Tropical Moduli Spaces of Rational Graphically Stable Curves

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Abstract

The tropical moduli space $\mathcal{M}_{0,n}^{\text{trop}}$ is a cone complex which parameterizes leaf-labeled metric trees called tropical curves. We introduce graphic stability and describe a refinement of the cone complex given by radial alignment. We prove that given a complete multipartite graph Γ, the moduli space of radially aligned Γ-stable tropical curves can be given the structure of a balanced fan. This fan structure coincides with the Bergman fan of the cycle matroid of Γ.

Mathematics Subject Classifications: 05E14, 14T15, 14D22, 05B35

1 Introduction

The tropical moduli space $\mathcal{M}_{0,n}^{\text{trop}}$ is a cone complex which parameterizes leaf-labeled metric trees. Its structure is obtained by gluing positive orthants of $\mathbb{R}^{n-3}$ corresponding to trivalent trees. Speyer and Sturmfels [SS04] give an embedding of this cone complex (in the context of phylogenetic trees) into a real vector space as a balanced fan where each top-dimensional cone is assigned weight 1. In [AK06], Ardila and Klivans study phylogenetic trees and show that the fan structure of $\mathcal{M}_{0,n}^{\text{trop}}$ has a refinement which coincides with the Bergman fan of the cycle matroid of K_{n-1}, the complete graph on $n-1$ vertices. As a generalization of Ardila and Klivans, it is shown by Cavalieri, Hampe, Markwig, and Ranganathan in [CHMR16] that the fan associated to the moduli space of rational heavy/light weighted stable tropical curves, $\mathcal{M}_{0,w}^{\text{trop}}$, and the Bergman fan of a graphic matroid have the same support.

This paper investigates stability conditions defined by a graph rather than a weight vector. We introduce *rational graphically stable tropical curves* (Definition 15) and write $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ for the moduli space of these curves. We define these moduli spaces so that if we begin with a graph that is also a reduced weight graph (Definition 2.13 of [CHMR16]) we recover the corresponding weighted moduli space.

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Keeping in mind that the Bergman fan of certain graphic matroids are a refinement of $\mathcal{M}_{0,n}^{\mathrm{trop}}$ (also $\mathcal{M}_{0,w}^{\mathrm{trop}}$), we add the extra condition of *radial alignment* to $\mathcal{M}_{0,n}^{\mathrm{trop}}$ to define the moduli space $\mathcal{M}_{0,n}^{\mathrm{trad}}$. Radial alignment refers to an ordered partition on the vertices of the combinatorial type of a curve.

The main result of this paper characterizes tropical moduli spaces given by a graphic matroid.

Theorem 29 Let $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ be the refinement of $\mathcal{M}_{0,\Gamma}^{\mathrm{trop}}$ given by radial alignment. It has the structure of a balanced fan and is naturally identified with $\mathcal{B}'(\Gamma)$ if and only if Γ is a complete multipartite graph.

Our motivation for this paper comes from the theory of tropical compactifications and log geometry. From works of Tevelev [Tev07] and Gibney and Maclagan [GM11] it has been shown that there is an embedding of $\mathcal{M}_{0,n}$ into the torus of a toric variety $X(\Sigma)$ where the tropicalization of $\mathcal{M}_{0,n}$ is a balanced fan $\Sigma \cong \mathcal{M}_{0,n}^{\mathrm{trop}}$. This embedding is special in the sense that the closure of $\mathcal{M}_{0,n}$ in $X(\Sigma)$ is $\overline{\mathcal{M}}_{0,n}$. Cavalieri et al. [CHMR16] show a similar embedding can be constructed for weighted moduli spaces when the weights are heavy/light spaces. In [RSPW19] Ranganathan, Santos-Parker, and Wise describe radial alignments of genus 1 tropical curves and show how this extra data can be used for desingularization. The subdivision given by radial alignments has been studied before in [AK06] and [FS05] but we use a rephrasing in order to relate it to log geometry and the results of Ranganathan et al. The main results of this paper provide the technical combinatorial conditions which is needed to classify which moduli spaces of graphically stable rational pointed curves give rise to tropical compactifications. The geometric aspects of this story will be formalized in a future paper.

The paper is organized as follows. In Section 2 we define a matroid using independence axioms. Then we restrict our attention to the cycle matroid where we define the relevant concepts and discuss necessary graph theory terminology.

In Section 3 we begin by defining the moduli space of rational n-marked tropical curves, $\mathcal{M}_{0,n}^{\mathrm{trop}}$. We also describe an embedding as a balanced fan into a real vector space as in [GKM09]. In Section 3.1, we define the Bergman fan of a matroid and also define radially aligned rational n-marked tropical curves by imposing a weak ordering on the vertices given by their distances from the root vertex.

Section 3.2 contains original work motivated by [CHMR16]. In subsection 3.2.1, we define a new tropical moduli space using graphic stability. We also investigate projections of $\mathcal{M}_{0,n}^{\mathrm{trad}}$ and the reduced Bergman fan $\mathcal{B}'(K_{n-1})$ by forgetting coordinates of rays corresponding unstable curves. We show that the fans coincide with $\mathcal{B}'(\Gamma)$ and relate it to the work of Shaw [Sha13]. In subsection 3.2.2 we investigate an obstruction that stops $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ from being embedded as a balanced fan. Finally, we prove our main result that states $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ and $\mathcal{B}'(\Gamma)$ are isomorphic as balanced fans only when Γ is a complete multipartite graph.

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2 Matroids

A *matroid* is a tuple $M = (E, I)$ where E is a finite set (called the *ground set*) and I is a collection of subsets of E such that (**I1**)–(**I3**) are satisfied.

(**I1**) $\emptyset \in I$

(**I2**) If $X \in I$ and $Y \subseteq X$, then $Y \in I$

(**I3**) If $U, V \in I$ with $|U| = |V| + 1$, then there exists $x \in U \setminus V$ such that $V \cup x \in I$.

The elements of I are called *independent sets* and we thusly call (**I1**), (**I2**), and (**I3**) the *independence axioms*. If a subset of E is not independent, then we call it *dependent*. More commonly (**I3**) is known as the *exchange property*. Other relevant matroid terms include circuit, rank, and flat but in this paper we only use cycle matroids so we define them in that context. For a more thorough introduction see [Oxl06] or [GM12].

We consider the matroid of a finite simple (no loops or multiedges) connected graph $\Gamma = (V, E)$ where V is the ordered vertex set and $E = E(\Gamma)$ is the edge set. We define $e_{ij} \in E$ to be an edge between vertices v_i and v_j. A graph is *complete* if each pair of distinct vertices has an edge between them. The complete graph on n vertices is denoted by K_n. A *clique* is a subgraph that is complete, denoted by K_S, where S is the set of vertices with edges between them. A disjoint union of complete graphs is called a *cluster graph*.

Often called the *cycle matroid*, a matroid defined by a graph is given by $M(\Gamma) = (E(\Gamma), I)$ where I is the collection of all forests of Γ. A *circuit* is a path in which the initial and terminal vertices are the same and no other vertices repeat. The *rank* of a set of edges E' is the number of edges in a spanning forest of $\Gamma_{E'}$, the subgraph induced by E'. Alternatively, the rank of a subgraph $G \subset \Gamma$ is $n - k$, where n is the number of non-isolated vertices in G and k is the number of connected components of among non-isolated vertices of G. An *isolated vertex* is a vertex that is not a part of an edge.

We restrict our attention to $\Gamma = K_n$ to examine flats. A *flat* of $M(K_n)$ is a cluster graph, $\coprod_{j=1}^{k} K_{I_j}$. For a subgraph G of K_n, whose connected components are given by vertex sets $V_1, \ldots, V_k$, the *closure of* G is the flat $\mathrm{cl}(G) = \coprod_{j=1}^{k} K_{V_j}$. The closure of a graph can be seen as *completing* each connected component.

One natural operation on a graph is to delete edges. The cycle matroid respects this operation in the sense that a subgraph induces a submatroid, called the *restriction matroid*. Rather than deleting edges, we may think of restricting the edge set to a subset of edges.

Lemma 1. *(Follows from Property 3.1.2 in [Oxl06]) Let G be a subgraph of K_n and denote $M(K_n) = (E, I)$. Let $I|G = \{X|X \subseteq E(G)$ and $X \in I\}$ be the restriction of forests of K_n to the edge set of G. Then $I|G$ is the set of independent sets of $M(G)$.*

Next we introduce some notation and write a technical lemma used for Proposition 26. The proof is purely graph theoretic so we prove it here.

Definition 2. Let Γ be a subgraph of K_n. The *restriction morphism* $r_\Gamma : M(K_n) \to M(\Gamma)$ is a map of flats given by

$$r_\Gamma(F) := F \cap \Gamma. \tag{1}$$

The previous definition is non-standard but useful for our purposes in Section 3.2. The definition is well-defined by the following discussion. Fix a graph Γ, a subgraph G of Γ, and a subset A of edges of $E(G)$. The forests of G can be obtained by intersecting a forest of Γ with G. Denote the closure operators for $M(\Gamma)$ and $M(G)$ as cl_Γ and cl_G, respectively. They are related by

$$\mathrm{cl}_G(A) = \mathrm{cl}_\Gamma(A) \cap G. \tag{2}$$

Unlike the closure operator, there is no ambiguity between the rank functions on $M(G)$ and $M(\Gamma)$ so we will denote both as $\mathrm{rk}(A)$.

Lemma 3. *Let Γ be a simple graph, not necessarily connected, and let G be a subgraph of Γ. Then $rk(\Gamma) = rk(G)$ if and only if G and Γ share a common spanning forest.*

Proof. The backwards direction follows from the definition of rank so let us assume that G and Γ have the same rank. Let T' be a spanning forest of G. Then there exists T a spanning forest of Γ such that $T \cap G = T'$. By assumption we know that $\mathrm{rk}(T) = \mathrm{rk}(T')$ and therefore they have the same number of edges. Since T' is a subgraph of T, they must be the equal. □

Here we define the complete multipartite graph and discuss some facts about it. A *k-partite graph* (or multipartite graph) is a graph on $n = \sum_{i=1}^{k} n_i$ vertices, partitioned into k sets (called *independent sets*) such that no two vertices from the same set are adjacent. A k-partite graph is a *complete k-partite graph* if each pair of vertices in different sets are adjacent. The complete k-partite graph on $n = \sum_{i=1}^{k} n_i$ vertices is denoted by $K_{n_1,\ldots,n_k}$. Alternatively, we may obtain a complete k-partite graph by removing from K_{n-1} the disjoint cliques on vertices given by the independent sets. Thus, the complement of a complete multipartite graph is a cluster graph.

The following lemma describes some useful characterizations of a complete multipartite graph.

Lemma 4. *Let G be a graph. The following are equivalent:*

1. *G is a complete multipartite graph.*

2. If e_{ij} is an edge of G, then for any vertex v_k, either e_{ik} or e_{jk} is an edge of G.

3. There do not exist 3 vertices whose induced subgraph has exactly 1 edge.

Proof. We can see that all three conditions express that the complement of G is a disjoint union of cliques. □

3 Tropical Moduli Spaces as Bergman Fans

We begin by introducing necessary background terminology on tropical moduli spaces. For a more thorough survey of tropical moduli spaces see [MS15]. Consider the space of genus 0, n-marked abstract tropical curves $\mathcal{M}_{0,n}^{\text{trop}}$. Points of $\mathcal{C} \in \mathcal{M}_{0,n}^{\text{trop}}$ are in bijection with metrized trees with *bounded edges* having finite length and n unbounded labeled edges called *ends*. By forgetting the lengths of the bounded edges of $\mathcal{C}$ we get a tree with labeled ends called the *combinatorial type* of $\mathcal{C}$. The space $\mathcal{M}_{0,n}^{\text{trop}}$ naturally has the structure of a cone complex where curves of a fixed combinatorial type with d bounded edges are parameterized by $\mathbb{R}_{>0}^{d}$. We obtain $\mathcal{M}_{0,n}^{\text{trop}}$ by gluing several copies of $\mathbb{R}_{\geqslant 0}^{n-3}$ via appropriate face morphisms, one for each trivalent combinatorial type.

The space $\mathcal{M}_{0,n}^{\text{trop}}$ may be embedded into a real vector space as a balanced, weighted, pure-dimensional polyhedral fan as in [GKM09]. We briefly recall this construction. A *weighted fan* (X, ω) is a fan X in $\mathbb{R}^n$ where each top-dimensional cone σ has a positive integer weight associated to it, denoted by $\omega(\sigma)$. A weighted fan is *balanced* if for all cones τ of codimension one, the weighted sum of primitive normal vectors of the top-dimensional cones $\sigma_i \supset \tau$ is 0, i.e.,

$$\sum_{\sigma_i \supset \tau} \omega(\sigma_i) \cdot u_{\sigma_i/\tau} = 0 \in V/V_\tau$$

where $u_{\sigma_i/\tau}$ is the primitive normal vector, V is the ambient real vector space, and V_τ is the smallest vector space containing the cone τ. See [GKM09, Construction 2.3] for a construction of the primitive normal vectors $u_{\sigma_i/\tau}$.

For a curve $\mathcal{C}$, define $\operatorname{dist}(i,j)$ as the sum of lengths of all bounded edges between the ends marked by i and j. Then the vector

$$d(\mathcal{C}) = (\operatorname{dist}(i,j))_{i<j} \in \mathbb{R}^{\binom{n}{2}}/\Phi(\mathbb{R}^n) = Q_n$$

identifies $\mathcal{C}$ uniquely, where $\Phi : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{2}}$ by $x \mapsto (x_i + x_j)_{i<j}$.

The combinatorial type of an abstract n-marked tropical curve $\mathcal{C}$ with one bounded edge *splits* the set of ends $[n]$ into $I \sqcup I^c$ where we adopt the convention that $1 \in I^c$. We denote the ray corresponding to $\mathcal{C}$ by $d(\mathcal{C}) = \rho_I = \rho_{I^c}$. In [KM09], Kerber and Markwig prove the relation

$$\sum_{S \in V_1} \rho_S = 0 \in Q_n \tag{3}$$

where $V_1 = \{I \mid 1 \notin I, |I| = 2\}$. They also show that for a subset $I \subset [n] \setminus \{1\}$

$$\sum_{S \in \binom{I}{2}} \rho_S = \rho_I \in Q_n \tag{4}$$

where $\binom{I}{2}$ is the set of all size-2 subsets of a set I.

Remark 5. Equation (3) tells us that any set of $\binom{n-1}{2} - 1$ combinatorial types of curves with one bounded edge and a trivalent vertex not containing the end 1 corresponds to a basis of Q_n. Equation (4) gives us the unique way to write any ray of $\mathcal{M}_{0,n}^{\text{trop}}$ as a linear combination of our basis.

Consider the combinatorial type of a tropical curve $\mathcal{C}$ with d bounded edges. We construct d splits, $I_1, \ldots, I_d$, in the following way. A split I_j is defined by the combinatorial type one obtains by contracting all but the jth bounded edge of $\mathcal{C}$. The cone corresponding to the combinatorial type of $\mathcal{C}$ is the span of rays $\rho_{I_1}, \ldots, \rho_{I_d}$. The combinatorial type of $\mathcal{C}$ may be uniquely reconstructed from the rays $\rho_{I_1}, \ldots, \rho_{I_d}$ (see "Tree Popping" [Ste16, Section 2.4]).

3.1 Tropical moduli spaces of rational stable curves as Bergman fans

Given any matroid M with ground set E we define a polyhedral fan called the *Bergman fan*

$$\mathcal{B}(M) := \{w \in \mathbb{R}^{|E|} \mid M_w \text{ is loop-free}\} \subseteq \mathbb{R}^{|E|},$$

where M_w is the matroid on E whose bases are all bases B of M which have maximal w-weight $\Sigma_{i \in B} w_i$. A *loop* of a matroid is an element whose rank is 0.

A more useful definition for our purposes is from Ardila and Klivans [AK06]. They show that $\mathcal{B}(M)$, as a polyhedral cone complex, coincides with the order complex of the lattice of flats of M. An *order complex* of a poset P is defined to be the simplicial complex whose vertices are the elements of P and whose faces are chains of elements of P. In other words, given a chain of flats $\mathcal{F}$ in M

$$\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq F_{r+1} = E,$$

we let $C_{\mathcal{F}}$ be the cone in $\mathbb{R}^{|E|}$ spanned by the rays $\rho_{F_1}, \ldots, \rho_{F_{r+1}}$, with lineality space spanned by ρ_E. Here $\rho_F = -\Sigma_{e \in F} v_e$, where v_e is a standard basis vector of $\mathbb{R}^{|E|}$.

Any Bergman fan contains the vector $(1, 1, \ldots, 1)$ as a ray. So rather than studying $\mathcal{B}(M)$ we quotient out the lineality space L, spanned by the vector $(1, 1, \ldots, 1)$, to get the reduced Bergman fan

$$\mathcal{B}'(M) := \mathcal{B}(M)/L.$$

Thus we identify a chain of flats $\mathcal{F}$ by its nontrivial flats $F_1, \ldots, F_r$ and set r to be its *length*. Note that a chain of flats of length r corresponds to a cone of dimension r in the Bergman fan. This polyhedral structure is known as the *fine subdivision* of $\mathcal{B}'(M)$.

Note 6. For the remainder of the paper we write Γ to represent both the simple connected graph and the cycle matroid of Γ (replacing $M(\Gamma)$).

It is shown in [AK06, Section 4] and [FR13, Example 7.2] that there is a linear isomorphism between $\mathbb{R}^{|E|}$ and Q_n that identifies the supports of $\mathcal{M}_{0,n}^{\text{trop}}$ and $\mathcal{B}'(K_{n-1})$. Ardila and Klivans and separately Feichtner in [FS05, Remark 3.4] confirm that cone complex of $\mathcal{B}'(K_{n-1})$ is a refinement of $\mathcal{M}_{0,n}^{\text{trop}}$.

We define the *root vertex* of a tropical curve $\mathcal{C}$ to be the vertex containing the end with marking 1, and we denote it $\mathcal{V}_0$. Given a labeling of the non-root vertices of $\mathcal{C}$, $\mathcal{V}_1, \ldots, \mathcal{V}_d$, we define ℓ_i to be the distance from the root vertex to $\mathcal{V}_i$. We set $\ell_0 = 0$.

Definition 7. A *radially aligned tropical curve* $\mathcal{C}$ is a tropical curve with the additional data of a weak ordering on the vertices given by the weak ordering on $\{\ell_i\}_{i=0}^{d}$. Define[1] $\mathcal{M}_{0,n}^{\text{trad}}$ as the moduli space of genus 0, n-marked radially aligned abstract tropical curves. The *radially aligned combinatorial type* is obtained by forgetting the lengths of the bounded edges but keeping the weak ordering on the vertices.

Remark 8. The choice of the name "radially aligned" is motivated by [RSPW19] and is discussed in Example 13. In [RSPW19], there is a natural choice of a "root" (the unique genus 1 component). In our setting, fixing a special marking is influenced algebraically. The algebraic moduli space $\mathcal{M}_{0,n}$ parameterizes n points on $\mathbb{P}^1$. By fixing one of the points to be ∞, $\mathcal{M}_{0,n}$ can be thought of as the complement of a coordinate hyperplane arrangement in $\mathbb{C}^{n-1}$. This is a useful property in many settings in algebraic geometry; see geometric tropicalization [Cue11], [HKT09]. For more details on $\mathcal{M}_{0,n}$ see [KV07].

Remark 9. A weak ordering on the vertices can be viewed as a partition of the vertices into disjoint subsets together with a total ordering on the subsets. Thus the number of cones of $\mathcal{M}_{0,n}^{\text{trad}}$ can be counted using ordered Bell numbers or Fubini numbers (highlighted in Example 11).

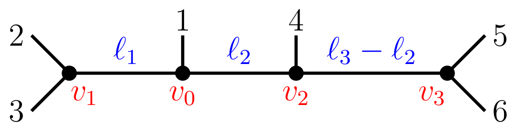
Although the supports of $\mathcal{M}_{0,n}^{\text{trop}}$ and $\mathcal{M}_{0,n}^{\text{trad}}$ in Q_n are the same, as a cone complex $\mathcal{M}_{0,n}^{\text{trad}}$ is different from $\mathcal{M}_{0,n}^{\text{trop}}$. It is a refinement called the *radially aligned subdivision.* The next two examples illustrate particular 3-dimensional cones of $\mathcal{M}_{0,n}^{\text{trop}}$ that become subdivided in the radially aligned subdivision.

Example 10. Consider the combinatorial type $\mathcal{C} \in \mathcal{M}_{0,6}^{\text{trop}}$ with splits $I_1 = \{2,3\},\ I_2 = \{4,5,6\},\ I_3 = \{5,6\}$; see Figure 1a. In $\mathcal{M}_{0,6}^{\text{trop}}$, the combinatorial type of such a curve corresponds to a single 3-dimensional cone with faces consisting of three 2-dimensional cones, and three rays. The 2-dimensional faces correspond to the combinatorial types obtained by shrinking the length of a bounded edge to 0. The rays correspond to contracting two bounded edges. In $\mathcal{M}_{0,6}^{\text{trad}}$, the radially aligned subdivision yields three distinct isomorphism classes, i.e., three 3-dimensional cones. By contracting the various bounded edges, there are seven 2-dimensional cones and five rays; see Figure 1b. The weak orderings are compiled in the 15 strings of inequalities listed below.

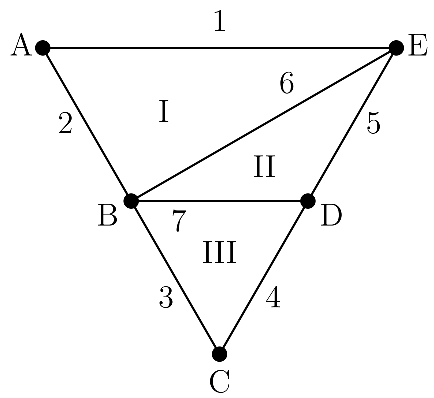
[1] "trad"="tropical radial" as "rad"="radial" in [RSPW19]

	$0 = \ell_1 < \ell_2 < \ell_3$	
$0 = \ell_1 < \ell_2 = \ell_3$	$0 < \ell_1 < \ell_2 = \ell_3$	
$0 < \ell_1 = \ell_2 = \ell_3$	$0 < \ell_2 = \ell_3 < \ell_1$	$0 < \ell_1 < \ell_2 < \ell_3$
$0 = \ell_2 = \ell_3 < \ell_1$	$0 = \ell_2 < \ell_3 < \ell_1$	$0 < \ell_2 < \ell_1 < \ell_3$
$0 = \ell_2 < \ell_1 = \ell_3$	$0 = \ell_2 < \ell_1 < \ell_3$	$0 < \ell_2 < \ell_3 < \ell_1$
$0 = \ell_2 = \ell_1 < \ell_3$	$0 < \ell_2 = \ell_1 < \ell_3$	
	$0 < \ell_2 < \ell_1 = \ell_3$	

The number of strict inequalities is the same as the dimension of the corresponding cone, i.e., the columns, from left to right, correspond to the rays, 2D cones, and 3D cones of $\mathcal{M}_{0,6}^{\text{trad}}$ which are ordered by A-E, 1-7, and (I, II, III), respectively.



(a) Tropical curve of $\mathcal{M}_{0,6}^{\text{trop}}$ with splits $I_1 = \{2,3\},\ I_2 = \{4,5,6\},\ I_3 = \{5,6\}$.



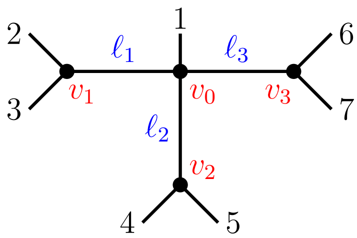
(b) A slice of a cone of $\mathcal{M}_{0,6}^{\text{trad}}$.

Figure 1

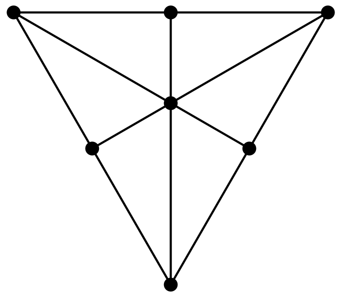
Example 11. Now consider the combinatorial type of a curve $\mathcal{C} \in \mathcal{M}_{0,7}^{\text{trop}}$ with splits $I_1 = \{2,3\},\ I_2 = \{4,5\},\ I_3 = \{6,7\}$; see Figure 2a. Similar to Example 10, in $\mathcal{M}_{0,7}^{\text{trop}}$, this combinatorial type corresponds to a single 3-dimensional cone with faces consisting of three 2-dimensional cones, and three rays. The radially aligned subdivision yields six 3-dimensional cones, twelve 2-dimensional cones, and seven rays; see Figure 2b. If we also consider the 0-dimensional cone which is the intersection of all of these cones there are 26 in total. We may also obtain 26 by doubling the ordered Bell number on a set of three elements. The factor of 2 is due to having a distinguished least element of $\ell_0 = 0$.

Lemma 12. *The cone complexes $\mathcal{M}_{0,n}^{trad}$ and $\mathcal{B}'(K_{n-1})$ are equal ([AK06, Section 4], [FR13, Example 7.2]), in the sense that there is a bijection Ψ between the set of chains of flats of K_{n-1} and the set of radially aligned combinatorial types of $\mathcal{M}_{0,n}^{trad}$ preserving the poset structure.*

Rather than presenting a tedious combinatorial proof of this lemma, we illustrate, in an example, the strategy that is used to construct the necessary explicit bijection.



(a) Tropical curve of $\mathcal{M}_{0,7}^{\mathrm{trop}}$ with splits $I_1 = \{2,3\},\ I_2 = \{4,5\},\ I_3 = \{6,7\}$.



(b) A slice of a cone of $\mathcal{M}_{0,7}^{\mathrm{trad}}$.

Figure 2

Example 13. The radially aligned tropical curve as pictured in $\mathcal{M}_{0,8}^{\mathrm{trad}}$ in Figure 3 corresponds to the following chain of flats of length 3,

$$K_{\{4,5\}} \sqcup K_{\{6,7\}} \subset K_{\{4,5,6,7\}} \sqcup K_{\{2,3\}} \subset K_{\{4,5,6,7\}} \sqcup K_{\{2,3,8\}}.$$

Beginning with a radially aligned tropical curve, consider a circle centered at the root vertex encompassing the tropical curve. We recover the chain of flats by shrinking the circle and recording a new flat each time the circle passes over a vertex of the tropical curve.

Starting with a chain of flats, we recover the radially aligned tropical curve by examining the chain of flats in the descending direction. Each time a label disappears or a flat splits up, we add structure to the tropical curve. For instance, having two components in the 3rd flat means that there are two bounded edges emanating from the root vertex.

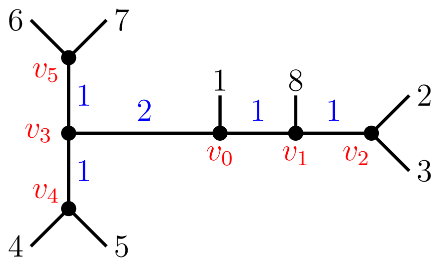


Figure 3: Tropical curve in $\mathcal{M}_{0,8}^{\mathrm{trad}}$.

Up to here we only compared the cone complex structures of $\mathcal{M}_{0,n}^{\mathrm{trad}}$ and $\mathcal{B}'(K_{n-1})$. Now we look at their fan structures. We say that two fans are *equivalent* if they share a common refinement. The bijection from Lemma 12 induces a linear isomorphism of vector spaces, also denoted by Ψ,

$$\begin{tikzcd} \mathcal{M}^{\text{trad}}_{0,n} \arrow[r, "="', "\Psi"'] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathcal{B}'(K_{n-1}) \arrow[d, hook] \\ Q_n \arrow[r, "\cong", "\Psi"'] & \mathbb{R}^{|E(K_{n-1})|}/L \end{tikzcd} \tag{5}$$

which preserves the cone complex structures of $\mathcal{M}^{\text{trad}}_{0,n}$ and $\mathcal{B}'(K_{n-1})$, and hence induces an equivalency of fans.

Example 14. It is known that the cone complex of $\mathcal{M}^{\text{trop}}_{0,5}$ is given by the cone over the Petersen graph. The Bergman fan $B'(K_4)$ is the refinement of that cone complex where 3 2D cones are subdivided; see Figure 4a. Label the lattice of flats of K_4 in the following way:

$$\begin{aligned} &\text{Rank } 1: \; F_1 = \{e_{23}\},\; F_2 = \{e_{24}\},\; F_3 = \{e_{34}\},\; F_4 = \{e_{35}\},\; F_5 = \{e_{45}\},\; F_6 = \{e_{25}\} \\ &\text{Rank 2 connected}: \; F_7 = \{e_{23}, e_{24}, e_{34}\},\; F_8 = \{e_{23}, e_{35}, e_{25}\},\; F_9 = \{e_{34}, e_{35}, e_{45}\}, \\ &\qquad\qquad\qquad\quad\; F_{10} = \{e_{24}, e_{45}, e_{25}\} \\ &\text{Rank 2 disconnected}: \; F_{11} = \{e_{23}, e_{45}\},\; F_{12} = \{e_{24}, e_{35}\},\; F_{13} = \{e_{25}, e_{34}\}. \end{aligned}$$

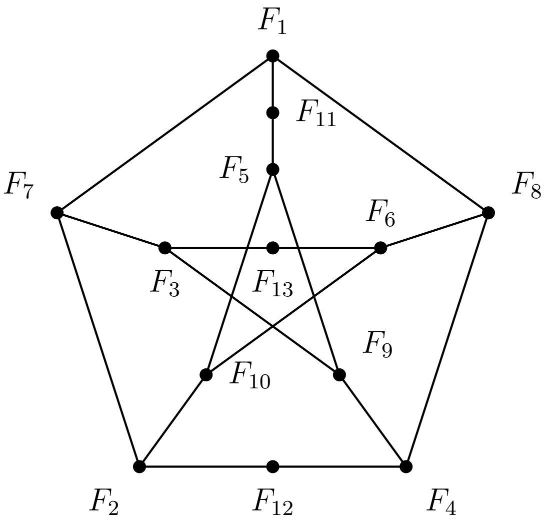
See Figure 5 for a visual representation of the flats F_1, F_7, and F_{11}.

The cone complexes can be embedded into a five dimensional real vector space. Let $\vec{e}_i$ be the standard basis vectors of $\mathbb{R}^5$. Given our labeling above and the discussion in Remark 5, we may choose which rank one flats form can be assigned to the standard basis vectors of $\mathbb{R}^5$. Hence, we write the rays associated to the connected flats of the Bergman fan $B'(K_4)$ as

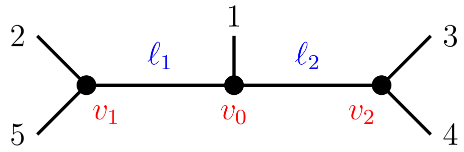
$$\begin{array}{lll} \vec{v}_{F_2} = \vec{v}_{\{2,4\}} = \vec{e}_1 & \vec{v}_{F_4} = \vec{v}_{\{3,5\}} = \vec{e}_4 & \vec{v}_{F_7} = \vec{v}_{\{2,3,4\}} = (0,-1,0,-1,-1) \\ \vec{v}_{F_6} = \vec{v}_{\{2,5\}} = \vec{e}_2 & \vec{v}_{F_5} = \vec{v}_{\{4,5\}} = \vec{e}_5 & \vec{v}_{F_8} = \vec{v}_{\{2,3,5\}} = (-1,0,-1,0,-1) \\ \vec{v}_{F_3} = \vec{v}_{\{3,4\}} = \vec{e}_3 & \vec{v}_{F_1} = \vec{v}_{\{2,3\}} = -\vec{1} & \vec{v}_{F_{10}} = \vec{v}_{\{2,4,5\}} = (1,1,0,0,1) \\ & & \vec{v}_{F_9} = \vec{v}_{\{3,4,5\}} = (0,0,1,1,1) \end{array}$$

Consider the top-dimensional cone σ in $\mathcal{M}^{\text{trop}}_{0,5}$ corresponding to a combinatorial type that has a root vertex $\mathcal{V}_0$ with two bounded edges and adjacent vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ with ends marked by $I_1 = \{2,5\}$ and $I_2 = \{3,4\}$. An abstract tropical curve $\mathcal{C}$ with this combinatorial type has edge lengths $\ell_1, \ell_2 \in \mathbb{R}^+$; see Figure 4b. Embedded in $Q_5 \cong \mathbb{R}^5$, σ is a two-dimensional cone spanned by $\vec{e}_2$ and $\vec{e}_3$.

In $B'(K_4)$, and therefore $\mathcal{M}^{\text{trad}}_{0,5}$, we see that this cone is subdivided into $\sigma_1 = \text{cone}(\rho_{F_6}, \rho_{F_{13}})$ and $\sigma_2 = \text{cone}(\rho_{F_{13}}, \rho_{F_3})$ with their intersection being a ray $\rho = \rho_{F_{13}}$. The ray ρ corresponds to $\mathcal{C}$ where $\ell_1 = \ell_2$ and σ_i is the cone corresponding to the abstract tropical curve $\mathcal{C}$ where $\ell_i > \ell_j$.

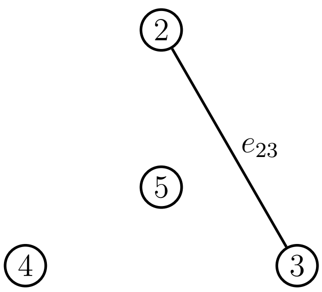


(a) A slice of the Bergman fan $\mathcal{B}'(K_4)$ with rays labeled by their corresponding flats.

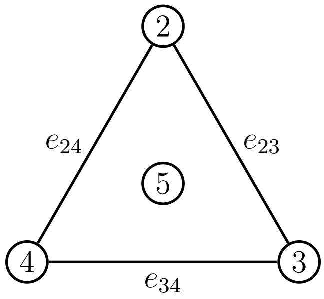


(b) A tropical curve of $\mathcal{M}_{0,5}^{\text{trop}}$ with splits $I_1 = \{2,5\}$ and $I_2 = \{3,4\}$.

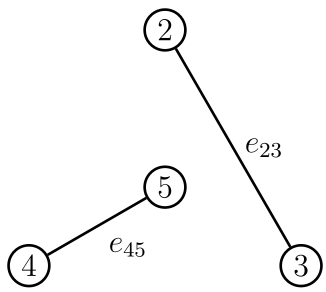
Figure 4



(a) Flat F_1.



(b) Flat F_7.



(c) Flat F_{11}.

Figure 5: Some flats of K_4.

3.2 Moduli spaces of rational graphically stable tropical curves as Bergman fans

The main point of Section 3.2.1 is to define Γ-stability and set up Theorem 29. We note that Lemma 22 and Lemma 23 are corollaries of Shaw's Proposition 2.22 in [Sha13] but they are useful in building up the context in this paper.

3.2.1 A projection of $\mathcal{M}_{0,n}^{\text{trad}}$ is isomorphic to $\mathcal{B}'(\Gamma)$

Motivated by [CHMR16] we define the space of graphically stable tropical curves and investigate its ability to be embedded as a balanced fan. In particular, we explore the relationship between $\mathcal{M}_{0,\Gamma}^{\text{trop}}$, $\mathcal{M}_{0,\Gamma}^{\text{trad}}$, and $\mathcal{B}'(\Gamma)$. Let Γ be a simple connected graph whose vertices are labeled $2, \ldots, n$.

Definition 15. A stable rational tropical curve $\mathcal{C}$ with n ends is Γ-*stable*, if at each vertex in $\mathcal{C}$ with exactly 1 bounded edge (and thus has two ends whose labels are i and j), there exists an edge $e_{ij} \in E(\Gamma)$. Define $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ to be the moduli space of all rational n-marked Γ-stable abstract tropical curves. Similarly, we define $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ to be the moduli space of rational n-marked Γ-stable radially aligned abstract tropical curves.

Just as in the stable case, $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ is a refinement of $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ for each Γ. These spaces are well-defined as cone complexes but do not necessarily admit an embedding into a vector space and a weight function making them balanced fans.

Definition 16. We define the *contraction morphism*

$$c_\Gamma : \mathcal{M}_{0,n}^{\text{trop}} \longrightarrow \mathcal{M}_{0,\Gamma}^{\text{trop}} \tag{6}$$

which successively contracts bounded edges adjacent to Γ-unstable vertices.

The previous definition is a well-defined map of sets (of curves and cone complexes) but typically is not induced by the restriction of a vector space homomorphism. We also define a section ι to be the natural inclusion map in the opposite direction by recalling that a Γ-stable curve is necessarily stable.

The next three examples show that the collection of moduli spaces of weighted stable rational tropical curves and the collection of moduli spaces of Γ-stable rational tropical curves have an intersection but that neither is contained in the other. For the reader that is not familiar with weighted spaces, see [Has03] for the construction of the algebraic moduli spaces and [CHMR16] for tropical moduli spaces.

Example 17. Let Γ be a path on three vertices. Then $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ is exactly the tropical moduli space of weighted stable rational tropical curves $\mathcal{M}_{0,\mathcal{A}}^{\text{trop}}$ with weight data $\mathcal{A} = (1, 1, 1/2, 1/2)$. The fan associated to this moduli space lives in $\mathbb{R}$ and contains a node at the origin and two rays pointing in opposite directions.

Example 18. Let Γ be the complete bipartite graph obtained from K_4 by removing two disjoint edges. In this case, $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ is not isomorphic to a tropical moduli space with weighted points.

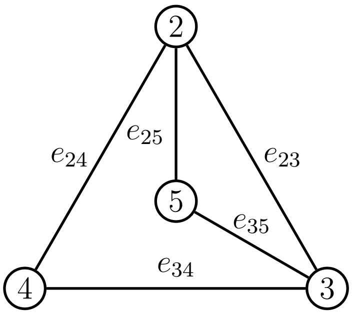
Example 19. Consider the weight data $\mathcal{A} = (1, 1, 1/2, 1/2, 1/2)$. We wish to find a graph with an equivalent weight data to $\mathcal{A}$. The graph obtained by deleting a single edge, e_{45}, has a weight data equivalent to $(1, 1, 1, 1/2, 1/2)$. The graph obtained by deleting a second edge, e_{35}, has a weight data equivalent to $(1, 1, 2/3, 2/3, 1/3)$. The graph Γ obtained by deleting a third edge, e_{34}, has a weight data equivalent to $(1, 1, \varepsilon, \varepsilon, \varepsilon)$. We can see the difference by looking at a curve with split $I = \{3, 4, 5\}$. The curve is $\mathcal{A}$-stable but not Γ-stable.

To relate the theory of Bergman fans to the graphically stable moduli spaces we define Γ-stability of flats and chains of flats of K_{n-1}.

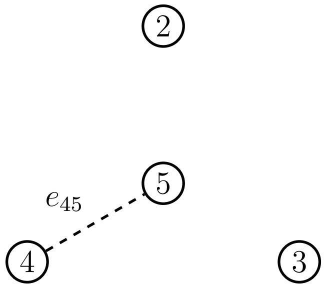
Definition 20. A flat F of K_{n-1} is Γ-*stable* if the associated combinatorial type of the radially aligned tropical curve is Γ-stable (see Lemma 12). A chain of flats is Γ-*stable* if each of its flats is Γ-stable.

Recall that a flat of Γ can be thought of as a flat of K_{n-1} restricted to the edge set of Γ, i.e., a flat of Γ is $F \cap \Gamma$ where F is a flat of K_{n-1}. Consider a combinatorial type $\mathcal{C}$ with splits $I_1, \ldots, I_d$ and its corresponding cluster graph $F_{\mathcal{C}} = \coprod_{j=1}^{d} K_{I_j}$. Then $\mathcal{C}$ and $F_{\mathcal{C}}$ are Γ-unstable if and only if there is a split I_j such that $K_{I_j} \cap \Gamma$ has no edges. In other words, Γ-unstability is caused by deleting entire cliques from K_{n-1}. This is illustrated in the following example.

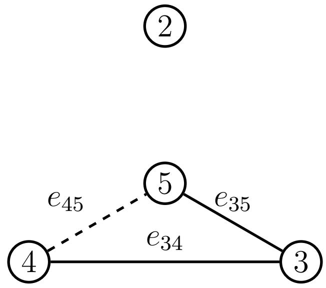
Example 21. Let Γ be the graph obtained from K_4 by removing edge e_{45}; see Figure 6a. Using the labeling from Example 14, the flat $F_5 = \{e_{45}\}$ is not Γ-stable but the flat $F_9 = \{e_{34}, e_{35}, e_{45}\}$ is Γ-stable. In Figure 6, the dashed edges are treated as normal edges or missing edges depending on whether we view F_5 and F_9 as flats of K_4 or Γ, respectively.



(a) Graph Γ.



(b) Flat F_5, not Γ-stable.



(c) Flat F_9, Γ-stable.

Figure 6: Figure for Example 21.

Now consider the map

$$\mathrm{pr}_\Gamma : \mathbb{R}^{|E(K_{n-1})|}/L \longrightarrow \mathbb{R}^{|E(K_{n-1})|}/(L+S) \tag{7}$$

where L is the lineality space spanned by the vector $(1,1,\ldots,1)$ and $S = \mathrm{span}\{v_e \mid e \notin \Gamma\}$ is the span of basis vectors corresponding to edges not in Γ. Note that pr_Γ is the natural projection map that forgets the coordinates corresponding to edges that are not in Γ.

Simultaneously, define

$$\widetilde{\mathrm{pr}}_\Gamma : Q_n \longrightarrow Q_n/U \tag{8}$$

where U is the linear span of Γ-unstable rays of $\mathcal{M}_{0,n}^{\mathrm{trop}}$. As described in Remark 5, a set of $\binom{n-1}{2} - 1$ combinatorial types of curves with splits I of size 2 correspond to a basis of Q_n. We also know that a split of size 2 corresponds to an edge of Γ. Thus U is generated by combinatorial types of curves corresponding to the edges removed from K_{n-1} when building Γ.

Lemma 22. *There is a linear isomorphism $\Psi' : Q_n/U \to \mathbb{R}^{|E(K_{n-1})|}/(L+S)$ that matches the supports of the underlying cone complexes of $\widetilde{pr}_\Gamma(\mathcal{M}_{0,n}^{trad})$, $\widetilde{pr}_\Gamma(\mathcal{M}_{0,n}^{trop})$, and $pr_\Gamma(\mathcal{B}'(K_{n-1}))$. Furthermore, via Ψ' $\widetilde{pr}_\Gamma(\mathcal{M}_{0,n}^{trad}) \cong pr_\Gamma(\mathcal{B}'(K_{n-1}))$ as fans.*

Proof. From the discussion leading up to this lemma the diagram in Figure 7 is commutative because both projections forget the coordinates corresponding to edges that are not in Γ. Since Ψ preserves the cone complex structures, we obtain an isomorphism Ψ' that also preserves the cone complex structures. $\square$

$$\begin{tikzcd} Q_n \arrow[r, "\cong", "\Psi"'] \arrow[d, "\widetilde{\mathrm{pr}}_\Gamma"] \arrow[dr, phantom, "\square"] & \mathbb{R}^{|E(K_{n-1})|}/L \arrow[d, "\mathrm{pr}_\Gamma"] \\ Q_n/U \arrow[r, "\cong", "\Psi'"'] & \mathbb{R}^{|E(K_{n-1})|}/(L+S) \end{tikzcd}$$

Figure 7: Diagram for Lemma 22.

Note that the support of $\mathcal{B}'(\Gamma)$ is a subset of $\mathbb{R}^{|E(\Gamma)|}/L$, where L is the lineality space spanned by the all ones vector. It is a straightforward computation to see that the dimension of $\mathbb{R}^{|E(K_{n-1})|}/(L+S)$ is the same as the dimension of $\mathbb{R}^{|E(\Gamma)|}/L$. There is a natural isomorphism between these two vector spaces given by underlying the matroidal structure of K_{n-1} and Γ. In other words, the standard basis vectors of each vector space can be given by a choice of all but one of the edges in Γ.

Lemma 23. *Let Γ be a connected graph on $n-1$ vertices. Then $\widetilde{pr}_\Gamma(\mathcal{M}_{0,\Gamma}^{trad}) \cong \mathcal{B}'(\Gamma)$ as balanced fans with constant weight function* 1.

Proof. This is an immediate corollary to [Sha13, Proposition 2.22]. $\square$

3.2.2 $\mathcal{B}'(\Gamma)$ equals $\mathcal{M}_{0,\Gamma}^{\mathbf{trad}}$ for Γ complete multipartite

In general, the cone complex structures of $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ and $\mathcal{B}'(\Gamma)$ do not coincide. Both $\mathcal{M}_{0,\Gamma}^{\mathrm{trop}}$ and $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ are well-defined as cone complexes but may not be embedded into Q_n/U as balanced fans. Geometrically, it may happen that these cone complexes contain cones which are adjacent to only one maximal cone, and thus their image cannot be balanced. We will characterize the obstruction by studying the relationship between the contraction morphism c_Γ (Equation 6) and the restriction morphism r_Γ (Equation 1).

Consider the locus in $\mathcal{M}_{0,n}^{\mathrm{trad}}$ of Γ-stable curves given by the section ι. Define $\Psi_\Gamma := r_\Gamma \circ (\Psi \circ \iota)$ to be the map between the set of Γ-stable radially aligned combinatorial types and the set of chains of flats of Γ where Ψ is the bijection from Lemma 12. The map of sets Ψ_Γ induces a map (denoted by the same name) of cone complexes between $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ and $\mathcal{B}'(\Gamma)$. Additionally, the map of cone complexes induced by r_Γ is the same map given by pr_Γ, so we have the following diagram.

$$\begin{tikzcd}
\mathcal{M}^{\text{trad}}_{0,n} \arrow[r, "="', "\Psi"'] \arrow[d, bend right, "c_\Gamma"'] & \mathcal{B}'(K_{n-1}) \arrow[d, "\text{pr}_\Gamma"] \\
\mathcal{M}^{\text{trad}}_{0,\Gamma} \arrow[u, bend right, "\iota"'] \arrow[r, dashed, "\Psi_\Gamma"'] & \mathcal{B}'(\Gamma)
\end{tikzcd}$$

Note that $\Psi \circ \iota$ is a bijection between the set of Γ-stable radially aligned combinatorial types and the set of Γ-stable chains of flats of K_{n-1}. Hence statements about Ψ_Γ are equivalent to statements about r_Γ restricted to Γ-stable flats.

By showing Ψ_Γ is a bijective map between the set of Γ-stable radially aligned combinatorial types and the set of chains of flats of Γ, we obtain an induced bijection of the cone complexes of $\mathcal{M}^{\text{trad}}_{0,\Gamma}$ and $\mathcal{B}'(\Gamma)$, preserving the poset structure. The next lemma shows that surjectivity of this map follows from the fact that flats of Γ are flats of K_{n-1} restricted to the edge set of Γ.

Lemma 24. *The map Ψ_Γ is surjective.*

Proof. Consider the chain of flats $\mathcal{F}$ of Γ given by $F_1 \subset \cdots \subset F_r$ where F_i has k_i connected components. Write the vertex set of each connected component of F_i as I^i_j. Construct the chain of flats $\mathcal{G}$ of K_{n-1} as

$$\mathcal{G} : \coprod_{j=1}^{k_1} K_{I^1_j} \subset \cdots \subset \coprod_{j=1}^{k_r} K_{I^r_j}.$$

Then we have $\text{pr}_\Gamma(\mathcal{G}) = \mathcal{F}$, and thus Ψ_Γ is surjective. □

The map Ψ_Γ is not always injective because flats of Γ are not always uniquely determined by Γ-stable flats. The obstruction is highlighted in the following example.

Example 25. Let Γ be the subgraph of K_4 with edges e_{35} and e_{45} removed; see Figure 8. In $\mathcal{M}^{\text{trop}}_{0,\Gamma}$, there are now eight combinatorial types with one bounded edge and nine combinatorial types with two bounded edges that are Γ-stable. This means that $\mathcal{M}^{\text{trop}}_{0,\Gamma}$, as a cone complex, has eight rays and nine 2-dimensional cones and $\mathcal{M}^{\text{trad}}_{0,\Gamma}$ has nine rays and ten 2-dimensional cones, as described in Example 14.

It is important to note that as cone complexes $\mathcal{B}'(\Gamma)$ is not equal to $\mathcal{M}^{\text{trop}}_{0,\Gamma}$ nor $\mathcal{M}^{\text{trad}}_{0,\Gamma}$; see Figures 9a and 9b. The obstruction lies in the ray $\rho = \rho_{\{3,4,5\}}$ and the cone $\sigma = \text{cone}(\rho_{\{3,4,5\}}, \rho_{\{3,4\}})$. Let $\mathcal{C}_\rho$ and $\mathcal{C}_\sigma$ be their corresponding combinatorial types. Geometrically, ρ is adjacent to only one Γ-stable maximal cell, meaning it is impossible to embed ρ and σ into any vector space as a balanced fan.

Write the lattice of flats of Γ with the same labels as in Example 14:

Rank 1 : $F_1 = \{e_{23}\},\ F_2 = \{e_{24}\},\ F_3 = \{e_{34}\},\ F_6 = \{e_{25}\}$
Rank 2 connected : $F_7 = \{e_{23}, e_{24}, e_{34}\},\ F_8 = \{e_{23}, e_{25}\},\ F_{10} = \{e_{24}, e_{25}\}$
Rank 2 disconnected : $F_{13} = \{e_{25}, e_{34}\}$

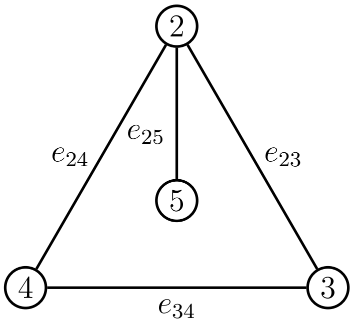


Figure 8: The graph Γ in Example 25.

The flat corresponding to $\rho_{\{3,4,5\}}$ in K_4, i.e., $(\Psi \circ \iota)(\mathcal{C}_\rho)$, is $K_{\{3,4,5\}} = F_9$. When restricting the edge set to $E(\Gamma)$ we have

$$K_{\{3,4,5\}} \cap \Gamma = K_{\{3,4\}} = F_3.$$

Similarly, $(\Psi \circ \iota)(\mathcal{C}_\sigma)$ is the chain of flats $K_{\{3,4\}} \subset K_{\{3,4,5\}}$, and this chain of flats reduces to the single flat $K_{\{3,4\}}$ when restricting the edge set. That is to say, there are three radially aligned combinatorial types of $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ whose cones all coincide in $\mathcal{B}'(\Gamma)$, namely

$$\Psi_\Gamma(\mathcal{C}_\rho) = \Psi_\Gamma(\mathcal{C}_\sigma) = \Psi_\Gamma(\mathcal{C}_{\rho_{\{3,4\}}}) = \rho_{F_3}.$$

The map Ψ_Γ sends the cone complex depicted in Figure 9a to the one in Figure 9b. Here we can see that the cone σ and ray ρ in $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ gets collapsed in $\mathcal{B}'(\Gamma)$ by Ψ_Γ.

For convenience, each of the rays in $\mathcal{B}'(\Gamma)$ are written coordinate-wise.

$$\begin{array}{lll}
\vec{v}_{F_2} = \vec{v}_{\{2,4\}} = (1,0,0) & \vec{v}_{F_6} = \vec{v}_{\{2,5\}} = (0,1,0) & \vec{v}_{F_3} = \vec{v}_{\{3,4\}} = (0,0,1) \\
\vec{v}_{F_1} = \vec{v}_{\{2,3\}} = (-1,-1,-1) & \vec{v}_{F_7} = \vec{v}_{\{2,3,4\}} = (0,-1,0) & \vec{v}_{F_8} = \vec{v}_{\{2,3,5\}} = (-1,0,-1) \\
\vec{v}_{F_{10}} = \vec{v}_{\{2,4,5\}} = (1,1,0) & \vec{v}_{F_9} = \vec{v}_{\{3,4,5\}} = (0,0,1) & \vec{v}_{F_{13}} = \vec{v}_{\{2,5\},\{3,4\}} = (0,1,1)
\end{array}$$

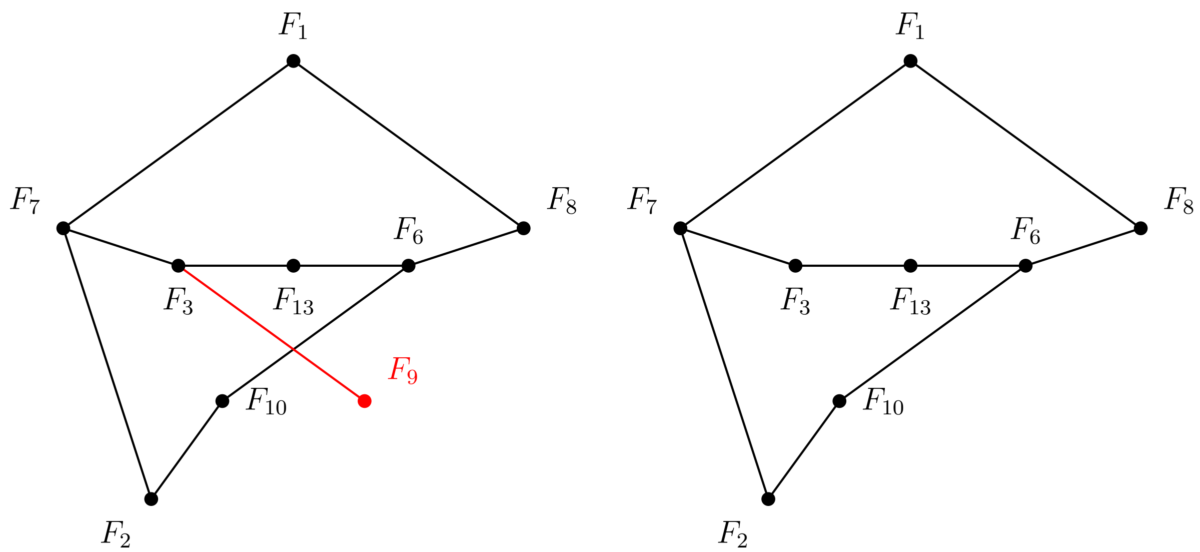
We see in Example 25 that r_Γ is not injective on flats which dropped in rank. It follows that the obstruction is Γ containing a K_3 subgraph which had two of its three edges deleted. So in order for r_Γ to be injective we can only allow a graph Γ if it has the property that if two edges of a K_3 subgraph are deleted, then the third edge must also be deleted (part (3) of Lemma 4).

The following series of results build up to the central result of the paper, namely Theorem 29.

Proposition 26. *The map r_Γ is injective on Γ-stable flats if and only if for any Γ-stable flat F, $rk(F) = rk(r_\Gamma(F))$.*

Proof. First assume that r_Γ is injective. Let F be a Γ-stable flat of K_{n-1} and let T be a spanning forest of $r_\Gamma(F) = F \cap \Gamma$. By way of contradiction, suppose that $\text{rk}(F \cap \Gamma) < \text{rk}(F)$. Consider $\text{cl}_{K_{n-1}}(T)$ as a flat of K_{n-1}. Then we have

$$r_\Gamma(F) = F \cap \Gamma = \text{cl}_\Gamma(T) = \text{cl}_{K_{n-1}}(T) \cap \Gamma = r_\Gamma(\text{cl}_{K_{n-1}}(T)).$$



(a) A slice of the cone complex of $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ with rays labeled by their corresponding flats.

(b) A slice of the cone complex of $\mathcal{B}'(\Gamma)$ with rays labeled by their corresponding flats.

Figure 9

But since $\text{rk}(\text{cl}_{K_{n-1}}(T)) = \text{rk}(T) < \text{rk}(F)$, $\text{cl}_{K_{n-1}}(T) \neq F$. This contradicts the injectivity of r_Γ on Γ-stable flats.

Now we prove the backwards direction. Suppose that for any Γ-stable flat F, $\text{rk}(F) = \text{rk}(r_\Gamma(F))$. Let F and G be Γ-stable flats of K_{n-1} with $r_\Gamma(F) = r_\Gamma(G)$. By our hypothesis, we deduce that $\text{rk}(F) = \text{rk}(G)$. By Lemma 3, F and G share a spanning forest; call it T. Then by definition, $\text{cl}_{K_{n-1}}(T) = F$ and $\text{cl}_{K_{n-1}}(T) = G$. This proves r_Γ is injective, completing the proof. $\square$

Lemma 27. *Suppose C is a clique of K_{n-1} and that Γ is a complete multipartite graph labeled by the same $n-1$ vertices. Then $rk(C) = rk(r_\Gamma(C))$ or $rk(r_\Gamma(C)) = 0$.*

Proof. Suppose that $\text{rk}(r_\Gamma(C)) \neq 0$, i.e., $r_\Gamma(C)$ is not the empty graph. Fix an edge e_{ij} of $\Gamma \cap C$ between vertices v_i and v_j of C. By Lemma 4, for any other vertex v_k of C, either e_{ik} or e_{jk} exists in Γ. So there is a path between any two vertices of $r_\Gamma(C)$ going through the edge e_{ij}. This means that $r_\Gamma(C)$ is connected and any spanning tree contains all $n-1$ vertices, proving the lemma. $\square$

Lemma 28. *The map r_Γ is injective on Γ-stable flats if and only if Γ is a complete multipartite graph.*

Proof. First we prove the backwards direction. Let F be a Γ-stable flat of K_{n-1}. Note that F is a disjoint union of cliques, C_i. By assumption, the image of each clique under

r_Γ is not empty. Using Lemma 27, we have

$$\mathrm{rk}(F) = \sum_{i=1}^{k} \mathrm{rk}(C_i) = \sum_{i=1}^{k} \mathrm{rk}(C_i \cap \Gamma) = \mathrm{rk}(F \cap \Gamma) = \mathrm{rk}(r_\Gamma(F)).$$

By Proposition 26, r_Γ is injective on Γ-stable flats.

Now suppose that r_Γ is injective on Γ-stable flats. It is enough that Γ satisfies (3) from Lemma 4. Let v_i and v_j be vertices of Γ such that e_{ij} is an edge of Γ. Fix another vertex v_k. Consider the flat $F = K_{\{v_i,v_j,v_k\}}$. We know that

$$K_{\{v_i,v_j\}} \subset r_\Gamma(F).$$

Since $\mathrm{rk}(r_\Gamma(F)) = \mathrm{rk}(F) = 2$, either e_{ik} or e_{jk} must exist as edges in Γ. □

Lemma 27 and Lemma 28 show that when Γ is a complete multipartite graph, the cone complex $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ will not contain a ray which is adjacent to only one maximal cell. Hence the tropical moduli space $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ can be embedded as a balanced fan into a real vector space because of the structure it shares with $\mathcal{B}'(\Gamma)$.

Theorem 29. *Via the embedding induced by $\widetilde{pr}_\Gamma \circ$ dist and the constant weight function 1, the cone complex underlying $\mathcal{M}_{0,\Gamma}^{trad}$ has the structure of a balanced fan if and only if Γ is a complete multipartite graph. The linear isomorphism Ψ' identifies it with $\mathcal{B}'(\Gamma)$.*

Proof. By Lemma 28 and Proposition 26, r_Γ induces a bijection between the set of Γ-stable flats of K_{n-1} and flats of Γ only when Γ is a complete multipartite graph. Only in this case is Ψ_Γ a bijection between the set of Γ-stable radially aligned combinatorial types of $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ and the set of flats of Γ. Thus the map Ψ_Γ induces an bijection of cone complexes preserving the poset structure. We finish the proof by noting that $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ is a balanced fan with constant weight function 1 by the fact that it has the same structure as the equivalent balanced fans $\mathrm{pr}_\Gamma(\mathcal{M}_{0,n}^{\mathrm{rad}})$ and $\mathcal{B}'(\Gamma)$. □

References

[AK06] Federico Ardila and Caroline J Klivans. The bergman complex of a matroid and phylogenetic trees. *Journal of Combinatorial Theory, Series B*, 96(1):38–49, 2006.

[CHMR16] Renzo Cavalieri, Simon Hampe, Hannah Markwig, and Dhruv Ranganathan. Moduli spaces of rational weighted stable curves and tropical geometry. In *Forum of Mathematics, Sigma*, volume 4. Cambridge University Press, 2016.

[Cue11] Maria Angelica Cueto. Implicitization of surfaces via geometric tropicalization. arXiv:1105.0509, 2011.

[FR13] Georges François and Johannes Rau. The diagonal of tropical matroid varieties and cycle intersections. *Collectanea Mathematica*, 64(2):185–210, 2013.

[FS05] Eva Maria Feichtner and Bernd Sturmfels. Matroid polytopes, nested sets and bergman fans. *Portugaliae Mathematica*, 62(4):437–468, 2005.

[GKM09] Andreas Gathmann, Michael Kerber, and Hannah Markwig. Tropical fans and the moduli spaces of tropical curves. *Compositio Mathematica*, 145(1):173–195, 2009.

[GM11] Angela Gibney and Diane Maclagan. Equations for chow and hilbert quotients. *Algebra & Number Theory*, 4(7):855–885, 2011.

[GM12] Gary Gordon and Jennifer McNulty. *Matroids: a geometric introduction.* Cambridge University Press, 2012.

[Has03] Brendan Hassett. Moduli spaces of weighted pointed stable curves. *Advances in Mathematics*, 173(2):316–352, 2003.

[HKT09] Paul Hacking, Sean Keel, and Jenia Tevelev. Stable pair, tropical, and log canonical compactifications of moduli spaces of del pezzo surfaces. *Inventiones mathematicae*, 178(1):173–227, 2009.

[KM09] Michael Kerber and Hannah Markwig. Intersecting psi-classes on tropical $\mathcal{M}_{0,n}$. *International Mathematics Research Notices*, 2009(2):221–240, 2009.

[KV07] Joachim Kock and Israel Vainsencher. *An invitation to quantum cohomology: Kontsevich's formula for rational plane curves*, volume 249. Springer Science & Business Media, 2007.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161. American Mathematical Soc., 2015.

[Oxl06] James G Oxley. *Matroid theory*, volume 3. Oxford University Press, USA, 2006.

[RSPW19] Dhruv Ranganathan, Keli Santos-Parker, and Jonathan Wise. Moduli of stable maps in genus one and logarithmic geometry, i. *Geometry & Topology*, 23(7):3315–3366, 2019.

[Sha13] Kristin M Shaw. A tropical intersection product in matroidal fans. *SIAM Journal on Discrete Mathematics*, 27(1):459–491, 2013.

[SS04] David Speyer and Bernd Sturmfels. The tropical grassmannian. *Advances in Geometry*, 4(3):389–411, 2004.

[Ste16] Mike Steel. *Phylogeny: discrete and random processes in evolution.* SIAM, 2016.

[Tev07] Jenia Tevelev. Compactifications of subvarieties of tori. *American Journal of Mathematics*, 129(4):1087–1104, 2007.