Roots of Descent Polynomials and an Algebraic Inequality on Hook Lengths

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Abstract

By reinterpreting the descent polynomial as a function enumerating standard Young tableaux of a ribbon shape, we use Naruse's hook-length formula to express the descent polynomial as a product of two polynomials: one is a trivial part which is a product of linear factors, and the other comes from the excitation factor of Naruse's formula. We expand the excitation factor positively in a Newton basis which arises naturally from Naruse's formula. Under this expansion, each coefficient is the weight of a certain combinatorial object, which we introduce in this paper. We introduce and prove the "Slice and Push Inequality", which compares the weights of such combinatorial objects. As a consequence, we establish a proof of a conjecture by Diaz-Lopez et al. that bounds the roots of descent polynomials.

Mathematics Subject Classifications: 05A17, 05A20

1 Introduction

1.1 Using Naruse's hook-length formula to develop combinatorial inequalities

In 2014, Naruse [16] announced a formula for $f^{\lambda/\mu}$, the number of standard Young tableaux of skew shape λ/μ. Later known as *Naruse's (hook-length) formula* in the literature, the formula expresses $f^{\lambda/\mu}$ as a sum over combinatorial objects called *excited (Young) diagrams*. These excited diagrams were introduced by Ikeda and Naruse [9] in the context of equivariant Schubert calculus a few years before Naruse's discovery of the skew-shape hook-length formula.

Since the inception of the hook-length formula by Frame–Robinson–Thrall [8], many have studied, re-proved, and generalized the formula. Similarly, many combinatorialists have been investigating Naruse's formula. Morales, Pak, and Panova have developed a

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series of papers studying the formula, in which new proofs, q-analogues, and many new properties of Naruse's formula have been presented; see e.g. [14]. Konvalinka [10] has also given a bijective proof of Naruse's formula. It is worth mentioning that, in addition to the hook-length formula for skew straight shapes, Naruse [16] also announced formulas for skew shifted shapes (types B and D), and for these, Konvalinka [11] gave bijective proofs as well.

For us, one of the main advantageous attributes of Naruse's hook-length formula is that it is cancellation-free. In [13], Morales, Pak, and Panova have exploited this positive summation property of Naruse's formula to establish asymptotic bounds on the number of standard Young tableaux of skew shapes. Combining the variational principle and Naruse's formula, Morales, Pak, and Tassy [15] prove precise limiting behaviors of the number of standard Young tableaux of shew shapes, proving and generalizing conjectures in [13]. From this point of view, Naruse's formula is an efficient tool to develop algebraic inequalities related to combinatorial objects.

1.2 The Slice and Push Inequality

In this paper, we present combinatorial objects which we call "○□-diagrams." These objects are closely related to the excited diagrams in Naruse's hook-length formula. By exploiting the cancellation-free property of Naruse's formula, we introduce and prove an algebraic inequality on ○□-diagrams, which we call the "Slice and Push Inequality" (Theorem 9). We briefly describe the diagrams and the inequality as follows.

The excitation factor from Naruse's hook-length formula (see Section 2 for details) is a sum over all excited diagrams of the weight of each excited diagram. The weight of our ○□-diagram is similar to the excitation factor from Naruse's formula: instead of allowing each cell to be excited many times as in Naruse's excitation, our ○□-diagram allows each ○-type cell to be excited *at most once*, and disallows any movement of the □-type cell. Similar to Naruse's excitation factor, this ○□ rule produces a collection of diagrams, and then we sum the weights of the diagrams over this collection to obtain the weight of a ○□-diagram (see Section 3 for details).

The Slice and Push Inequality says that if we start with a ○□-diagram, consider any vertical line between any two consecutive columns, push all the ○-type cells to the right of the line one unit to the right, change all the pushed cells to □, and obtain a new ○□-diagram, then the weight of the original diagram is greater than or equal to the weight of the new diagram. In the notation of Theorem 9, this bound can be written concisely:

$$\mathrm{wt}(D;F) \geqslant \mathrm{wt}(D|_k;\ F \sqcup ({|_k}D)^{\rightarrow}).$$

See Figure 6 for a depiction of the Slice and Push Inequality. Also, see Example 11 for an explicit calculation. The Slice and Push Inequality has an application to the studies of descent polynomials, as we shall see below.

1.3 Descent polynomials

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, \ldots, n\}$. For a permutation $\pi = \pi_1 \cdots \pi_n$, the *descent set* of π is the set of positions $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. Given a finite set of positive integers $I \subseteq \mathbb{Z}_{>0}$, the *descent polynomial* $d_I(z)$ is the unique polynomial such that $d_I(n)$ is the number of elements of $\mathfrak{S}_n$ whose descent set is I (assuming $n > \max(I \cup \{0\})$). MacMahon introduced the descent polynomial in [12], where polynomiality of this function was proved by an inclusion-exclusion argument. One can also deduce this from Naruse's formula, as we show in Subsection 2.3.

In this paper, we reinterpret descent polynomials as functions enumerating standard Young tableaux of a family of skew shapes known as *ribbons*. Using Naruse's formula, we find that the descent polynomial can be written as a product of two polynomials: the "trivial" part and the "excitation factor" (see Subsection 2.4 for details). The trivial part is easy to analyze, as it is simply a product of linear polynomials. Interesting combinatorics of the descent polynomial is then encoded in the excitation factor, which under Naruse's formula is expanded naturally in a certain Newton basis (see Subsection 2.2).

Oftentimes in algebraic combinatorics, as one expresses a polynomial in a certain linear basis, a natural question which arises is whether the coefficients in the expression are positive. In our situation however, the positive answer to this question is immediate: it follows by construction that the coefficients obtained in the expansion using Naruse's formula are weights of certain ○□-diagrams, and thus are positive. Furthermore, we are able to use the Slice and Push Inequality to derive bounds between these coefficients (see Proposition 6 and Corollary 13).

Diaz-Lopez et al. [7] used a recurrence for descent polynomials to prove that $d_I(z)$ is a degree $m = \max(I \cup \{0\})$ polynomial with $d_I(i) = 0$ for all $i \in I$. We let $|z|$, $\Re(z)$, and $\Im(z)$ denote the complex modulus, the real part, and the imaginary part of z, respectively. Diaz-Lopez et al. [7] conjectured the following bounds on the roots of the descent polynomial.

Theorem 1 ([7], Conjecture 4.3). *If z_0 is a complex number such that $d_I(z_0) = 0$, then*

1. *$|z_0| \leqslant m$, and*
2. *$\Re(z_0) \geqslant -1$.*

Stronger inequalities in the special case where $I = \{m\}$ were proved in [7, Theorem 4.7, Corollary 4.8]. A plot of the roots of the descent polynomial $d_{\{10\}}$ is given in Figure 1. The shaded region is determined by the inequalities in Theorem 1.

By expanding the descent polynomial in the Newton basis from Naruse's formula, analyzing the relations between the coefficients from the expansion, and invoking some technical lemmas which we prove in the appendices, we obtain a proof of Theorem 1 (see Section 4).

We remark that Bencs [3] has independently discovered a separate proof of Theorem 1. Bencs' proof relies on inequalities satisfied by the coefficients of $d_I(z)$ in different bases for polynomials of degree $\leqslant m$ than the ones we consider.

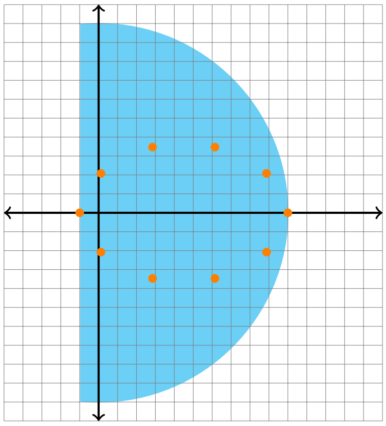

Figure 1: A plot of the roots of $d_{\{10\}}(z)$

1.4 Outline

This paper is organized as follows. In Section 2, we review Naruse's hook-length formula. We establish a connection between Naruse's formula and the descent polynomial. In Section 3, we investigate the Slice and Push Inequality, and prove inequalities between the coefficients obtained from expanding the descent polynomial. In Section 4, we give a proof of the conjecture of Diaz-Lopez, Harris, Insko, Omar, and Sagan. In Section 5, we collect potential future research directions. Technical analytic lemmas used in the proof of the conjecture of Diaz-Lopez et al. are proved in Appendices A and B.

2 Naruse's formula

2.1 Excited diagrams

We use the convention that a partition is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ such that $\lim_{i\to\infty} \lambda_i = 0$. Its conjugate partition is denoted $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$. A *Young diagram* $\mathbb{D}(\lambda)$ for a partition λ is a left-justified array of cells, with λ_1 cells in the first (top) row, λ_2 cells in the next row, and so on. For a diagram $\mathbb{D}$, we let $c_{i,j}$ be the cell in the i-th row and the j-th column. If $\mu \subseteq \lambda$, we draw the *skew Young diagram* for λ/μ by shading in the cells contained in μ. When considering a fixed skew shape λ/μ, we typically let n be the *size* of the shape; i.e.,

$$n = |\lambda| - |\mu| = \sum_{i=1}^{\infty} (\lambda_i - \mu_i).$$

A *standard Young tableau* of shape λ/μ is a bijective filling of the unshaded cells of λ with numbers $\{1, \ldots, n\}$ such that values increase to the right along any row and going down along any column. The six standard tableaux of shape $(3,3,3)/(2,2)$ are shown in Figure 2 (left).

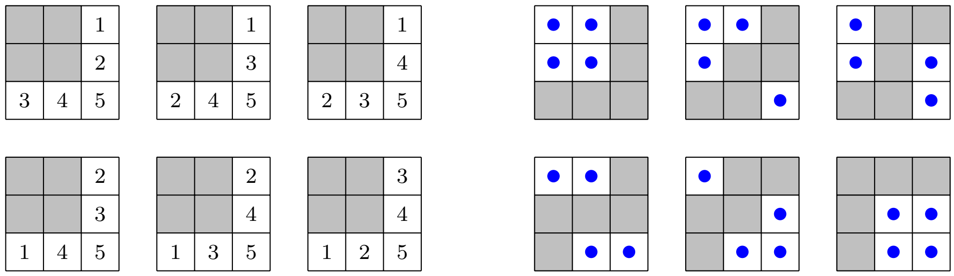

Figure 2: (left) Standard tableaux (right) Excited diagrams

Let $f^{\lambda/\mu}$ be the number of standard Young tableaux of shape λ/μ, setting $f^{\lambda/\mu} = f^{\lambda}$ if $\mu = \overline{0} := (0, 0, \ldots)$. For a cell $c \in \mathbb{D}(\lambda)$, its *hook length* $h(c) = h(c; \lambda)$ is the sum of the number of cells that lie in the same row, weakly to the right of c, and the number of cells that lie in the same column, strictly below c. The Frame–Robinson–Thrall "hook length formula" is a product formula for the number of standard Young tableaux of a (non-skew) shape:

$$f^{\lambda} = |\lambda|! \prod_{c \in \mathbb{D}(\lambda)} \frac{1}{h(c)}.$$

For some skew shapes λ/μ, the number $f^{\lambda/\mu}$ has large prime factors, removing the possibility for such a simple product formula in general. We refer to the recent survey [1] for a wide array of formulas for $f^{\lambda/\mu}$.

We use a formula recently discovered by Naruse [16], recalled in Theorem 2. Fix a skew shape λ/μ. Divide the cells of $\mathbb{D}(\lambda)$ into collections $\ldots, X_{-1}, X_0, X_1, \ldots$ according to their *contents*; that is,

$$X_k = \{c_{i,j} \in \mathbb{D}(\lambda) \mid k = j - i\}.$$

We consider cells to be partially ordered so that $c \leqslant c'$ if c is weakly northwest of c'; that is, $c_{i,j} \leqslant c_{i',j'}$ if $i \leqslant i'$ and $j \leqslant j'$. An *excited diagram* D of type λ/μ is a subset of cells of $\mathbb{D}(\lambda)$ such that

- there exists a bijection $\eta : \mathbb{D}(\mu) \to D$ with $\eta(c_{i,j}) \in X_{j-i}$ for all $c_{i,j} \in \mathbb{D}(\mu)$, and
- for each $k \in \mathbb{Z}$, the restriction of η to $\mathbb{D}(\mu) \cap (X_k \cup X_{k+1})$ is order-preserving.

Equivalently, one may imagine each excited diagram as a result of a (possibly empty) sequence of local moves, defined as follows. Imagine that we initially have one pebble in each cell of μ. A local move consists of moving a pebble to its immediate southeast cell as long as the cell to the east, the one to the south, and the one to the southeast are all unoccupied. We let $\mathbb{E}(\lambda/\mu)$ be the set of excited diagrams of type λ/μ. The six excited diagrams of shape $(3, 3, 3)/(2, 2)$ are shown in Figure 2 (right).

Theorem 2 (Naruse [16])**.** *For a skew shape λ/μ of size n,*

$$f^{\lambda/\mu} = \frac{n!}{\prod_{c \in \mathbb{D}(\lambda)} h(c)} \sum_{D \in \mathbb{E}(\lambda/\mu)} \prod_{c' \in D} h(c').$$

Applying this formula to the shape $(3,3,3)/(2,2)$, we get the identity

$$6 = \frac{5!}{5 \cdot 4^2 \cdot 3^3 \cdot 2^2 \cdot 1} \left(5 \cdot 4^2 \cdot 3 + 5 \cdot 4^2 \cdot 1 + 5 \cdot 4 \cdot 2 \cdot 1 + 5 \cdot 4 \cdot 2 \cdot 1 + 5 \cdot 2^2 \cdot 1 + 3 \cdot 2^2 \cdot 1\right).$$

For proofs of the formula and pointers to the literature, we recommend [14].

2.2 Skew shapes with varying first row

For $t \in \mathbb{Z}_{>0}$, we let $\lambda^{(t)}$ be the partition obtained from λ by replacing the first part λ_1 with $\lambda_1 + t - 1$. For a fixed shape λ/μ, we define the size of $\lambda^{(t)}/\mu$ to be $n + t - 1$ so that n does not depend on t. We consider the function

$$p(t; \lambda/\mu) = f^{\lambda^{(t)}/\mu}.$$

If the skew shape λ/μ is understood, we simply write $p(t)$ for this function. For the remainder of this section, we will assume that $\lambda_1 = \lambda_2$. We are free to make this assumption since $p(t; \lambda^{(u)}/\mu) = p(t + u - 1; \lambda/\mu)$ for $u \geqslant 1$. To clarify, we are *not* making λ_2 equal to $\lambda_1 + t - 1$. Rather, we mean that the first two rows in the partition $\lambda^{(1)}$ have the same length, and in $\lambda^{(t)}$ we extend only the first row.

We fix some additional parameters:

- $r = \lambda_1$,
- $s = \mu_1$, and
- $\alpha_i = h(c_{1,i}; \lambda) - 1$ for $1 \leqslant i \leqslant r$.

We are going to use Theorem 2 to give a formula for $p(t)$ in Lemma 5. Before doing so, we first make a few observations.

The only cells whose hook lengths vary with t are those in the first row. For each excited diagram $D \in \mathbb{E}(\lambda^{(t)}/\mu)$, let $\overline{D}$ be the subdiagram of D obtained by removing all cells from the first row of $\mathbb{D}(\lambda^{(t)})$.

Observe that the shapes $\lambda^{(t)}/\mu$ and λ/μ have the *same* set of excited diagrams $\mathbb{E}(\lambda/\mu)$. Furthermore, if D is an excited diagram with d cells in the first row, then its first row must be $\{c_{1,1}, \ldots, c_{1,d}\}$. We let $E(t)$ be the *excitation factor* of $\lambda^{(t)}/\mu$; i.e.,

$$\begin{aligned} E(t) &= \sum_{D \in \mathbb{E}(\lambda^{(t)}/\mu)} \left(\prod_{c \in \overline{D}} h(c) \right) \left(\prod_{c_{1,i} \in D} (t + \alpha_i) \right) \\ &= \sum_{d=0}^{s} \left(\prod_{i=1}^{d} (t + \alpha_i) \right) \left(\sum_{\substack{D \in \mathbb{E}(\lambda/\mu) \\ \#(D - \overline{D}) = d}} \prod_{c \in \overline{D}} h(c) \right) \end{aligned}$$

When $d = s$, the inner sum is nonempty, so $E(t)$ has degree *exactly* s. Let $C_0, \ldots, C_s$ be the nonnegative integers for which

$$E(t) = C_0(t+\alpha_1)\cdots(t+\alpha_s) + C_1(t+\alpha_1)\cdots(t+\alpha_{s-1}) + \cdots + C_{s-1}(t+\alpha_1) + C_s$$

so that $C_d(t+\alpha_1)\cdots(t+\alpha_{s-d})$ is the "contribution" to the excitation factor from those excited diagrams with $s-d$ cells in the first row.

Remark 3. The list of polynomials $1,\ t+\alpha_1,\ (t+\alpha_1)(t+\alpha_2), \ldots,\ (t+\alpha_1)\cdots(t+\alpha_s)$ is an example of a Newton basis for the space of polynomials in t of degree $\leqslant s$. A sequence of polynomials $(p_k(t))_{k=0}^s$ is a *Newton basis* if there exist complex numbers $\beta_1, \ldots, \beta_s \in \mathbb{C}$ and $\lambda_0, \ldots, \lambda_s \in \mathbb{C} \setminus \{0\}$ such that $p_k(t) = \lambda_k \prod_{i=1}^k (t+\beta_i)$, for $k = 0, 1, \ldots, s$. Using inequalities on the coefficients of polynomials with respect to a Newton basis is a common approach to proving bounds on the roots of those polynomials. This approach is taken for bounding the roots of descent polynomials in [3] and [7] using the falling factorial basis.

Remark 4. After the preprint version of this present paper became available, Cai [6] investigated this sequence $(C_0, C_1, \ldots, C_s)$ of coefficients. Cai calls these coefficients "Naruse-Newton coefficients," and examines various properties, such as log-concavity, unimodality, and limiting behavior. In Appendix A of [6], Cai gives tables listing the values of the coefficients for all nonempty descent sets $I \subseteq \{1, 2, \ldots, 7\}$.

We now calculate

$$p(t) = (n+t-1)! \left(\prod_{c \in \overline{\mathbb{D}(\lambda)}} \frac{1}{h(c)} \right) \left(\prod_{i=1}^r \frac{1}{t+\alpha_i} \right) \frac{1}{(t-1)!} E(t). \tag{1}$$

For convenience, we will assume that $\mathbb{D}(\lambda/\mu)$ is connected; i.e., $\mu_i < \lambda_{i+1}$ whenever $\mu_i \neq 0$. In particular, this implies that $\alpha_1 \leqslant n-1$ holds. By canceling common factors in the expression above, we obtain a useful factorization of the polynomial $p(t)$ in Lemma 5. We collect some of the factors that do not depend on t as a constant C. The first, the third, and the fourth factors combine to be a polynomial in t.

Lemma 5. *If $\mathbb{D}(\lambda/\mu)$ is connected, then there exists a positive real number C not depending on t such that*

$$p(t) = C \cdot E(t) \prod_{\substack{\beta \in \{0,1,\ldots,n-1\} \\ \beta \notin \{\alpha_1, \ldots, \alpha_r\}}} (t+\beta).$$

Lemma 5 allows us to reduce the problem of bounding the roots of $p(t)$ to bounding the roots of the lower degree polynomial $E(t)$, which we do in Section 4.

2.3 Ribbons and descent polynomials

A *ribbon* is a (nonempty) connected skew Young diagram $\mathbb{D}$ that does not contain a 2×2 block of cells. If T is a standard filling of $\mathbb{D}$, then T determines a permutation $\pi(T) = \pi_1 \cdots \pi_n$ whose entries appear in order along the ribbon, starting from the bottom left corner to the upper right. The positions of the descents of $\pi(T)$ are determined by the shape of $\mathbb{D}$, as illustrated in Figure 3, where the shape of the ribbon forces the permutation $\pi(T)$ to have descent set $I = \{3, 5, 8, 9, 11\}$. Namely, there is a descent at i if and only if the i-th cell of the ribbon is below the $(i+1)$-st cell. Conversely, if $I \subseteq [n-1]$ we may construct a ribbon $\mathbb{D}$ for which the permutations π with descent set I are of the form $\pi = \pi(T)$ for some standard filling T of $\mathbb{D}$. Recall that when I is a descent set, we let m denote $\max(I \cup \{0\})$.

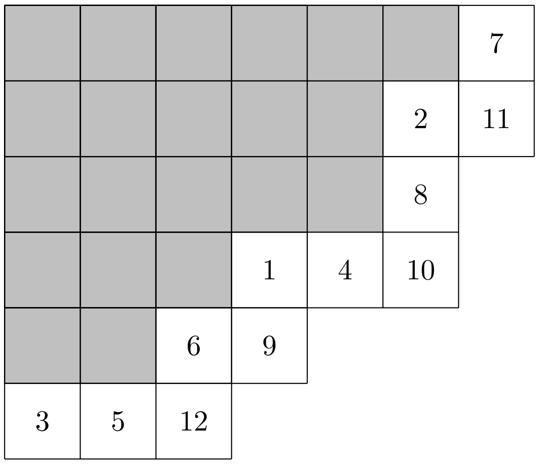

Figure 3: A tableau of ribbon shape

Now suppose $\mathbb{D}(\lambda/\mu)$ is a ribbon. Combined with the assumption that $\lambda_1 = \lambda_2$, this implies $\mu_1 = \lambda_1 - 1$. Hence, $s = r - 1,\ m = n - 1$, and the polynomial $p(t)$ has degree m.

In this dictionary between standard fillings of a ribbon and permutations with a given descent set, the addition of cells to the first row of $\mathbb{D}$ corresponds to taking longer permutations without changing the descent set. So if $\mathbb{D}(\lambda/\mu)$ is a ribbon shape corresponding to the descent set I, we have the identity of polynomials $p(t-m) = d_I(t)$. Furthermore, one may observe that the ascents in the permutation are either in $\{m+1, m+2, \ldots\}$ or $\{m - \alpha_i :\ i \in \{2, \ldots, r\}\}$.

To prove suitable bounds on the roots of the excitation factor $E(t)$, we show that the sequence of coefficients $(C_0, C_1, \ldots, C_s)$ does not "grow too quickly," in the following sense.

Proposition 6. *For ribbons, with $(C_0, \ldots, C_s)$ defined as above,*

$$\frac{C_0}{0!} \geqslant \frac{C_1}{1!} \geqslant \frac{C_2}{2!} \geqslant \cdots \geqslant \frac{C_s}{s!}.$$

The proof of Proposition 6 will be given in Section 3. These inequalities are used in Section 4 to prove Theorem 1.

Let us add two remarks here. First, observe that for any descent set I, when we fill in the hook lengths inside the cells of the Young diagram of $\lambda^{(t)}$, the hook lengths that appear in the cells on the first column but strictly below the first row are exactly the elements of I. See, for example, Figure 4, where the descent set is $I = \{3, 5\}$. Note the hook lengths of 5 and 3 on the first column. Second, it is known that the descent polynomial $d(I; x)$ (using the notation of [7]) may have some negative integer coefficients in general, when we expand in the usual polynomial basis $1, x, x^2, \ldots$. By the analysis using Naruse's formula we show here, we obtain as an immediate consequence that for any nonempty descent set I, all the coefficients of $d(I; x + \alpha_1)$ in the usual polynomial basis $1, x, x^2, \ldots$ are strictly positive. (Recall that when I is nonempty, $\alpha_1 = m = \max(I)$.)

2.4 Computing the descent polynomial via Naruse's formula

In this subsection, we show how to use Naruse's hook-length formula to compute the descent polynomial. Fix a descent set I. From the Young diagram of the corresponding skew shape λ/μ, we determine $\alpha_1, \alpha_2, \ldots, \alpha_s$. From Equation (1) in Subsection 2.2, we write the descent polynomial $p(t)$ as a product

$$p(t) = T(t) \cdot E(t),$$

where $T(t)$ is the *trivial* part, and $E(t)$ is the excitation factor. The trivial part has an easy description:

$$T(t) = \frac{1}{B_\lambda} \cdot \prod_{i \in I} (t + \alpha_1 - i),$$

where B_λ denotes the product of hook lengths below the first row of λ. The excitation factor $E(t)$ is computed using the combinatorial formula in Subsection 2.2.

We remark that our notation $p(t)$ for the descent polynomial is slightly different from the notation $d(I; -)$, used in [7]. To compute explicitly the polynomial $d(I; N)$, the number of permutations in $\mathfrak{S}_N$ whose descent sets are exactly I, we simply note the translation $N = t + \alpha_1$. As a result, we have

$$d(I; N) = p(N - \alpha_1).$$

Using our formula for $p(t)$ above, we obtain the desired descent polynomial.

One interesting immediate consequence of the expression $p(t) = T(t) \cdot E(t)$ is that we may recover Lemma 3.8 of [7]. Observe that $d(I; 0) = p(-\alpha_1) = T(-\alpha_1) \cdot E(-\alpha_1)$. We know that

$$T(-\alpha_1) = \frac{1}{B_\lambda} \cdot \prod_{i \in I} (-i) = \frac{(-1)^{\#I} \cdot \prod_{i \in I} i}{B_\lambda}$$

On the other hand, $E(-\alpha_1) = C_s$, which is the product of hook lengths below the first row, strictly to the right of the first column, of λ. Since the hook lengths below the first row on the first column of λ are exactly the elements of I, we find that

$$E(-\alpha_1) = \frac{B_\lambda}{\prod_{i \in I} i},$$

whence $d(I; 0) = (-1)^{\#I}$, which is precisely Lemma 3.8 of [7].

On a similar note, we can establish another result from [7] as follows. Using Lemma 5, we see that an integer $\gamma \in \{-m, \ldots, 0\}$ is among the roots of $p(t)$ whenever $\gamma \neq -\alpha_i$ for any $i \in [r]$. This means γ is a root of $p(t)$ whenever $m + \gamma$ is in I. These are precisely the roots of the descent polynomials indicated in Theorem 4.1 of [7].

$t+5$	$t+4$	$t+3$	$t+1$	$t-1$	$t-2$	$\cdots$	2	1
5	4	3	1					
3	2	1						

Figure 4: The hook lengths for $\lambda^{(t)}$ when $\lambda = (4, 4, 3)$

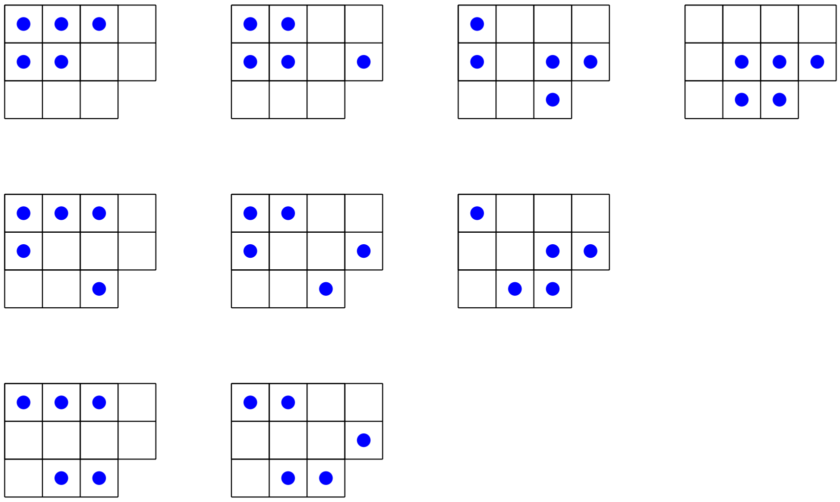

Figure 5: The nine excited diagrams of λ/μ when $\lambda = (4, 4, 3)$ and $\mu = (3, 2)$

Example 7. In this example, we compute the descent polynomial for the descent set $I = \{3, 5\}$. The ribbon corresponding to $I = \{3, 5\}$ is λ/μ, where $\lambda = (4, 4, 3)$ and $\mu = (3, 2)$. In Figure 4, the partition $\lambda^{(t)}$ is shown along with the hook length in each cell. The five cells of μ are shaded. In this case, we have $r = 4$ and $s = 3$. The $\boldsymbol{\alpha}$-vector is $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (5, 4, 3, 1)$. In general, the hook lengths on the first row of $\lambda^{(t)}$ are always

$$t + \alpha_1, t + \alpha_2, \ldots, t + \alpha_r, t - 1, t - 2, \ldots, 3, 2, 1.$$

There are 9 excited diagrams of λ/μ (see Figure 5). Three of them (on the first column in Figure 5) contribute scalar multiples of $(t + 5)(t + 4)(t + 3)$. Another three (on the second column) contribute multiples of $(t+5)(t+4)$. Two (on the third column) contribute multiples of $(t + 5)$. The other one contributes a scalar. We write

$$E(t) = C_0(t + 5)(t + 4)(t + 3) + C_1(t + 5)(t + 4) + C_2(t + 5) + C_3.$$

These coefficients can be computed directly:

$$\begin{aligned}
C_0 &= 5 \cdot 4 + 5 \cdot 1 + 2 \cdot 1 = 27, \\
C_1 &= (5 \cdot 4 + 5 \cdot 1 + 2 \cdot 1) \cdot 1 = 27, \\
C_2 &= (5 + 2) \cdot 3 \cdot 1 \cdot 1 = 21, \\
C_3 &= 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 24.
\end{aligned}$$

The trivial part of the descent polynomial is

$$T(t) = \frac{1}{5 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 1 \cdot 1} \cdot t(t+2) = \frac{t(t+2)}{360}.$$

Therefore, the descent polynomial is

$$p(t) = \frac{t(t+2)}{360} \left(27(t+5)(t+4)(t+3) + 27(t+5)(t+4) + 21(t+5) + 24\right).$$

To translate to the notation of [7], we simply use the shift $N = t + 5$ to obtain

$$d(\{3,5\}; N) = \frac{(N-5)(N-3)}{360} \left(27N(N-1)(N-2) + 27N(N-1) + 21N + 24\right).$$

For readers interested in computational data, we recommend Cai's work [6], especially the table of coefficients in the appendix.

3 The Slice and Push Inequality

To prove Proposition 6, we apply an inductive argument to a slightly more general statement. For this, we consider a more general class of subdiagrams of a Young diagram $\mathbb{D}(\lambda)$. Recall that we divide the diagram $\mathbb{D}(\lambda)$ into diagonals $\ldots, X_{-1}, X_0, X_1, \ldots$ by their contents.

Recall that a *multiset* is, informally, a set in which each element may appear more than once. We say a multiset D is a *multi-subset* of a set X if every element of D is in X. For instance, $\{2, 3, 3\}$ is a multi-subset of $\{1, 2, 3\}$. If D and F are multisets, the *multiset union* $D \sqcup F$ is the multiset where the multiplicity of each element is the sum of its multiplicities in D and F; e.g., $\{2, 3\} \sqcup \{3\} = \{2, 3, 3\}$. A *subdiagram* of a diagram $\mathbb{D}(\lambda)$ is a finite subset of cells of $\mathbb{D}(\lambda)$. More generally, if D is a finite multi-subset of cells of $\mathbb{D}(\lambda)$, we call it a *multi-subdiagram* of $\mathbb{D}(\lambda)$. The *weight* $\mathrm{wt}(D)$ of a multi-subdiagram is the product of the hook lengths of its cells taken with multiplicity. The following formula is easy to verify.

Lemma 8. *If F is any multi-subdiagram of $\mathbb{D}(\lambda)$, and $\mathbb{D}(\lambda)$ contains the collection of cells $\{c_{i,j}, c_{i',j}, c_{i,j'}, c_{i',j'}\}$, then*

$$\mathrm{wt}(F \sqcup \{c_{i,j}\}) + \mathrm{wt}(F \sqcup \{c_{i',j'}\}) \;=\; \mathrm{wt}(F \sqcup \{c_{i,j'}\}) + \mathrm{wt}(F \sqcup \{c_{i',j}\}).$$

We consider pairs $(D;F)$ where D is a subdiagram and F is a multi-subdiagram of $\mathbb{D}(\lambda)$ such that for every cell $c_{i,j} \in D$, the cell $c_{i+1,j+1}$ exists in $\mathbb{D}(\lambda)$. We refer to this pair as a ○□-*diagram*, depicted by labeling each cell in D by a circle and each cell in F by a square. The *weight* of a ○□-diagram $(D;F)$ is the sum of the weights of the multi-subdiagrams $D' \sqcup F$, where the sum ranges over diagrams D' such that

- there exists a bijection $\eta : D \to D'$ with $\eta(c_{i,j}) \in \{c_{i,j},\ c_{i+1,j+1}\}$, and
- for each k, the restriction of η to $D \cap (X_k \cup X_{k+1})$ is order-preserving.

In other words, one is allowed to move a circle up to one step southeast as long as it does not interfere with the cells from neighboring diagonals. We let $\mathrm{wt}(D;F)$ be the weight of the ○□-diagram $(D;F)$.

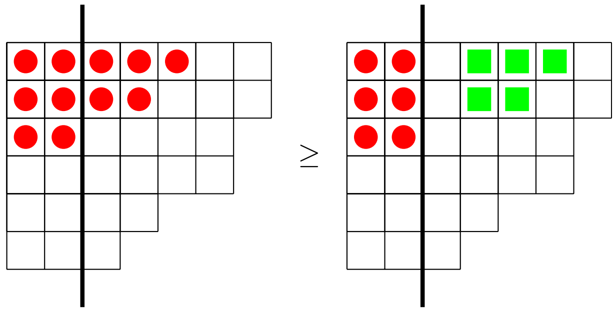

Figure 6: A depiction of the Slice and Push Inequality

We will use the following notations for constructing new diagrams from D when D is a subset of cells of $\mathbb{D}(\lambda)$.

- Let $|_k D$ be the subdiagram of D obtained by removing the first k columns from $\mathbb{D}(\lambda)$, and let $D|_k$ be the subdiagram of D contained in the first k columns of $\mathbb{D}(\lambda)$. We think of the bar $|_k$ as a "knife" placed between columns k and $k+1$ where $D|_k$ is the portion of the diagram to the left and $|_k D$ is the portion to the right of the knife.
- Similarly, let $\overline{D}^i$ be the subdiagram obtained by removing the first i rows, and let $\underline{D}_i$ the subdiagram contained in the first i rows of $\mathbb{D}(\lambda)$. These subdiagrams are constructed by placing the knife horizontally instead of vertically.
- Let $D^{\rightarrow}$ be the diagram obtained by replacing each cell $c_{i,j}$ in D with $c_{i,j+1}$; that is, we "push" every cell one step to the right.
- Similarly, let $D^{\searrow}$ be the diagram with every cell pushed one step down and to the right.
- For a diagram D, let i_0 (resp. j_0) be the first row (resp. column) occupied by at least one cell in D, and set

$$D^{\circ} := \{c_{i-i_0+1,j-j_0+1} \mid c_{i,j} \in D\}.$$

In other words, D° is the diagram obtained by translating D as far north and west as possible while remaining inside $\mathbb{D}(\lambda)$.

Theorem 9 (The Slice and Push Inequality)**.** *If $(D;F)$ is a ○□-diagram such that $D^\circ = \mathbb{D}(\mu)$ for some nonempty partition $\mu \subseteq \lambda$, then for any k,*

$$\mathrm{wt}(D;F) \geqslant \mathrm{wt}(D|_k;\ F \sqcup (|_k D)^{\rightarrow}). \tag{2}$$

We remark that the assumption $D^\circ = \mathbb{D}(\mu)$ in the theorem requires the cells of D to "look like" the cells from a straight shape μ.

Proof. We first observe that the multi-subdiagram F contributes the same multiplicative factor to each side of the inequality, so we may assume $F = \emptyset$. We also assume that $|_k D$ only contains cells in a single column, namely the $(k+1)$-st column. To deduce the inequality for an arbitrary number of columns in $|_k D$, we may iteratively slice off and push the rightmost column several times.

Let μ be the partition for which $D^\circ = \mathbb{D}(\mu)$. We proceed by induction on $|\mu|$. If D only contains cells in column $k+1$, then

$$\mathrm{wt}(D;\emptyset) > \mathrm{wt}(\emptyset;D) > \mathrm{wt}(\emptyset;D^{\rightarrow}) = \mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow}).$$

Suppose $i+1$ is the last row occupied by D and j is the first column occupied by D. We claim that (Figure 7)

$$\mathrm{wt}(D;\emptyset) \ = \ \mathrm{wt}(|_j D;\ D|_j) \ + \ \mathrm{wt}(\underline{D}_i;\ (\overline{D}^i)^{\searrow})$$

Indeed, for any diagram D' in the sum on the left, the same diagram appears in one of the two summands on the right depending on whether the cell $c_{i+1,j}$ is in D'. Now assume that μ is *not* a rectangle. This assumption ensures that the slices $|_k D$ and $\overline{D}^i$ are disjoint. By induction, we may apply the inequality (2) to each of the summands on the right to obtain:

$$\begin{aligned} \mathrm{wt}(D;\emptyset) &\geqslant \mathrm{wt}\,(|_j D|_k;\ D|_j \sqcup (|_k D)^{\rightarrow}) \\ &\quad + \mathrm{wt}\left(\underline{(D|_k)}_i;\ (\overline{D}^i)^{\searrow} \sqcup (|_k D)^{\rightarrow}\right) \\ &= \mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow}). \end{aligned}$$

Lastly, we assume that μ is a rectangle with at least two columns. Let i (resp. i') be the first (resp. last) row occupied by a cell in D. Then by Lemma 8, (first row of Figure 8)

$$\begin{aligned} &\mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow}) \\ &= \mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow} \sqcup \{c_{i,k+1}\} \setminus \{c_{i,k+2}\}) \\ &\quad + \mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow} \sqcup \{c_{i'+1,k+2}\} \setminus \{c_{i,k+2}\}) \\ &\quad - \mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow} \sqcup \{c_{i'+1,k+1}\} \setminus \{c_{i,k+2}\}) \end{aligned}$$

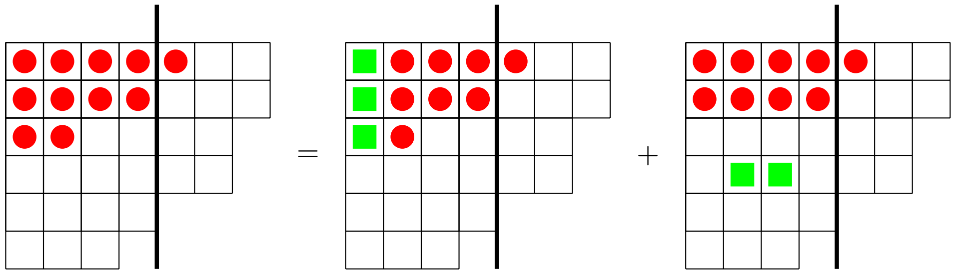

Figure 7: The non-rectangle case

Applying a similar division as in the non-rectangle case, we have (second row of Figure 8)

$$\begin{aligned}\mathrm{wt}(D;\emptyset) &= \mathrm{wt}\left(\overline{D}^i;\ \underline{D}_i\right)\\ &\quad + \mathrm{wt}(D|_k;\ (|_k D)^{\searrow})\\ &= \mathrm{wt}\left(\overline{D}^i;\ \underline{D}_i\right)\\ &\quad + \mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow} \sqcup \{c_{i'+1,k+2}\} \setminus \{c_{i,k+2}\})\end{aligned}$$

In the same manner, (third row of Figure 8)

$$\begin{aligned}&\mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow} \sqcup \{c_{i,k+1}\} \setminus \{c_{i,k+2}\})\\ &= \mathrm{wt}\left(\overline{D|_k}^i;\ \underline{D}_i \sqcup (\overline{|_k D}^i)^{\rightarrow}\right)\\ &\quad + \mathrm{wt}\left(D|_{k-1};\ (|_{k-1} D)^{\rightarrow} \sqcup \{c_{i'+1,k+1}\} \setminus \{c_{i,k+2}\}\right)\end{aligned}$$

Putting this together, we have

$$\begin{aligned}&\mathrm{wt}(D;\emptyset) - \mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow})\\ &= \mathrm{wt}\left(\overline{D}^i;\ \underline{D}_i\right)\\ &\quad - \mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow} \sqcup \{c_{i,k+1}\} \setminus \{c_{i,k+2}\})\\ &\quad + \mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow} \sqcup \{c_{i'+1,k+1}\} \setminus \{c_{i,k+2}\})\\ &= \mathrm{wt}\left(\overline{D}^i;\ \underline{D}_i\right)\\ &\quad - \mathrm{wt}\left(\overline{D|_k}^i;\ \underline{D}_i \sqcup (\overline{|_k D}^i)^{\rightarrow}\right)\\ &\quad + \mathrm{wt}(D|_k;\ (|_k D)^{\rightarrow} \sqcup \{c_{i'+1,k+1}\} \setminus \{c_{i,k+2}\})\\ &\quad - \mathrm{wt}\left(D|_{k-1};\ (|_{k-1} D)^{\rightarrow} \sqcup \{c_{i'+1,k+1}\} \setminus \{c_{i,k+2}\}\right)\end{aligned}$$

In the last expression, the difference between the first two terms and the difference between the last two terms are both nonnegative by the inductive hypothesis. This completes the proof of the rectangle case. □

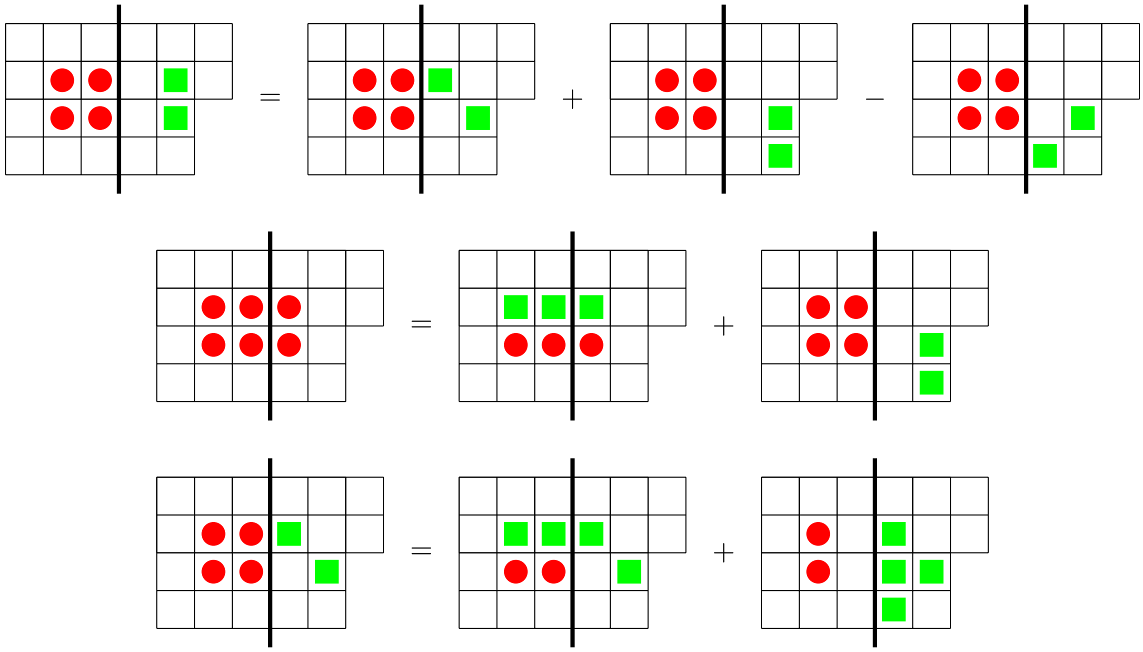

Figure 8: The rectangle case

Remark 10. One may wonder whether the stronger inequality

$$\mathrm{wt}(D;F) \geqslant \mathrm{wt}(D|_k;\ F \sqcup (|_k D))$$

holds; i.e., if you "slice" but do not "push." In fact, this inequality does not hold in general. For example, consider $\lambda = (4,3)$, $D = \{c_{1,1}, c_{1,2}\}$, $F = \emptyset$, and $k = 1$. We note that

$$\mathrm{wt}(D;F) = 5 \cdot 4 + 5 \cdot 1 + 2 \cdot 1 = 27,$$

while if $k = 1$,

$$\mathrm{wt}(D|_k;\ F \sqcup (|_k D)) = 5 \cdot 4 + 2 \cdot 4 = 28.$$

Example 11. Here we give an illustration of the Slice and Push Inequality. Suppose that the partition λ is $\lambda = (5,5,4,3,2)$. Consider the subset D of cells of $\mathbb{D}(\lambda)$ given by $D = \{c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}\}$. In Figure 9 (left), we show the four cells of D as the four circles. The set $D|_1$ consists of two cells: the two circles in Figure 9 (right). Once we slice D by a knife between columns 1 and 2 into a left portion and a right portion, and then push the right portion one unit to the right, the moved right portion is now $(|_1 D)^{\rightarrow}$. The set $(|_1 D)^{\rightarrow}$ is shown as the two squares in Figure 9 (right).

The Slice and Push Inequality says that $\mathrm{wt}(D;\emptyset)$ is at least $\mathrm{wt}(D|_1; (|_1 D)^{\rightarrow})$; i.e., the weight of the ○□-diagram on the left of Figure 9 is greater than or equal to the weight of the ○□-diagram on the right.

Indeed, we can compute the two weights explicitly. We observe that $\mathrm{wt}(D;\emptyset) = 9 \cdot 8 \cdot 8 \cdot 7 + 9 \cdot 8 \cdot 8 \cdot 3 + 9 \cdot 8 \cdot 5 \cdot 3 + 9 \cdot 8 \cdot 5 \cdot 3 + 9 \cdot 5 \cdot 5 \cdot 3 + 7 \cdot 5 \cdot 5 \cdot 3 = 9120$, and that $\mathrm{wt}(D|_1; (|_1 D)^{\rightarrow}) = (9 \cdot 8 + 9 \cdot 5 + 7 \cdot 5) \cdot 6 \cdot 5 = 4560$. In this example, the weight of the left diagram is exactly twice that of the right diagram.

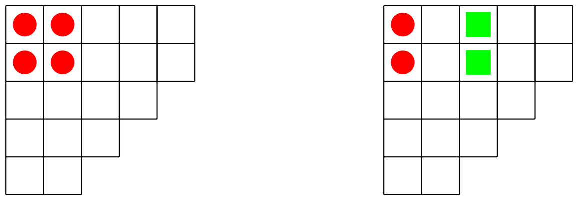

Figure 9: Two ○□-diagrams on the Young diagram of the partition $\lambda = (5, 5, 4, 3, 2)$

For the rest of this section, we let $D = \mathbb{D}(\mu)$ where $\mathbb{D}(\lambda/\mu)$ is a ribbon. The weight of $(D; \emptyset)$ is equal to the excitation factor of $f^{\lambda/\mu}$. Likewise, the coefficients that appear in the polynomial

$$E(t) = \sum_{d=0}^{s} C_{s-d} \prod_{i=1}^{d} (t + \alpha_i)$$

are the weights of some ○□-diagrams.

Lemma 12.

$$C_{s-i} = \mathrm{wt}(\overline{D}^1|_i;\ (|_i\overline{D}^1)^{\rightarrow}) \prod_{j=1}^{s-i} h(c_{\lambda'_{i+j+1}, i+j+1})$$

Proof. The quantity $C_{s-i} \prod_{j=1}^{i} (t + \alpha_j)$ is the sum of the weights of excited diagrams D' with i cells in the first row. Dividing by the weights of the cells in the first row, we have

$$C_{s-i} = \mathrm{wt}(\overline{D}^1|_i;\ (|_i D)^{\searrow}).$$

We may rewrite the weight of the ○□-diagram as follows.

$$\begin{aligned} C_{s-i} &= \mathrm{wt}(\overline{D}^1|_i;\ (|_i\overline{D}^1)^{\rightarrow} \sqcup \{c_{\lambda'_{i+2}, i+2},\ \ldots,\ c_{\lambda'_{s+1}, s+1}\}) \\ &= \mathrm{wt}(\overline{D}^1|_i;\ (|_i\overline{D}^1)^{\rightarrow}) \cdot h(c_{\lambda'_{i+2}, i+2}) \cdots h(c_{\lambda'_{s+1}, s+1}) \end{aligned}$$

This completes the proof. □

The hook length of a cell at the bottom of its column satisfies

$$h(c_{\lambda'_i, i}) = \begin{cases} h(c_{\lambda'_{i+1}, i+1}) + 1 \text{ if } \lambda'_i = \lambda'_{i+1} \\ 1 \text{ otherwise.} \end{cases}$$

Here, we set $\lambda'_i = 0$ if $i > \lambda_1$.

Corollary 13. *If $\lambda'_{i+1} > \lambda'_{i+2}$ for some $i \leqslant s$, then*

$$\frac{C_{s-i}}{0!} \geqslant \frac{C_{s-i+1}}{1!} \geqslant \cdots \geqslant \frac{C_s}{i!}.$$

Proof. Applying the Slice and Push Inequality with Lemma 12, it follows that when $\ell < k$

$$C_{s-\ell} \leqslant C_{s-k} \prod_{j=1}^{k-\ell} h(c_{\lambda'_{\ell+j+1},\ell+j+1}).$$

If $\lambda'_{i+1} > \lambda'_{i+2}$, then $h(c_{\lambda'_{i+1},i+1}) = 1$. By induction, we have for $\ell < k \leqslant i$ that

$$\prod_{j=1}^{k-\ell} h(c_{\lambda'_{\ell+j+1},\ell+j+1}) \leqslant \frac{(i-\ell)!}{(i-k)!}.$$

This completes the proof. □

Proposition 6 now follows from Corollary 13 by taking $i = s$. We note that by assumption, $\lambda'_{s+2} = 0$ and $\lambda'_{s+1} \geqslant 2$, so the conditions of Corollary 13 are satisfied. We will also make use of the case in which $(|_i\overline{D}^1)$ is empty.

Corollary 14. *If $\lambda'_{i+1} = \lambda'_{s+1} = 2$ for some $i \leqslant s$, then*

$$\frac{C_0}{0!} = \frac{C_1}{1!} = \cdots = \frac{C_{s-i}}{(s-i)!}.$$

Proof. By Lemma 12, if $k \in \{0, 1, \ldots, s-i\}$ then

$$\begin{aligned} C_{s-i-k} &= \mathrm{wt}(\overline{D}^1|_{i+k};\ (|_{i+k}\overline{D}^1)^{\to}) \prod_{j=1}^{s-i-k} h(c_{\lambda'_{i+j+k+1},i+j+k+1}) \\ &= \mathrm{wt}(\overline{D}^1; \emptyset) \cdot (s-i-k)! \end{aligned}$$

The corollary is proved. □

4 Proof of the Conjecture of Diaz-Lopez, Harris, Insko, Omar, and Sagan

Theorem 1 is implied by the following two inequalities: $|z+m| \leqslant m$ and $|z+1| \leqslant m$, for any $z \in \mathbb{C}$ such that $E(z) = 0$. In this section, we will use the Slice and Push Inequality and lemmas from the appendices to prove the two desired inequalities.

Theorem 15. *Let z be a complex number such that $E(z) = 0$. Then $|z+m| \leqslant m$.*

Proof. Recall that $\alpha_{s-i} \geqslant i+2$, for all $i = 0, 1, \ldots, s-1$. The case in which $\alpha_{s-i} = i+2$ for all i was proved in [7, Theorem 4.4]. Let us suppose there exists some i such that $\alpha_{s-i} \geqslant i+3$. Let $\kappa \geqslant 0$ be the smallest index such that $\alpha_{s-\kappa} \geqslant \kappa+3$. By Corollary 14,

$$\frac{C_0}{0!} = \frac{C_1}{1!} = \cdots = \frac{C_\kappa}{\kappa!}.$$

We argue that

$$\frac{C_{\kappa+1}}{1!} \geqslant \frac{C_{\kappa+2}}{2!} \geqslant \cdots \geqslant \frac{C_s}{(s-\kappa)!}. \tag{$*$}$$

If $\kappa = 0$, then $(*)$ follows from Proposition 6. If $\kappa \geqslant 1$, then note that $\lambda'_{s-\kappa} > \lambda'_{s-\kappa+1}$, and therefore, Corollary 13 gives

$$\frac{C_{\kappa+1}}{0!} \geqslant \frac{C_{\kappa+2}}{1!} \geqslant \frac{C_{\kappa+3}}{2!} \geqslant \cdots \geqslant \frac{C_s}{(s-\kappa-1)!},$$

which is a stronger inequality than $(*)$.

If $z = 0$ or $z = -\alpha_i$ for some i, it is easy to see that the desired inequality holds. Suppose $z \neq 0$ and $z + \alpha_i \neq 0$ for all i. We have

$$\begin{aligned} 0 &= \frac{E(z)}{C_0 \prod_{i=1}^s (z+\alpha_i)} \\ &= 1 + \frac{1!}{z+2} + \frac{2!}{(z+2)(z+3)} + \cdots + \frac{\kappa!}{(z+2)\cdots(z+\kappa+1)} \\ &\quad + \frac{C_{\kappa+1}/C_0}{(z+2)\cdots(z+\kappa+1)} \cdot \left(\frac{1}{z+\alpha_{s-\kappa}} + \frac{C_{\kappa+2}/C_{\kappa+1}}{(z+\alpha_{s-\kappa-1})(z+\alpha_{s-\kappa})} \right. \\ &\qquad \left. + \cdots + \frac{C_s/C_{\kappa+1}}{(z+\alpha_1)\cdots(z+\alpha_{s-\kappa})} \right). \end{aligned}$$

By induction, one may show that

$$1 + \frac{1!}{z+2} + \cdots + \frac{\kappa!}{(z+2)\cdots(z+\kappa+1)} = \frac{(z+1)(z+2)\cdots(z+\kappa+1) - (\kappa+1)!}{z(z+2)(z+3)\cdots(z+\kappa+1)}.$$

Inserting this into the previous equation gives:

$$\begin{aligned} &\left| \frac{(z+1)(z+2)\cdots(z+\kappa+1) - (\kappa+1)!}{z} \right| \\ &\quad = \left| \frac{C_{\kappa+1}}{C_0} \left(\frac{1}{z+\alpha_{s-\kappa}} + \cdots + \frac{C_s/C_{\kappa+1}}{(z+\alpha_1)\cdots(z+\alpha_{s-\kappa})} \right) \right|. \end{aligned}$$

Using $C_{\kappa+1}/C_0 \leqslant (\kappa+1)!$ along with $(*)$ and the triangle inequality, we obtain

$$\begin{aligned} &\left| \frac{(z+1)(z+2)\cdots(z+\kappa+1) - (\kappa+1)!}{z(\kappa+1)!} \right| \\ &\leqslant \frac{1!}{|z+\alpha_{s-\kappa}|} + \frac{2!}{|z+\alpha_{s-\kappa-1}||z+\alpha_{s-\kappa}|} + \cdots + \frac{(s-\kappa)!}{|z+\alpha_1||z+\alpha_2|\cdots|z+\alpha_{s-\kappa}|}. \end{aligned}$$

Suppose instead that $|z+m| > m$. Then, for $i = \kappa, \ldots, s-1$, we have $|z+\alpha_{s-i}| \geqslant |z+m| - (m-\alpha_{s-i}) > i+3$. Therefore,

$$\begin{aligned} &\frac{1!}{|z+\alpha_{s-\kappa}|} + \frac{2!}{|z+\alpha_{s-\kappa-1}||z+\alpha_{s-\kappa}|} + \cdots + \frac{(s-\kappa)!}{|z+\alpha_1||z+\alpha_2|\cdots|z+\alpha_{s-\kappa}|} \\ &< \frac{1!}{\kappa+3} + \frac{2!}{(\kappa+3)(\kappa+4)} + \cdots + \frac{(s-\kappa)!}{(\kappa+3)(\kappa+4)\cdots(s+2)} < \frac{1}{\kappa+1}. \end{aligned}$$

On the other hand, $|z+(\kappa+2)| \geqslant |z+m|-(m-\kappa-2) > \kappa+2$. Thus, Lemma 18 gives

$$\left|\frac{(z+1)(z+2)\cdots(z+\kappa+1)-(\kappa+1)!}{z(\kappa+1)!}\right| \geqslant \frac{1}{\kappa+1},$$

a contradiction. We have proved that $|z+m| \leqslant m$. □

Theorem 16. *Let z be a complex number such that $E(z)=0$. Then, $|z+1| \leqslant m$.*

Proof. When $s=0$, the result holds vacuously. Assume $s \geqslant 1$. This theorem is a consequence of the Polynomial Perturbation Lemma (Lemma 19). We use the lemma for when $(g_1, g_2, \ldots, g_k)$ is $\left(\frac{C_1}{C_0}, \frac{C_2}{C_1}, \ldots, \frac{C_s}{C_{s-1}}\right)$ and $(a_1, a_2, \ldots, a_k)$ is $(\alpha_s-1, \alpha_{s-1}-1, \ldots, \alpha_1-1)$. From Proposition 6, we have that for all i, $g_i \leqslant i$, and therefore, the lemma applies. We have $|z+1| \leqslant m$. □

5 Potential Future Directions

5.1 Tighter bounds on the convex hull of roots of descent polynomials

It might be possible to use our expansion of the descent polynomial to obtain tighter bounds on the convex hull of roots of descent polynomials. Numerical data (see e.g. the plots of roots in Section 5 of [3]) seem to suggest that for each fixed m, the convex hull of roots of a descent polynomial $d_I(z)$ when $\max I = m$ occupies a much smaller space than the region in the conjecture of Diaz-Lopez et al. To proceed in this direction, one might try to improve the inequalities on $C_0, \ldots, C_s$ given in Proposition 6 and Corollary 13, or to find sharper versions of our technical bounds given in the appendices.

5.2 Further applications of the bounds in the appendices

The bounds in the appendices are crucial ingredients in our proof of the conjecture by Diaz-Lopez et al. These results are less combinatorial and more analytic. The reader can enjoy Appendices A and B separately from the rest of the paper. We would be interested to see further applications of these inequalities to other problems.

5.3 Generalizing the Slice and Push Inequality

The Slice and Push Inequality gives a relationship between the weights of two ○□-diagrams. To calculate the weight of a ○□-diagram, we consider all excited diagrams, where we allow each ○-type cell to be excited *at most once*, and do not allow any □-type cell to move. One simple generalization of these diagrams is by introducing another type of cells where we allow them to be excited at most twice. It might be interesting to see how the Slice and Push Inequality would generalize to such a setting.

5.4 Peak polynomials

We remark that there is a related family of polynomials, the *peak polynomials* $P_I(z)$, defined by the property that $2^{n-|I|-1}P_I(n)$ is the number of permutations of $\mathfrak{S}_n$ with peak set I. Here, the *peak set* of a permutation $\pi = \pi_1 \cdots \pi_n$ is the set of positive integers $i \in \{2, \ldots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$. Polynomiality of this function was proved in [4]. It has been observed that peak polynomials and descent polynomials have many similar properties. In particular, [7, Conjecture 4.3] was motivated by a similar conjecture bounding the roots of peak polynomials [5, Conjecture 1.6]. Supporting the connection, Oğuz [17] proved that descent polynomials can be expressed as a sum of peak polynomials, and conversely, each peak polynomial is an alternating sum of descent polynomials. Our approach to bounding the roots of descent polynomials does not seem to have a clear application to peak polynomials, however. In particular, we would be interested in finding a basis for polynomials of small degree for which the coefficients of the peak polynomial $P_I(z)$ satisfy the conditions of the Polynomial Perturbation Lemma in Appendix B.

5.5 Type B

In the work [17], Oğuz reinterpreted the work in Type B of Aguiar, Bergeron, and Nyman [2]. It might be interesting to generalize our results in the present paper to other types.

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A A Complex Analytic Lemma

In this section, we prove a lemma used in the proof of Theorem 15.

Let k be a fixed positive integer. Consider the meromorphic function

$$P(z) := \frac{(z+1)(z+2)\cdots(z+k)-k!}{z} \in \mathbb{C}[z].$$

Since the numerator is divisible by z, the function $P(z)$ may be regarded as a polynomial function on the whole complex plane. We hereafter refer to $P(z)$ as a polynomial.

Lemma 17. *If z_0 is a root of $P(z)$, then $|z_0+1| \leqslant k$ and $|z_0+k| < k$.*

Proof. This proof is similar to that of [7, Theorem 4.4].

When $k=1$, we have $P(z)=1$ for all z, so we may assume $k>1$. The polynomial $P(z)$ has a nonzero constant term $\sum_{i=1}^{k} \frac{k!}{i}$, so 0 is not a root of $P(z)$.

If $|z_0+1| > k$ then $|z_0+i| > k+1-i$ for all $i \in \{1,\dots,k\}$. This implies

$$|(z_0+1)\cdots(z_0+k)-k!| \geqslant |(z_0+1)\cdots(z_0+k)| - k! > 0,$$

so $P(z_0) \neq 0$.

Similarly, if $|z_0+k| > k$, then $|z_0+i| > i$ for all $i \in \{1,\dots,k\}$. Again, we have $P(z_0) \neq 0$.

Finally, if $|z_0+k| = k$ and $z_0 \neq 0$ then $|z_0+i| > i$ for $i \in \{1,\dots,k-1\}$, and we again deduce that $P(z_0)$ is nonzero. □

Lemma 17 implies that $\frac{1}{P(z)}$ is holomorphic in the domain $\{z \in \mathbb{C} : |z+(k+1)| > k+1\}$. The *maximum modulus principle* states that the modulus of any non-constant holomorphic function does not have a local maximum in any open, connected domain. As $\frac{1}{P(z)}$ vanishes at infinity, the maximum value of $\frac{1}{|P(z)|}$ in the domain $\{z : |z+(k+1)| \geqslant k+1\}$ is attained in the compact subset $\{z : M \geqslant |z+(k+1)| \geqslant k+1\}$ for any sufficiently large value of M. Hence, this maximum value is achieved at the boundary $\{z : |z+(k+1)| = k+1\}$.

Lemma 18. *If z is a complex number such that $|z+(k+1)| \geqslant k+1$, then $|P(z)| \geqslant (k-1)!$.*

It is easy to see that the inequality holds when $z=0$. From now on, assume $z \neq 0$. Let us suppose $k \geqslant 8$ and deal with small values of k later. As stated above, we can assume $|z+(k+1)| = k+1$.

Without loss of generality, assume $\Im(z) \geqslant 0$ and write $z = -a+bi$, where $a,b \geqslant 0$. Note that $(k+1)^2 - |z|^2 = (k+1-a)^2 - a^2$, so

$$a = \frac{|z|^2}{2(k+1)}. \tag{3}$$

We consider two cases.

<u>Case 1.</u> $|z| \geqslant \frac{2}{k}$.

We have

$$|(z+1)\cdots(z+k)| = \sqrt{\prod_{j=1}^{k}((j-a)^2+b^2)}$$
$$= k!\cdot\sqrt{\prod_{j=1}^{k}\left(\left(1-\frac{a}{j}\right)^2+\left(\frac{b}{j}\right)^2\right)}.$$

Note that $\left(1-\frac{a}{j}\right)^2+\left(\frac{b}{j}\right)^2 > 1$ because $|z+j| > j$. For $x_1,\ldots,x_k > 0$, we have the inequality $\prod_{j=1}^{k}(1+x_j) > 1+\sum_{j=1}^{k}x_j$. Using this inequality with $k \geqslant 8$, $|z| \geqslant \frac{2}{k}$, and (3), we have

$$|(z+1)\cdots(z+k)| > k!\cdot\sqrt{1+\sum_{j=1}^{k}\left[-\frac{2a}{j}+\frac{a^2+b^2}{j^2}\right]}$$
$$> k!\cdot\sqrt{1-\left(\frac{\log k+1}{k+1}\right)|z|^2+\frac{3}{2}|z|^2}$$
$$> k!\cdot\sqrt{1+\frac{2|z|}{k}+\frac{|z|^2}{k^2}}$$
$$= \left(1+\frac{|z|}{k}\right)\cdot k!.$$

Therefore, $|(z+1)\cdots(z+k)-k!| \geqslant |(z+1)\cdots(z+k)|-k! > |z|\cdot(k-1)!$.

Case 2. $|z| < \frac{2}{k}$.

We claim that

$$\frac{9999}{10000}|z| \leqslant b \leqslant a+b \leqslant \frac{73}{72}|z| < \frac{73}{36k}. \tag{4}$$

The last inequality is immediate from the hypothesis on $|z|$. Note that

$$a = \frac{|z|^2}{2(k+1)} \leqslant \frac{2}{k}\cdot\frac{|z|}{2(k+1)} = \frac{|z|}{k(k+1)} \leqslant \frac{|z|}{72}.$$

Therefore, $a+b \leqslant \frac{|z|}{72}+|z| = \frac{73}{72}|z|$. We also have

$$b = \sqrt{|z|^2-a^2} = |z|\cdot\sqrt{1-\frac{|z|^2}{4(k+1)^2}} \geqslant |z|\cdot\sqrt{1-\frac{1}{(k(k+1))^2}} \geqslant \frac{9999}{10000}|z|,$$

as claimed.

The following useful trigonometric inequalities can be verified by single variable calculus:

- For all $x \geqslant 0$,
$$x - x^2 \leqslant \arctan(x) \leqslant x. \tag{T1}$$
- For all $x \geqslant 0$,
$$x - \frac{x^3}{3} \leqslant \sin(x). \tag{T2}$$

For each $j = 1, 2, \dots, k$, let $\theta_j := \arg(z + j) \in [0, 2\pi)$. Let $\theta = \theta_1 + \cdots + \theta_k$. We have that
$$(z+1)\cdots(z+k) = |(z+1)\cdots(z+k)| \cdot e^{i\theta}.$$

Now, we claim that
$$(\log k)b \leqslant \theta \leqslant (1 + \log k)(a + b) < \frac{\pi}{4}.$$

Since $k \geqslant 8$, (4) implies that $a + b < 1$, from which it follows that
$$\frac{b}{j-a} \leqslant \frac{a+b}{j}$$
whenever $j \geqslant 1$. Then for $j = 1, 2, \dots, k$,
$$\theta_j = \arctan\left(\frac{b}{j-a}\right) \leqslant \arctan\left(\frac{a+b}{j}\right).$$
Therefore,
$$\theta \leqslant \sum_{j=1}^{k} \arctan\left(\frac{a+b}{j}\right) \overset{\text{(T1)}}{\leqslant} \sum_{j=1}^{k} \frac{a+b}{j} \leqslant (1 + \log k)(a + b).$$
On the other hand,
$$\theta_j = \arctan\left(\frac{b}{j-a}\right) \geqslant \arctan\left(\frac{b}{j}\right).$$
Hence,
$$\theta \geqslant \sum_{j=1}^{k} \arctan\left(\frac{b}{j}\right) \overset{\text{(T1)}}{\geqslant} \sum_{j=1}^{k} \left(\frac{b}{j} - \frac{b^2}{j^2}\right) \geqslant \left(\frac{1}{2} + \log k\right) b - \frac{\pi^2}{6} b^2 \geqslant (\log k)b.$$
Since $k \geqslant 8$, we also have
$$(1 + \log k)(a + b) \leqslant (1 + \log k) \cdot \frac{73}{36k} < \frac{\pi}{4},$$
as claimed. We remark that the last inequality does not hold for $k = 7$.

The function $x \mapsto x - \frac{x^3}{3}$ is increasing on $\left[0, \frac{\pi}{4}\right]$. Thus,
$$\sin\theta \overset{\text{(T2)}}{\geqslant} \theta - \frac{\theta^3}{3} \geqslant (\log k)b - \frac{(\log k)^3 b^3}{3} \geqslant \left(\log k - \frac{4(\log k)^3}{3k^2}\right) \frac{9999}{10000}|z| \geqslant |z|.$$

For the last inequality, we used the fact that $k \geqslant 8$. Therefore,

$$\begin{aligned}|(z+1)\cdots(z+k)-k!| &\geqslant \Im\left((z+1)\cdots(z+k)-k!\right)\\ &= |(z+1)\cdots(z+k)|\cdot\sin\theta \geqslant k!\cdot|z| \geqslant (k-1)!\cdot|z|.\end{aligned}$$

This finishes the proof of the inequality for $k \geqslant 8$.

Finally, we consider the cases with $k \leqslant 7$. Since the seven cases for k may all be proved in roughly the same manner, we give a proof for $k = 7$ and leave the other cases to the reader.

Let $k = 7$ and $w = z + 5$. We have $|w| \geqslant 5$.

$$\begin{aligned}&\left|\frac{(z+1)(z+2)(z+3)(z+4)(z+5)(z+6)(z+7)-7!}{z}\right|\\ &= \left|(z+5)^6 - 2(z+5)^5 - 3(z+5)^4 + 20(z+5)^3 + 44(z+5)^2 + 192(z+5) + 1008\right|\\ &= \left|w^6 - 2w^5 - 3w^4 + 20w^3 + 44w^2 + 192w + 1008\right|\\ &\geqslant |w|^6 - 2|w|^5 - 3|w|^4 - 20|w|^3 - 44|w|^2 - 192|w| - 1008 \geqslant 1932 > 6!.\end{aligned}$$

In each of the remaining cases, one may use a similar argument where w is defined as in the following table.

k	w
1	z
2	$z+3$
3	$z+3$
4	$z+4$
5	$z+4$
6	$z+\frac{17}{4}$

B The Polynomial Perturbation Lemma

In this section, we prove a lemma on polynomial perturbation. Starting with a certain polynomial with distinct real roots, we obtain a new polynomial by perturbing it in a certain bounded manner. The lemma gives an upper bound on the moduli of the roots of the resulting polynomial. We also note that our result is sharp.

Lemma 19 (Polynomial Perturbation)**.** *Suppose that k is a positive integer. Let $a_1 < a_2 < \cdots < a_k$ be a strictly increasing sequence of positive integers. Let $g_1, \ldots, g_k \geqslant 0$ satisfy $g_i \leqslant i$ for all $i = 1, \ldots, k$. Define*

$$\begin{aligned}P(z) := {}&(z+a_k)(z+a_{k-1})\cdots(z+a_1) + g_1(z+a_k)(z+a_{k-1})\cdots(z+a_2)\\ &+ g_1g_2(z+a_k)(z+a_{k-1})\cdots(z+a_3) + \cdots + g_1g_2\cdots g_{k-1}(z+a_k) + g_1g_2\cdots g_k.\end{aligned}$$

If z is a complex root of $P(z)$, then $|z| \leqslant a_k + 1$.

We think of the first term $(z+a_k)\cdots(z+a_1)$ as the *main* term and the rest as the perturbation. The original roots of the main term are $-a_k,\ldots,-a_1$, which all lie inside the closed ball $\{z : |z| \leqslant a_k\}$. The lemma says that the roots of the perturbed polynomial are inside a slightly larger closed ball $\{z : |z| \leqslant a_k + 1\}$.

To show this lemma, we prove the following stronger statement. This strategy is predictable as we have the picture that the main term should dominate the rest.

Claim 20. *Let $a_1,\ldots,a_k,g_1,\ldots,g_k$ be as in the above lemma. If $|z| > a_k+1$, then*

$$|(z+a_k)\cdots(z+a_1)| > \left|\sum_{i=1}^{k} g_1\cdots g_i(z+a_{i+1})\cdots(z+a_k)\right|. \tag{5}$$

We will first assume that $k \geqslant 3$, and then work on $k=1,2$ later.

Step I. Reduction of g_i's.

In the first step, we will reduce the problem to the case in which $g_i = i$ for all i. For this purpose, we consider z and $a_1,\ldots,a_k$ fixed within this step. Define

$$F(g_1,\ldots,g_k) := \left|\sum_{i=1}^{k} g_1\cdots g_i(z+a_{i+1})\cdots(z+a_k)\right|.$$

Note that F is a convex function for each $g_i \in [0,i]$. Therefore,

$$\begin{aligned}&F(g_1,\ldots,g_{i-1},g_i,g_{i+1},\ldots,g_k)\\ &\leqslant \max\left\{F(g_1,\ldots,g_{i-1},0,g_{i+1},\ldots,g_k), F(g_1,\ldots,g_{i-1},i,g_{i+1},\ldots,g_k)\right\}.\end{aligned}$$

Using the inequality above for all $i \in [k]$, we find that there exist $\widehat{g}_1,\ldots,\widehat{g}_k \in \mathbb{R}$ such that

$$F(g_1,\ldots,g_k) \leqslant F(\widehat{g}_1,\ldots,\widehat{g}_k),$$

where $\widehat{g}_i \in \{0,i\}$ for each $i=1,2,\ldots,k$. If $\widehat{g}_1 = 0$, then $F(\widehat{g}_1,\ldots,\widehat{g}_k)=0$ and (5) follows immediately. Suppose that $\widehat{g}_1 = 1$. Let $k' \in [k]$ be the largest index such that $\widehat{g}_i = i$ for all $1 \leqslant i \leqslant k'$. By the definition of F, we note that

$$F(\widehat{g}_1,\ldots,\widehat{g}_k) = F(1,2,\ldots,k',0,\ldots,0).$$

It suffices to show

$$|(z+a_{k'})\cdots(z+a_1)| > \left|\sum_{i=1}^{k'} i!(z+a_{i+1})\cdots(z+a_{k'})\right|.$$

Since $\{z : |z| > a_k+1\} \subseteq \{z : |z| > a_{k'}+1\}$, it suffices to prove the claim for the case where k is replaced by k' and $g_i = i$ for all $i \in [k']$.

From now on we assume $g_i = i$ for all $i = 1,2,\ldots,k$.

Step II. Reduction of a_i's.

We consider the vector of first differences of a_i's

$$\Delta := (a_k - a_{k-1}, \ldots, a_3 - a_2, a_2 - a_1) \in (\mathbb{Z}_{>0})^{k-1}.$$

Define

$$\mathcal{A} := \left\{(1^{k-1}), (1^{k-2}, 2), (1^{k-2}, 3), (1^{k-2}, 4), (1^{k-3}, 2, 1), (1^{k-3}, 2, 2)\right\}.$$

Here, 1^m denotes m copies of 1's for each $m \geqslant 0$. The goal of this step is to prove the claim when $\Delta \notin \mathcal{A}$.

Assume $\Delta \notin \mathcal{A}$. Let $z \in \mathbb{C}$ such that $|z| > a_k + 1$. There are two cases.

Case 1. Suppose that there is some index j such that $4 \leqslant j \leqslant k$ and $a_j - a_{j-1} \geqslant 2$. In particular, we must have $k \geqslant 4$ for this case to happen. By the triangle inequality,

$$|z + a_i| > k + 1 - i \tag{6}$$

holds for $i = 1, 2, \ldots, k$. In this case, we obtain better inequalities for $i = 1, 2, 3$: $|z+a_1| > k + 1$, $|z + a_2| > k$, and $|z + a_3| > k - 1$.

Recall that we want to show that

$$|(z + a_k) \cdots (z + a_1)| > \left| \sum_{i=1}^{k} i!(z + a_{i+1}) \cdots (z + a_k) \right|.$$

That is,

$$1 > \left| \sum_{i=1}^{k} \frac{i!}{(z + a_1) \cdots (z + a_i)} \right|.$$

By the triangle inequality, it suffices to prove that

$$\sum_{i=1}^{k} \frac{i!}{|z + a_1| \cdots |z + a_i|} < 1.$$

Using the bounds for $|z + a_i|$ in (6), we obtain

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{i!}{|z + a_1| \cdots |z + a_i|} \\
&< \frac{1!}{k+1} + \frac{2!}{(k+1)k} + \frac{k-2}{k+1} \left(\frac{3!}{k(k-1)(k-2)} + \frac{4!}{k(k-1)(k-2)(k-3)} + \cdots + \frac{k!}{k!} \right) \\
&= \frac{2}{k^2 - 1} + \frac{k-2}{k+1} \sum_{i=1}^{k} \binom{k}{i}^{-1} < 1,
\end{aligned}$$

as desired. To see why the last inequality holds, one may use the bound

$$\sum_{i=1}^{k} \binom{k}{i}^{-1} < 1 + \frac{3}{k},$$

for $k \geqslant 8$, and check the cases where $4 \leqslant k \leqslant 7$ by hand.

Case 2. Suppose that $a_i - a_{i-1} = 1$ for all $i \geqslant 4$. In this case, because $\Delta \notin \mathcal{A}$, we have

$$(a_3 - a_2, a_2 - a_1) \notin \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2)\}.$$

Note that $A := |z+a_1| > k-2+(a_3-a_1)$ and $B := |z+a_2| > k-2+(a_3-a_2)$. By some easy casework, we obtain the bound $(A-1)B > k^2+2k-3$. With the same argument as in the case above, we want to show that

$$\sum_{i=1}^{k} \frac{i!}{|z+a_1|\cdots|z+a_i|} < 1.$$

Observe that

$$\sum_{i=1}^{k} \frac{i!}{|z+a_1|\cdots|z+a_i|} < \frac{1}{A} + \frac{k(k-1)}{AB} \sum_{i=2}^{k} \binom{k}{i}^{-1}.$$

Thus, it suffices to show

$$k(k-1)\sum_{i=2}^{k} \binom{k}{i}^{-1} \leqslant (A-1)B.$$

Using $(A-1)B > k^2+2k-3$ and $\sum_{i=2}^{k} \binom{k}{i}^{-1} \leqslant 1 + \frac{21}{10k}$, we obtain the desired inequality.

Step III. The case in which $\Delta \in \mathcal{A}$.

Still assuming that $k \geqslant 3$ is a fixed integer, we work on the six remaining cases of Δ.

Case 1. $a_3 - a_2 = a_2 - a_1 = 1$. In this case, $(a_k, \ldots, a_1) = (k+\alpha, k-1+\alpha, \ldots, 1+\alpha)$ for some integer $\alpha \geqslant 0$. Note that if we prove the desired claim for the case when $\alpha = 0$, all other cases when $\alpha > 0$ will follow. Thus, we may now assume $(a_k, \ldots, a_1) = (k, k-1, \ldots, 1)$. The inequality we want to prove becomes

$$|(z+1)(z+2)\cdots(z+k)| > \left|\sum_{i=1}^{k} i!(z+i+1)(z+i+2)\cdots(z+k)\right|. \tag{7}$$

By induction, we can show that

$$\sum_{i=0}^{k} i!(z+i+1)(z+i+2)\cdots(z+k) = \frac{z(z+1)(z+2)\cdots(z+k)-(k+1)!}{z-1}.$$

Therefore, the inequality (7) is equivalent to

$$\left|\frac{1}{z-1} - \frac{(k+1)!}{(z-1)(z+1)(z+2)\cdots(z+k)}\right| < 1.$$

By the triangle inequality, it suffices to show that

$$\frac{1}{|z-1|}\left(1+\frac{(k+1)!}{|z+1||z+2|\cdots|z+k|}\right)<1. \tag{8}$$

Moreover, by the triangle inequality, we know that $|z+j| \geqslant |z|-j > k+1-j$, for $j=1,2,\ldots,k$, and also $|z-1| \geqslant |z|-1 > k$.

If $|z-1| \geqslant k+2$, it is easy to see that the bound (8) holds. Assume $|z-1| < k+2$. Let $w = z+k+1$. Write $w = a+bi$, where $a, b \in \mathbb{R}$. From $|z-1| < k+2$, we have $(a-(k+2))^2+b^2 < (k+2)^2$, and so

$$a > \frac{a^2+b^2}{2(k+2)}.$$

In particular, we have $a > 0$.

From $|z| > k+1$, we have $(a-(k+1))^2+b^2 > (k+1)^2$, and so

$$\frac{a^2+b^2}{2a} > k+1. \tag{9}$$

The inequality (8) is equivalent to

$$|z+1||z+2|\cdots|z+k|+(k+1)! < |z-1||z+1||z+2|\cdots|z+k|,$$

which is

$$\begin{aligned}&\left|1-\frac{w}{k}\right|\left|1-\frac{w}{k-1}\right|\cdots|1-w|+(k+1)\\ &<(k+2)\left|1-\frac{w}{k+2}\right|\left|1-\frac{w}{k}\right|\left|1-\frac{w}{k-1}\right|\cdots|1-w|.\end{aligned} \tag{10}$$

Note that for $j=1,2,\ldots,k$, we have $\left|1-\frac{w}{j}\right| > 1$, while $\left|1-\frac{w}{k+2}\right| < 1$. Therefore, we can write

$$\prod_{j=1}^{k}\left|1-\frac{w}{j}\right| = 1+\varepsilon,$$

and $\left|1-\frac{w}{k+2}\right| = 1-\varepsilon'$, where $\varepsilon, \varepsilon' > 0$. Now, (10) is equivalent to

$$\varepsilon' < \frac{k+1}{k+2}\cdot\frac{\varepsilon}{1+\varepsilon}. \tag{11}$$

From $|z-1| > k$, we know $\varepsilon' < \frac{2}{k+2}$. If $\varepsilon \geqslant \frac{2}{k-1}$, then (11) is clear. Assume $\varepsilon < \frac{2}{k-1}$. Thus, $\frac{\varepsilon}{1+\varepsilon} > \frac{k-1}{k+1}\cdot\varepsilon$. To show (11), it suffices to show that

$$\frac{\varepsilon}{\varepsilon'} \geqslant \frac{k+2}{k-1}. \tag{12}$$

Using a trick similar to one in the proof of Lemma A.2, we find

$$(1+\varepsilon)^2 = \prod_{j=1}^{k}\left(1 - \frac{2a}{j} + \frac{a^2+b^2}{j^2}\right) \geqslant 1 + \sum_{j=1}^{k}\left(-\frac{2a}{j} + \frac{a^2+b^2}{j^2}\right).$$

This shows

$$\varepsilon(2+\varepsilon) \geqslant \sum_{j=1}^{k}\left(-\frac{2a}{j} + \frac{a^2+b^2}{j^2}\right). \tag{13}$$

On the other hand, since $(1-\varepsilon')^2 = \left|1 - \frac{w}{k+2}\right|^2 = \left(1 - \frac{a}{k+2}\right)^2 + \left(\frac{b}{k+2}\right)^2$, we have

$$\varepsilon'(2-\varepsilon') = \frac{2a}{k+2} - \frac{a^2+b^2}{(k+2)^2}. \tag{14}$$

Therefore, we have

$$\begin{aligned}
&\frac{\varepsilon(2+\varepsilon) - 5\varepsilon'(2-\varepsilon')}{2a}\\
&\overset{(13),\,(14)}{\geqslant} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{5}{(k+2)^2}\right)\frac{a^2+b^2}{2a} - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \frac{5}{k+2}\right)\\
&\overset{(9)}{>} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{5}{(k+2)^2}\right)(k+1) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \frac{5}{k+2}\right) > 0.
\end{aligned}$$

This shows that $\varepsilon(2+\varepsilon) > 5\varepsilon'(2-\varepsilon')$, and thus

$$\frac{\varepsilon}{\varepsilon'} > 5\cdot\frac{2-\varepsilon'}{2+\varepsilon} > 5\cdot\frac{2-\frac{2}{k+2}}{2+\frac{2}{k-1}} = 5\cdot\frac{(k+1)(k-1)}{k(k+2)} > \frac{k+2}{k-1}.$$

It is easy to see that the last inequality $5\cdot\frac{(k+1)(k-1)}{k(k+2)} > \frac{k+2}{k-1}$ holds, for $k \geqslant 3$. This proves (12) and we have finished the proof for this case.

Case 2. $a_3 - a_2 = 1$ and $d := a_2 - a_1 \in \{2,3,4\}$. Write

$$(a_k, \ldots, a_1) = (u+k+1, u+k, \ldots, u+3, u+3-d),$$

where $u \geqslant 0$ is an integer. The inequality we want to prove becomes

$$\begin{aligned}
&|(z+u+k+1)(z+u+k)\cdots(z+u+3)(z+u+3-d)|\\
&> \left|\sum_{i=1}^{k} i!(z+u+k+1)(z+u+k)\cdots(z+u+i+2)\right|.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{i=1}^{k} i!(z+u+k+1)(z+u+k)\cdots(z+u+i+2)\\
&= \frac{(z+u+k+1)(z+u+k)\cdots(z+u+2) - (k+1)!}{z+u}.
\end{aligned}$$

Therefore, it suffices to show that

$$|z+u| > \left|1 + \frac{d-1}{z+u+3-d} - \frac{(k+1)!}{(z+u+k+1)\cdots(z+u+3)(z+u+3-d)}\right|. \tag{15}$$

Recall that $|z| > a_k + 1 = u + k + 2$. Therefore, by the triangle inequality, we have

$$\begin{aligned}
RHS_{(15)} &\leqslant 1 + \frac{d-1}{|z+u+3-d|} + \frac{(k+1)!}{|z+u+k+1|\cdots|z+u+3||z+u+3-d|} \\
&< 1 + \frac{d-1}{k+d-1} + \frac{(k+1)!}{(k-1)!(k+d-1)} \\
&= 1 + \frac{d-1}{k+d-1} + \frac{k(k+1)}{k+d-1} \\
&\leqslant k+2 < |z+u| = LHS_{(15)}.
\end{aligned}$$

Case 3. $a_3 - a_2 = 2$ and $d := a_2 - a_1 \in \{1, 2\}$. Write

$$(a_k, \ldots, a_1) = (u+k+1, u+k, \ldots, u+4, u+2, u+2-d),$$

where $u \geqslant 0$ is an integer. The inequality we want to prove becomes

$$\begin{aligned}
&|(z+u+k+1)(z+u+k)\cdots(z+u+4)(z+u+2)(z+u+2-d)| \\
&> \Bigg|1!(z+u+k+1)(z+u+k)\cdots(z+u+4)(z+u+2) \\
&\qquad + \sum_{i=2}^{k} i!(z+u+k+1)(z+u+k)\cdots(z+u+i+2)\Bigg|.
\end{aligned}$$

Note that

$$\begin{aligned}
&1!(z+u+k+1)(z+u+k)\cdots(z+u+4)(z+u+2) \\
&+ \sum_{i=2}^{k} i!(z+u+k+1)(z+u+k)\cdots(z+u+i+2) \\
&= \frac{(z+u+k+1)(z+u+k)\cdots(z+u+4)\left((z+u)^2+4(z+u)+6\right) - (k+1)!}{z+u}.
\end{aligned}$$

Therefore, it suffices to show that

$$\begin{aligned}
|z+u| > \Bigg|1 &+ \frac{d}{z+u+2-d} + \frac{2}{(z+u+2)(z+u+2-d)} \\
&- \frac{(k+1)!}{(z+u+k+1)(z+u+k)\cdots(z+u+4)(z+u+2)(z+u+2-d)}\Bigg|.
\end{aligned} \tag{16}$$

Again, by the triangle inequality, we have

$$\begin{aligned}
RHS_{(16)} &\leqslant 1 + \frac{d}{|z+u+2-d|} + \frac{2}{|z+u+2||z+u+2-d|} \\
&\quad + \frac{(k+1)!}{|(z+u+k+1)(z+u+k)\cdots(z+u+4)(z+u+2)(z+u+2-d)|} \\
&< 1 + \frac{d}{k+d} + \frac{2}{k(k+d)} + \frac{(k+1)!}{(k-2)!k(k+d)} \\
&= 1 + \frac{2}{k(k+d)} + \frac{k^2+d-1}{k+d} \\
&\leqslant k+2 < |z+u| = LHS_{(16)}.
\end{aligned}$$

This finishes Step III.

Step IV. Small *k*.

Finally, we work with the cases $k = 1, 2$.

$\boxed{k=1}$ Suppose $|z| > a_1 + 1$. Then, $|z + a_1| \geqslant |z| - a_1 > 1 \geqslant |g_1|$.

$\boxed{k=2}$ For this case, we prove one little lemma.

Lemma 21. *Let $u, v \in \mathbb{R}_{\geqslant 0}$ such that $u \geqslant v + 1$. For any $z \in \mathbb{C}$ with $|z| > u + 1$, we have*

$$|z+u||z+v| > |z+u+2|.$$

Proof. By the triangle inequality, we have $|z+v| > (u+1)-v \geqslant 2$. By classical geometry, the locus of all points $\zeta \in \mathbb{C}$ satisfying

$$2\,|\zeta+u| \leqslant |\zeta+u+2|$$

is the closed disk enclosed by the circle of Apollonius centered at $-u + \frac{2}{3}$ of radius $\frac{4}{3}$. Since the whole of this disk lies inside $\{\zeta : |\zeta| \leqslant u+1\}$, we finish the proof. ☐

Applying the lemma for $(u, v) = (a_2, a_1)$, we find that for $z \in \mathbb{C}$ such that $|z| > a_2 + 1$,

$$|z+a_2||z+a_1| > |z+a_2+2|.$$

Since $|z + a_1| > 2$, we also have $|z + a_2||z + a_1| > |z + a_2|$. Therefore,

$$|z+a_2||z+a_1| > \max\{|z+a_2+2|, |z+a_2|\} \geqslant |z+a_2+g_2| \geqslant |g_1(z+a_2)+g_1g_2|,$$

by the convexity argument as we did earlier. This finishes the proof.

This also concludes our proof of the Polynomial Perturbation Lemma.

It is worth noting that the Polynomial Perturbation Lemma is sharp. When k is odd, $\Delta = (1^{k-1})$, and $g_i = i$ for all $i \in [k]$, observe that $-a_k - 1$ is a root of $P(z)$. This implies that the upper bound on the moduli of the roots of the perturbed polynomial can indeed be attained.