - |\mathcal{P}_M| - |\mathcal{O}_M|,$$

where $T_{S(M)}$ is the set of induced subgraphs of $S(M)$ isomorphic to K_3.

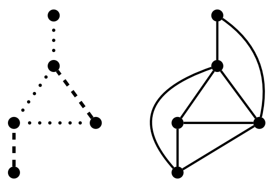


Figure 2: Λ (right) constructed from M (left)

Proof. We proceed considering the existence of a counterexample. Let n be the least integer such that there exists a mixed 2-edge-coloured graph with n vertices such that the statement of the theorem is false. Among all counterexamples on n vertices, let $M = (G, F)$ be a counterexample that maximizes the number of edges in $S(M)$.

As in the proof of Theorem 2 we may assume $\mathcal{P}_M = \emptyset$.

We claim there exists a pair of vertices u and v such that u and v are not adjacent in $S(M)$. If u and v do not exist, then $S(M) \cong K_n$ and so $P(M, \lambda) = P(K_n, \lambda)$. The coefficient of λ^{n-2} in $P(K_n, \lambda)$ is given by$\binom{\binom{n}{2}}{2} - \binom{n}{3}$ [17]. For M we observe

$$\begin{aligned}\binom{\binom{n}{2}}{2} - \binom{n}{3} &= \binom{|R_G| + |B_G| + |\mathcal{P}_M| + |F_M|}{2} - |T_{S(M)}| \\ &= \binom{|R_G| + |B_G| + |\mathcal{P}_M| + |F_M|}{2} - |T_{S(M)}| - |\mathcal{P}_M| - |\mathcal{O}_M|.\end{aligned}$$

This last equality follows by observing $\mathcal{P}_M = \emptyset$ (by hypothesis) and $\mathcal{O}_M = \emptyset$ when M is complete. This equality contradicts our choice of M, and so we conclude that such vertices u and v exist.

Thus we consider $u, v \in V(M)$ such that u and v are not adjacent in $S(M)$. Let c_{n-2} be the coefficient of λ^{n-2} in $P(M, \lambda)$. By Theorem 2, our choice of M and Equation (2) we have

$$\begin{aligned}c_{n-2} = &\binom{|R_G| + |B_G| + |\mathcal{P}_M| + |F_{M+uv}|}{2} - \left(|T_{S(M+uv)}| + |\mathcal{P}_M| + |\mathcal{O}_{M+uv}|\right) \\ &- \left(|R_{G_{uv}}| + |B_{G_{uv}}| + |F_{M_{uv}}| + |\mathcal{P}_{M_{uv}}|\right).\end{aligned}$$

Observe $|F_{M+uv}| = |F_M| + 1$. Let C be the set of common neighbours of u and v in $S(M)$. We observe $|T_{S(M+uv)}| = |T_{S(M)}| + |C|$ and $|R_{G_{uv}}| + |B_{G_{uv}}| + |F_{M_{uv}}| = |R_G| + |B_G| + |F_M| - |C|$. Thus

$$\begin{aligned}c_{n-2} = &\binom{|R_G| + |B_G| + |\mathcal{P}_M| + |F_M| + 1}{2} - \left(|T_{S(M)}| + |\mathcal{P}_M| + |\mathcal{O}_{M+uv}|\right) \\ &- \left(|R_G| + |B_G| + |F_M| + |\mathcal{P}_{M_{uv}}|\right).\end{aligned}$$

Since $\mathcal{P}_M = \emptyset$, a pair of obstructing edges, (ux, vy) in M, is not obstructing in $M + uv$ if and only if $xy \notin F$ and one of uy or vx is contained in F. Let $\mathcal{O}_M^{uv}$ be the set of such obstructing edges. Therefore $|\mathcal{O}_{M+uv}| = |\mathcal{O}_M| - |\mathcal{O}_M^{uv}|$. Notice now that every element of $\mathcal{O}_M^{uv}$ contributes an element of $\mathcal{P}_{G_{uv}}$ that was not an element of $\mathcal{P}_M$. Thus $|\mathcal{P}_{M_{uv}}| = |\mathcal{P}_M| + |\mathcal{O}_M^{uv}|$.

Substituting yields

$$\begin{aligned}
c_{n-2} &= \binom{|R_G| + |B_G| + |\mathcal{P}_M| + |F_M| + 1}{2} - \left(|T_{S(M)}| + |\mathcal{P}_M| + \mathcal{O}_M - |\mathcal{O}_M^{uv}|\right) \\
&- (|R_G| + |B_G| + |F_M| + |\mathcal{P}_M| + |\mathcal{O}_M^{uv}|) \\
&= \binom{|R_G| + |B_G| + |\mathcal{P}_M| + |F_M| + 1}{2} - (|R_G| + |B_G| + |F_M| + |\mathcal{P}_M|) \\
&- \left(|T_{S(M)}| + |\mathcal{P}_M| + |\mathcal{O}_M|\right) \\
&= \binom{|R_G| + |B_G| + |\mathcal{P}_M| + |F_M|}{2} - \left(|T_{S(M)}| + |\mathcal{P}_M| + |\mathcal{O}_M|\right).
\end{aligned}$$

Thus M is not a counterexample. And so by choice of M, no counterexample exists. □

Corollary 5. *For a 2-edge-coloured graph $G = (\Gamma, R, B)$, the coefficient of λ^{n-2} in $P(G, \lambda)$ is given by*

$$\binom{|R_G| + |B_G| + |\mathcal{P}_G|}{2} - |T_\Gamma| - |\mathcal{P}_G| - |\mathcal{O}_G|.$$

Alternatively, like the chromatic polynomial of a graph, one may approach the construction of the chromatic polynomial of a mixed 2-edge-coloured graph by the considering sums of chromatic polynomials of graphs whose chromatic number is equal to the number of vertices.

Recall that in the third line of Figure 1, each of the mixed 2-edge-coloured graphs has the property that two distinct vertices are end vertices of either an edge or an induced bichromatic 2-path. From this we notice the chromatic polynomial of a mixed 2-edge-coloured graph on n vertices can be expressed in terms of chromatic polynomials of mixed 2-edge-coloured graphs having this property:

$$P(M, \lambda) = \sum_{t=1}^{n} w(t) P(K_t, \lambda), \tag{3}$$

where $w(t)$ is the number of partitions of V into t sets of vertices such that vertices in the same set can receive the same colour in a vertex colouring of M. With this approach, the results above can be obtained directly by counting various obstructing substructures.

Consider computing c_{n-1} with this method. From Equation 3 we have

$$c_{n-1} = [\lambda^{n-1}] \sum_{t=1}^{n} w(t) P(K_t, \lambda) = w(n)[\lambda^{n-1}] P(K_n, \lambda) + w(n-1)[\lambda^{n-1}] P(K_{n-1}, \lambda).$$

We compute $w(n-1)$ by counting the number of ways to partition the vertices of M into $n-1$ sets of vertices that can receive the same colour in a vertex colouring of M. As there are n vertices in M, a partition into $n-1$ sets consists of $n-2$ singletons and a set with two elements. The set of two elements can consist of any pair of vertices of M that are non-adjacent and are not the ends of a bichromatic 2-path. There are $|R_G|+|B_G|+|\mathcal{P}_M|+|F_M|$ such pairs. And so $w(n-1) = \binom{n}{2} - (|R_G| + |B_G| + |\mathcal{P}_M| + |F_M|)$. Since $w(n) = 1$ and $[\lambda^{n-1}]P(K_n, \lambda) = -\binom{n}{2}$ we have

$$\begin{aligned}
c_{n-1} &= w(n)[\lambda^{n-1}]P(K_n, \lambda) + \left(\binom{n}{2} - (|R_G| + |B_G| + |\mathcal{P}_M| + |F_M|)\right)[\lambda^{n-1}]P(K_{n-1}, \lambda) \\
&= -\binom{n}{2} + \binom{n}{2} - (|R_G| + |B_G| + |\mathcal{P}_M| + |F_M|) \\
&= -(|R_G| + |B_G| + |\mathcal{P}_M| + |F_M|)
\end{aligned}$$

Using this method one can similarly obtain Theorem 4.

3 Chromatically Invariant 2-edge-coloured Graphs

Consider a 2-edge-coloured graph $G = (\Gamma, R, B)$. By definition every colouring of G is necessarily a colouring of Γ. Thus for each integer $k \geqslant 1$ we have $P(\Gamma, k) \geqslant P(G, k)$. In this section we study the structure of 2-edge-coloured graphs G for which $P(\Gamma, \lambda) = P(G, \lambda)$. We refer to such 2-edge-coloured graphs as *chromatically invariant*. Trivially, every 2-edge-coloured graph in which $R = \emptyset$ or $B = \emptyset$ is chromatically invariant. We refer to those chromatically invariant 2-edge-coloured graphs with $R, B \neq \emptyset$ as *non-trivially chromatically invariant*. We begin by providing a forbidden subgraph characterisation of chromatically invariant 2-edge-coloured graphs.

Lemma 6. *Let G be a 2-edge-coloured graph. If $\mathcal{P}_G = \mathcal{O}_G = \emptyset$, then G is chromatically invariant.*

Proof. Let $G = (\Gamma, R, B)$ be a 2-edge-coloured graph such that $\mathcal{P}_G = \mathcal{O}_G = \emptyset$. For each $k \geqslant 1$, let $C_{G,k}$ be the set of k-colourings of G. Similarly, let $C_{\Gamma,k}$ be the set of k-colourings of Γ. Recalling the definition of k-colouring of a 2-edge-coloured graph, it follows directly that $C_{G,k} \subseteq C_{\Gamma,k}$. To complete the proof it suffices to show $C_{\Gamma,k} \subseteq C_{G,k}$.

Let c be a k-colouring of Γ. Consider $ux \in R$ and $vy \in B$ such that $c(u) = c(v)$. If $u = v$, then $xy \in E(\Gamma)$ as $\mathcal{P}_G = \emptyset$. Thus $c(x) \neq c(y)$.

Consider now the case where $u \neq v$. Since $\mathcal{O}_G = \emptyset$ it follows that $|\{c(u), c(x), c(v), c(y)\}| \in \{3, 4\}$. Thus $c(x) \neq c(y)$. Therefore $c \in C_{G,k}$ and so it follows $C_{\Gamma,k} \subseteq C_{G,k}$. □

Theorem 7. *A 2-edge-coloured graph $G = (\Gamma, R, B)$ is chromatically invariant if and only if G contains no induced bichromatic 2-path and G contains no induced bichromatic copy of $2K_2$.*

Proof. Let $G = (\Gamma, R, B)$ be a 2-edge-coloured graph.

Assume G is non-trivial chromatically invariant. For a contradiction, we first assume $\mathcal{P}_G \neq \emptyset$. Consider $uvw \in \mathcal{P}_G$. Let k be the least integer such that there is a k-colouring c of Γ for which $c(u) = c(v)$. Let $C_{\Gamma,k}$ be the set of k-colourings of Γ. Let $C_{G,k}$ be the set of k-colourings of G. By construction, $c \in C_{\Gamma,k}$ but $c \notin C_{G,k}$. By the argument in the proof of Lemma 6, we have $C_{G,k} \subseteq C_{\Gamma,k}$. Therefore $|C_{G,k}| < |C_{\Gamma,k}|$. Thus $P(G, k) < P(\Gamma, k)$, which implies $P(G, \lambda) \neq P(\Gamma, \lambda)$.

Assume now G contains an induced bichromatic copy of $2K_2$. Let $ux \in R$ and $vy \in B$ be such that $G[u, x, v, y]$ is a bichromatic copy of $2K_2$. Let k be the least integer such that there is a k colouring c of Γ for which $c(u) = c(v)$ and $c(x) = c(y)$. A contradiction follows as in the previous paragraph.

Assume $\mathcal{P}_G = \emptyset$ and G contains no induced bichromatic copy of $2K_2$. By Lemma 6 it suffices to show $\mathcal{O}_G = \emptyset$. Consider $ux \in R$ and $vy \in B$ such that $u \neq v, y$ and $x \neq v, y$. Since G contains no induced bichromatic copy of $2K_2$, there exists an edge with an end in $\{u, x\}$ and an end in $\{v, y\}$. Without loss of generality, assume $uv \in R$. Since $\mathcal{P}_G = \emptyset$ and $vy \in B$ it follows that $uy \in E(\Gamma)$. Therefore $\Gamma[u, v, x, y]$ contains a copy of K_3. Thus $\{ux, vy\}$ is not a pair of obstructing edges. Therefore $\mathcal{O}_G = \emptyset$. The result follows by Lemma 6. □

Theorem 7 gives a full characterization of chromatically invariant 2-edge-coloured graphs. This characterization allows us to further characterize these 2-edge-coloured graphs by way of pairs of independent sets.

Theorem 8. *A 2-edge-coloured graph $G = (\Gamma, R, B)$ is chromatically invariant if and only if for every disjoint pair of non-empty independent sets I_1 and I_2 in Γ, the 2-edge-coloured subgraph induced by I_1 and I_2 is monochromatic.*

Proof. Let $G = (\Gamma, R, B)$ be a 2-edge-coloured graph and let I_1 and I_2 be disjoint non-empty independent sets of Γ.

Assume G is chromatically invariant. Thus by Theorem 7, it follows that G has no induced bichromatic 2-path and no induced bichromatic copy of $2K_2$. If $G[I_1 \cup I_2]$ has at most one edge, then the result holds – necessarily this subgraph is monochromatic. Otherwise, assume $e = u_1u_2$ and $f = v_1v_2$ are edges of $G[I_1 \cup I_2]$ with $u_1, v_1 \in I_1$ and $u_2, v_2 \in I_2$. Assume, without loss of generality, $e \in R$ and $f \in B$.

We first show that e and f have a common end point. Recall G contains no induced bichromatic copy of $2K_2$. Thus if e and f do not have a common end point, then, without loss of generality, we have $u_1v_2 \in E(\Gamma)$. If $u_1v_2 \in R$ then $u_1v_2v_1 \in \mathcal{P}$. Similarly, if $u_1v_2 \in B$, then $v_2u_1v_1 \in \mathcal{P}$. However, we have $\mathcal{P} = \emptyset$. Therefore e and f have a common end point.

If e and f have a common endpoint, then $ef \in \mathcal{P}$. However, we have $\mathcal{P} = \emptyset$. Therefore e and f do not have a common end point, a contradiction. Therefore the 2-edge-coloured subgraph induced by I_1 and I_2 is monochromatic.

Assume G is not chromatically invariant. By Theorem 7 we have that $P \neq \emptyset$ or G contains an induced bichromatic copy of $2K_2$. In either case we find a pair of disjoint independent sets I_1, I_2 such that $G[I_1 \cup I_2]$ is not monochromatic. □

Corollary 9. *If G is a 2-edge-coloured chromatically invariant graph, then every induced subgraph of G is chromatically invariant.*

We turn now to the problem of classifying those graphs which admit a non-trivial chromatically invariant 2-edge-colouring. We begin by fully classifying those graphs that admit a chromatically invariant 2-edge-colouring in which every vertex is incident with both a red and blue edge. We use this classification to then give a full classification of graphs that admit a non-trivial chromatically invariant 2-edge-colouring.

Recall a graph Γ is a *join* when $V(\Gamma)$ has a partition $\{X, Y\}$ such that $xy \in E(\Gamma)$ for all $x \in X$ and $y \in Y$. For $\Gamma_1 = \Gamma[X]$ and $\Gamma_2 = \Gamma[Y]$ we say that Γ *is the join of* Γ_1 *and* Γ_2 and we write $\Gamma = \Gamma_1 \vee \Gamma_2$. We call an edge $uv \in V(\Gamma_1 \vee \Gamma_2)$ a *joining edge* when $u \in V(\Gamma_1)$ and $v \in V(\Gamma_2)$.

Lemma 10. *Let Γ_1 and Γ_2 be graphs such that each of Γ_1 and Γ_2 have no isolated vertices. The graph $\Gamma_1 \vee \Gamma_2$ admits a non-trivial chromatically invariant 2-edge colouring in which every vertex is incident with at least one red edge and one blue edge.*

Proof. Let J be the set of joining edges of $\Gamma_1 \vee \Gamma_2$. Let $R = E(\Gamma_1) \cup E(\Gamma_2)$ and $B = J$. Since each of Γ_1 and Γ_2 have no isolated vertices, each vertex of $\Gamma_1 \vee \Gamma_2$ is incident with at least one red edge and one blue edge. For any pair of disjoint independent sets $I_1, I_2 \subset V(\Gamma)$ we have $I_1, I_2 \subset V(\Gamma_1)$ or $I_1, I_2 \subset V(\Gamma_2)$ or $I_1 \subset V(\Gamma_1)$ and $I_2 \subset V(\Gamma_2)$. In the latter case all edge between I_1 and I_2 are monochromatic. For the former cases the result follows by observing that the 2-edge-coloured graph (Γ, R, B) satisfies the hypothesis of Theorem 8. □

Lemma 11. *Let $G = (\Gamma, R, B)$ be a non-trivial chromatically invariant 2-edge-coloured graph. If there exists graphs Γ_1 and Γ_2 such that $\Gamma = \Gamma_1 \vee \Gamma_2$ and neither of Γ_1 or Γ_2 is a join, then all of the joining edges of $\Gamma_1 \vee \Gamma_2$ have the same colour in G.*

Proof. Let Γ_1 and Γ_2 be graphs that are not joins. Let $\Gamma = \Gamma_1 \vee \Gamma_2$. Let $G = (\Gamma, R, B)$ be non-trivial chromatically invariant 2-edge-coloured graph.

We first show that for any pair $y_1, y_2 \in V(\Gamma_2)$ there is a sequence of independent sets $I_1, I_2, \ldots, I_\ell$ such that $y_1 \in I_1$, $y_2 \in I_\ell$ and $I_i \cap I_{i+1} \neq \emptyset$ for all $1 \leqslant i \leqslant \ell - 1$. Since Γ_2 is not a join, its complement, $\overline{\Gamma_2}$, is connected. Therefore there is a path from y_1 to y_2 in $\overline{\Gamma_2}$. The edges of such a path form a sequence of independent sets in Γ_2: $I_1, I_2, \ldots, I_\ell$ such that $y_1 \in I_1$, $y_2 \in I_\ell$ and $I_i \cap I_{i+1} \neq \emptyset$ for all $1 \leqslant i \leqslant \ell$.

Consider $v \in V(\Gamma_1)$ and $y_1, y_2 \in V(\Gamma_2)$. Since $\Gamma = \Gamma_1 \vee \Gamma_2$, we have $vy_1, vy_2 \in E(\Gamma)$. Let $I_1, I_2, \ldots, I_\ell$ be a sequence of independent sets in such that $y_1 \in I_1$, $y_2 \in I_\ell$ and $I_i \cap I_{i+1} \neq \emptyset$ for all $1 \leqslant i \leqslant \ell - 1$. By Lemma 8 edges between I_i and v all have the same colour. Since $I_i \cap I_{i+1} \neq \emptyset$, all of the edges between I_{i+1} and v all have that same colour. Since v is adjacent to every vertex in Γ_2 it follows that the edges between I_1 and v have the same colour as those between I_ℓ and v. Therefore every joining edge with an end at v has the same colour.

Similarly, for any $u \in V(\Gamma_2)$, every joining edge with an end at u has the same colour in G. Thus it follows that all of the joining edges of $\Gamma_1 \vee \Gamma_2$ have the same colour in G. □

Lemma 12. *If $G = (\Gamma, R, B)$ is a chromatically invariant 2-edge-coloured graph in which every vertex is incident with a red edge and a blue edge, then Γ is a join.*

Proof. Let $G = (\Gamma, R, B)$ be a minimum counterexample with respect to number of vertices. We first show that there exists a vertex $v \in V(\Gamma)$ such that every vertex of $G - v$ is incident with both a red edge and a blue edge.

If no such vertex exists, then for every $x \in V(\Gamma)$ there exists $y \in V(\Gamma)$ such that $xy \in E(\Gamma)$ and the edge xy is the only one of its colour incident with y. We proceed in cases based on the existence of a pair $u, v \in V(\Gamma)$ such that uv is the only one of its colour incident with u and the only one of its colour incident with v.

If such a pair exists, then, without loss of generality, let uv be red. Since G has no induced bichromatic 2-path, $N_{G[B]}(u) = N_{G[B]}(v)$. If $V(\Gamma) = \{u, v\} \cup N_{G[B]}(u)$, then Γ is a join. As such the set $Q = V(\Gamma) \setminus \big(\{u, v\} \cup N_{G[B]}(u)\big)$ is non-empty. Notice that as G has no induced bichromatic 2-path, all edges between Q and $N_{G[B]}(u)$ are blue. Since every vertex of G is incident with both a red and a blue edge, it follows that every vertex of Q is incident with a red edge in the 2-edge-coloured graph $G[Q]$. As Γ is not a join, there exists $q \in Q$ and $x \in N_{G[B]}(u)$ such that $qx \notin E(\Gamma)$. Let rq be a red edge in $G[Q]$. Notice $rx \notin E(\Gamma)$, as otherwise such an edge is blue in G and so xrq is an induced bichromatic 2-path in G. Therefore the subgraph induced by $\{u, x, q, r\}$ is a bichromatic copy of $2K_2$. This is a contradiction as G is chromatically invariant. Therefore no such pair u, v exists.

Since no such pair u, v exists, there is a maximal sequence of vertices of Γ: $u_1, u_2, \ldots u_k$ such that $u_i u_{i+1} \in E(\Gamma)$ and the edge $u_i u_{i+1}$ is the only one of its colour incident with u_{i+1} for all $1 \leqslant i \leqslant k - 1$. We further note that, without loss of generality, vertices with an even index are adjacent with a single red edge and vertices with an odd index (other that u_1) are incident with a single blue edge. Thus $u_1, u_2, \ldots, u_k$ is an path whose edges are alternately red and blue. If $k \geqslant 4$, then since G has no induced bichromatic 2-path, we have $u_2 u_4 \in E(\Gamma)$. However, this edge is either a second red edge incident with u_2 or a second blue edge incident with u_4. This is a contradiction, and so $k = 3$.

Since this path was chosen to be maximal, it follows that the edge between u_1 and u_3 is the only one of its colour incident with u_1. Since $u_1 u_2$ is red, it follows that $u_1 u_3$ is blue. But then u_3 is incident with two blue edges: $u_2 u_3$ and $u_1 u_3$. This is a contradiction. And so there exists a vertex $v \in V(\Gamma)$ such that every vertex of $G - v$ is incident with both a red edge and a blue edge.

Consider $v \in V(\Gamma)$ such that every vertex of $G - v$ is incident with both a red edge and a blue edge. By Lemma 11, $G - v$ is a chromatically invariant 2-edge-coloured graph. By the minimality of G, we have that $(\Gamma - v)$ is a join. Therefore $V(\Gamma - v)$ admits a partition $\{X_1, X_2, \ldots, X_k\}$ such that $(\Gamma - v)\,[X_i]$ is not a join for each $1 \leqslant i \leqslant k$ and for each $1 \leqslant i < j \leqslant k$ we have $(\Gamma - v)\,[X_i \cup X_j] = (\Gamma - v)\,[X_i] \vee (\Gamma - v)\,[X_j]$.

Notice for any $1 \leqslant i \leqslant k$ that if v is adjacent to every vertex of X_i, then Γ is necessarily a join. Thus, for every for every $1 \leqslant i \leqslant k$ vertex v is not adjacent to at least one vertex of X_i.

By hypothesis, v in incident with a red edge and a blue edge. Let vr and vb be such edges for some $r, b \in V(\Gamma)$. We proceed in cases based on the location of r and b within the partition $\{X_1, X_2, \ldots, X_k\}$ of $V(\Gamma - v)$.

Consider, without loss of generality, $r, b \in X_1$. Notice that by Theorem 9 and Lemma 11, for every $1 \leqslant i < j \leqslant k$, the joining edges of $(\Gamma - v)[X_i \cup X_j] = (\Gamma - v)[X_i] \vee (\Gamma - v)[X_j]$ are the same colour. Since each of r and b are adjacent to every vertex of X_2, then either vry or vby is an induced bichromatic 2-path for every $y \in X_2$. (Whether or not vry or vby is an induced bichromatic 2-path depends on the colour of the joining edges between X_1 and X_2). Since G is chromatically invariant, by Theorem 7 no such path can be exist And so v is adjacent to every vertex in X_2, a contradiction.

Consider, without loss of generality, $r \in X_1$ and $b \in X_2$. Further assume without loss of generality that all of the joining edges of $(\Gamma - v)[X_1 \cup X_2] = (\Gamma - v)[X_1] \vee (\Gamma - v)[X_2]$ are red. Since Γ is not a join, there is at least one vertex of X_1, say x_1, that is not adjacent to v. However, x_1bv is an induced bichromatic 2-path. Since G is chromatically invariant, by Theorem 7 no such path can be exist. This is a contradiction. $\square$

Together Lemmas 10 and 12 imply the following characterization of those graphs which admit non-trivial chromatically invariant 2-edge-colourings in which every vertex is incident with at least one edge of each colour.

Theorem 13. *A graph Γ admits a non-trivial chromatically invariant 2-edge-colouring in which every vertex is incident with both a red edge and a blue edge if and only if Γ is the join of two graphs having no isolated vertices.*

Using this characterization we find a full characterization for the set of graphs that admit a non-trivial chromatically invariant 2-edge-colouring. To do so we require the following lemmas.

Lemma 14. *Let $G = (\Gamma, R, B)$ be a non-trivial chromatically invariant 2-edge-coloured graph with no isolated vertices. Let V_B be the set of vertices that are incident with only blue edges and whose neighbours are only incident with blue edges. The set V_B is independent in Γ.*

Proof. Let G be a non-trivial chromatically invariant 2-edge-coloured graph with no isolated vertices. Let $u, v \in V_B$. Notice

$$\emptyset = N_{G[R]}(u) = N_{G[R]}(v) = \bigcup_{w \in N(u)} N_{G[R]}(w) = \bigcup_{w \in N(v)} N_{G[R]}(w)$$

Since G is non-trivially chromatically invariant, there exists $xy \in E$ such that $xy \in R$. By hypothesis, $x, y \notin N_{G[R]}(u) \cup N_{G[R]}(v)$. By hypothesis, if $uv \in E$, then $uv \in B$. However, in this case $\{uv, xy\} \in \mathcal{O}_G$, contradicting the statement of Theorem 7. $\square$

Lemma 15. *Let $G = (\Gamma, R, B)$ be a non-trivial chromatically invariant 2-edge-coloured graph with no isolated vertices. Let $u \in V$. If $N_{G[R]}(u) = \emptyset$ and there exists $v \in N(u)$ such that $N_{G[R]}(v) \neq \emptyset$, then u is adjacent to all vertices of Γ that are in the same component of $G[R]$ as v.*

Proof. Let G be a non-trivial chromatically invariant 2-edge-coloured graph with no isolated vertices. Let $u \in V$ such that $N_{G[R]}(u) = \emptyset$ with $v \in N(u)$ such that $N_{G[R]}(v) \neq \emptyset$. Let J be the component of $G[R]$ that contains v.

Consider $x \in J$, and let $y_1, \ldots, y_{\ell-1}, y_\ell = x$ be a path of vertices from v to x in J. Note that every edge in this path is red. We will now show through induction that there is a blue edges from u to every vertex in this path. First, suppose $uy_1 \notin B$. Then $uvy_1 \in \mathcal{P}_G$, contradicting the statement of Theorem 7. Now suppose for some $k \geqslant 1$ that $uy_k \in B$. To reach a contradiction, suppose $uy_{k+1} \notin B$. Then $uy_ky_{k+1} \in \mathcal{P}_G$, contradicting the statement of Theorem 7. Thus by induction, u is adjacent to each of $y_1, \ldots, y_{\ell-1}, y_\ell$. Therefore u is adjacent to x. □

Theorem 16. *A graph Γ admits a non-trivial chromatically invariant 2-edge-colouring if and only if there exists non-empty $V_1, V_2 \subset V(\Gamma)$ such that*

- $V_1 \cap V_2 = \emptyset$,
- $\Gamma[V_1]$ *has at least one edge,*
- $\Gamma[V_1 \cup V_2] = \Gamma[V_1] \vee \Gamma[V_2]$, *and*
- $V \setminus (V_1 \cup V_2)$ *is an independent set and has no neighbours in* V_1.

Proof. First, we show that every graph with this structure admits a non-trivial chromatically invariant 2-edge-colouring. Assume Γ has the structure as outlined in the theorem. Without loss of generality, assume Γ has no isolated vertices. Consider the 2-edge-coloured graph G obtained by colouring all edges in $\Gamma[V_1]$ red and all other edges blue. Using Theorem 8 we prove such a 2-edge-coloured graph is chromatically invariant.

Let I_1 and I_2 be a disjoint pair of non-empty independent sets in Γ. And consider a pair of edges u_1u_2, v_1v_2 with $u_1, v_1 \in I_1$ and $u_2, v_2 \in I_2$. We claim $c_G(v_1v_2) = c_G(u_1u_2)$. By symmetry, it suffices to prove that if $c_G(u_1u_2) = R$, then $c_G(v_1v_2) = R$.

If $c_G(u_1u_2) = R$, then $u_1, u_2 \in V_1$. Since $\Gamma[V_1 \cup V_2] = \Gamma[V_1] \vee \Gamma[V_2]$, then $v_1, v_2 \notin V_2$. Since $V \setminus (V_1 \cup V_2)$ is an independent set and has no neighbours in V_1, it follows that $v_1, v_2 \in V_1$. Therefore $c_G(v_1v_2) = R$.

By Theorem 8, G is chromatically invariant.

We now show that every non-trivial chromatically invariant 2-edge-coloured graph has the structure outlined in the theorem. For a fixed Γ, Let $\mathcal{G}_\Gamma$ be the set of non-trivial chromatically invariant 2-edge-coloured graphs whose underlying graph is Γ. Assume $\mathcal{G}_\Gamma$ is non-empty.

Consider $G \in \mathcal{G}_\Gamma$. If all vertices are incident with a red edge and a blue edge in G, then the result follows from Theorem 13 with $V \setminus (V_1 \cup V_2) = \emptyset$.

Otherwise assume all elements of $\mathcal{G}_\Gamma$ have a vertex incident with edges of exactly one colour. Without loss of generality, assume this colour is blue. That is, every non-trivial chromatically invariant 2-edge-colouring of Γ in $\mathcal{G}_\Gamma$ has a vertex incident with only blue edges.

Consider $G \in \mathcal{G}_\Gamma$ that minimizes the number of components of $G[R_G]$. That is, let G be a non-trivial chromatically invariant 2-edge-coloured graph that minimizes the number of components in the subgraph induced by the red edges of G among all 2-edge-coloured graphs whose underlying graph is Γ. Define the following sets, which form a partition of V.

- $V_{G,s}$ – the set of vertices incident with only blue edges and whose neighbours are incident with only blue edges
- $V_{G,w}$ – the set of vertices incident with only blue edges, but have at least one neighbour incident with a red edge
- $V_{G,r}$ – the set of vertices incident with a red edge

By Lemma 14, $V_{G,s}$ is an independent set. By construction, $V_{G,s}$ has no neighbours in $V_{G,r}$.

If $\Gamma[V_{G,w} \cup V_{G,r}] = \Gamma[V_{G,w}] \vee \Gamma[V_{G,r}]$, then the result follows by letting $V_1 = V_{G,r}$ and $V_2 = V_{G,w}$.

By hypothesis, $G[R_G]$ has at least one component. Suppose $G[R_G]$ has one component. By definition, each vertex in $V_{G,w}$ is adjacent to some vertex in $V_{G,r}$. Therefore, each vertex in $V_{G,w}$ is adjacent to some vertex in the one component of $G[R_G]$. By Lemma 15 each vertex in $V_{G,w}$ is adjacent to every vertex in the one component of $G[R_G]$. Thus $\Gamma[V_{G,w} \cup V_{G,r}] = \Gamma[V_{G,w}] \vee \Gamma[V_{G,r}]$ and the result follows.

We claim that if $\Gamma[V_{G,w} \cup V_{G,r}] \neq \Gamma[V_{G,w}] \vee \Gamma[V_{G,r}]$ then G does not minimize the number of components of $G[R_G]$. That is, if $\Gamma[V_{G,w} \cup V_{G,r}] \neq \Gamma[V_{G,w}] \vee \Gamma[V_{G,r}]$ then there exists $G' \in \mathcal{G}_\Gamma$ such that $G'[R_{G'}]$ has fewer components than $G[R_G]$.

Assume $G[R_G]$ has $k > 1$ components and $\Gamma[V_{G,w} \cup V_{G,r}] \neq \Gamma[V_{G,w}] \vee \Gamma[V_{G,r}]$. Since $\Gamma[V_{G,w} \cup V_{G,r}] \neq \Gamma[V_{G,w}] \vee \Gamma[V_{G,r}]$ there exists a pair of non-adjacent vertices $u \in V_{G,w}$ and $v \in V_{G,r}$.

Let $J_1, J_2, \ldots J_k$ be the set of components of $G[R_G]$. Since u has at least one neighbour incident to a red edge, we may order these components such that there exists $1 < t < k$ such that in Γ u has a neighbour in each of $J_1, J_2, \ldots J_t$ and no neighbour in each of $J_{t+1}, J_{t+2}, \ldots J_k$. In other words, in Γ, u has neighbour in each of $J_1, J_2, \ldots J_t$ and none of $J_{t+1}, J_{t+2}, \ldots J_k$. By Lemma 15 u is adjacent to every vertex in each of $J_1, J_2, \ldots J_t$.

Consider $1 \leqslant i \leqslant t$, $t+1 \leqslant j \leqslant k$ and vertices $x \in J_i$ and $y \in J_j$. We claim $xy \in E(\Gamma)$, which in turn implies $\Gamma[J_i \cup J_j] = J_i \vee J_j$.

By hypothesis $ux \in B$. By hypothesis, y is incident with a red edge $e \in E(J_j)$. Since G is chromatically invariant, there is an edge between at least one end of e and one end of ux. By construction, such an edge must be blue and must have an end at x. This edge is in a bichromatic 2-path with e. Since G is chromatically invariant, by Theorem 7 this bichromatic 2-path is not induced. And so there must be a blue edge between both ends of e and x. Therefore for every vertex $x \in J_i$ and $y \in J_j$ there is a blue edge between them. Therefore $\Gamma[J_i \cup J_j] = J_i \vee J_j$.

Consider the 2-edge-coloured graph G' formed from G as follows:

- $c_{G'}(e) = B$ for all $e \in J_1, J_2, \ldots J_t$
- $c_{G'}(e) = c_G(e)$, otherwise.

Notice that the only red edges of G' are contained in $J_{t+1}, J_{t+2}, \ldots J_k$. And so G' has fewer components in $G'[R_{G'}]$ than does $G[R_G]$. We show G' is a non-trivially chromatically invariant 2-edge-colouring of Γ.

Let xyz be a 2-path in G' such that $c_{G'}(xy) = R$ and $c_{G'}(yz) = B$. Therefore there exists $t+1 \leqslant j \leqslant k$ such that $x, y \in J_j$. Since G is chromatically invariant and all edges other than those in $J_1, J_2, \ldots J_t$ have the same colour in G and in G', we may assume there exists $1 \leqslant i \leqslant t$ such that $z \in J_i$. Recall, $\Gamma[J_i \cup J_j] = J_i \vee J_j$. And so $zx \in E$. Therefore G' has no induced bichromatic 2-path.

Let e, f be a $2K_2$ with e red and f blue in G'. Arguing as above, we may assume there exists $t+1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant t$ such that $e \in J_j$ and $f \in J_i$. However, as above, $\Gamma[J_i \cup J_j] = J_i \vee J_j$. And so this bichromatic copy of $2K_2$ is not induced.

By Theorem 7, G' is a non-trivial chromatically invariant 2-edge-colouring of Γ. However, $G' \in \mathcal{G}_\Gamma$ and $G'[R_{G'}]$ has fewer components than $G[R_G]$. This contradicts the assumption that G was the element of $\mathcal{G}_\Gamma$ that minimized the number of components of $G[R_G]$. □

4 Roots of Chromatic Polynomials of 2-edge-coloured Graphs

The coefficients of a graph polynomial such as the chromatic polynomial capture structural properties of the graph (see e.g., Theorems 2 and 4 above for two coefficients of the chromatic polynomial of a 2-edge-coloured graph). Locating the roots of a polynomial in the complex plane gives information about its coefficients (see e.g., [10]), which motivates the study of how the roots of chromatic polynomials are distributed. A root of a chromatic polynomial of a graph is called a *chromatic root*.

See Figure 4 for the chromatic roots of all connected graphs on six vertices obtained by computer search. We call a root of a chromatic polynomial of a 2-edge-coloured graph a *monochromatic root* if the graph is monochromatic and a *bichromatic root* if the graph is bichromatic. See Figure 3 for the bichromatic roots of all connected 2-edge-coloured graphs on 6 vertices obtained by computer search. Recall the chromatic polynomial of a monochromatic graph is simply the chromatic polynomial of the underlying graph. Therefore the collection of all chromatic roots is exactly the collection of all monochromatic roots. In this section we provide results on bichromatic roots.

We begin with a study of the real roots. The real chromatic roots are always positive [17] as the coefficients of the chromatic polynomial of a graph alternate in sign and there are no real roots in $(0,1) \cup (1, \frac{32}{27}]$. In contrast we will show the bichromatic roots are dense in $\mathbb{R}$ and the collection of all rational bichromatic roots is $\mathbb{Z}$. For $n > 1$, let K_n^r and K_n^b denote monochromatic copies of K_n with red and blue edges, respectively. For a pair of 2-edge-coloured graphs $G = (\Gamma_G, R_G, B_G)$ and $H = (\Gamma_H, R_H, B_H)$, let $G \cup H$ denote the disjoint union of G and H.

Theorem 17. *Let $G = (\Gamma, R, B)$ be a 2-edge-coloured graph on n vertices such that $\chi(G) = n$. We have*

$$P(G \cup K_2^r, \lambda) = \lambda(\lambda - 1) \cdots (\lambda - n + 1)(\lambda^2 - \lambda - 2|B|).$$

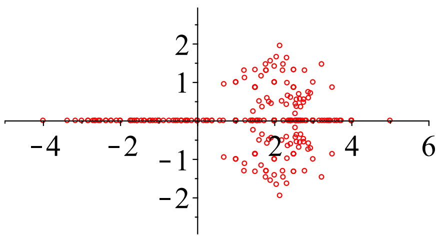


Figure 3: Bichromatic roots of all connected 2-edge-coloured graphs on six vertices

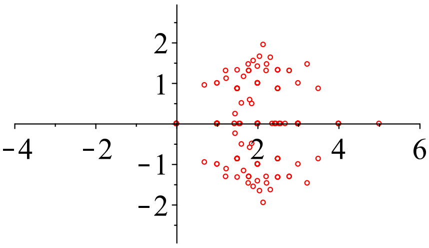


Figure 4: Chromatic roots of all connected graphs on six vertices

Proof. Let $G = (\Gamma, R, B)$ be a 2-edge-coloured graph on n vertices such that $\chi(G) = n$. For fixed $k > 0$, we construct a k-colouring of $G \cup K_2^r$ by first colouring vertices G and then those of K_2^r. As $\chi(G) = n$, any k-colouring assigns each vertex of G a unique colour. There are $k(k-1)\cdots(k-|V|+1)$ such colourings. We can then colour vertices of K_2^r with any two different colours unless there exists $b \in B$ whose ends have been assigned those two colours. Thus each $b \in B$ prohibits two possible colourings of the K_2^r. Therefore given any k-colouring of G there are $k^2 - k - 2|B|$ such colourings of K_2^r. And so

$$P(G \cup K_2^r, \lambda) = \lambda(\lambda-1)\cdots(\lambda-n+1)(\lambda^2-\lambda-2|B|). \qquad \square$$

Corollary 18. *Let $n > 1$ be an integer. We have*

$$P(K_n^b \cup K_2^r, \lambda) = \lambda(\lambda-1)\cdots(\lambda-n)(\lambda+n-1).$$

Theorem 19. *The closure of the rational bichromatic roots is $\mathbb{Z}$.*

Proof. By Theorem 1, for any 2-edge-coloured graph G, the leading coefficient of $P(G, \lambda)$ is 1. Therefore by the rational root theorem, any rational root of a 2-edge-coloured graph chromatic polynomial must be an integer. Let $m > 0$ be an integer By Corollary 18, we have

$$P(K_{m+1}^b \cup K_2^r, \lambda) = \lambda(\lambda-1)\cdots(\lambda-(m+1))(\lambda+m).$$

We observe $P(K_{m+1}^b \cup K_r^2, \lambda)$ has roots at $\lambda = -m, 0, 1, \ldots, m, m+1$. $\square$

Theorem 20. *The closure of the real bichromatic roots is $\mathbb{R}$.*

Proof. For any $r \in \mathbb{R}$ let $d(r) = r - \lfloor r \rfloor$. Furthermore let

$$A = \left\{ \frac{1-\sqrt{1+8m}}{2} : m \in \mathbb{N} \right\}.$$

Let G be a 2-edge-colouring of a complete graph such that $|B| = m$. By Theorem 17 each element of A is a real root of $P(G \cup K_2^r, \lambda)$. We first show that for any $r \in \mathbb{R}$ and $\epsilon > 0$, there exists an $a \in A$ such that $|d(r) - d(a)| < \epsilon$.

Let $f(m) = \frac{1-\sqrt{1+8m}}{2}$ and $M = \frac{2}{\epsilon^2}$. Note that $|f(m+1) - f(m)| < \epsilon$ for $m \geqslant M$. Furthermore $f(m) \to -\infty$ as $m \to \infty$. Thus for any $s \leqslant f(M)$, there exists an $m \geqslant M$ such that $f(m+1) \leqslant s \leqslant f(m) \leqslant r$. This implies $|s - f(m)| < \epsilon$. Furthermore $|d(s) - d(f(m))| < \epsilon$. By choosing $s \leqslant f(M)$ such that $d(s) = d(r)$ and let $a = f(m) \in A$. It then follows that $|d(r) - d(a)| < \epsilon$.

Let G_a be a 2-edge-coloured graph such that $P(G_a, \lambda)$ has a real root at a. Let H_a be the 2-edge-coloured graph formed the join of G_a and any 2-edge-coloured K_n, where all of the joining edges are red. We have

$$P(H_a, \lambda) = \lambda(\lambda - 1) \cdots (\lambda - n + 1) P(G_a, \lambda - n).$$

Consider $n = \lfloor r \rfloor - \lfloor a \rfloor$. As $r = d(r) + \lfloor r \rfloor$ and $a + n = d(a) + \lfloor a \rfloor + n = d(a) + \lfloor r \rfloor$ we have

$$|r - (a + n)| = |d(r) + \lfloor r \rfloor - (d(a) + \lfloor r \rfloor)| = |d(r) - d(a)| < \epsilon.$$

Thus H_a has a root at $a + n$. $\square$

We turn now to study complex roots of chromatic polynomials of 2-edge-coloured graphs. We show they may have arbitrarily large modulus. To do this we study the limit of the complex roots of a 2-edge-coloured complete bipartite graph.

Let $p_n(z) = \sum_{j=1}^{k} \alpha_j(z) \lambda_j(z)^n$. Beraha, Kahne and Weiss studied the limits of the complex roots of such functions (as arising in recurrences). They fully classified those values that occur as limits of roots of a family of polynomials. See [3] for a full statement of the Bereha-Kahane-Weiss Theorem.

A limit of roots of a family of polynomials P_n is a complex number, z, for which there are sequences of integers (n_k) and complex numbers (z_k) such that z_k is a zero of P_{n_k}, and $z_k \to z$ as $k \to \infty$. The Bereha-Kahane-Weiss Theorem requires non-degeneracy conditions: no α_i is identically 0, and $\lambda_i \neq \omega \lambda_k$ for any $i \neq k$ and any root of unity ω. The Bereha-Kahane-Weiss Theorem implies that the limit of roots of $P_n(z)$ are precisely those complex numbers z such that one of the following hold:

- one of the $|\lambda_i(z)|$ exceeds all others and $\alpha_i(z) = 0$; or
- $|\lambda_1(z)| = |\lambda_2(z)| = \cdots = \lambda_\ell(z) > |\lambda_j(z)|$ for $\ell + 1 \leqslant j \leqslant k$ for some $\ell \geqslant 2$.

Theorem 21. *Non-real bichromatic roots can have arbitrarily large modulus.*

Proof. Let $n \geqslant 5$ be an integer. Consider $K_{2,n-2}$ with partition $\{X, Y\}$ with $X = \{u, v\}$. Let $G = (K_{2,n-2}, R, B)$ such that three of the edges incident with v are blue and all other edges are red. Let $x, y, z \in Y$ be the vertices of G that are adjacent to v by a blue edge. In any k-colouring c, we have $c(u) \neq c(v)$. Further, for each $w \in Y \setminus \{x, y, z\}$ we must have $c(w) \neq c(x), c(y), c(z)$. We proceed to count the number of k-colourings of G based on the cardinality of $|\{c(x), c(y), c(z)\}|$.

When $|\{c(x), c(y), c(z)\}| = 3$, there are

$$k(k-1)(k-2)(k-3)(k-4)(k-5)^{n-5}$$

k-colourings of G. When $|\{c(x), c(y), c(z)\}| = 2$, there are

$$3k(k-1)(k-2)(k-3)(k-4)^{n-5}$$

k-colourings of G. Finally, when $|\{c(x), c(y), c(z)\}| = 1$, there are

$$k(k-1)(k-2)(k-3)^{n-5}$$

k-colourings of G. Thus

$$\begin{aligned} P(G,\lambda) &= (\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)^{n-5} + 3(\lambda-2)(\lambda-3)(\lambda-4)^{n-5} + \lambda(\lambda-1)(\lambda-2)(\lambda-3)^{n-5} \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)\left((\lambda-3)^{n-6} + 3(\lambda-4)^{n-5} + (\lambda-4)(\lambda-5)^{n-5}\right). \end{aligned}$$

Consider the polynomial $g(n,\lambda) = (\lambda-3)^{n-6} + 3(\lambda-4)^{n-5} + (\lambda-4)(\lambda-5)^{n-5}$. We may express this polynomial as:

$$p(n,z) = \alpha_1(z)(\lambda_1(z))^{n-6} + \alpha_2(z)(\lambda_2(z))^{n-5} + \alpha_3(z)(\lambda_3(z))^{n-5}.$$

Here the non-degeneracy conditions hold for $p(n,z)$. Applying the Bereha-Kahne-Weiss Theorem and setting $|\lambda_1(z)| = |\lambda_3(z)| > |\lambda_4(z)|$ we solve for $z = a+bi$ such that $|z-3| = |z-5| > |z-4|$. One can verify when $a = 4$ we have $|z-3| = |z-5|$ and $|z-5| > |z-4|$ for all values of b. Thus the curve $z = 4+bi$ is a limit of the roots for 2-edge-coloured chromatic polynomial of $K_{2,n-2}$. As there are no restrictions on b, it then follows that $P(G,\lambda)$ can have complex roots of arbitrarily large modulus. □

Consider our above $K_{2,\ell-2}$ with partition $\{X, Y\}$ with $X = \{u, v\}$ and the 2-edge-coloured graph $G_\ell = (K_{2,\ell-2}, R, B)$ such that three of the edges incident with v are blue and all other edges are red. Let $H_{n,\ell}$ be the 2-edge-coloured graph formed by G_ℓ by joining G_ℓ with a copy of K_n^r such that all joining edges are blue. Every vertex of K_n^r requires a distinct colour and the joining edges are all blue. Therefore no vertex of G can be assigned any of the n colours appearing on the vertices of K_n^r. Thus

$$P(H_{n,\ell},\lambda) = \lambda(\lambda-1)\cdots(\lambda-(n-1))P(G_\ell,\lambda-n)$$

Taken with Theorem 21, this implies that the curve $f(n,b) = 4+n+bi$ is also limit of the roots for $n \geqslant 1$. See Figures 5 and 6 for a plot of the roots of these polynomials.

From the plots in Figure 5 and Figure 6 one can see that the closure of the roots contain an infinite number of vertical curves crossing the real axis at integer values of at least 4. The real and complex chromatic roots (and hence monochromatic roots) are dense in the complex plane [19]. It remains to be seen if bichromatic roots are dense in the complex plane.

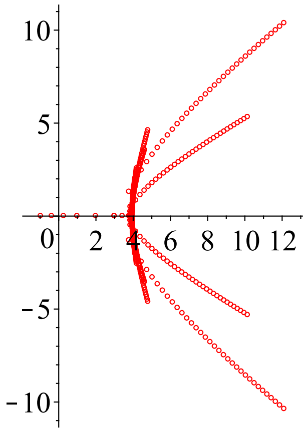


Figure 5: Complex chromatic roots of the 2-edge coloured $K_{2,n-2}$

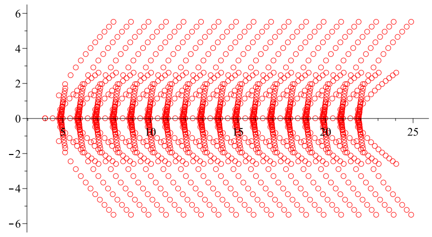


Figure 6: Complex chromatic roots of $H_{n,\ell}$ for $\ell = 6, \ldots, 18$, $n = 1, \ldots, 18$

5 Further Remarks

The results and methods in Section 2 closely resemble results and methods for the chromatic polynomial – Equations 1 and 2 both hold for the chromatic polynomial of a graph. We note, however, that the standard *delete and contract* technique for the chromatic polynomial of a graph does not apply, in general, for 2-edge-coloured graphs. Deleting a coloured edge, in some sense, forgets the colour of the adjacency between a pair of vertices – important information for a vertex colouring a 2-edge-coloured graph. For example, proceeding via deletion/contraction for a bichromatic 2-path results in polynomial $x(x-1)^2$, instead of the correct polynomial $(x)(x-1)(x-2)$. Thus the chromatic polynomial of a 2-edge-coloured graph does not arise as an evaluation of the Tutte polynomial of its underlying graph. A modified notion of deletion/contraction would be needed, and perhaps an augmenting of the 2-edge-coloured graphs by weights, for it to be possible for the chromatic polynomial of a 2-edge-coloured graph to satisfy a full deletion-contraction recurrence, applicable to any edge. (Compare the vertex weights that enable a deletion-contraction relation to be derived for calculating the U-polynomial of a graph via the W-polynomial of a vertex-weighted graph in [14])

We note however that in a mixed 2-edge-coloured graphs one can apply the standard *delete and contract* technique to the set F to reduce the chromatic polynomial of a mixed 2-edge-coloured graph to a sum of chromatic polynomials of 2-edge-coloured graphs.

The results and methods in Section 2 closely mirror those for the oriented chromatic polynomial in [7]. Such a phenomenon has been observed in past study of the chromatic number oriented graphs and 2-edge-coloured graphs, thus motivating our study of the chromatic polynomial of 2-edge-coloured graphs. In [16] Raspaud and Sopena give an upper bound for the chromatic number of an orientation of a planar graph. And in [1] Alon and Marshall use the same techniques to derive the same upper bound for the chromatic number of a 2-edge-coloured planar graph. In this latter work, Alon and Marshall profess the similarity of their techniques to those appearing in [16], yet see no way to derive one set of results from the other. In the following years Nešetřil and Raspaud [13] showed these results were in fact special cases of a more general result for (m,n)-coloured mixed graphs – graphs in which there are m different arc colours and n different edge colours. Note this use of the term "mixed" is different than that introduced above.

As opposed to the uncoloured adjacency permitted above, every adjacency in an (m,n)-coloured mixed graph is assigned a colour. Ordinary graphs are $(0,1)$-mixed graphs, oriented graphs are $(1,0)$-mixed graphs and 2-edge-coloured graphs $(0,2)$-mixed graphs. By way of homomorphism, one can define, for each $(m,n) \neq (0,0)$, a notion of proper vertex colouring for (m,n)-mixed graphs that generalizes graph colouring, oriented graph colouring and 2-edge-coloured graph colouring.

As our results in Section 2 closely mirror those in [7] for oriented graphs, we expect that the results in Section 2 are in fact special cases of a more general result for the, to be defined, chromatic polynomial of an (m,n)-mixed graph. Showing such a result would require successfully generalizing the notions of obstructing arcs/edges as well as the notions of 2-dipath and bichromatic 2-path. This latter problem is considered in [2].

Unlike past work unifying oriented and 2-edge-coloured graphs, we do not see how to take an approach that is common to both types of graph when considering chromatic invariance. The classification of chromatically invariant oriented graphs given in [7] bears little resemblance to the statement of Theorem 16. This is due to the fact that the family of graphs that admit an orientation with no induced 2-dipath differs drastically from the family that admit a non-trivial 2-edge-colouring with no induced bichromatic 2-path (see [9] and [8])

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