Powersum Bases in Quasisymmetric Functions and Quasisymmetric Functions in Non-commuting Variables

Anthony Lazzeroni

Submitted: Dec 11, 2022; Accepted: Nov 29, 2023; Published: Dec 15, 2023

Abstract

We introduce a new powersum basis for the Hopf algebra of quasisymmetric functions that refines the powersum symmetric basis. Unlike the quasisymmetric powersums of types 1 and 2, our basis is defined combinatorially: its expansion in quasisymmetric monomial functions is given by fillings of matrices. This basis has a shuffle product, a deconcatenate coproduct, and has a change of basis rule to the quasisymmetric fundamental basis by using tuples of ribbons. We lift our powersum quasisymmetric P basis to the Hopf algebra of quasisymmetric functions in non-commuting variables by introducing fillings with disjoint sets. This new basis has a shifted shuffle product and a standard deconcatenate coproduct, and certain basis elements agree with the fundamental basis of the Malvenuto-Reutenauer Hopf algebra of permutations. Finally we discuss how to generalize these bases and their properties by using total orders on indices.

Mathematics Subject Classifications: 05E05

1 Introduction

The Hopf algebra of symmetric functions, denoted Sym whose bases are indexed by integer partitions λ, is a very well known space for its connections in representation theory and other areas of mathematics. Some of the well studied bases are the *monomial* basis m_λ, *powersum* basis p_λ, and *Schur* basis s_λ. In [14] the change of basis from p_λ to m_μ is illustrated by fillings. In this work, a *filling* F is a rectangular matrix with exactly one non-zero entry in every row and no zero columns. The column (respectively, row) sum is a composition recording the sum of all entries in each column (respectively, row) denoted as $\mathsf{col}(\mathsf{F})$ (respectively, $\mathsf{row}(\mathsf{F})$). The change of basis is combinatorially defined as

$$p_\lambda = \sum_{\mathsf{F}\in\mathcal{A}(\lambda)} m_{\mathsf{col}(\mathsf{F})} \tag{1}$$

Department of Mathematics, Hong Kong Baptist University, Hong Kong SAR (alazzeroni@hkbu.edu.hk).

where $\mathcal{A}(\lambda)$ is the set of all distinct fillings with row sum λ and the column sum a partition. For example, $p_{(2,2,1)} = 2m_{(2,2,1)} + 2m_{(3,2)} + m_{(4,1)} + m_{(5)}$ since the fillings of $\mathcal{A}(2,2,1)$ are

$$\begin{array}{c|ccc} 2 & 2 & \cdot & \cdot \\ 2 & \cdot & 2 & \cdot \\ 1 & \cdot & \cdot & 1 \\ \hline & 2 & 2 & 1 \end{array} \qquad \begin{array}{c|ccc} 2 & \cdot & 2 & \cdot \\ 2 & 2 & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \\ \hline & 2 & 2 & 1 \end{array} \qquad \begin{array}{c|ccc} 2 & 2 & \cdot & \cdot \\ 2 & \cdot & 2 & \cdot \\ 1 & 1 & \cdot & \cdot \\ \hline & 3 & 2 & \end{array}$$

$$\begin{array}{c|ccc} 2 & \cdot & 2 & \cdot \\ 2 & 2 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ \hline & 3 & 2 & \end{array} \qquad \begin{array}{c|ccc} 2 & 2 & \cdot & \cdot \\ 2 & 2 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ \hline & 4 & 1 & \end{array} \qquad \begin{array}{c|ccc} 2 & 2 & \cdot & \cdot \\ 2 & 2 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ \hline & 5 & & \end{array}.$$

One of the important properties of the powersum basis is the Murnaghan-Nakayama rule which illustrates the product rule of s_λ and p_μ expanded in terms of Schur functions. This expansion gives, as corollary, the change of basis formula from the powersum basis to the Schur basis, which is important for its connection to the character table of $\mathfrak{S}_n$.

A space that contains Sym is the Hopf algebra of quasisymmetric functions, QSym whose bases are indexed by integer compositions α. This space was defined in [10] by using P-partitions, which gives rise to the fundamental quasisymmetric functions F_α. In [5] the authors studied two powersum quasisymmetric bases (i.e. a basis of QSym that refines p_λ) Φ and Ψ whose duals are the $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ bases in NSym, the algebra of non-commutative symmetric functions, which is defined in [9]. These bases of NSym aren't defined combinatorially, but via formal series. The Ψ basis is most notable for the change of basis to the fundamental basis by using P-partitions as defined in [2]. In [3] the authors introduced the Shuffle basis S_α, which also refines p_λ and is notable because S_α is an eigenvector under the theta map Θ.

In this paper we define a powersum quasisymmetric basis P_α combinatorially by using fillings in an analog of (1). Alternatively, P_α can be defined using a subposet of the refinement poset $\mathcal{P}$ on compositions. This basis has a shuffle product, a deconcatenate coproduct, refines the powersum symmetric basis and is dual (up to scaling) to the Zassenhaus functions Z_α in NSym as defined in [12]. All of these results are stated in Sections 3 and 4. We note that this powersum basis was independently defined in [4] using weighted generating functions of P-partitions. In Section 5, we show a Murnaghan-Nakayama like change of basis rule from the powersum quasisymmetric basis to the quasisymmetric fundamental basis by use of tuples of ribbons.

In Section 6, we note that both the filling and the subposet notions have generalizations that depend on a total order $\preceq$ on the parts of the compositions. Hence we can define a whole family of powersum quasisymmetric bases, denoted as $P_\alpha^{\succeq}$ for different choices of $\succeq$, so that all the properties described are held for any choice of $\succeq$.

Finally, in Section 12 we show how one might use this method to define a powersum basis in other algebras.

Figure 1 summarizes the relationships between our new powersum bases for QSym and NCQSym and existing powersum bases in other algebras.

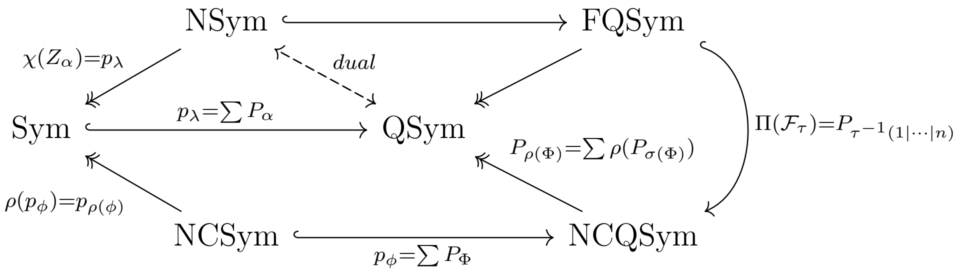


Figure 1: Diagram of some Hopf Algebras related to QSym

2 Preliminaries for QSym

Given an integer composition of n of length k, $\alpha = (a_1, a_2, \ldots, a_k)$, let $m_i(\alpha)$ denote the multiplicities of the part size i within α, and denote the weak composition of size multiplicities by $m(\alpha) = (m_1(\alpha), \ldots, m_n(\alpha))$. Let $\mathcal{P}$ be a poset on compositions with the cover relation $\alpha \lessdot \beta$ if β is obtained from α by summing two adjacent parts, i.e. $\beta = (a_1, \ldots, a_i + a_{i+1}, \ldots, a_k)$. In this paper $\alpha \leqslant \beta$ will refer to $\mathcal{P}$, unless stated otherwise.

For the purpose of several proofs in Section 10, the standard bijection between compositions of n and subsets of $[n-1]$ will be useful:

$$(a_1, \ldots, a_k) \longleftrightarrow \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{k-1}\}.$$

Using this identification, we see $\alpha \lessdot \beta$ if and only if as sets $\alpha = \beta \cup \{k\}$ for some $k \in [n-1]$.

Let $X = \{x_1, x_2, \ldots\}$ be a set of variables and $\alpha = (a_1, \ldots, a_k)$ be a composition of n. Then a formal power series of bounded degree is quasisymmetric if, for every k and $i_1 < i_2 < \cdots < i_k$, the coefficient of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ is equal to the coefficient of $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$. When $|X| = \infty$, the set of all quasisymmetric functions is a graded Hopf algebra denoted as QSym. One of the more natural bases of QSym is the *quasisymmetric monomial basis* M_α, defined as

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}.$$

QSym is most notable for the *quasisymmetric fundamental basis* F_α which proves useful for countless enumeration problems and also corresponds to the characters of the 0-Hecke algebra. It is defined as

$$F_\alpha = \sum_{\beta \leqslant \alpha} M_\beta.$$

3 Powersum functions in QSym

3.1 Filling Interpretations

A *filling* of a composition $\alpha = (a_1, \ldots, a_k)$ is a $k \times l$ matrix where each a_i can be placed anywhere in row i and omit all columns that doesn't have an entry a_i (which implies

$l \leqslant k$). Given a filling F, the *column sum* of a filling, denoted $\mathsf{col}(\mathsf{F})$, is the composition $\beta = (b_1, \ldots, b_l)$ such that the sum of all the entries in column j is b_j. The *row sum*, $\mathsf{row}(\mathsf{F})$, has a similar definition, i.e. $\mathsf{row}(\mathsf{F}) = \alpha$.

Then $\sigma \in \mathfrak{S}_{m_1(\alpha)} \times \cdots \times \mathfrak{S}_{m_n(\alpha)} = \mathfrak{S}_{m(\alpha)}$ is a *row permutation* and acts on fillings by permuting two rows that contain parts of like sizes. Given a filling F of a composition, $\mathfrak{S}_{\mathsf{F}}$ is the set of row permutations between different columns. Note that $\mathfrak{S}_{\mathsf{F}}$ is a set of coset representatives in $\mathfrak{S}_{m(\alpha)}$ (for the subgroup which permute only within columns). Also note that $\mathsf{col}(\sigma\mathsf{F}) = \mathsf{col}(\mathsf{F})$, and $\mathfrak{S}_{\mathsf{F}}$ is dependent only on $\mathsf{row}(\mathsf{F})$ and $\mathsf{col}(\mathsf{F})$.

Example 1. Let F be the left filling below. Then $\mathfrak{S}_{\mathsf{F}} = \{\mathsf{id}, (1,4), (2,4)\}$. The middle and right filling are $(1,4)\mathsf{F}$ and $(2,4)\mathsf{F}$ respectively.

$$\begin{array}{|ccc} 2 & \cdot & \cdot \\ 2 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 2 \\ \hline 4 & 1 & 2 \end{array} \qquad \begin{array}{|ccc} \cdot & \cdot & 2 \\ 2 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ 2 & \cdot & \cdot \\ \hline 4 & 1 & 2 \end{array} \qquad \begin{array}{|ccc} 2 & \cdot & \cdot \\ \cdot & \cdot & 2 \\ \cdot & 1 & \cdot \\ 2 & \cdot & \cdot \\ \hline 4 & 1 & 2 \end{array}$$

Remark 2. Fillings can be defined more generally to generate other bases in Sym, see [19], but for the subject of this paper the definition above suffices.

The following two fillings are important to the sequel.

Definition 3. A filling F of $(a_1, \ldots, a_k)$ is *diagonal descending* if there exists a row permutation $\sigma \in \mathfrak{S}_{\mathsf{F}}$ such that the entries of $\sigma\mathsf{F}$ follow:

1. Entry a_1 is in the upper leftmost corner of the matrix.
2. Entry a_i is in row i.
3. (a) If $a_{i-1} \geqslant a_i$ then a_i is directly below a_{i-1} or in the southeast position of a_{i-1}.
   (b) If $a_{i-1} < a_i$ then a_i is in the southeast position of a_{i-1}.

Denote the set of all diagonal descending fillings with row sum α as $\mathcal{DD}(\alpha)$.

Example 4. $\mathcal{DD}(212)$ consists of the following four fillings.

$$\begin{array}{|ccc} 2 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 2 \\ \hline 2 & 1 & 2 \end{array} \qquad \begin{array}{|ccc} 2 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 2 & \cdot \\ \hline 3 & 2 & \end{array} \qquad \begin{array}{|ccc} \cdot & \cdot & 2 \\ \cdot & 1 & \cdot \\ 2 & \cdot & \cdot \\ \hline 2 & 1 & 2 \end{array} \qquad \begin{array}{|ccc} \cdot & 2 & \cdot \\ 1 & \cdot & \cdot \\ 2 & \cdot & \cdot \\ \hline 3 & 2 & \end{array}$$

Definition 5. Let α be a composition of n. Then the *descending powersum quasisymmetric function* is defined as

$$P_\alpha = \sum_{\mathsf{F} \in \mathcal{DD}(\alpha)} M_{\mathsf{col}(\mathsf{F})}. \tag{2}$$

Thus from Example 4, $P_{212} = 2M_{212} + 2M_{32}$. It is clear that these powersum quasisymmetrics form a basis of QSym as they have a triangular change of basis from the monomials.

It will be convenient to group fillings by row permutation type.

Definition 6. A *strict diagonal* filling is a filling such that Conditions 1-3 of Definition 3 are satisfied, i.e. the permutation that sastisfies Conditions 1-3 is id. Denote the set of strict diagonal fillings with $\mathsf{row}(\mathsf{F}) = \alpha$ as $\mathcal{SD}(\alpha)$.

The first two fillings of Example 4 are all the fillings in $\mathcal{SD}(212)$.

Notice that we can define a $\mathcal{DD}$ filling as a pair of $\mathcal{SD}$ filling and a row permutation, which leads us to an alternate definition,

$$P_\alpha = \sum_{\substack{\mathsf{F}\in\mathcal{SD}(\alpha)\\ \sigma\in\mathfrak{S}_\mathsf{F}}} M_{\mathsf{col}(\sigma(\mathsf{F}))} = \sum_{\mathsf{F}\in\mathcal{SD}(\alpha)} |\mathfrak{S}_\mathsf{F}| M_{\mathsf{col}(\mathsf{F})}, \tag{3}$$

where the last equality comes from the fact that $\mathsf{col}(\sigma\mathsf{F}) = \mathsf{col}(\mathsf{F})$.

In [4, Definition 5.1], the authors independently defined a powersum basis using weighted P-partitions, that, when translated to fillings, is precisely this one: define push to be a map of fillings by a sorting of rows such that the reading word is a partition. Then the image of $\mathcal{DD}(\alpha)$ with column sum β under push is the set of fillings $\mathfrak{R}_{\alpha\beta}$ as described in [4, Theorem 5.12].

Example 7. Let T be the matrix transpose map. Let F_1 be the top left $\mathcal{DD}$ filling, and F_2 be the bottom left $\mathcal{DD}$ filling. Then $T(\mathsf{push}(\mathsf{F}_1))$ and $T(\mathsf{push}(\mathsf{F}_2))$ are respectively the second and third filling of [4, Example 5.13].

$$\begin{array}{|ccc} \cdot & \cdot & 1 \\ \cdot & 2 & \cdot \\ 1 & \cdot & \cdot \\ \hline 1 & 2 & 1 \end{array} \quad \xrightarrow{\mathsf{push}} \quad \begin{array}{|ccc} \cdot & 2 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \hline 1 & 2 & 1 \end{array} \quad \xrightarrow{T} \quad \begin{pmatrix} \cdot & \cdot & 1 \\ 2 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}$$

$$\begin{array}{|cc} 1 & \cdot \\ \cdot & 2 \\ \cdot & 1 \\ \hline 1 & 3 \end{array} \quad \xrightarrow{\mathsf{push}} \quad \begin{array}{|cc} \cdot & 2 \\ 1 & \cdot \\ \cdot & 1 \\ \hline 1 & 3 \end{array} \quad \xrightarrow{T} \quad \begin{pmatrix} \cdot & 1 & \cdot \\ 2 & \cdot & 1 \end{pmatrix}$$

They also prove the following result.

Theorem 8. *The powersum quasisymmetric functions refine the powersum symmetric functions. In other words,*

$$p_\lambda = \sum_{\alpha\colon \mathsf{sort}(\alpha)=\lambda} P_\alpha. \tag{4}$$

We remark that one can prove this refinement property by changing p_λ and P_α into the quasimonomial basis and showing that the two expressions are equivalent. An analogue of this proof is in Section 9.1.

3.2 Subposet Interpretation

Notice that we can define these powersum quasisymmetrics by subposets as all fillings only coarsen according to the row sum. Given compositions $\alpha = (a_1, \ldots, a_k) \leqslant \beta = (b_1, \ldots, b_l)$, there exists $1 \leqslant k_1 \leqslant \ldots \leqslant k_j \leqslant k$ such that

$$b_1 = \sum_{i=1}^{k_1} a_i, \; b_2 = \sum_{i=k_1+1}^{k_2} a_i, \; \ldots, \; b_l = \sum_{i=k_j+1}^{k} a_i.$$

Given a positive integer L, define a composition $c_L(\alpha, \beta)$ as

$$c_L(\alpha, \beta) = \left(\sum_{i=1}^{k_1} \delta_{La_i}, \; \sum_{i=k_1+1}^{k_2} \delta_{La_i}, \; \ldots \; \sum_{i=k_j+1}^{k} \delta_{La_i} \right)$$

where δ_{ij} is the Kronecker delta. Let $c_L(\alpha, \beta)!$ be the product of the factorials of the parts of $c_L(\alpha, \beta)$.

Example 9. Let $\alpha = (3, 2, 1, 1, 3, 1, 1, 1, 2, 1)$ and $\beta = (6, 1, 3, 1, 2, 3)$. Then

$$\begin{aligned} c_1(\alpha, \beta) &= (0 + 0 + 1, 1, 0, 1, 1 + 1, 0 + 1) = (1, 1, 1, 2, 1) \\ c_2(\alpha, \beta) &= (0 + 1 + 0, 0, 0, 0, 0 + 0, 1 + 0) = (1, 1) \\ c_3(\alpha, \beta) &= (1 + 0 + 0, 0, 1, 0, 0 + 0, 0 + 0) = (1, 1) \\ c_1(\alpha, \beta)! &= 1! \times 1! \times 1! \times 2! \times 1! = 2. \end{aligned}$$

The following proposition is needed in proving formulas for the product, coproduct, and that the dual basis is the Zassenhaus basis which are Theorems 14, 17, and 23 respectively.

Proposition 10. *Let $\alpha, \alpha', \beta, \beta'$ be compositions such that $\alpha \leqslant \beta$ and $\alpha' \leqslant \beta'$. Then*

$$c_i(\alpha|\alpha', \beta|\beta') = c_i(\alpha, \beta)|c_i(\alpha', \beta')$$

where $\alpha|\alpha'$ denotes the concatenation of α and α'.

This proposition follows from the definition and Example 9. For example, let $\alpha = (3, 2, 1, 1)$, $\alpha' = (3, 1, 1, 1, 2, 1)$, $\beta = (6, 1)$, and $\beta' = (3, 1, 2, 3)$. Then,

$$c_1(\alpha|\alpha', \beta|\beta') = (1, 1, 1, 2, 1) = (1, 1)|(1, 2, 1) = c_1(\alpha, \beta)|c_1(\alpha', \beta').$$

Let $\mathcal{D}$ be the subposet of $\mathcal{P}$ with the cover relation $\lessdot_{\mathcal{D}}$ given by: $\alpha \lessdot_{\mathcal{D}} \beta$ if β is obtained from α by summing two descending adjacent parts, i.e. $\beta = (a_1, \ldots, a_i + a_{i+1}, \ldots, a_k)$ for some i satisfying $a_i \geqslant a_{i+1}$. Then define $C(\alpha)$ to be the maximal composition of the subposet $\mathcal{D}$ bigger than or equal to α. From Example 9, $C(3, 2, 1, 1, 3, 1, 1, 1, 2, 1) = (7, 6, 3)$.

Let F be a $\mathcal{SD}$ filling. The number of permutations of rows with entry i is $m_i(\mathsf{row}(\mathsf{F}))!$. Furthermore, the number of permutations that only permute a_j and a_k if they are in the same column and $a_j = a_k = i$ is $c_i(\mathsf{row}(\mathsf{F}), \mathsf{col}(\mathsf{F}))!$. Thus,

$$|\mathfrak{S}_{\mathsf{F}}| = \prod_{i=1}^{m} \frac{m_i(\mathsf{row}(\mathsf{F}))!}{c_i(\mathsf{row}(\mathsf{F}), \mathsf{col}(\mathsf{F}))!}.$$

This leads to an alternate definition of a descending powersum quasisymmetric.

Definition 11. Let α and β be partitions of n and

$$C_{\alpha\beta} = \prod_{i=1}^{m} \frac{m_i(\alpha)!}{c_i(\alpha, \beta)!}. \tag{5}$$

Then an alternate definition of the *(descending) powersum quasisymmetric* function is

$$P_\alpha = \sum_{\alpha \leqslant \beta \leqslant C(\alpha)} C_{\alpha\beta} M_\beta = \sum_{\alpha \leqslant_{\mathcal{D}} \beta} C_{\alpha\beta} M_\beta. \tag{6}$$

Example 12.

$$\begin{aligned} P_{1211} &= \left(\frac{3!}{1!1!1!}\right)\left(\frac{1!}{1!}\right) M_{1211} + \left(\frac{3!}{1!2!}\right)\left(\frac{1!}{1!}\right) M_{122} \\ &\quad + \left(\frac{3!}{1!1!1!}\right)\left(\frac{1!}{1!}\right) M_{131} + \left(\frac{3!}{1!2!}\right)\left(\frac{1!}{1!}\right) M_{14} \\ &= 6M_{1211} + 3M_{122} + 6M_{131} + 3M_{14} \end{aligned}$$

3.3 Product and Coproduct

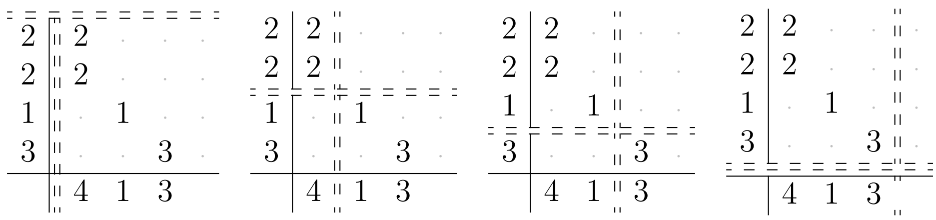
A *deconcatenation* of an $\mathcal{SD}$ filling F, denoted $\mathsf{F} = \mathsf{F}_1|\mathsf{F}_2$, is achieved by drawing a vertical line in between two columns to get two $\mathcal{SD}$ fillings F_1 and F_2. Notice that (for $\mathcal{SD}$ fillings) there is also a horizontal line that we could draw to deconcatenate the filling. Moreover, $\mathsf{col}(\mathsf{F}_1)|\mathsf{col}(\mathsf{F}_2) = \mathsf{col}(\mathsf{F})$ and $\mathsf{row}(\mathsf{F}_1)|\mathsf{row}(\mathsf{F}_2) = \mathsf{row}(\mathsf{F})$.

Evidently the coproduct of a quasimonomial function can be expressed as

$$\Delta(M_{\mathsf{col}(\mathsf{F})}) = \sum_{\mathsf{F}_1|\mathsf{F}_2=\mathsf{F}} M_{\mathsf{col}(\mathsf{F}_1)} \otimes M_{\mathsf{col}(\mathsf{F}_2)}, \tag{7}$$

where F is an $\mathcal{SD}$ filling.

Example 13. The deconcatenation of a filling mimics the deconcatenation of a quasimonomial.



Likewise, $\Delta(M_{413}) = 1 \otimes M_{413} + M_4 \otimes M_{13} + M_{41} \otimes M_3 + M_{413} \otimes 1$.

Theorem 14. *Let γ be a composition of n. Then a powersum quasisymmetric has the deconcatenation coproduct:*

$$\Delta(P_\gamma) = \sum_{\alpha|\beta=\gamma} \prod_{i=1}^{n} \frac{m_i(\gamma)!}{m_i(\alpha)!m_i(\beta)!} P_\alpha \otimes P_\beta. \tag{8}$$

Proof. By (3) and (7)

$$\Delta(P_\gamma) = \Delta\left(\sum_{\mathsf{F}\in\mathcal{SD}(\gamma)} |\mathfrak{S}_\mathsf{F}| M_{\mathsf{col}(\mathsf{F})}\right) = \sum_{\mathsf{F}\in\mathcal{SD}(\gamma)} \sum_{\mathsf{F}_1|\mathsf{F}_2=\mathsf{F}} |\mathfrak{S}_\mathsf{F}| M_{\mathsf{col}(\mathsf{F}_1)} \otimes M_{\mathsf{col}(\mathsf{F}_2)}.$$

Let m be the degree of $\mathsf{col}(\mathsf{F}_1)$, then fix m. Let F_1' be the filling with $\mathsf{row}(\mathsf{F}_1) = \mathsf{row}(\mathsf{F}_1')$, say $\mathsf{row}(\mathsf{F}_1') = (\gamma_1, \dots, \gamma_j)$. Then every filling of degree m comes from moving entry γ_{i+1} to the column of γ_i when $\gamma_i \geqslant \gamma_{i+1}$ for certain choices of i. In other words, the fillings of degree m is precisely $\mathcal{SD}(\gamma_1, \dots, \gamma_j)$. The same is said for the right side, and thus

$$\sum_{\mathsf{F}\in\mathcal{SD}(\gamma)} \sum_{\mathsf{F}_1|\mathsf{F}_2=\mathsf{F}} |\mathfrak{S}_\mathsf{F}| M_{\mathsf{col}(\mathsf{F}_1)} \otimes M_{\mathsf{col}(\mathsf{F}_2)} = \sum_{\alpha|\beta=\gamma} \sum_{\substack{\mathsf{F}_1\in\mathcal{SD}(\alpha)\\ \mathsf{F}_2\in\mathcal{SD}(\beta)}} \left|\mathfrak{S}_{\mathsf{F}_1|\mathsf{F}_2}\right| M_{\mathsf{col}(\mathsf{F}_1)} \otimes M_{\mathsf{col}(\mathsf{F}_2)}.$$

Finally to express this in the powersum quasisymmetric basis we divide the coefficient by $|\mathfrak{S}_{\mathsf{F}_1}||\mathfrak{S}_{\mathsf{F}_2}|$. We simplify the coefficient through (5) and Proposition 10 then we get

$$\frac{\left|\mathfrak{S}_{\mathsf{F}_1|\mathsf{F}_2}\right|}{|\mathfrak{S}_{\mathsf{F}_1}||\mathfrak{S}_{\mathsf{F}_2}|} = \prod_{i=1}^{n} \frac{\frac{m_i(\gamma)!}{c_i(\gamma,\mathsf{col}(\mathsf{F}_1|\mathsf{F}_2))!}}{\frac{m_i(\alpha)!}{c_i(\alpha,\mathsf{col}(\mathsf{F}_1))!}\frac{m_i(\beta)!}{c_i(\beta,\mathsf{col}(\mathsf{F}_2))!}} = \prod_{i=1}^{n} \frac{m_i(\gamma)!}{m_i(\alpha)!m_i(\beta)!}. \tag{9}$$

The coefficient is interpreted as all the permutations of $\mathfrak{S}_\mathsf{F}$ that permuted a row in F_1 to a row in F_2. □

Let $\alpha = (a_1, \dots, a_k)$ and $\beta = (b_1, \dots, b_l)$ be compositions. The *shuffle* of α and β, denoted by $\alpha \sqcup\!\sqcup \beta$, is the multiset of compositions defined recursively by

1. $\emptyset \sqcup\!\sqcup \alpha = \alpha \sqcup\!\sqcup \emptyset = \{\alpha\}$
2. $\alpha \sqcup\!\sqcup \beta = (a_1)|\big((a_2, \dots, a_k) \sqcup\!\sqcup \beta\big) + (b_1)|\big(\alpha \sqcup\!\sqcup (b_2, \dots, b_l)\big)$,

using $+$ for disjoint (multiset) union. If we add to the recursion condition (2) the term $(a_1 + b_1)|\big((a_2, \dots, a_k) \sqcup\!\sqcup (b_2, \dots, b_l)\big)$, then this is called the *quasi-shuffle* and is denoted as $\widetilde{\sqcup\!\sqcup}$. Recall that the product rule of the quasimonomial basis is a quasishuffle product:

$$M_\alpha M_\beta = \sum_{\gamma\in\alpha\widetilde{\sqcup\!\sqcup}\beta} M_\gamma.$$

Likewise, the quasishuffle of fillings $\mathsf{F}_1 = (c_1, \dots, c_k)$ and $\mathsf{F}_2 = (d_1, \dots, d_l)$, where c_i and d_i are the columns of F_1 and F_2 respectively, is defined recursively

1. $\emptyset \sqcup\!\sqcup \mathsf{F}_1 = \mathsf{F}_1 \sqcup\!\sqcup \emptyset = \{\mathsf{F}_1\}$ where $\emptyset$ is the empty filling
2. $\mathsf{F}_1 \sqcup\!\sqcup \mathsf{F}_2 = c_1 | \big((c_2, \ldots, c_k) \sqcup\!\sqcup \mathsf{F}_2\big) + d_1 | \big(\mathsf{F}_1 \sqcup\!\sqcup (d_2, \ldots, d_l)\big) + (c_1 + d_1) | \big((c_2, \ldots, c_k) \sqcup\!\sqcup (d_2, \ldots, d_l)\big)$

where: concatenating columns c_i and c_j means starting c_j southeast of c_i; and addition $c_i + c_j$ builds one column by sort combining the two columns.

Example 15. Let $c_i = (5, 3, 2)$ and $c_j = (4, 2, 1)$. Then the concatenation of columns c_i and c_j is

$$\begin{array}{|cc} 5 & \cdot \\ 3 & \cdot \\ 2 & \cdot \\ \cdot & 4 \\ \cdot & 2 \\ \cdot & 1 \\ \hline \end{array}\,.$$

Example 16. The following illustrates that $M_{23}M_1 = M_{231} + M_{213} + M_{24} + M_{123} + M_{33}$.

$$\begin{array}{c|cc} 2 & 2 & \cdot \\ 1 & \cdot & 1 \\ \hline & 2 & 1 \end{array} \;\widetilde{\sqcup\!\sqcup}\; \begin{array}{c|c} 3 & 3 \\ \hline & 3 \end{array} = \begin{array}{c|c} 2 & 2 \\ \hline & 2 \end{array} \Bigg| \left(\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \;\widetilde{\sqcup\!\sqcup}\; \begin{array}{c|c} 3 & 3 \\ \hline & 3 \end{array} \right) + \begin{array}{c|c} 3 & 3 \\ \hline & 3 \end{array} \Bigg| \left(\begin{array}{c|cc} 2 & 2 & \cdot \\ 1 & \cdot & 1 \\ \hline & 2 & 1 \end{array} \;\widetilde{\sqcup\!\sqcup}\; \emptyset \right)$$

$$+ \begin{array}{c|c} 3 & 3 \\ 2 & 2 \\ \hline & 5 \end{array} \Bigg| \left(\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \;\widetilde{\sqcup\!\sqcup}\; \emptyset \right)$$

$$= \begin{array}{c|cc} 2 & 2 & \cdot \\ 1 & \cdot & 1 \\ \hline & 2 & 1 \end{array} \Bigg| \left(\emptyset \;\widetilde{\sqcup\!\sqcup}\; \begin{array}{c|c} 3 & 3 \\ \hline & 3 \end{array} \right) + \begin{array}{c|cc} 2 & 2 & \cdot \\ 3 & \cdot & 3 \\ \hline & 2 & 3 \end{array} \Bigg| \left(\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \;\widetilde{\sqcup\!\sqcup}\; \emptyset \right)$$

$$+ \begin{array}{c|cc} 2 & 2 & \cdot \\ 3 & \cdot & 3 \\ 1 & \cdot & 1 \\ \hline & 2 & 4 \end{array} + \begin{array}{c|ccc} 3 & 3 & \cdot & \cdot \\ 2 & \cdot & 2 & \cdot \\ 1 & \cdot & \cdot & 1 \\ \hline & 3 & 2 & 1 \end{array} + \begin{array}{c|cc} 3 & 3 & \cdot \\ 2 & 2 & \cdot \\ 1 & \cdot & 1 \\ \hline & 5 & 1 \end{array}$$

$$= \begin{array}{c|ccc} 2 & 2 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ 3 & \cdot & \cdot & 3 \\ \hline & 2 & 1 & 3 \end{array} + \begin{array}{c|ccc} 2 & 2 & \cdot & \cdot \\ 3 & \cdot & 3 & \cdot \\ 1 & \cdot & \cdot & 1 \\ \hline & 2 & 3 & 1 \end{array} + \begin{array}{c|cc} 2 & 2 & \cdot \\ 3 & \cdot & 3 \\ 1 & \cdot & 1 \\ \hline & 2 & 4 \end{array} + \begin{array}{c|ccc} 3 & 3 & \cdot & \cdot \\ 2 & \cdot & 2 & \cdot \\ 1 & \cdot & \cdot & 1 \\ \hline & 3 & 2 & 1 \end{array} + \begin{array}{c|cc} 3 & 3 & \cdot \\ 2 & 2 & \cdot \\ 1 & \cdot & 1 \\ \hline & 5 & 1 \end{array}$$

Observe from the last example that if $\mathsf{F} \in \mathsf{F}_1 \widetilde{\sqcup\!\sqcup} \mathsf{F}_2$, then $\mathsf{row}(\mathsf{F}) \in \mathsf{row}(\mathsf{F}_1) \sqcup\!\sqcup \mathsf{row}(\mathsf{F}_2)$ and $\mathsf{col}(\mathsf{F}) \in \mathsf{col}(\mathsf{F}_1) \widetilde{\sqcup\!\sqcup} \mathsf{col}(\mathsf{F}_2)$.

Theorem 17. *Let α and β be compositions of n. Then the powersum quasisymmetrics have a shuffle product, in other words*

$$P_\alpha P_\beta = \sum_{\gamma \in \alpha \sqcup\!\sqcup \beta} \prod_i \frac{m_i(\alpha)! m_i(\beta)!}{m_i(\gamma)!} P_\gamma.$$

Proof. From (3)

$$\begin{aligned} P_\alpha P_\beta &= \left(\sum_{\mathsf{F}_1 \in \mathcal{SD}(\alpha)} |\mathfrak{S}_{\mathsf{F}_1}| M_{\mathsf{col}(\mathsf{F}_1)} \right) \left(\sum_{\mathsf{F}_2 \in \mathcal{SD}(\beta)} |\mathfrak{S}_{\mathsf{F}_2}| M_{\mathsf{col}(\mathsf{F}_2)} \right) \\ &= \sum_{\substack{\mathsf{F}_1 \in \mathcal{SD}(\alpha), \mathsf{F}_2 \in \mathcal{SD}(\beta) \\ \mathsf{F} \in \mathsf{F}_1 \widetilde{\sqcup\!\sqcup} \mathsf{F}_2}} |\mathfrak{S}_{\mathsf{F}_1}||\mathfrak{S}_{\mathsf{F}_2}| M_{\mathsf{col}(F)}. \end{aligned}$$

In order to express the above in the powersum basis, we have to switch the sums, i.e. use $\mathcal{SD}(\alpha \sqcup\!\sqcup \beta) = \bigcup_{\gamma \in \alpha \sqcup\!\sqcup \beta} \mathcal{SD}(\gamma)$ instead of $\mathcal{SD}(\alpha) \widetilde{\sqcup\!\sqcup} \mathcal{SD}(\beta)$. We will first show that these two are equivalent as sets, then show that multiplying by the reciprocal of the double count that comes from the quasishuffle, denoted O_{F}, will result in them being equal as multisets. Finally, we will reduce the coefficient.

Notice that any filling in $\mathcal{SD}(\alpha) \widetilde{\sqcup\!\sqcup} \mathcal{SD}(\beta)$ is also in $\mathcal{SD}(\alpha \sqcup\!\sqcup \beta)$ due to the fact that every filling $\mathsf{F} \in \mathsf{F}_1 \widetilde{\sqcup\!\sqcup} \mathsf{F}_2$ has the property that $\mathsf{row}(\mathsf{F}) \in \mathsf{row}(\mathsf{F}_1) \sqcup\!\sqcup \mathsf{row}(\mathsf{F}_2)$. Conversely, we can also construct any filling $\mathsf{F} \in \mathcal{SD}(\alpha \sqcup\!\sqcup \beta)$ as a quasishuffle of two fillings F_1 and F_2 of $\mathcal{SD}(\alpha)$ and $\mathcal{SD}(\beta)$ respectively. Fix a filling F, then if the first row is from α, make a new filling F_1' where the first row is F_1' is the first row of F (and likewise for β) and remove the first row from F. Continue to do this for all row to get two fillings F_1 and F_2 from $\mathcal{SD}(\alpha)$ and $\mathcal{SD}(\beta)$. Thus $\mathcal{SD}(\alpha) \widetilde{\sqcup\!\sqcup} \mathcal{SD}(\beta)$ and $\mathcal{SD}(\alpha \sqcup\!\sqcup \beta)$ are equivalent as sets, but not as multisets.

In order to switch the sum from $\mathcal{SD}(\alpha) \widetilde{\sqcup\!\sqcup} \mathcal{SD}(\beta)$ to $\mathcal{SD}(\alpha \sqcup\!\sqcup \beta)$, we must consider the double count that comes from the quasishuffle, which we illustrate this with an example.

Example 18. Let $\mathsf{F}_1 \in \mathcal{SD}(2,1,1,1)$ and $\mathsf{F}_2 \in \mathcal{SD}(1,1,3)$ be the fillings shown in Figure 2. One of the resulting fillings is when we combine the second column of F_1 and the first column of F_2. Notice that there is only one way to make this filling. However if we consider the number of ways that this filling is generated by $\mathcal{SD}((2,1,1,1) \sqcup\!\sqcup (1,1,3))$, then that would be $\binom{5}{3}$.

$$\begin{array}{c|cc} 2 & 2 & \cdot \\ 1 & \cdot & 1 \\ 1 & \cdot & 1 \\ 1 & \cdot & 1 \\ \hline & 2 & 3 \end{array} \quad \widetilde{\sqcup\!\sqcup} \quad \begin{array}{c|cc} 1 & 1 & \cdot \\ 1 & 1 & \cdot \\ 3 & \cdot & 3 \\ \hline & 2 & 3 \end{array} \quad \ni \quad \begin{array}{c|ccc} 2 & 2 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & \cdot \\ 1 & \cdot & 1 & \cdot \\ 3 & \cdot & \cdot & 3 \\ \hline & 2 & 5 & 3 \end{array}$$

Figure 2: The quasishuffle of fillings

Let $\alpha = (a_1, \ldots, a_k)$, $\beta = (b_1, \ldots, b_l)$, and fix $\mathsf{F} \in \mathcal{SD}(\alpha \sqcup\!\sqcup \beta)$, $\mathsf{F}_1 \in \mathcal{SD}(\alpha)$, and $\mathsf{F}_2 \in \mathcal{SD}(\beta)$ such that $\mathsf{F} \in \mathsf{F}_1 \widetilde{\sqcup\!\sqcup} \mathsf{F}_2$. Then we define the partitions α_j and $\beta_{j'}$ to be the entries of column j of F and column j' of F_2 respectively. Then it is easy to see that $\mathsf{col}(\mathsf{F}_1) = (|\alpha_1|, \ldots, |\alpha_{k'}|)$ and $\alpha = \alpha_1|\cdots|\alpha_{k'}$. Let J be a column of F such that the combination of columns j and j' from fillings F_1 and F_2 results in the column J. The number of ways building the column J of F from the shuffle of α_j and $\beta_{j'}$ is $\binom{m_i(\alpha_j)+m_i(\beta_{j'})}{m_i(\alpha_j)}$. Then we define O_{F} as the reciprocal of all such ways to build J for all columns with such J. To switch the sum as described above, we multiply O_{F} such that for any two columns that combine in result of the quasishuffle, say α_j and $\beta_{j'}$, then

$$O_{\mathsf{F}} = \prod_i \frac{1}{\prod_{j,j'} \binom{m_i(\alpha_j)+m_i(\beta_{j'})}{m_i(\alpha_j)}}$$

where the product runs over all j, j' where α_j is combined with $\beta_{j'}$. From the example above, $O_{\mathsf{F}} = 1/\binom{3+2}{3}$.

Finally the coefficient that is obtained from changing the expression into the powersum basis is as follows

$$O_{\mathsf{F}} \frac{|\mathfrak{S}_{\mathsf{F}_1}||\mathfrak{S}_{\mathsf{F}_2}|}{|\mathfrak{S}_{\mathsf{F}}|} = \prod_{i=1}^{n} \frac{1}{\prod_{j,j'} \binom{m_i(\alpha_j)+m_i(\beta_{j'})}{m_i(\alpha_j)}} \frac{c_i(\gamma, \mathsf{col}(\mathsf{F}_1 \widetilde{\sqcup} \mathsf{F}_2))!}{c_i(\alpha, \mathsf{col}(\mathsf{F}_1))! c_i(\beta, \mathsf{col}(\mathsf{F}_2))!} \left[\frac{m_i(\alpha)! m_i(\beta)!}{m_i(\gamma)!}\right].$$

It suffices to show that this expression reduces to what is inside the bracket. By Proposition 10 we can write

$$c_i((a_1, \ldots, a_l), (|\alpha_1|, \ldots, |\alpha_{k'}|))! = c_i((a_1, \ldots, a_{i_1}), |\alpha_1|)! \cdots c_i((a_{i_{k'-1}+1}, \ldots, a_k), |\alpha_{k'}|)!,$$

and similarly for β. Consequently everything outside of the bracket can be considered column by column, while what is inside the bracket comes from the whole of a filling. Since F is a quasishuffle of F_1 and F_2 there are three ways a column of F is made.

First consider the case when a column a_j isn't combined. Fix i, then O_{F} doesn't contribute to the coefficient which leaves us with

$$\frac{c_i((a_{i_{j-1}+1}, \ldots, a_j), |\alpha_j|)!}{c_i((a_{i_{j-1}+1}, \ldots, a_j), |\alpha_j|)!}$$

which is obviously 1. The same argument is made when $\beta_{j'}$ isn't combined with any other column.

The third case is when two columns α_j and $\beta_{j'}$ combine. Fix i, and we get the expression

$$\frac{1}{\binom{m_i(\alpha_j)+m_i(\beta_{j'})}{m_i(\beta_{j'})}} \frac{c_i((a_{i_{j-1}+1}, \ldots, a_j) \sqcup (b_{i_{j'-1}+1}, \ldots, b_{j'}), |\alpha_j| + |\beta_{j'}|)!}{c_i((a_{i_{j-1}+1}, \ldots, a_j), |\alpha_j|)! c_i((b_{i_{j'-1}+1}, \ldots, b_{j'}), |\beta_{j'}|)!}.$$

Recall that the formula of c_i counts the number of i's in the composition

$$(a_{i_{j-1}+1}, \ldots, a_j)|(b_{i_{j'-1}+1}, \ldots, b_{j'}),$$

which is, in this case, the same as the function m_i. Thus using this information and writing out the choose function yields

$$\frac{m_i(\alpha_j)!(m_i(\alpha_j)+m_i(\beta_{j'})-m_i(\alpha_j))!}{(m_i(\alpha_j)+m_i(\beta_{j'}))!}\frac{m_i((a_{i_{j-1}+1},\ldots,a_j)\sqcup(b_{i_{j'-1}+1},\ldots,b_{j'}))!}{m_i((a_{i_{j-1}+1},\ldots,a_j))!m_i((b_{i_{j'-1}+1},\ldots,b_{j'})))!}.$$

Recall that α_j are the entries of column j and $(a_{i_{j-1}+1},\ldots,a_j)$ are also the entries of column j, thus the expression is 1, which completes the proof. □

We with hold a formula for antipode for now.

Remark 19. Theorems 14 and 17 can be proven (perhaps more straightforwardly) without using fillings. However, our chosen approach, reveals a Hopf algebra structure on fillings. We have defined a product and a coproduct that satisfies the conditions for being a Hopf algebra and because this is graded connected, there exists an antipode.

4 The Scaled Powersum Quasisymmetric Basis and its Dual

4.1 Scaled Powersum Quasisymmetric Basis

We will now look at a scaled version of this basis that will end up being the dual of the Zassenhauss basis and will have a nice product and coproduct. Recall the scalar

$$z_\lambda=\prod_i i^{m_i(\lambda)}m_i(\lambda)!.$$

We extend this definition to compositions by $z_\alpha=z_{\mathsf{sort}(\alpha)}$. Powersum symmetrics form an orthogonal basis for Sym, to be more exact, $\langle p_\lambda,p_\mu\rangle=z_\lambda\delta_{\lambda\mu}$ where $\langle,\rangle$ is the Hall inner product. Thus the *scaled powersum symmetrics* is defined as $\widetilde{p}_\lambda=z_\lambda^{-1}p_\lambda$, so that $\langle\widetilde{p}_\lambda,p_\mu\rangle=\delta_{\lambda\mu}$. Though QSym is not self-dual, we define scaled powersum quasisymmetrics in the same way.

Definition 20. A *Scaled Powersum Quasisymmetric* functions are defined as

$$\tilde{P}_\alpha=\frac{1}{z_\alpha}P_\alpha. \tag{10}$$

By multiplying both sides of (4) by z_λ^{-1} it is evident that the scaled powersum quasisymmetric functions refines $\widetilde{p}_\lambda$. However with this new scaling there is a nicer product and coproduct.

Corollary 21. *The scaled powersum quasisymmetric functions has a deconcatenate coproduct and a shuffle product, in other words*

$$\Delta(\widetilde{P}_\gamma)=\sum_{\alpha|\beta=\gamma}\widetilde{P}_\alpha\otimes\widetilde{P}_\beta\qquad\text{and}\qquad\widetilde{P}_\alpha\otimes\widetilde{P}_\beta=\sum_{\gamma\in\alpha\sqcup\beta}\widetilde{P}_\gamma.$$

Proof. Substituting (10) into (8) yields

$$\Delta(z_\gamma \widetilde{P}_\gamma) = \sum_{\alpha|\beta=\gamma} \prod_{i=1}^{n} \frac{m_i(\gamma)!}{m_i(\alpha)! m_i(\beta)!} z_\alpha \widetilde{P}_\alpha \otimes z_\beta \widetilde{P}_\beta.$$

The coefficient reduces to

$$\prod_{i=1}^{n} \frac{m_i(\gamma)!}{m_i(\alpha)! m_i(\beta)!} \frac{z_\alpha z_\beta}{z_\gamma} = \prod_{i=1}^{n} \frac{i^{m_i(\alpha)+m_i(\beta)}}{i^{m_i(\gamma)}} = 1.$$

Likewise we do the same for the coefficient of the shuffle product. □

Finally, we look at the antipode of QSym, denoted S. Since $\widetilde{P}$ has shuffle product and deconcatenate coproduct, $\widetilde{P}$ makes QSym a shuffle algebra. By Theorem 3.1 of [6] the antipode acting on $\widetilde{P}$ is defined as:

Corollary 22. *Let $\alpha = (a_1, \ldots, a_k)$ be a composition. The antipode of QSym acts on a powersum quasisymmetric function as*

$$S(\widetilde{P}_\alpha) = (-1)^{\mathsf{len}(\alpha)} \widetilde{P}_{\overleftarrow{\alpha}} \tag{11}$$

where $\overleftarrow{\alpha} = (a_k, a_{k-1}, \ldots, a_2, a_1)$ is the reverse composition and $\mathsf{len}(\alpha) = k$ is the length of α.

Moreover the previous corollary implies that $S(P_\alpha) = (-1)^{\mathsf{len}(\alpha)} P_{\overleftarrow{\alpha}}$.

4.2 Zassenhaus Basis

The dual space of QSym is the space of noncommutative symmetric functions, or NSym. The dual basis of the monomial quasisymmetric basis is the complete basis, denoted $\mathbf{S}_\alpha$. The complete basis and the Zassenhaus basis of NSym, denoted $\mathbf{Z}_\alpha$, are both multiplicative and in [12] the Zassenhaus basis is defined as

$$\sum_k t^k \mathbf{S}_k = \cdots \exp \frac{\mathbf{Z}_i}{i} t^i \cdots \exp \frac{\mathbf{Z}_2}{2} t^2 \exp \frac{\mathbf{Z}_1}{1} t^1. \tag{12}$$

It would be in our interest to rewrite this definition using posets. Thus expand (12) as

$$\sum_k t^k \mathbf{S}_k = \cdots \left(1 + \frac{\mathbf{Z}_i}{i} t^i + \frac{\left(\frac{\mathbf{Z}_i}{i} t^i\right)^2}{2!} + \cdots\right) \cdots$$
$$\cdots \left(1 + \frac{\mathbf{Z}_2}{2} t^2 + \frac{\left(\frac{\mathbf{Z}_2}{2} t^2\right)^2}{2!} + \cdots\right) \left(1 + \frac{\mathbf{Z}_1}{1} t^n + \frac{\left(\frac{\mathbf{Z}_1}{1} t^1\right)^2}{2!} + \cdots\right).$$

In the change of basis from $\mathbf{S}_k$ to the Zassenhaus basis every term β has the property that $\beta \leqslant_{\mathcal{D}} (k)$. For a fixed β, all parts of size i come from the factor $\exp \frac{\mathbf{Z}_i}{i} t^i$. Pick a term j in $\exp \frac{\mathbf{Z}_i}{i} t^i$, then the coefficient of $Z_{i^j} t^j$ in $\exp \frac{\mathbf{Z}_i}{i} t^i$ is $\frac{1}{i^j j!}$. Note that $i^j = i^{m_i(\beta)}$ and that $j!$ is the number of parts with size i that is next to each other, and since β is a composition of k, $j! = c_i(\beta, k)!$. Then (12) is written as

$$\mathbf{S}_k = \sum_{\beta \leqslant_{\mathcal{D}} (k)} \prod_i \frac{1}{i^{m_i(\beta)} c_i(\beta, k)!} \mathbf{Z}_\beta.$$

Then by Proposition 10 we get an alternate definition,

$$\mathbf{S}_\alpha = \sum_{\beta \leqslant_{\mathcal{D}} \alpha} \prod_i \frac{1}{i^{m_i(\beta)} c_i(\beta, \alpha)!} \mathbf{Z}_\beta. \tag{13}$$

Theorem 23. *The dual of the scaled powersum quasisymmetric basis is the Zassenhaus basis.*

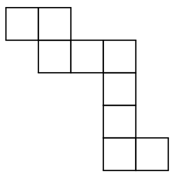
Proof. Combining (11) and (10) yields

$$\widetilde{P}_\alpha = \sum_{\alpha \leqslant_{\mathcal{D}} \beta} \prod_i \frac{1}{i^{m_i(\beta)} c_i(\beta, \alpha)!} M_\beta. \tag{14}$$

Comparing this equation and (13), and recalling that $\langle \mathbf{S}_\alpha, M_\beta \rangle = \delta_{\alpha\beta}$, it is apparent that $\langle \mathbf{Z}_\alpha, \widetilde{P}_\beta \rangle = \delta_{\alpha\beta}$. □

5 Change of Basis

Recall that a ribbon is a skew Ferrers diagram with no 2×2 boxes; for this paper we will allow ribbons to be disconnected, however adjacent boxes at least share a corner (if not an edge). There is a natural ribbon to associate to a composition, which we denote $R(\alpha)$:

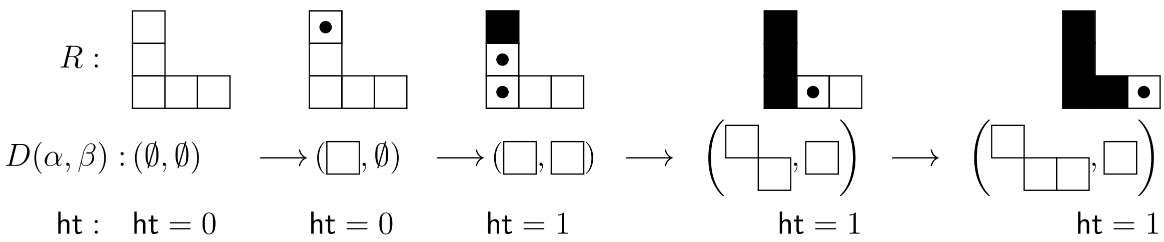
$\alpha = (2, 3, 1, 1, 2) \longleftrightarrow$ 

Let R be a ribbon and j a positive integer, then $R - j$ removes the first j boxes, in Example 24 the first j boxes are bulleted. Let $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_l)$ be compositions of n. The *height* of a ribbon $R(\alpha)$, denoted $\mathsf{ht}(R(\alpha))$, is the length of α. We build the tuple of ribbons $D(\beta, \alpha) = (R_1, R_2, \ldots, R_n)$ from the following algorithm, where R_i is nonempty if and only if $m_i(\alpha) \neq 0$.

Example 24. Let $\alpha = (1, 2, 1, 1)$ and $\beta = (1, 1, 3)$. We find $D(\beta, \alpha)$ using the algorithm. In a given step, the black has been removed from R and the $\bullet$ are about to be removed from R, which are denoted a_i' in the algorithm. Then to find $D(\beta, \alpha)$:

Algorithm 1 Descent Ribbons

Initialize $\mathsf{ht}=0$, $R=R(\beta)$, $D(\beta,\alpha)=(\emptyset,\dots,\emptyset)$
for i in $[1,\dots,k]$ **do**: ▷ Extend ribbon R_{a_i} at it's bottom right end by adding a box in one of three ways
  if $a_{i-1}\neq a_i$ **or** $a_i=1$ **then**
    Extend R_{a_i} by adding a box in the southeast position
  if $a_{i-1}=a_i$ **and** there exists a j such that $R((b_j,\dots,b_l))=R$ **then**
    Extend R_{a_i} by adding a box in the southeast position
  if $a_{i-1}=a_i$ **and** there doesn't exist a j such that $R((b_j,\dots,b_l))=R$ **then**
    Extend R_{a_i} by adding a box in the east position
  $\mathsf{ht}=\mathsf{ht}-1+\mathsf{ht}(a'_i)$ where a'_i has a_i boxes and $a'_i+A=R$ for some A
  $R=R-a_i$



We denote the height in the algorithm as $\mathsf{ht}(\beta,\alpha)$.

Definition 25. Let α be a composition of n and R the ribbon of α. A *standard ribbon filling* of R is a filling of R where the numbers from 1 to n each appear once. A standard ribbon filling of R is a *standard descent ribbon* if the entries are increasing from left to right, and decreasing from top to bottom.

Denote the set of all standard descent ribbon fillings of $D(\alpha,\beta)$ as $\mathsf{SDR}(\beta,\alpha)$. Let $\alpha=(a_1,\dots,a_k)$ be a composition of n. Then break α into strictly decreasing compositions such that $\alpha=\gamma_1|\cdots|\gamma_l$ where $\gamma_i=(a_{j_1},\dots,a_{j_1+j_2})$ and $a_{j_1-1}\leqslant a_{j_1}>\dots>a_{j_1+j_2}\leqslant a_{j_1+j_2+1}$. Then $T(\alpha)=\gamma'_1|\cdots|\gamma'_l$ where γ'_i is the transpose of γ_i (i.e. the composition of the ribbon formed when the ribbon of γ'_i is reflected about the $y=-x$ diagonal). For example, $T(1,2,1,1)=(1,1,2,1)$ and $C(1,2,1,1)=(1,4)$. Note that, in subset notation, $\alpha=C(\alpha)\cup D(\alpha)$, and $T(\alpha)=C(\alpha)\cup D(\alpha)^C$.

Theorem 26. *Let α be a composition, then*

$$P_\alpha=\sum_{T(\alpha)\leqslant\beta\leqslant C(\alpha)}(-1)^{\mathsf{ht}(\beta,\alpha)}|\mathsf{SDR}(\beta,\alpha)|F_\beta. \tag{15}$$

The proof of this theorem requires a basis of NCQSym to be defined in Section 8, so the proof of this theorem is in Section 10.1.

Example 27. Continuing from Example 24, the coefficient of $F_{(1,1,3)}$ when $\alpha = (1,2,1,1)$ is -3 since the $\mathsf{ht}(113, 1211) = 1$ and the fillings of $\mathsf{SDR}(113, 1211)$ are

$$\left(\begin{array}{lll}\boxed{1} & & \\ & \boxed{2} & \boxed{3}\end{array}, \boxed{1}\right), \left(\begin{array}{lll}\boxed{2} & & \\ & \boxed{1} & \boxed{3}\end{array}, \boxed{1}\right), \left(\begin{array}{lll}\boxed{3} & & \\ & \boxed{1} & \boxed{2}\end{array}, \boxed{1}\right).$$

Repeat for compositions in the interval $[T(\alpha), C(\alpha)] = [(1,1,2,1), (1,4)]$ to get $P_{(1,2,1,1)} = -3F_{(1,1,2,1)} - 3F_{(1,1,3)} + 3F_{(1,3,1)} + 3F_{(1,4)}$.

6 Powersum quasisymmetric Families and Involutions

Let $\prec$ denote any total ordering on positive integers. Let $\mathcal{R}$ be the subposet of $\mathcal{P}$ with the cover relation $\alpha \lessdot_{\mathcal{R}} \beta$ if β is obtained from α by summing two descending adjacent parts with respect to $\preceq$, i.e. $\beta = (a_1, \ldots, a_i + a_{i+1}, \ldots, a_k)$ *only if* $a_i \succeq a_{i+1}$. Define $C^{\mathcal{R}}(\alpha)$ to be the maximal composition of the subposet $\mathcal{R}$ bigger than or equal to α. Recall that $C(\alpha)$ is the maximal composition of the subposet $\mathcal{D}$ bigger than or equal to α. Then $\mathcal{D}$ is a special case of $\mathcal{R}$ where $\succeq$ is defined by size, it also follows that $C^{\mathcal{D}}(\alpha) = C(\alpha)$. Then we can define a whole family of powersum quasisymmetrics as

$$P^{\prec}_{\alpha} = \sum_{\alpha \leqslant \beta \leqslant C^{\mathcal{R}}(\alpha)} C_{\alpha\beta} M_{\beta} = \sum_{\alpha \leqslant_{\mathcal{R}} \beta} C_{\alpha\beta} M_{\beta}. \tag{16}$$

Example 28. Consider the total order on positive integers $a_i \succeq_{\mathcal{E}} a_{i+1}$ if a_i is even and a_{i+1} is odd, or if a_i and a_{i+1} is both even or odd and $a_i \succeq a_{i+1}$. Then define the subposet on compositions $\mathcal{E}$ with the cover relation $\alpha \lessdot_{\mathcal{E}} \beta$ if $\beta = (a_1, \ldots, a_i + a_{i+1}, \ldots, a_k)$ *only if* $a_i \succeq_{\mathcal{E}} a_{i+1}$. Thus we get another type of powersum quasisymmetric

$$P^{\mathcal{E}}_{23426} = 2M_{23426} + 2M_{5426} + 2M_{2366} + 2M_{566}.$$

Similarly, we can adapt these powersum quasisymmetrics to be generated by fillings by switching $\leqslant$ to $\preceq$ in Definition 6. Thus these types of powersum quasisymmetrics have the property that it refines the powersum symmetrics, has a shuffle product, and has a deconcatenate coproduct, and the dual up to scaling is of the form

$$\sum_k t^k \mathbf{S}_k = \cdots \exp \frac{\mathbf{Z}^{\prec}_{n_i}}{n_i} t^{n_i} \cdots \exp \frac{\mathbf{Z}^{\prec}_{n_2}}{n_2} t^{n_2} \exp \frac{\mathbf{Z}^{\prec}_{n_1}}{n_1} t^{n_1}$$

where $N = \{\ldots, n_i, \ldots, n_2, n_1\}$ such that $n_i >_{\mathcal{R}} n_{i-1}$ and $n_i \in \mathbb{N}$, simply by switching $\leqslant$ to $\preceq$ in the proofs of Theorems 43, 14, and Section 4.2. For example, $\mathbf{S}_{23426} = 2\mathbf{Z}^{\mathcal{E}}_{23426} + 2\mathbf{Z}^{\mathcal{E}}_{5426} + 2\mathbf{Z}^{\mathcal{E}}_{2366} + 2\mathbf{Z}^{\mathcal{E}}_{566}$ where $\langle P^{\mathcal{E}}_{\alpha}, \mathbf{Z}^{\mathcal{E}}_{\beta} \rangle = z_{\alpha} \delta_{\alpha\beta}$.

Let $\alpha = \gamma_1 | \cdots | \gamma_l$ be broken into strictly decreasing compositions (according to $\prec$), then $T^{\mathcal{R}}(\alpha) = \gamma'_1 | \cdots | \gamma'_l$ and $C^{\mathcal{R}}(\alpha) = |\gamma'_1| | \cdots | |\gamma'_l|$. Just as with $P^{\mathcal{D}}$, there is a change of basis rule to the fundamental basis that mimics the Murnaghan-Nakayama rule.

Theorem 29. *Let $\prec$ be a total order on integers and $P^{\mathcal{R}}$ be the powersum basis induced by $\prec$. Then*

$$P^{\prec}_{\alpha} = \sum_{T^{\mathcal{R}}(\alpha) \leqslant \beta \leqslant C^{\mathcal{R}}(\alpha)} (-1)^{\mathsf{ht}(\beta,\alpha)} |\mathsf{SDR}(\beta, \alpha)| F_{\beta}.$$

The proof of this theorem is in Section 11 as the proof requires bases of NCQSym that are defined in Section 11.

For a subposet $\mathcal{R}$, induced by $\prec$, as defined above, we define the reverse subposet of $\mathcal{R}$, denoted as $\overleftarrow{\mathcal{R}}$, with the cover relation $\alpha \lessdot_{\overleftarrow{\mathcal{R}}} \beta$ if β is obtained from α by summing two ascending adjacent parts with respect to $\preceq$, i.e. $\beta = (a_1, \ldots, a_i + a_{i+1}, \ldots, a_k)$ *only if* $a_i \preceq a_{i+1}$.

Example 30. Let $\mathcal{A} = \overleftarrow{\mathcal{D}}$ be the reverse subposet of $\mathcal{D}$. Then define the ascending powersum quasisymmetric function as

$$P^{\mathcal{A}}_{\alpha} = \sum_{\alpha \leqslant \beta \leqslant C^{\mathcal{A}}(\alpha)} C_{\alpha\beta} M_{\beta}.$$

Let $\overleftarrow{\alpha}$ denote the reverse composition, then it is easy to see that $\overleftarrow{C^{\mathcal{D}}(\alpha)} = C^{\mathcal{A}}(\overleftarrow{\alpha})$.

In [13], the star involution ($\star$), omega involution (ω) and ψ involution are defined as

- $(M_{\alpha})^{\star} = M_{\overleftarrow{\alpha}}$
- $\omega(F_{\alpha}) = F_{\alpha^t}$
- $\psi = \star \circ \omega = \omega \circ \star$

Theorem 31. *Let $\succ$ be a total order on integers and $P^{\mathcal{R}}$ be the powersum basis induced by $\succ$. Then,*

1. $(P^{\prec}_{\alpha})^{\star} = P^{\overleftarrow{\prec}}_{\overleftarrow{\alpha}}$
2. $\omega(P^{\prec}_{\alpha}) = \varepsilon_{\alpha} P^{\prec}_{\overleftarrow{\alpha}}$
3. $\psi(P^{\prec}_{\alpha}) = \varepsilon_{\alpha} P^{\overleftarrow{\prec}}_{\alpha}$

where $\varepsilon_{\alpha} = (-1)^{n - \mathsf{len}(\alpha)}$

Proof. (1) By definition

$$(P^{\prec}_{\alpha})^{\star} = \left(\sum_{\alpha \leqslant \beta \leqslant C^{\mathcal{R}}(\alpha)} C_{\alpha\beta} M_{\beta} \right)^{\star}$$

Note that $C_{\alpha\beta} = C_{\overleftarrow{\alpha}\overleftarrow{\beta}}$ because $m_i(\alpha)$ is the same no matter the order of the parts in α, and $c_L(\overleftarrow{\alpha}, \overleftarrow{\beta}) = \overleftarrow{c_L(\alpha, \beta)}$, but we are only interested in the product of the factorial of all the parts. Then,

$$(P^{\prec}_{\alpha})^{\star} = \sum_{\overleftarrow{\alpha} \leqslant \beta \leqslant C^{\overleftarrow{\mathcal{R}}}(\overleftarrow{\alpha})} C_{\overleftarrow{\alpha}\beta} M_{\beta} = P^{\overleftarrow{\prec}}_{\overleftarrow{\alpha}}.$$

(2) Recall that the alternate definition of the omega involution is $\omega(f) = (-1)S(f)$, where f is a quasisymmetric function and S is the antipode. Substituting f with a powersum quasisymmetric function and using Theorem 22 yields the above.

(3) This follows from (1) and (2). □

Remark 32. Recall that the quasisymmetric analogue of the forgotten basis of Sym is the Essential Basis E_α and is defined as $E_\alpha = \sum_{\beta \geqslant \alpha} M_\beta$. Since $\psi(M_\alpha) = -E_\alpha$, the change of basis to the Essential basis is

$$P^{\prec}_{\alpha} = \sum_{\alpha \leqslant_{\widetilde{\mathcal{R}}} \beta} \varepsilon_\beta C_{\alpha\beta} E_\beta.$$

7 Preliminaries for NCSym and NCQSym

7.1 NCSym

The Hopf algebra of symmetric functions in non-commuting variables, denoted NCSym, was first defined by Wolf in [20]. This space should not be confused with NSym from Subsection 4.2. Bases in NCSym are indexed by set partitions. A *set partition* $\phi = \{\mathrm{B}_1, \ldots, \mathrm{B}_k\}$ of $[n]$ is a set of nonempty disjoint subsets of $[n]$ where their union is $[n]$. Of course being a partition means that order doesn't matter, but we write our set partition (for convenience) as $\mathrm{B}_1/\cdots/\mathrm{B}_k$ where $|\mathrm{B}_1| \geqslant \cdots \geqslant |\mathrm{B}_k|$ and $\min(\mathrm{B}_i) > \min(\mathrm{B}_{i+1})$ whenever $|\mathrm{B}_i| = |\mathrm{B}_{i+1}|$. Let $\bar{\mathcal{P}}$ be the refinement poset on set partitions with the covering relation $\phi \lessdot \psi$ if $\psi = \mathrm{B}_1/\cdots/\mathrm{B}_i \cup \mathrm{B}_j/\cdots/\mathrm{B}_k$.

In [18], Rosas and Sagan defined many bases analogous to those in Sym, here we review two. The *symmetric monomial basis in non-commuting variables* is defined as

$$m_\phi = \sum_{\substack{i_j = i_{j'} \text{ iff} \\ j,j' \in B_l}} x_{i_1} \cdots x_{i_n}$$

Then the change of basis formulae of a *powersum symmetric function in noncommuting variables* to a symmetric monomial function in noncommuting variables is

$$p_\phi = \sum_{\phi \leqslant \psi} m_\psi. \tag{17}$$

7.2 NCQSym and Morphisms

Bases in NCQSym are indexed by set compositions of $[n]$: a list $\Phi = (B_1, \ldots, B_k)$ of disjoint nonempty sets satisfying $\bigcup_i B_i = [n]$. When there is no danger of confusion, we compress the notation for set compositions as usual, e.g., $(\{5\}, \{1,3\}, \{2\}, \{4\})$ is written as $5|13|2|4$. The refinement poset $\widetilde{\mathcal{P}}$ on set compositions has the cover relation $\Phi \lessdot \Psi$ if $\Psi = (B_1, \ldots, B_i \cup B_{i+1}, \ldots, B_k)$.

Let $\Phi = B_1|\cdots|B_k$ be a set composition of $[n]$. Then the quasisymmetric functions in non-commuting variables $X = (x_1, x_2, \ldots)$, denoted NCQSym, is the space spanned by the formal series

$$M_\Phi = \sum x_{i_1} x_{i_2} \cdots x_{i_n},$$

where the sum is over all monomials such that $i_j = i_{j'}$ if and only if $j, j' \in B_i$ and $i_j < i_{j'}$ if and only if $j \in B_i$ and $j' \in B_{i'}$ such that $i < i'$. For example, let $\Phi = 5|13|2|4$, then $M_\Phi = x_2x_3x_2x_4x_1 + x_2x_3x_2x_5x_1 + x_2x_4x_2x_5x_1 + \cdots$.

Denote by ρ the map from set compositions of $[n]$ to compositions of n that records the cardinality of each block. For example $\rho(5|13|2|4) = (1,2,1,1)$. The map ρ from set compositions to compositions induces a Hopf algebra surjection that we will also denote as ρ with $\rho : \mathrm{NCQSym} \to \mathrm{QSym}$ given by $\rho(M_\Phi) = M_{\rho(\Phi)}$. Note that the map ρ lets the variables commute.

Let Φ be a set composition, then sort is a map from set compositions to set partitions by forgetting the order of the blocks of Φ. Analogous to the commutative story, there is a Hopf inclusion of NCSym to NCQSym given by

$$m_\phi = \sum_{\Phi\colon \mathsf{sort}(\Phi)=\phi} M_\Phi. \tag{18}$$

8 Basic Definitions

Given a set composition $\Phi = (B_1, \ldots, B_l)$, a (matrix) filling F_Φ places a block B_i in some column along row i for every i. Define the set compositions $\mathsf{row}(\mathsf{F}_\Phi)$ and $\mathsf{col}(\mathsf{F}_\Phi)$ by analogy with the integer composition story. For example, $\mathsf{col}(\mathsf{F}_\Phi) = (A_1, \ldots, A_k)$, then A_1 is the union of all blocks B_j of Φ appearing in the first column of F_Φ.

We want to define a powersum quasisymmetric function in non-commuting variables based on fillings that compare blocks. To this end, let $\min(B_i)$ be the smallest integer in B_i, and define a total order on blocks, B_i and B_j, of a set composition of $[n]$, $B_i >_{\widetilde{D}} B_j$ if and only if either $|B_i| > |B_j|$, or $|B_i| = |B_j|$ and $\min(B_i) < \min(B_j)$.

Definition 33. A *labelled diagonal descending ($\mathcal{LDD}$) filling* of $\Phi = (B_1, \ldots, B_l)$ is a filling such that

1. B_1 is in the first column.
2. B_i is in row i.
3. B_{i+1} can be in the same column as B_i if $B_i >_{\widetilde{D}} B_{i+1}$. Otherwise, B_{i+1} is in the column to the right of B_i.

Let $\mathcal{LDD}(\Phi)$ be the set of all $\mathcal{LDD}$ fillings with the row sum of Φ. Note that if F and F' are two $\mathcal{LDD}$ fillings such that $\mathsf{col}(\mathsf{F}) = \mathsf{col}(\mathsf{F}')$ and $\mathsf{row}(\mathsf{F}) = \mathsf{row}(\mathsf{F}')$, then $\mathsf{F} = \mathsf{F}'$. In other words every filling is unique to its row and column sum.

Example 34. The $\mathcal{LDD}$ fillings with $\mathsf{row}(\mathsf{F}_\Phi) = 14|2|3$ are

$$\begin{array}{|ccc} 14 & \cdot & \cdot \\ \cdot & 2 & \cdot \\ \cdot & \cdot & 3 \\ \hline 14| & 2| & 3 \end{array} \qquad \begin{array}{|ccc} 14 & \cdot & \cdot \\ 2 & \cdot & \cdot \\ \cdot & 3 & \cdot \\ \hline 124| & 3 & \end{array} \qquad \begin{array}{|ccc} 14 & \cdot & \cdot \\ \cdot & 2 & \cdot \\ \cdot & 3 & \cdot \\ \hline 14| & 23 & \end{array} \qquad \begin{array}{|ccc} 14 & \cdot & \cdot \\ 2 & \cdot & \cdot \\ 3 & \cdot & \cdot \\ \hline 1234 & & \end{array}$$

Now for a more exciting example, $\mathcal{LDD}(5|13|4|2)$ has the fillings

$$\begin{array}{|cccc} 5 & \cdot & \cdot & \cdot \\ \cdot & 13 & \cdot & \cdot \\ \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ \hline 5| & 13| & 4| & 2 \end{array} \qquad \begin{array}{|cccc} 5 & \cdot & \cdot & \cdot \\ \cdot & 13 & \cdot & \cdot \\ \cdot & 4 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot \\ \hline 5| & 134| & 2| & \end{array}$$

Definition 35. Powersum quasisymmetric function in NCQSym is defined as

$$P_\Phi = \sum_{\mathsf{F}_\Phi \in \mathcal{LDD}(\Phi)} M_{\mathsf{col}(\mathsf{F}_\Phi)}. \tag{19}$$

Example 36. The previous example yields $P_{14|2|3} = M_{14|2|3} + M_{124|3} + M_{14|23} + M_{1234}$ and $P_{5|13|4|2} = M_{5|13|4|2} + M_{5|134|2}$.

Let $\widetilde{\mathcal{D}}$ be a subposet of $\widetilde{\mathcal{P}}$ on set compositions with the cover relation

$$(B_1, \dots, B_i, B_{i+1}, \dots, B_k) \lessdot_{\widetilde{\mathcal{D}}} (B_1, \dots, B_i \cup B_{i+1}, \dots, B_k)$$

only if $B_i >_{\widetilde{\mathcal{D}}} B_{i+1}$. Just as in the commuting variables case, there is a poset interpretation for the powersum basis of NCQSym.

Definition 37. Powersum quasisymmetric function in NCQSym is defined as

$$P_\Phi = \sum_{\Phi \leqslant_{\widetilde{\mathcal{D}}} \Psi} M_\Psi = \sum_{\Phi \leqslant \Psi \leqslant C(\Phi)} M_\Psi$$

where $C(\Phi)$ is the greatest element in the poset $\widetilde{\mathcal{D}}$ bigger than or equal to Φ.

Example 38. Let $\Phi = 1|3|25|6|4$, then $P_{1|3|25|6|4} = M_{1|3|25|6|4} + M_{13|25|6|4} + M_{1|3|256|4} + M_{13|256|4}$.

9 Basic Theorems and Properties

From Definition 37 it is clear that P_Φ is a basis of NCQSym. In the coming sections we will show that P_Φ refines the powersum symmetric basis of NCSym and projects to the powersum quasisymmetric basis P_α. Finally we'll develop product and coproduct formulas, and generalizations $P_\Phi^{\triangleright}$ of P_Φ given any total order on disjoint sets.

9.1 Noncommuting powersum quasisymmetric functions refine the Noncommuting powersums functions

Let us expand a powersum symmetric function in NCSym in terms of quasisymmetric monomial functions in NCQSym by combining (17) and (18)

$$p_\phi = \sum_{\phi \leqslant \psi} \sum_{\Psi\colon \mathsf{sort}(\Psi)=\psi} M_\Psi. \tag{20}$$

We want to rewrite (20) using fillings, thus we need a filling such that $\mathsf{row(F)}$ is a set partition and $\mathsf{col(F)}$ is a set composition. A *labelled single row filling*, denoted as $\mathcal{LSR}$, as a filling where the row sum is a set partition and every row has only one entry and the column sum is a set composition. Denote $\mathcal{LSR}(\phi)$ as the set of all $\mathcal{LSR}$ fillings with the row sum of ϕ. $\mathcal{LSR}$ fillings have the property that for any set partition ϕ and set composition Ψ there is at most one filling with the row sum of ϕ and column sum of Ψ.

Example 39. $\mathcal{LSR}(13/2)$ has the fillings

$$\begin{array}{|cc} 13 & \cdot \\ \cdot & 2 \\ \hline 13| & 2 \end{array} \qquad \begin{array}{|cc} \cdot & 13 \\ 2 & \cdot \\ \hline 2| & 13 \end{array} \qquad \begin{array}{|cc} 13 & \cdot \\ 2 & \cdot \\ \hline 123| & \end{array}$$

Thus we define the change of basis as

$$p_\phi = \sum_{\mathsf{F} \in \mathcal{LSR}(\phi)} M_{\mathsf{col(F)}}. \tag{21}$$

This is equivalent to (20) because the union of multiple blocks (or none) follows from the first summand, and allowing the entry of a block to be anywhere follows from the second summand.

Let $\mathsf{diag} : \mathcal{LSR}(\phi) \to \bigcup_\Phi \mathcal{LDD}(\Phi)$ where the union is over all Φ with $\mathsf{sort}(\Phi) = \phi$ be the map of fillings that sorts the rows of a filling so that every entry has an $\mathcal{LDD}$ filling. Then let $\mathsf{push} : \bigcup_\Phi \mathcal{LDD}(\Phi) \to \mathcal{LSR}(\phi)$ where the union is over all Φ with $\mathsf{sort}(\Phi) = \phi$ be a map of fillings such that push sorts the rows, so that the row sum is a set partition.

Example 40. Below is an example of the map diag with an $\mathcal{LSR}$ filling and push with an $\mathcal{LDD}$ filling, and notice here that the fillings are unique and have the same column sum.

$$\begin{array}{|ccccc} \cdot & \cdot & 45 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 67 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & 3 & \cdot & \cdot & \cdot \\ \hline 1| & 3| & 245| & 67| & \end{array} \quad \xrightarrow{\mathsf{diag}} \quad \begin{array}{|ccccc} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 45 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 67 & \cdot \\ \hline 1| & 3| & 245| & 67| & \end{array}$$

$$\begin{array}{|ccc} 24 & \cdot & \cdot \\ 5 & \cdot & \cdot \\ \cdot & 7 & \cdot \\ \cdot & \cdot & 168 \\ \cdot & \cdot & 3 \\ \hline 245| & 7| & 1368| \end{array} \quad \xrightarrow{\mathsf{push}} \quad \begin{array}{|ccc} \cdot & \cdot & 168 \\ 24 & \cdot & \cdot \\ \cdot & \cdot & 3 \\ 5 & \cdot & \cdot \\ \cdot & 7 & \cdot \\ \hline 245| & 7| & 1368| \end{array}$$

Theorem 41. *The powersum quasisymmetric functions in NCQSym refines the powersum symmetric function in NCSym. In other words*

$$p_\phi = \sum_{\Phi\colon \mathsf{sort}(\Phi)=\phi} P_\Phi. \tag{22}$$

Proof. By substituting (21) and (19) into (22), it is equivalent to prove

$$\sum_{\widetilde{\mathsf{F}} \in \mathcal{LSR}(\phi)} M_{\mathsf{col}(\widetilde{\mathsf{F}})} = \sum_{\Phi\colon \mathsf{sort}(\Phi)=\phi} \sum_{\mathsf{F} \in \mathcal{LDD}(\Phi)} M_{\mathsf{col}(\mathsf{F})}.$$

We will prove that the above is true by showing that push and diag are column-reading-preserving bijections between the fillings of $\mathcal{LSR}(\phi)$ and $\bigcup_\phi \mathcal{LDD}(\Phi)$ since $\mathsf{diag} \circ \mathsf{push} = \mathsf{push} \circ \mathsf{diag} = id$.

Fix a filling $\widetilde{\mathsf{F}} \in \mathcal{LSR}(\Phi)$. diag maps $\widetilde{\mathsf{F}}$ to a $\mathcal{LDD}$ filling such that $\mathsf{col}(\widetilde{\mathsf{F}}) = \mathsf{col}(\mathsf{diag}(\widetilde{\mathsf{F}}))$ and the rows may be permuted. push maps $\mathsf{diag}(\widetilde{\mathsf{F}})$ by placing the row with the greatest entry and permutes it with the top row, then takes the row with the second greatest entry and permutes that with the second row and so on, thus resulting in the row sum of $\mathsf{push}(\mathsf{diag}(\widetilde{\mathsf{F}}))$ being a set partition. Since the rows don't change entries (only permutes) under the maps, $\mathsf{row}(\mathsf{push}(\mathsf{diag}(\widetilde{\mathsf{F}}))) = \mathsf{row}(\widetilde{\mathsf{F}})$. Knowing that the column sum is preserved under both maps, i.e. $\mathsf{col}(\mathsf{push}(\mathsf{diag}(\widetilde{\mathsf{F}}))) = \mathsf{col}(\widetilde{\mathsf{F}})$, there is at most one unique $\mathcal{LSR}$ filling for a given row sum and column sum. Thus $\mathsf{push}(\mathsf{diag}(\widetilde{\mathsf{F}})) = \widetilde{\mathsf{F}}$. The same sort of argument is used for the other direction. □

9.2 Product and Coproduct

Let $B_1, \ldots, B_k, A_1, \ldots, A_l$ be disjoint sets. Then the *shuffle* of lists of disjoint sets $(B_1, \ldots, B_k)$ and $(A_1, \ldots, A_l)$ is defined recursively by

1. $\emptyset \sqcup\!\sqcup (B_1, \ldots, B_k) = (B_1, \ldots, B_k) \sqcup\!\sqcup \emptyset = (B_1, \ldots, B_k)$
2. $(B_1, \ldots, B_k) \sqcup\!\sqcup (A_1, \ldots, A_l) = B_1 \big| \big[(B_2, \ldots, B_k) \sqcup\!\sqcup (A_1, \ldots, A_l)\big] + A_1 \big| \big[(B_1, \ldots, B_k) \sqcup\!\sqcup (A_2, \ldots, A_l)\big]$.

A *quasishuffle* of lists of disjoint sets, denoted $\widetilde{\sqcup\!\sqcup}$, uses the same initial condition 1, but adds the term $+(A_1 \cup B_1) \big| \big[(B_2, \ldots, B_k) \widetilde{\sqcup\!\sqcup} (A_2, \ldots, A_l)\big]$ to recursion 2. Let Ψ be a set composition of $[m]$. Let $\Psi^{\uparrow n}$ denote the operation of adding n to every element of every block of Ψ, a set composition of $\{n+1, \ldots, n+m\}$. Then the *shifted shuffle* of two set partitions Φ and Ψ, of $[n]$ and $[m]$ respectively, is $\Phi \sqcup\!\sqcup \Psi^{\uparrow n}$. Every element in $\Phi \sqcup\!\sqcup \Psi^{\uparrow n}$ is then a set composition of $[n+m]$. Finally, let Γ be a sequence of disjoint subsets of $I = \{i_1, \ldots, i_n\}$ where $i_j < i_{j+1}$. The *standardization* of Γ, denoted $\Gamma^{\downarrow}$, is the set composition of $[n]$ received by replacing the element i_j with j.

We extend the notions of shifting and standardizing to fillings, denoted $\mathsf{F}^{\uparrow n}$ and $\mathsf{F}^{\downarrow}$. Explicitly, let $\mathsf{row}(\mathsf{F}) = B_1|\cdots|B_k$ and let $(B_1|\cdots|B_k)^{\uparrow n} = A_1|\cdots|A_k$. Then $\mathsf{F}^{\uparrow n}$ is obtained by replacing the entries B_i with A_i. Likewise, let $(B_1|\cdots|B_k)^{\downarrow} = C_1|\cdots|C_k$, then $\mathsf{F}^{\downarrow}$ is obtained by replacing the entries B_i with C_i. When the shifting factor n is clear from context, we suppress it. Furthermore, given two $\mathcal{LDD}$ fillings F_1 and F_2, we define a quasishuffle of fillings on F_1 and $\mathsf{F}_2^{\uparrow}$ just like for integer matrix fillings, by taking unions of blocks in place of sums of integers.

Example 42.

$$\left|\begin{array}{cc} 13 & \cdot \\ \cdot & 2 \\ \hline 13| & 2| \end{array}\right. \;\widetilde{\sqcup\!\sqcup}\; \left|\begin{array}{c} 45 \\ \hline 45| \end{array}\right. = \left|\begin{array}{c} 13| \\ \hline 13| \end{array}\right. \left(\left|\begin{array}{c} 2 \\ \hline 2| \end{array}\right. \;\widetilde{\sqcup\!\sqcup}\; \left|\begin{array}{c} 45 \\ \hline 45| \end{array}\right. \right) + \left|\begin{array}{c} 45 \\ \hline 45| \end{array}\right. \left(\left|\begin{array}{cc} 13 & \cdot \\ \cdot & 2 \\ \hline 13| & 2| \end{array}\right. \;\widetilde{\sqcup\!\sqcup}\; \emptyset \right)$$

$$+ \left|\begin{array}{c} 13 \\ 45 \\ \hline 1345| \end{array}\right. \left(\left|\begin{array}{c} 2 \\ \hline 2| \end{array}\right. \;\widetilde{\sqcup\!\sqcup}\; \emptyset \right)$$

$$= \left|\begin{array}{ccc} 13 & \cdot & \cdot \\ \cdot & 2 & \cdot \\ \cdot & \cdot & 45 \\ \hline 13| & 2| & 45| \end{array}\right. + \left|\begin{array}{ccc} 13 & \cdot & \cdot \\ \cdot & 45 & \cdot \\ \cdot & \cdot & 2 \\ \hline 13| & 45| & 2| \end{array}\right. + \left|\begin{array}{ccc} 45 & \cdot & \cdot \\ \cdot & 13 & \cdot \\ \cdot & \cdot & 2 \\ \hline 13| & 45| & 2| \end{array}\right.$$

$$+ \left|\begin{array}{cc} 13 & \cdot \\ 45 & \cdot \\ \cdot & 2 \\ \hline 1345| & 2| \end{array}\right. + \left|\begin{array}{cc} 13 & \cdot \\ \cdot & 45 \\ \cdot & 2 \\ \hline 13| & 245| \end{array}\right.$$

Let Φ and Ψ be set compositions of n and m respectively. Recall the quasisymmetric monomial basis in non-commuting variables has a shifted quasishuffle product and a standardized deconcatenate coproduct,

$$M_\Phi M_\Psi = \sum_{\Upsilon \in \Phi \widetilde{\sqcup\!\sqcup} \Psi^{\uparrow n}} M_\Upsilon, \quad \Delta(M_\Phi) = \sum_{i=0}^{k} M_{(B_1|\cdots|B_i)^\downarrow} \otimes M_{(B_{i+1}|\cdots|B_k)^\downarrow},$$

and the fillings analogue

$$M_{\mathsf{col}(\mathsf{F}_1)} M_{\mathsf{col}(\mathsf{F}_2)} = \sum_{\mathsf{F} \in \mathsf{F}_1 \widetilde{\sqcup\!\sqcup} \mathsf{F}_2^{\uparrow}} M_{\mathsf{col}(\mathsf{F})}, \quad \Delta(M_{\mathsf{col}(\mathsf{F})}) = \sum_{\mathsf{F}=\mathsf{F}_1|\mathsf{F}_2} M_{\mathsf{col}(\mathsf{F}_1^\downarrow)} \otimes M_{\mathsf{col}(\mathsf{F}_2^\downarrow)}. \tag{23}$$

Theorem 43. *Let Φ and Ψ be set compositions of n and m respectively. The powersum basis of* NCQSym *has a shifted shuffle product, i.e.*

$$P_\Phi P_\Psi = \sum_{\Upsilon \in \Phi \sqcup\!\sqcup \Psi^{\uparrow n}} P_\Upsilon.$$

Proof. From (19) and (23),

$$\begin{aligned} P_\Phi P_\Psi &= \left(\sum_{\mathsf{F}_1 \in \mathcal{LDD}(\Phi)} M_{\mathsf{col}(\mathsf{F}_1)} \right) \left(\sum_{\mathsf{F}_2 \in \mathcal{LDD}(\Psi)} M_{\mathsf{col}(\mathsf{F}_2)} \right) \\ &= \sum_{\substack{\mathsf{F}_1 \in \mathcal{LDD}(\Phi), \mathsf{F}_2 \in \mathcal{LDD}(\Psi)^\uparrow \\ \mathsf{F} \in \mathsf{F}_1 \widetilde{\sqcup\!\sqcup} \mathsf{F}_2}} M_{\mathsf{col}(\mathsf{F})}. \end{aligned}$$

The proof will be carried out the same way as in Section 3.3, however in this case we need not worry about the coefficient since the coefficient is either 1 or 0. Thus, with abuse of notation, we will show that the sets $\mathcal{LDD}(\Phi)\widetilde{\sqcup\!\sqcup}\mathcal{LDD}(\Psi)^{\uparrow}$ and $\mathcal{LDD}(\Phi \sqcup\!\sqcup \Psi^{\uparrow n})$ are equivalent.

Fix $\mathsf{F} \in \mathsf{F}_1\widetilde{\sqcup\!\sqcup}\mathsf{F}_2$ where $\mathsf{F}_1 \in \mathcal{LDD}(\Phi)$ and $\mathsf{F}_2 \in \mathcal{LDD}(\Psi)^{\uparrow}$, then $\mathsf{row}(\mathsf{F}) \in \mathsf{row}(\mathsf{F}_1) \sqcup\!\sqcup \mathsf{row}(\mathsf{F}_2^{\uparrow})$, which by definition $\mathsf{row}(\mathsf{F}) \in \Phi \sqcup\!\sqcup \Psi^{\uparrow n}$. Since the quasishuffle of two $\mathcal{LDD}$ fillings is a $\mathcal{LDD}$ filling, $\mathsf{F} \in \mathcal{LDD}(\Phi \sqcup\!\sqcup \Psi^{\uparrow n})$. Conversely fix a filling $\mathsf{F} \in \mathcal{LDD}(\Phi \sqcup\!\sqcup \Psi^{\uparrow n})$, then from this filling we construct two fillings: F_1 by removing all entries that has an integer greater than n (then remove all empty columns and rows), and F_2 by removing all entries that has an integer less than or equal to n (then remove all empty columns and rows) and subtracting n from all integers. Thus $\mathsf{row}(\mathsf{F}_1) = \Phi$ and F_1 is an $\mathcal{LDD}$ filling, thus $\mathsf{F}_1 \in \mathcal{LDD}(\Phi)$ and likewise for F_2, thus $\mathsf{F} \in \mathsf{F}_1 \sqcup\!\sqcup \mathsf{F}_2$. Thus the sets are equivalent,

$$\sum_{\substack{\mathsf{F}_1\in\mathcal{LDD}(\Phi),\mathsf{F}_2\in\mathcal{LDD}(\Psi)^{\uparrow}\\ \mathsf{F}\in\mathsf{F}_1\widetilde{\sqcup\!\sqcup}\mathsf{F}_2}} M_{\mathsf{col}(\mathsf{F})} = \sum_{\mathsf{F}\in\mathcal{LDD}(\Phi\sqcup\!\sqcup\Psi^{\uparrow n})} M_{\mathsf{col}(\mathsf{F})} = \sum_{\Upsilon\in\Phi\sqcup\!\sqcup\Psi^{\uparrow n}} P_{\Upsilon} \qquad \square$$

Theorem 44. *Let $\Phi = B_1|\cdots|B_k$ be a set composition of $[n]$. Then the powersum quasisymmetric basis in non-commuting variables has a standardized deconcatenation coproduct,*

$$\Delta(P_\Phi) = \sum_{i=0}^{k} P_{(B_1|\cdots|B_i)^{\downarrow}} \otimes P_{(B_{i+1}|\cdots|B_k)^{\downarrow}}.$$

Proof. Definition 37 and (23) yields

$$\Delta(P_\Phi) = \Delta\left(\sum_{\mathsf{F}\in\mathcal{LDD}(\Phi)} M_{\mathsf{col}(\mathsf{F})}\right) = \sum_{\mathsf{F}\in\mathcal{LDD}(\Phi)} \sum_{\mathsf{F}=\mathsf{F}_1|\mathsf{F}_2} M_{\mathsf{col}(\mathsf{F}_1^{\downarrow})} \otimes M_{\mathsf{col}(\mathsf{F}_2^{\downarrow})}.$$

Let m be the degree of $\mathsf{col}(\mathsf{F}_1)$ and fix m. Let F_1', such that $\mathsf{row}(\mathsf{F}_1') = \mathsf{col}(\mathsf{F}_1') = (B_1, \ldots, B_j)$ (which means F_1' is a "diagonal" filling), then every filling of degree m comes from moving entry B_{i+1} to the column of B_i when $B_i >_{\widetilde{\mathcal{D}}} B_{i+1}$. Thus the set of fillings is $\mathcal{LDD}(B_1, \ldots, B_j)$ and likewise for the right side, thus

$$\sum_{\mathsf{F}\in\mathcal{LDD}(\Phi)} \sum_{\mathsf{F}=\mathsf{F}_1|\mathsf{F}_2} M_{\mathsf{col}(\mathsf{F}_1^{\downarrow})} \otimes M_{\mathsf{col}(\mathsf{F}_2^{\downarrow})} = \sum_{i=0}^{k} \sum_{\substack{\mathsf{F}_1\in\mathcal{LDD}((B_1|\cdots|B_i)^{\downarrow})\\ \mathsf{F}_2\in\mathcal{LDD}((B_{i+1}|\cdots|B_k)^{\downarrow})}} M_{\mathsf{col}(\mathsf{F}_1)} \otimes M_{\mathsf{col}(\mathsf{F}_2)}. \qquad \square$$

Remark 45. The method of proving the theorems above could be solved more straight forward and perhaps easily if we used the subposet definition instead of the fillings definition. However we choose this approach to demonstrate that one can make a Hopf algebra out of the fillings using this approach and that there may not always be a subposet definition for different Hopf algebas.

10 The Projection of the powersum quasisymmetric Basis

Since there is a powersum basis in Sym, NCSym, QSym, and NCQSym, it is natural to explore the projection of P_Φ onto QSym. As we'll see in Theorem 47, unlike the story in the monomial basis, it's not the case that $\rho(P_\Phi) = P_{\rho(\Phi)}$.

We first look to when two powersum quasisymmetric function in NCQSym map to the same function in QSym under ρ.

Proposition 46. *Let $\Phi = B_1|\cdots|B_k$ and $\Psi = A_1|\cdots|A_k$ be set compositions such that*

1. *for all $1 \leqslant i \leqslant k$, $|B_i| = |A_i|$, and*
2. *for all $1 \leqslant i < k$, $A_i >_{\widetilde{\mathcal{D}}} A_{i+1}$ if and only if $B_i >_{\widetilde{\mathcal{D}}} B_{i+1}$.*

Then $\rho(P_\Phi) = \rho(P_\Psi)$.

Proof. Recall that M_Φ projects to QSym by,

$$\rho(M_\Phi) = M_{\rho(\Phi)}. \tag{24}$$

It follows that

$$\rho(P_\Phi) = \rho\left(\sum_{\Phi \leqslant_{\widetilde{\mathcal{D}}} \hat{\Phi}} M_{\hat{\Phi}}\right) = \sum_{\Phi \leqslant_{\widetilde{\mathcal{D}}} \hat{\Phi}} M_{\rho\left(\hat{\Phi}\right)} \quad \text{and} \quad \rho(P_\Psi) = \sum_{\Psi \leqslant_{\widetilde{\mathcal{D}}} \hat{\Psi}} M_{\rho\left(\hat{\Psi}\right)}. \tag{25}$$

For every $\hat{\Phi}$ there exists one and only one $\hat{\Psi}$ such that $\rho(\hat{\Phi}) = \rho(\hat{\Psi})$. Fix a $\hat{\Phi}$, then $\Phi \leqslant \hat{\Phi}$ by combining certain blocks B_j and B_{j+1}, since condition 2 holds, then there exists a $\Psi \leqslant \hat{\Psi}$ by combining blocks A_j and A_{j+1}. $\rho(\hat{\Phi}) = \rho(\hat{\Psi})$ holds due to condition 1. □

Let $\Phi = B_1|\ldots|B_k$ be a set composition of $[n]$. Let $\mathfrak{B}_m = \{i : |B_i| = m\}$. Define

$$\mathfrak{S}_\Phi = \mathfrak{S}_{\mathfrak{B}_1}\mathfrak{S}_{\mathfrak{B}_2}\cdots\mathfrak{S}_{\mathfrak{B}_n}.$$

Every $\sigma \in \mathfrak{S}_\Phi$ acts on Φ as follows,

$$\sigma(B_1|B_2|\ldots|B_k) = B_{\sigma(1)}|B_{\sigma(2)}|\ldots|B_{\sigma(k)}.$$

For example, if $\Phi = 4|25|7|13|6$, then $\mathfrak{B}_1 = \{1,3,5\}$, $\mathfrak{B}_2 = \{2,4\}$, and $\mathfrak{B}_m = \emptyset$ for $2 < m \leqslant 7$. Therefore, $\mathfrak{S}_\Phi = \mathfrak{S}_{\{1,3,5\}}\mathfrak{S}_{\{2,4\}}$. Now, $(3,5)(2,4) \in \mathfrak{S}_\Phi$, thus

$$(3,5)(2,4)(4|25|7|13|6) = 4|13|6|25|7.$$

Note that $\rho(\Phi) = \rho(\sigma(\Phi))$. This leads to the following theorem.

Theorem 47. *For any composition α and set composition Φ such that $\rho(\Phi) = \alpha$,*

$$P_\alpha = \sum_{\sigma \in \mathfrak{S}_\Phi} \rho(P_{\sigma(\Phi)}). \tag{26}$$

Let $\Phi = (B_1, \ldots, B_l)$ be a set composition of $[n]$. We call Φ *strict* if for all $i < j$ with $|B_i| = |B_j|$, we have $B_i >_{\widetilde{\mathcal{D}}} B_j$. For example $\Phi = 2|13|4|5$ is a strict set composition of $[n]$.

Definition 48. Let a *Strict Labelled Diagonal* filling, denoted $\mathcal{SLD}$, be an $\mathcal{LDD}$ filling such that the row sum is a strict set composition.

It is important to note that $\mathcal{SLD}$ fillings follow the rule that B_i and B_{i+1} can be in the same column if $|B_i| \geqslant |B_{i+1}|$.

Example 49. The first two are $\mathcal{SLD}$ fillings, but the third is only a $\mathcal{LDD}$ filling.

$$\begin{array}{|cccc} 2 & \cdot & \cdot & \cdot \\ \cdot & 13 & \cdot & \cdot \\ \cdot & 4 & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot \\ \hline 2| & 134| & 5| & \end{array} \qquad \begin{array}{|cc} 2 & \cdot \\ \cdot & 13 \\ \cdot & 4 \\ \cdot & 5 \\ \hline 2| & 1345| \end{array} \qquad \begin{array}{|cc} 4 & \cdot \\ \cdot & 13 \\ \cdot & 2 \\ \cdot & 5 \\ \hline 4| & 1235| \end{array} .$$

Note, for a fixed $\bar{\mathsf{F}} \in \mathcal{LDD}$, some $\sigma \in \mathfrak{S}_\Phi$ preserves the *diagonal descent property*: $\sigma\bar{\mathsf{F}} \in \mathcal{LDD}$. Next we will show how to generate $\mathcal{LDD}$ fillings from permutations and $\mathcal{SLD}$ fillings.

Example 50. Let $\bar{\mathsf{F}}$ be a filling of $\mathcal{SLD}(2|13|4|5)$ on the left. Then the permutations of $\mathfrak{S}_\Phi$ that results in an $\mathcal{LDD}$ filling are $\{id, (13)(4)(2), (134)(2)\}$ which are shown below from left to right:

$$\begin{array}{|cccc} 2 & \cdot & \cdot & \cdot \\ \cdot & 13 & \cdot & \cdot \\ \cdot & 4 & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot \\ \hline 2| & 1345| & & \end{array} \qquad \begin{array}{|cccc} 4 & \cdot & \cdot & \cdot \\ \cdot & 13 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot \\ \hline 4| & 1235| & & \end{array} \qquad \begin{array}{|cccc} 5 & \cdot & \cdot & \cdot \\ \cdot & 13 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \\ \cdot & 4 & \cdot & \cdot \\ \hline 2| & 1345| & & \end{array}$$

The permutation $(14)(3)(2)$ is an example of a permutation that does not result in a $\mathcal{LDD}$ filling.

$$\begin{array}{|cccc} 5 & \cdot & \cdot & \cdot \\ \cdot & 13 & \cdot & \cdot \\ \cdot & 4 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \\ \hline 5| & 1234| & & \end{array}$$

The group $\mathfrak{S}_\Phi$ acts analogously on fillings, permuting matrix entries. For a fixed filling F, we are interested in a subset $\mathfrak{S}_\mathsf{F}$ of $\mathfrak{S}_\Phi$ described as follows. Say $\sigma \in \mathfrak{S}_\mathsf{F}$ if for every σB_{i_1}, σB_{i_2}, and σB_{i_3} in the same column, if $|B_{i_1}| = |B_{i_2}| = |B_{i_3}|$ and σB_{i_3} above σB_{i_1} above σB_{i_2} (where there is no restriction on i_1, i_2, and i_3), then $B_{i_3} >_{\widetilde{\mathcal{D}}} B_{i_1} >_{\widetilde{\mathcal{D}}} B_{i_2}$.

Let $\Phi = B_1| \ldots |B_k$ be a set composition of $[n]$ and F be a filling of Φ. Let $\mathfrak{B}_{m,j} = \{i : |B_i| = m, B_i \text{ is in column } j\}$. Define a subgroup of $\mathfrak{S}_\Phi$

$$\mathfrak{H}_\mathsf{F} = \mathfrak{S}_{\mathfrak{B}_{1,1}} \cdots \mathfrak{S}_{\mathfrak{B}_{1,j}} \cdots \mathfrak{S}_{\mathfrak{B}_{2,1}} \cdots \mathfrak{S}_{\mathfrak{B}_{n,n}}.$$

In other words, $\mathfrak{H}_{\mathsf{F}}$ is the group of all σ that preserves the columns and placements of block sizes. Thus if F is an $\mathcal{LDD}$ filling, then for $\sigma \in \mathfrak{H}_{\mathsf{F}}$, $\sigma\mathsf{F}$ is not an $\mathcal{LDD}$ filling unless $\sigma = id$. It is easy to see that $|\mathfrak{H}_{\mathsf{F}}| = \prod_i c_i(\mathsf{row}(\mathsf{F}), \mathsf{col}(\mathsf{F}))$. For example, let F be the left filling in Example 50, then $\mathfrak{B}_{1,1} = \{1\}$, $\mathfrak{B}_{1,2} = \{3,4\}$, $\mathfrak{B}_{2,1} = \emptyset$, $\mathfrak{B}_{2,2} = \{2\}$, and $\mathfrak{B}_{m,j} = \emptyset$ for $j = 1,2$ and $2 < m \leqslant 5$. Thus, $\mathfrak{H}_{\mathfrak{F}} = \mathfrak{S}_{\{1\}}\mathfrak{S}_{\{3,4\}}\mathfrak{S}_{\{2\}}$.

Note that the set $\mathfrak{S}_{\mathsf{F}}\mathsf{F}$ contains a unique $\mathcal{SLD}$ filling, $\bar{\mathsf{F}}$, for any $\mathcal{LDD}$ filling F. Let $\bar{\mathsf{F}}$ be a $\mathcal{SLD}$ filling with row sum Φ. Consider the coset $\mathfrak{S}_{\Phi}/\mathfrak{H}_{\bar{\mathsf{F}}}$, all permutations must involve permutations across columns such that the order in columns is preserved, which is precisely $\mathfrak{S}_{\bar{\mathsf{F}}}$. Thus,

$$|\mathfrak{S}_{\bar{\mathsf{F}}}| = \frac{|\mathfrak{S}_{\Phi}|}{|\mathfrak{H}_{\bar{\mathsf{F}}}|} = \prod_i \frac{m_i(\alpha)!}{c_i(\mathsf{row}(\bar{\mathsf{F}}), \mathsf{col}(\bar{\mathsf{F}})))!}. \tag{27}$$

Proof of Theorem 47. Note that for the set $\mathfrak{S}_{\Phi}\Phi$ of all set compositions resulting from the action of $\mathfrak{S}_{\Phi}$ must contain a strict set composition. Since the action is transitive, we let Φ be a strict set composition for the rest of this proof. We take the right hand side of (26) and use Definition 35 to get

$$\rho\left(\sum_{\sigma\in\mathfrak{S}_{\Phi}} P_{\sigma(\Phi)}\right) = \rho\left(\sum_{\sigma\in\mathfrak{S}_{\Phi}}\sum_{\mathsf{F}\in\mathcal{LDD}(\sigma(\Phi))} M_{\mathsf{col}(\mathsf{F})}\right).$$

Fix a permutation $\sigma \in \mathfrak{S}_{\Phi}$ and a filling $\mathsf{F} \in \mathcal{LDD}(\sigma(\Phi))$. Then we can permute the entries of F by σ^{-1}, this results in a $\mathcal{SLD}$ filling $\bar{\mathsf{F}}$. This means that $\sigma \in \mathfrak{S}_{\bar{\mathsf{F}}}$ because $\mathfrak{S}_{\bar{\mathsf{F}}}$ are all the permutations that permute the entries of a $\mathcal{SLD}$ filling to result in a $\mathcal{LDD}$ filling, and we know that $\sigma^{-1}(\mathsf{F}) = \bar{\mathsf{F}}$ which implies that $\mathsf{F} = \sigma(\bar{\mathsf{F}})$. Thus,

$$\rho\left(\sum_{\sigma\in\mathfrak{S}_{\Phi}}\sum_{\mathsf{F}\in\mathcal{LDD}(\sigma(\Phi))} M_{\mathsf{col}(\mathsf{F})}\right) = \rho\left(\sum_{\bar{\mathsf{F}}\in\mathcal{SLD}(\Phi)}\sum_{\bar{\sigma}\in\mathfrak{S}_{\bar{\mathsf{F}}}} M_{\mathsf{col}(\bar{\sigma}(\bar{\mathsf{F}}))}\right).$$

Recall that $\rho(\Phi) = \rho(\sigma(\Phi))$, thus

$$\rho\left(\sum_{\bar{\mathsf{F}}\in\mathcal{SLD}(\Phi)}\sum_{\bar{\sigma}\in\mathfrak{S}_{\bar{\mathsf{F}}}} M_{\mathsf{col}(\bar{\sigma}(\bar{\mathsf{F}}))}\right) = \sum_{\bar{\mathsf{F}}\in\mathcal{SLD}(\Phi)} |\mathfrak{S}_{\bar{\mathsf{F}}}| M_{\rho(\mathsf{col}(\bar{\mathsf{F}}))}.$$

Consider when ρ acts on a filling $\bar{\mathsf{F}}$ by replacing the entry B_i with $|B_i|$. ρ can be seen as a bijection between $\mathcal{SLD}(\Phi)$ and $\mathcal{SD}(\alpha)$ because $\rho(\Phi) = \alpha$ and the rule "B_i is in the same column as B_{i+1} when $|B_i| \geqslant |B_{i+1}|$" maps to the rule "$a_i$ is in the same column as a_{i+1} when $a_i \geqslant a_{i+1}$". Finally, since $\mathfrak{S}_{\bar{\mathsf{F}}}$ depends only on the sizes of the entries of $\bar{\mathsf{F}}$, $|\mathfrak{S}_{\bar{\mathsf{F}}}| = |\mathfrak{S}_{\widetilde{\mathsf{F}}}|$ such that $\rho(\bar{\mathsf{F}}) = \widetilde{\mathsf{F}}$. Thus we get the following,

$$\sum_{\bar{\mathsf{F}}\in\mathcal{SLD}(\Phi)} |\mathfrak{S}_{\bar{\mathsf{F}}}| M_{\rho(\mathsf{col}(\bar{\mathsf{F}}))} = \sum_{\widetilde{\mathsf{F}}\in\mathcal{SD}(\alpha)} |\mathfrak{S}_{\widetilde{\mathsf{F}}}| M_{\mathsf{col}(\widetilde{\mathsf{F}})} = P_{\alpha}. \qquad \square$$

Now we determine the image of P_Φ in QSym in the quasisymmetric fundamental basis. We start by defining two operations on set compositions. Let $\Phi = (B_1, \dots, B_k)$ be a set composition of $[n]$ with $\rho(\Phi) = (b_1, \dots, b_k)$. Break Φ at nondescents, $B_i <_{\widetilde{\mathcal{D}}} B_{i+1}$ to build the sequence $\bar{\Phi} = (\Phi_1, \dots, \Phi_l)$ such that for every $\Phi_i = B_{j_1}|\cdots|B_{j_1+j_2}$ satisfies $B_{j_1-1} <_{\widetilde{\mathcal{D}}} B_{j_1} >_{\widetilde{\mathcal{D}}} B_{j_1+1} >_{\widetilde{\mathcal{D}}} \cdots >_{\widetilde{\mathcal{D}}} B_{j_1+j_2} <_{\widetilde{\mathcal{D}}} B_{j_1+j_2+1}$. Let $\rho(\Phi_i) = \alpha_i$ where $\alpha_i = (b_{j_1}, \dots, b_{j_1+j_2})$ is a subsequence of $\rho(\Phi)$. Associate to ρ the functions ρ_C and ρ_T defined by

$$\rho_C(\Phi) = (|\alpha_1|, |\alpha_2|, \dots, |\alpha_l|) \quad \text{and} \quad \rho_T(\Phi) = \alpha_1'|\alpha_2'|\cdots|\alpha_l'$$

where α_i' denotes ribbon transpose. These yield minimal and maximal compositions associated to $\rho(\Phi)$ in a way that will become clear shortly. For example, let $\Phi = 2|5|14|36|7$, then $\bar{\Phi} = (2|5, 14|36|7)$, $\rho_C(\Phi) = (|(1,1)|, |(2,2,1)|) = (2,5)$, and $\rho_T(\Phi) = (1,1)'|(2,2,1)' = (2,1,2,2)$. This is the last theorem needed to prove Theorem 26.

Remark 51. In subset notation, $\rho(\Phi) = \rho_C(\Phi) \cup D$, and $\rho_T(\Phi) = \rho_C(\Phi) \cup D^{\complement}$, where D represents the blocks where the next block descends with respect to $\widetilde{\mathcal{D}}$.

Theorem 52. *Let Φ be a set composition and $\mu_{\mathcal{P}}$ be the Möbius function on the refinement poset $\mathcal{P}$ on compositions, then*

$$\rho(P_\Phi) = \sum_{\rho_T(\Phi) \leqslant \beta \leqslant \rho_C(\Phi)} \mu_{\mathcal{P}}(\beta, \rho_C(\Phi)) F_\beta. \tag{28}$$

Proof. One can see that

$$\rho(P_\Phi) = \rho\left(\sum_{\Phi \leqslant \Psi \leqslant C(\Phi)} M_\Psi\right) = \sum_{\rho(\Phi) \leqslant \gamma \leqslant \rho_C(\Phi)} M_\gamma.$$

Using the change of basis from the monomial basis to the fundamental basis we get

$$\rho(P_\Phi) = \sum_{\rho(\Phi) \leqslant \gamma \leqslant \rho_C(\Phi)} \sum_{\gamma \geqslant \beta} \mu_{\mathcal{P}}(\beta, \gamma) F_\beta.$$

We gather all the F_β terms and write

$$\rho(P_\Phi) = \sum_{\beta \leqslant \rho_C(\Phi)} \left(\sum_{\gamma \in G(\Phi, \beta)} \mu_{\mathcal{P}}(\beta, \gamma)\right) F_\beta$$

where $G(\Phi, \beta)$ is the interval $[\rho(\Phi) \wedge \beta, \rho_C(\Phi)]$. We note that this a boolean lattice so the cardinality of this set is either 1 or an even number. Let $\widetilde{\gamma} = \min(G(\Phi, \beta))$, where $\widetilde{\gamma} \neq \rho_C(\Phi)$. It takes j coarsenings to go from $\widetilde{\gamma}$ to $\rho_C(\Phi)$, thus we can relate all compositions in $G(\Phi, \beta)$ relative to $\widetilde{\gamma}$ by the tuple $(g_1, \dots, g_j)$ where $g_i = 1$ if the two entries are coarsened and $g_i = 0$ if the two entries are not coarsened. This means that there are 2^j compositions in $G(\Phi, \beta)$ and there is natural pairing of two tuples $(0, g_2, \dots, g_j)$ and $(1, g_2, \dots, g_j)$. The resulting compositions have a length difference of 1, which means that $\mu_{\mathcal{P}}(\beta,\gamma_1)+\mu_{\mathcal{P}}(\beta,\gamma_2)=0$ where γ_1 and γ_2 are the compositions that come from $(0,g_2,\ldots,g_j)$ and $(1,g_2,\ldots,g_j)$ respectively. Since the cardinality of $G(\Phi,\beta)$ is even and the sum of every pair is 0, the coefficient of F_α is 0.

Now consider the case $\rho(\Phi)\wedge\beta=\rho_C(\Phi)$, then the coefficient is $\mu_{\mathcal{P}}((\beta,\rho_C(\Phi))$. To find the interval for all such β's: since $\rho(\Phi)\wedge\beta=\rho_C(\Phi)$, in subset notation $\rho(\Phi)\cap\beta=\rho_C(\Phi)$. This means $\beta\supseteq\rho_C(\Phi)$ and, for every $A\in\rho(\Phi)-\rho_C(\Phi)$, then $A\notin\beta$. Since we want the smallest β in the interval, i.e. the largest subset, we should take $\beta=\rho_C(\Phi)\cup(\rho(\Phi)-\rho_C(\Phi))^{\complement}=\rho_T(\Phi)$. □

10.1 Proof of Theorem 26

Notice that we can express a powersum quasisymmetric function in the fundamental basis by substituting (28) into (26)

$$P_\alpha=\sum_{\sigma\in\mathfrak{S}_\Phi}\rho(P_{\sigma(\Phi)})=\sum_{\sigma\in\mathfrak{S}_\Phi}\sum_{\rho_T(\sigma(\Phi))\leqslant\beta\leqslant\rho_C(\sigma(\Phi))}\mu_{\mathcal{P}}(\beta,\rho_C(\sigma(\Phi)))F_\beta. \tag{29}$$

This expression will be needed in the proof of Theorem 26, and fortunately it turns out that the intervals $[\rho_T(\sigma(\Phi)),\rho_C(\sigma(\Phi))]$ are disjoint for different σ, which means that it is quite easy to find the coefficient of F_β.

Lemma 53. *Let Φ be a set composition and $\sigma\in\mathfrak{S}_\Phi$. Fix β such that $\rho_T(\sigma(\Phi))\leqslant\beta\leqslant\rho_C(\sigma(\Phi))$. If there exists a $\sigma'\in\mathfrak{S}_\Phi$ such that $\rho_C(\sigma(\Phi))\neq\rho_C(\sigma'(\Phi))$, then β is not in the interval $[\rho_T(\sigma'(\Phi)),\rho_C(\sigma'(\Phi))]$.*

Proof. We'll do a proof by contradiction and use set notation. If $\rho_C(\sigma(\Phi))\neq\rho_C(\sigma'(\Phi))$ then without loss of generality there exists $A\in\rho_C(\sigma'(\Phi))$, $A\notin\rho_C(\sigma(\Phi))$. Thus if β is less than $\rho_C(\sigma(\Phi))$ and $\rho_C(\sigma'(\Phi))$ then $\beta\supseteq\rho_C(\sigma'(\Phi))\ni A$. Since $A\in\rho_C(\sigma'(\Phi))$, we must have $A\in\rho(\sigma'(\Phi))=\rho(\Phi)$, but $A\notin\rho_C(\sigma(\Phi))$ so A is a "descending division" so $\rho_T(\sigma(\Phi))\not\ni A$. However if $\beta\geqslant\rho_T(\sigma(\Phi))$ then $\beta\subseteq\rho_T(\sigma(\Phi))$ so $\beta\not\ni A$, thus a contradiction. □

Proof of Theorem 26. Note that the largest (with respect to $\mathcal{P}$) composition in the interval of (29) is $C(\alpha)$ due to the fact that there is a $\Psi\in\mathfrak{S}_\Phi(\Phi)$ such that Ψ is a strict composition thus $\rho_C(\Psi)=C(\alpha)$. Equivalently, the smallest composition in the interval comes from the opposite of a strict set composition which is $T(\alpha)$.

Throughout, we use subset notation for compositions and appeal to Remark 51. Lemma 53 implies that all intervals are disjoint, which means that in (29) the coefficient of F_β is (up to sign) the number of $\sigma\in\mathfrak{S}_\Phi$ that results in the same $\rho_C(\sigma(\Phi))$, we denote this set as Σ. Fix β and $\bar{\sigma}$ such that $\beta\leqslant\rho_C(\bar{\sigma}(\Phi))$ and let $\Sigma=\{\sigma\in\mathfrak{S}_\Phi:\bar{\sigma}(B_i)>\bar{\sigma}(B_{i+1})\iff\sigma(B_i)>\sigma(B_{i+1})\}$. Notice that according to $\widetilde{\mathcal{D}}$, if B_i and B_{i+1} are two different sizes and $B_i>B_{i+1}$, then $\sigma(B_i)>\sigma(B_{i+1})$, due to the fact that $\widetilde{\mathcal{D}}$ compares sizes. We observe that, in Algorithm 1, the conditions for whether lines 3, 5, or 7 happens at the ith iteration are equivalent to three conditions on $\bar{\sigma}(B_{i-1})$ and $\bar{\sigma}(B_i)$. For the first case, $|\bar{\sigma}(B_{i-1})|\neq|\bar{\sigma}(B_i)|$ if and only if $a_{i-1}\neq a_i$. In the second case, if

$\bar{\sigma}(B_{i-1}) <_{\mathcal{D}} \bar{\sigma}(B_i)$ where $|\bar{\sigma}(B_{i-1})| = |\bar{\sigma}(B_i)|$, then obviously $a_{i-1} = a_i$, but this implies that the composition $\rho_C(\bar{\sigma}(\Phi))$ contains A (written in subset notation). This implies $A \in \beta$, since $\rho_C(\bar{\sigma}(\Phi)) \geqslant \beta$. Thus the A^{th} box is above the $(A+1)^{\text{st}}$ box in the original ribbon of β and since R begins at the $(A+1)^{\text{st}}$ box, there exists a b_j such that $(b_j, \ldots, b_l)$ forms the ribbon R. For the last case, let $\bar{\sigma}(B_{i-1}) >_{\mathcal{D}} \bar{\sigma}(B_i)$ where $|\bar{\sigma}(B_{i-1})| = |\bar{\sigma}(B_i)|$, then $A \notin \rho_T(\bar{\sigma}(\Phi))$ which means that $A \notin \beta$ because $\rho_T(\bar{\sigma}(\Phi)) \leqslant \beta$. This implies that the A^{th} and $(A+1)^{\text{st}}$ box are in the same row of the ribbon of β. Since R starts at the $(A+1)^{\text{st}}$ box, there does not exist a b_j such that $(b_j, \ldots, b_l)$ forms the ribbon R.

Recall $\mathfrak{B}_k$, the set of all blocks B of Φ such that $|B| = k$, then there is a natural bijection, O from $\mathfrak{B}_k$ to the interval $[|\mathfrak{B}_k|]$, satisfying $B >_{\tilde{\mathcal{D}}} B'$ if and only if $O(B) > O(B')$. Thus Σ is also enumerated by the number of standard fillings of $D(\beta, \rho(\bar{\sigma}(\Phi)))$ where the entries of a filling, when read from left to right, are increasing (due to the alternate condition of RTa2) and, when read from top to bottom, are decreasing (due to the alternate condition of RTa3). Thus $|\Sigma| = \mathsf{SDR}(\beta, \alpha)$.

Finally the Möbius function has a $(-1)^{\mathsf{len}(\beta)-\mathsf{len}(\rho_C(\sigma(\Phi))}$ factor and we will show by induction that $(-1)^{\mathsf{ht}(\beta,\rho(\sigma(\Phi)))} = (-1)^{\mathsf{len}(\beta)-\mathsf{len}(\rho_C(\sigma(\Phi))}$ when $\rho_T(\sigma(\Phi)) \leqslant \beta \leqslant \rho_C(\sigma(\Phi))$. Consider the base case when $\beta = \rho_C(\sigma(\Phi))$. Then $\rho(\sigma(\Phi))$ refines $\rho_C(\sigma(\Phi))$, thus $\mathsf{ht}(\rho_C(\sigma(\Phi)), \rho(\sigma(\Phi))) = 0$ and obviously $\mathsf{len}(\rho_C(\sigma(\Phi)) - \mathsf{len}(\rho_C(\sigma(\Phi)) = 0$. Now suppose that $\mathsf{ht}(\beta, \rho(\sigma(\Phi))) = \mathsf{len}(\beta) - \mathsf{len}(\rho_C(\sigma(\Phi))) = t$. Let $\mathsf{len}(\gamma) = \mathsf{len}(\beta) + 1$ where $\rho_T(\sigma(\Phi)) \leqslant \gamma < \beta \leqslant \rho_C(\sigma(\Phi))$, then there exists an element B such that $B \notin \beta$ and $B \in \gamma$, since $\rho_T(\sigma(\Phi)) \leqslant \gamma$, $B \notin D$. Let A be maximal such that $A < B$ and $a_1 + \cdots + a_k = A$. Then suppose that in the kth iteration of RT for β the height equals s, then in the kth iteration of RT for γ, the height equals $s+1$ because the ith box is the end of a row of the ribbon γ. Thus at the final iteration we will get $\mathsf{ht}(\gamma, \rho(\sigma(\Phi))) = t+1$. Obviously $\mathsf{len}(\gamma) - \mathsf{len}(\rho_C(\sigma(\Phi))) = t+1$, thus $\mathsf{ht}(\gamma, \rho(\sigma(\Phi))) = \mathsf{len}(\gamma) - \mathsf{len}(\rho_C(\sigma(\Phi))$. □

11 Generalizations of P_Φ Using Total Orders

With careful work we can generalize a $\mathcal{LDD}$ filling by replacing $>_{\tilde{\mathcal{D}}}$ by a different total ordering $\rhd$ on disjoint integer sets. The resulting powersum $P^{\rhd}$ refines the powersum symmetrics in non-commuting variables. Only certain total orders $\rhd$ induce $P^{\rhd}$ that have a shifted shuffle product and a standardized deconcatenate coproduct.

Definition 54. A total order $\rhd$ on disjoint sets A and B is *shift-invariant* if it has the property that if $A \rhd B$, then $A^{\uparrow k} \rhd B^{\uparrow k}$ for all k. A total order $\rhd$ is *standard-invariant* if it has the property that if $A \rhd B$, then $A^{\downarrow} \rhd B^{\downarrow}$.

Studying carefully Definition 37 and the proofs of Theorem 43 and 44, we see that the only crucial properties of $>_{\tilde{\mathcal{D}}}$ used are its shift and standard invariance. We conclude:

Proposition 55. *If the total order $\rhd$ is shift-invariant, then $P^{\rhd}$ has a shifted shuffle product. Likewise, if $\rhd$ is standard-invariant, then $P^{\rhd}$ has a standardized deconcatenation coproduct.*

Example 56. Let $>_{Med}$ be the total order on disjoint sets A and B by the covering relation

$$A >_{Med} B = \begin{cases} |A| > |B| & |A| \neq |B| \\ \mathrm{Med}(A) > \mathrm{Med}(B) & |A| = |B| \end{cases}$$

where $\mathrm{Med}(A)$ is the median (rounded up) of A. It follows that $>_{\mathrm{Med}}$ is a shift-invariant and standard-invariant total order, which means that P^{Med} has a shifted shuffle product and a standardized deconcatenate coproduct. An interesting question is whether a standard invariant total order is also shift invariant (and vice versa), we leave this question to the reader.

Let B and A be two disjoint sets of different sizes. We say that a total ordering *projects under* ρ if there exists a total ordering on integers $\succ$, such that $B \rhd A$ implies $|B| \succ |A|$.

Example 57. Define the total order on sets $>_{\widetilde{M}}$ as $B > A$ if $\min(B) > \min(A)$. Then $>_{\widetilde{M}}$ does not project under ρ because $\{3,4\} >_{\widetilde{M}} \{2\}$ and $\{2,4\} <_{\widetilde{M}} \{3\}$. This means that $2 >_M 1$ and $2 <_M 1$, however this implies that $<_M$ is not a total order. We note that $>_{\widetilde{M}}$ is a standard invariant and shift invariant total order.

Let $\Phi = (B_1, \ldots, B_k)$ be a set composition of n with $\rho(\Phi) = (b_1, \ldots, b_k)$. Break $\rho(\Phi)$ when $B_i \rhd B_{i+1}$ so that $\rho(\Phi) = \beta_1|\beta_2|\cdots|\beta_l$ where $\beta_i = (b_{j_1}, \ldots, b_{j_1+j_2})$ with $B_{j_1-1} \lhd B_{j_1} \rhd B_{j_1+1} \rhd \cdots \rhd B_{j_1+j_2} \lhd B_{j_1+j_2+1}$. Then we define $\rho_C^{\rhd}(\Phi) = (|\beta_1|, |\beta_2|, \ldots, |\beta_l|)$ and $\rho_T^{\rhd}(\Phi) = \beta_1'|\beta_2'|\cdots|\beta_l'$, where β_i' is the transpose of β_i.

Theorem 58. *Let $\rhd$ be a total order that projects to $\prec$. Then*

- $P_\alpha^{\prec} = \sum_{\sigma \in \mathfrak{S}_\Phi} \rho(P_{\sigma(\Phi)}^{\rhd})$ *where* $\alpha = \rho(\Phi)$,
- $\rho(P_\Phi^{\rhd}) = \sum_{\rho_T^{\rhd}(\Phi) \leqslant \alpha \leqslant \rho_C^{\rhd}(\Phi)} \mu_{\mathcal{P}}(\alpha, \rho_C^{\rhd}(\Phi)) F_\alpha$.

This follows from the proofs of Theorems 47 and 52 by changing the total order.

Proof of Theorem 29. The proof in Section 10.1 requires Lemma 53, Theorems 47 and 52, and for the total order $\rhd$ to project under ρ to $\prec$. We define a total order on set compositions, $\rhd$, that projects to $\prec$ by

$$A \rhd B = \begin{cases} |A| \prec |B| & \text{if } |A| \neq |B| \\ \min(A) < \min(B) & \text{if } |A| = |B| \end{cases}.$$

$\rhd$ is a standard invariant projective total order, thus a total order analogue of Lemma 53 and Theorem 58 holds. □

In Section 6 we could naturally pair total orders (and thereby pairing powersums) using the star involution. Now we will show how to do this generally.

Recall that the *algebraic complement involution*, denoted $\leftarrow$, sends M_Φ to $M_{\overleftarrow{\Phi}}$, where $\Phi = (B_1, \ldots, B_k)$ and $\overleftarrow{\Phi} = (B_k, \ldots, B_1)$ is the reverse set composition of Φ.

Theorem 59. *The algebraic complement of a powersum quasisymmetric is the reverse powersum*

$$\overleftarrow{P^{\rhd}_{\Phi}} = P^{\overleftarrow{\rhd}}_{\overleftarrow{\Phi}} \tag{30}$$

where $\overleftarrow{\rhd}$ is the reverse of the total order $\rhd$.

Proof. By definition, $\overleftarrow{P^{\rhd}_{\Phi}} = \overleftarrow{\sum_{\Psi \rhd \Phi} M_{\Psi}} = \sum_{\Psi \rhd \Phi} M_{\overleftarrow{\Psi}}$. Denote the reverse composition as $\overleftarrow{\Phi} = A_1 | \cdots | A_k$. Note that for blocks B_i, B_{i+1} in Φ, if $B_i \rhd B_{i+1}$, then $\overleftarrow{B_i} \rhd \overleftarrow{B_{i+1}}$, which means that $A_{k+1-i} \lhd A_{k+1-i+1}$, which is rewritten as $A_{k+1-i+1} \overleftarrow{\rhd} A_{k+1-i}$. Thus, $\sum_{\Psi \rhd \Phi} M_{\overleftarrow{\Psi}} = \sum_{\overleftarrow{\Phi} \overleftarrow{\rhd} \Psi} M_{\Psi} = P^{\overleftarrow{\rhd}}_{\overleftarrow{\Phi}}$. □

Let Φ be a set composition of $[n]$. Recall that the *coalgebraic complement involution* is defined as the map $\overline{M_{\Phi}} = M_{\overline{\Phi}}$, where $\overline{\Phi} = (\overline{B_1}, \ldots, \overline{B_k})$ and $\overline{B_i} = \{n+1-b_{i_1}, \ldots, n+1-b_{i_j}\}$. For example, $\overline{M_{13|2|45}} = M_{35|4|12}$. Let A and B be disjoint subsets of n, we define the complement of a shift-invariant total order $\rhd$, denoted $\overline{\rhd}$, as

$$A \overline{\rhd} B = \begin{cases} \overline{A} \lhd \overline{B} & \text{if } |A| = |B| \\ A \rhd B & \text{if } |A| \neq |B| \end{cases}.$$

We similarly define the complement of shift-invariant total order $\rhd$ as $A \rhd B = \overline{A} \lhd \overline{B}$.

Proposition 60. *If $\rhd$ projects to $\prec$ under ρ, then $\overline{\rhd}$ does as well. In other words complementation is an involution on such total orders.*

This follows from the definition that $A \overline{\rhd} B$ when $A \rhd B$. This goes to say that if a total order $\rhd$ compares to disjoint sets based off the sets' size then the complement total order will still compare the sets by using the sets' size.

Theorem 61. *Let Φ be a set composition of $[n]$. If $\rhd$ is a total order on set compositions, then*

$$\overline{P^{\rhd}_{\Phi}} = P^{\overline{\rhd}}_{\overline{\Phi}}$$

where $\overline{\rhd}$ is the complement of the total order $\rhd$.

Proof. Let $\Phi = (B_1, \ldots, B_k)$. This proof will be done in two cases, when $\rhd$ is projective, and when $\rhd$ is not projective. However for both cases $\overline{P^{\rhd}_{\Phi}} = \overline{\sum_{\Psi \rhd \Phi} M_{\Psi}} = \sum_{\Psi \rhd \Phi} M_{\overline{\Psi}}$.

Let $\rhd$ be projective. If $B_i \rhd B_{i+1}$, then $\overline{B_i} \lhd \overline{B_{i+1}}$ when $|B_i| = |B_{i+1}|$, which means that $B_i \overline{\rhd} B_{i+1}$. And when $|B_i| \neq |B_{i+1}|$, if $B_i \rhd B_{i+1}$, then $B_i \overline{\rhd} B_{i+1}$. Thus it is equivalent to write $\sum_{\Psi \rhd \Phi} M_{\overline{\Psi}} = \sum_{\Psi \overline{\rhd} \overline{\Phi}} M_{\Psi} = P^{\overline{\rhd}}_{\overline{\Phi}}$.

Finally if $\rhd$ is not projective, then $B_i \rhd B_{i+1}$, which implies $\overline{B_i} \lhd \overline{B_{i+1}}$ for all B_i. Thus $\sum_{\Psi \rhd \Phi} M_{\overline{\Psi}} = \sum_{\Psi \overline{\rhd} \overline{\Phi}} M_{\Psi} = P^{\overline{\rhd}}_{\overline{\Phi}}$. □

12 Powersums in Other Algebras

In [7], the authors define a way to define a monomial basis for a combinatorial Hopf algebra given certain conditions. In some of the examples below we sketch the relationship that P has with other combinatorial algebras using fillings and the monomial basis.

12.1 FQSym

FQSym is the Hopf algebra of permutations first introduced by Malvenuto and Reutenauer [15], which is a subalgebra of the Hopf algebra of Word Quasisymmetric function, WQSym, whose bases are indexed by packed words [8, Section 3] and [17, Section 2.4]. It can be seen by the map Π that WQSym and NCQSym are isomorphic, which gives a polynomial realization to WQSym. In this section we will show that the subalgebra FQSym maps nicely, via Π, to the powersum basis indexed by singleton set compositions, a *singleton set composition* is a set composition whose all parts have cardinality 1.

A *packed word* of n is a word $w = w_1 \cdots w_n$ such that for any w_i either the letter $w_i - 1$ is in w or $w_i = 1$. Then *packing* a word u, $\mathsf{pack}(u)$, is the packed word w such that all letters keep the same relative order as u, meaning $u_i > u_{i+1}$ (respectively $u_i < u_{i+1}$, $u_i = u_{i+1}$) if and only if $w_i > w_{i+1}$ (respectively $w_i < w_{i+1}$, $w_i = w_{i+1}$). For example, $\mathsf{pack}(2426) = 1213$. The space of WQSym is spanned by the monomial basis where $M_w = \sum_{pack(u)=w} u$. The map $\mathsf{setcomp}(w)$ bijects packed words to a set composition by $w = w_1 \cdots w_n \to \Phi = B_1 | \cdots | B_k$ where $i \in B_{w_i}$. For example, let 21243 be a packed word, then $\mathsf{setcomp}(21243) = 2|13|5|4$. The map $\Pi : M_w \to M_{\mathsf{setcomp}(w)}$ is a Hopf isomorphism between WQSym and NCQSym, more details in [16].

A permutation of n, $\tau\mathfrak{S}_n$, acts on a set composition of length n, $\Phi = (B_1, \ldots, B_n)$, by $\tau(\Phi) = B_{\tau(1)} | \ldots | B_{\tau(n)}$. The *standardization* of a word, $st(w)$ (which is not to be confused with standardization of a set composition $\Phi^{\downarrow}$), is the permutation obtained from reading the word from left to write and label $1, 2, \ldots$ when $w_i = 1$ and repeat for $w_i = 2$ and so on. For example 132 and 121 are both packed words with $st(132) = st(121) = 132$. Let $\Phi = (B_1, \ldots, B_k)$ and $\Psi = (A_1, \ldots, A_l)$ be a set composition and a singleton set composition respectively, $st(\Phi) = \Psi$ means that for every i, $B_i = A_j \cup \cdots \cup A_{j+j'}$ such that $A_j < \cdots < A_{j+j'}$ where $<$ compares the only integer in the block.

In [17, Section 2.4], FQSym is a sub Hopf algebra of WQSym by the map $\mathcal{G}_\tau \to \sum_{st(u)=\tau} M_u$ where $\mathcal{G}_\tau$ is the *dual fundamental* basis and τ is a permutation. Thus for NCQSym, $\Pi(\mathcal{G}_\tau) \to \sum_{st(\Phi)=\tau(1|\cdots|n)} M_\Phi$. We note that the Φ in this sum are precisely the set compositions satisfying $\Phi \geqslant_{\widetilde{\mathcal{D}}} \tau(1|\cdots|n)$, thus through Definition 37 we get the following theorem.

Theorem 62. *The image of the $\mathcal{G}$ basis under Π on* NCQSym *is the powersum quasisymmetric function in non-commuting variables indexed by a singleton set composition, i.e. for a permutation τ*

$$\Pi(\mathcal{G}_\tau) = P_{\tau(1|2|\cdots|n)}. \tag{31}$$

12.2 QSym^r

Hivert defined the space of r-quasisymmetric functions, denoted QSym^r where r is some nonnegative integer, as the invariant space of certain local actions in [11]. A local action is a permutation from $\mathfrak{S}_n$, where n is the number of variables, that acts on the polynomial ring by

$$\sigma_i(x_i^a x_{i+1}^b) = \begin{cases} x_i^b x_{i+1}^a & \text{if } a < r \text{ or } b < r \\ x_i^a x_{i+1}^b & \text{else} \end{cases} \tag{32}$$

The spaces of r-quasisymmetric functions are nested sub Hopf algebras of $\mathbb{Q}[X]$, in other words $\mathbb{Q}[X] = \mathrm{QSym}^0 \supset \mathrm{QSym} = \mathrm{QSym}^1 \supset \mathrm{QSym}^2 \supset \cdots \supset \mathrm{QSym}^r \supset \cdots \supset \mathrm{Sym}$.

Let $\alpha = (a_1, \ldots, a_k)$ be a composition of n such that $a_i \geqslant r$ and let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of m such that $\lambda_i < r$. Then we define an *r-composition* of $n + m$ as the pairing of α and λ denoted as $(\alpha; \lambda)$. Let β be a composition of n, then $\mathsf{sort}_r(\beta)$ is the r-composition that comes from moving all parts less than r to the end. For example $\mathsf{sort}_3((3, 5, 2, 1, 3, 4, 5, 2)) = (3, 5, 3, 4, 5; 2, 2, 1)$.

The r-quasimonomial function, denoted as $M^r_{(\alpha;\lambda)}$ is a basis of QSym^r and is defined as

$$M^r_{(\alpha;\lambda)} = \sum_{\beta\colon \mathsf{sort}_r(\beta)=(\alpha;\lambda)} M_\beta.$$

In much the same way as above, we can define fillings for QSym^r, denoted $\mathcal{SD}^r$, to define a combinatorial way to change the basis from P to M.

$$P^r_{(\alpha;\lambda)} = \sum_{\substack{\mathsf{F}\in\mathcal{SD}^r(\alpha;\lambda)\\ \sigma\in\mathfrak{S}_{\mathsf{F}}}} M^r_{\mathsf{col}(\sigma(\mathsf{F}))}.$$

This also has the property that $P^r_{(\alpha;\lambda)} = \sum_{\mathsf{sort}_r(\beta)=(\alpha;\lambda)} P_\beta$. This will be explored more in a future paper.

12.3 Cyclic Quasisymmetric Functions

In [1] the authors defined a different action on QSym called the cyclic shift, denoted ς_i. The invariant space of QSym under the cyclic shift is the space of cyclic quasisymmetric functions, cQSym, note that this isn't a Hopf algebra, due to the fact that cQSym is not preserved under the coproduct. In [1], the authors use set notation, however out of familiarity we will use composition notation. Let $\alpha = (a_1, \ldots, a_k)$ be a composition, then the cyclic shift of α is $\varsigma(\alpha) = (a_k, a_1, \ldots, a_{k-1})$. Thus the monomial basis of cQSym is

$$\hat{M}_\alpha = \sum_{i=1}^{k} M_{\varsigma^i(\alpha)}, \tag{33}$$

where $\varsigma^i(\alpha) = \varsigma^{i-1} \circ \varsigma(\alpha)$.

For example, $\hat{M}_{(3,1,2)} = M_{(3,1,2)} + M_{(2,3,1)} + M_{(1,2,3)}$ and $\hat{M}_{(2,1,2,1)} = 2M_{(2,1,2,1)} + 2M_{(1,2,1,2)}$. The powersums for cQSym can also be defined in terms of fillings (with an equivalence relation on fillings by cycling the columns) and row permutations

$$\hat{P}_\alpha = \sum_{\substack{\mathsf{F}\in\hat{\mathcal{SD}}(\alpha)\\ \sigma\in\hat{\mathfrak{S}}_{\mathsf{F}}}} M_{\mathsf{col}(\sigma(\mathsf{F}))}.$$

For example, $\hat{P}_{(2,1,2,1)} = 2\hat{M}_{(2,1,2,1)} + 4\hat{M}_{(3,1,2)} + 2\hat{M}_{(3,3)}$. Contrary to the scaling of (33), the cyclic powersum quasisymmetrics has either a coefficient of 1 or 0 when expanding a cyclic powersum quasisymmetric in terms of powersum quasisymmetrics. From the example above, $\hat{P}_{(2,1,2,1)} = P_{(2,1,2,1)} + P_{1,2,1,2}$. Thus the alternate definition is

$$\hat{P}_\alpha = \sum_{i \in I} P_{\varsigma^i(\alpha)}.$$

where $I = [j]$ where j is the minimum integer such that $\varsigma^{j+1}(\alpha) = \varsigma(\alpha)$.

Acknowledgements

The author would like to thank his advisor Amy Pang for all the help and guidance with the development of this project and the development of the author as a mathematician. The author would like to thank Aaron Lauve for this project, a lot of the code used, editing and guidance. The author would also like to thank Travis Scrimshaw for all of his helpful comments. Finally the author would like to thank the Sage community.

References

[1] R. M. Adin, I. M. Gessel, V. Reiner, and Y. Roichman. Cyclic quasi-symmetric functions. *Israel Journal Mathematics*, 243(1), 437–500, 2021.

[2] P. Alexandersson and R. Sulzgruber. P-Partitions and p-Positivity. *International Mathematics Research Notices*, 2021(14): 10848–10907, 2021.

[3] F. Aliniaeifard and S. Li. Peak algebras in combinatorial Hopf algebras. arXiv:2110.05648, 2021.

[4] F. Aliniaeifard, V. Wang, and S. van Willigenburg. P-partition power sums. *European Journal of Combinatorics*, 110: 103688, 2023.

[5] C. Ballantine, Z. Daugherty, A. Hicks, S. Mason, and E. Niese. On quasisymmetric power sums. *Journal of Combinatorial Theory, Series A*, 175: 105273, 2020.

[6] C. Benedetti and B. E. Sagan. Antipodes and involutions. *Journal of Combinatorial Theory, Series A*, 148: 275–315, 2017.

[7] N. Bergeron, R. S. González D'León, S. Li, C. Y. A. Pang, and Y. Vargas. Hopf algebras of parking functions and decorated planar trees. *Advances in Applied Mathematics*, 143: 102436, 2023.

[8] G. Duchamp, F. Hivert, and J.-Y. Thibon. Noncommutative symmetric functions VI: Free quasi-symmetric functions and related algebras. *International Journal of Algebra and Computation*, 12(05): 671–717, 2002.

[9] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J.-Y. Thibon. Noncommutative symmetric functions. *Advances in Mathematics*, 112(2): 218–348, 1995.

[10] I. M. Gessel. Multipartite P-partitions and inner products of skew Schur functions. *Contemporary Mathematics*, 34: 289–317, 1984.

[11] F. Hivert. Local action of the symmetric group and generalizations of quasi-symmetric functions. *FPSAC Proceedings*, 179–193, 2005.

[12] D. Krob, B. Leclerc, and J.-Y. Thibon. Noncommutative symmetric functions II: Transformations of alphabets. *International Journal of Algebra and Computation*, 7(02): 181–264, 1997.

[13] K. Luoto, S. Mykytiuk, and S. van Willigenburg. An introduction to quasisymmetric Schur functions: Hopf algebras, quasisymmetric functions, and Young composition tableaux. Springer Science & Business Media, 2013

[14] I. G. Macdonald. Symmetric functions and Hall polynomials. Oxford University Press, 1998.

[15] C. Malvenuto and C. Reutenauer. Duality between quasi-symmetrical functions and the Solomon descent algebra. *Journal of Algebra*, 177(3): 967–982, 1995.

[16] J.-C. Novelli and J.-Y. Thibon. Polynomial realizations of some trialgebras. arXiv:math/0605061, 2006.

[17] J.-C. Novelli, J.-Y. Thibon, and L. K. Williams. Combinatorial Hopf algebras, noncommutative Hall–Littlewood functions, and permutation tableaux. *Advances in Mathematics*, 224(4): 1311–1348, 2010.

[18] M. Rosas and B. E. Sagan. Symmetric functions in noncommuting variables. *Transactions of the American Mathematical Society*, 358(1): 215–232, 2006.

[19] R. Stanley. Enumerative Combinatorics Volume 2. Cambridge Studies in Advanced Mathematics, 1999.

[20] M. C. Wolf. Symmetric functions of non-commutative elements. *Duke Mathematical Journal*, 2(4): 626–637, 1936.